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A Possible Symmetry in Sakata's Model for Bosons-Baryons System. III

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In the full symmetry theory a physical system can be characterized by six quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. Now, following the original intention of the composite model, we try to replace them by other quantum numbers that are easier to understand by intuition. This study leads us to a new picture of particle which presents a striking contrast to the usual one in the perturbation theoretical treatment. We hope that the realistic aspects developed in the present paper not only help a deeper understanding of the full symmetry theory of the composite model but also throw a new light on the structure of elementary particles.

§ 1. Introduction

On the basis of the full symmetry theory, there have been made some studies[1)~4)] of Sakata's composite model of baryons and mesons.[5)] It is expected that some deviation from the full symmetry will arise in nature. Nevertheless, in view of the results so far achieved, our theory seems to have considerable implications for understanding new aspects of the particle physics. In the theory of full symmetry the whole scheme is mathematically expressed as the invariance of the theory under the unitary group $U(3)$ of degree three. Then a physical state realized in nature may be expressed by a vector of an irreducible representation space of $U(3)$ and be specified by six quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. The first three of these numbers define the representation space and the next three indicate the representative vector.

Now, turning our attention back to the original intention of the composite model,[5)] we are informed that physical quantities should be understood on a realistic ground. In fact, Sakata has shown that the strangeness quantum number S is comprehensible in terms of the number n_Λ of Λ-particles.†† Of the above six

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†† To avoid the confusion of terminology we shall use "particle" or "antiparticle" when the discrimination is necessary between a positive and a negative (hole) state. The number of baryons (nucleons, protons, etc.) should be understood in the usual way, that is, as the number of particles minus that of antiparticles.

quantum numbers, n_B, S and I_3 can be given an intuitive meaning in a similar fashion: n_B is obvious, $S=-n_\Lambda$ and $I_3=(n_p-n_n)/2$, where n_p and n_n are the number of protons and of neutrons respectively. On the other hand, we have never gained any intuitive meaning of the remaining three (s_0, i_0, I), although there is no room for doubt as to their mathematical role. Thus, to fit the purpose of the composite model, it is desirable to reexamine these three quantum numbers from a realistic point of view.

It is the first aim of this paper to show that (s_0, i_0, I), can be consistently replaced by other quantum numbers for which the physical meaning is intuitively comprehensible on the ground of composite particle model. We hope that this will give a fresh turn to a further development of our theory.

Now we shall explain our line of thought in brief. As already stated,[4] in the theory of full symmetry a "physical system" is represented by the totality of configurations corresponding to the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. The configuration realized by a system should be dependent on dynamical conditions imposed on the system, though we have little knowledge of such dynamics at present. But, as far as the kinematical character is concerned, all the configurations with the same quantum numbers are on an equal footing. Of these configurations there exists the "lowest", which is composed of the minimum number of basic particles and that of basic antiparticles. The sum of these two numbers, the "*total particle number*" n_T, plays an important role in our following discussions. What is important is the symmetry character in which particles and antiparticles are combined to form the lowest configuration. The following are the numbers which express such a symmetry character: the "*number of alternating pairs among particles*"[3] $\hat{n}_A$, the "*number of alternating pairs among antiparticles*" $\check{n}_A$ and the "*number of alternating triplets*" n_A among particles and among antiparticles. Only two of n_T, $\hat{n}_A$, $\check{n}_A$ and n_A are essential, and we can use the four numbers in place of s_0 and i_0 for the specification of the system as a whole (not only of the particular configuration). Therefore, we may consider n_T as the total particle number of the system itself, and similarly for $\hat{n}_A$, $\check{n}_A$ and n_A. It is to be noted that $n_T(\hat{n}_A, \check{n}_A, n_A)$ thus defined must not be taken as an apparent number in an individual high configuration. They are the quantities kinematically characterizing a physical system which may contain high configurations.

The above arguments about physical systems should be valid especially for basic particles and antiparticles. This new picture of physical basic particles (or antiparticles) plays a very important role throughout the present paper and enables us to understand a composite system intuitively. To be more precise, a physical system with the quantum number n_T can be interpreted as composed of the physical basic particles and antiparticles whose sum is n_T in total, and a similar interpretation is possible also for other quantum numbers that specify the system. Moreover, in order to be persistent in our particle picture, some properties must be possessed by the physical basic particles and antiparticles. It is the second but no less

important aim of the present paper to show how such a particle picture can be set up unambiguously and to study its distinctive features.

Our ideas will be explained in detail in the next section by citing the case of proton and neutron systems. In § 3 it will be shown how a physical system in the composite model can be specified by quantum numbers which have a fairly intuitive meaning. We shall illustrate in § 4 some interesting features of composite systems from the new point of view. In § 5 we shall reexamine the results of our previous papers in the light of the new quantum numbers. In the final section some discussions will be given on the perspective of our further investigation.

§ 2. Proton and neutron system

In this section we shall study a system composed of only protons and neutrons. For the system we require the conservation of the nucleon number as well as charge independence. Then the whole scheme of the system is mathematically described in terms of the unitary group $U(2)$ of degree two. Its irreducible representation can be specified by two quantum numbers, for which we usually take the nucleon number n_N and the isospin I.

It is possible to understand these quantum numbers intuitively under the original spirit of the composite model. We explain our reasoning by taking the case of the system with $n_N=1$ and $I=3/2$. To deal with this system in the composite model we need *at least* two particles and one antiparticle. In fact, when we proceed from the so-called one-nucleon system, we can never reach the system with $I=3/2$ no matter how operative a charge independent interaction is made. What is essential to a p-n system is the minimum number of particles and that of antiparticles necessary for the composition of the system. The difference of these two numbers is obviously the nucleon number n_N, but their sum is important although it is usually concealed behind the concept of isospin. We call this sum *total nucleon number* and denote it by n_{NT}. Thus the system with $(n_N, I)=(1, 3/2)$ and that with $(1, 1/2)$ are distinguished by $n_{NT}=3$ and $=1$ respectively.

One may be embarrassed with this situation, since the number of particles (or antiparticles) in the perturbation theoretical image does not hold its definite meaning when a charge independent interaction is switched on. In this connection we emphasize the following fact: A basic particle (antiparticle) should be the totality of the states which are connected through charge independent interactions with the one-particle (-antiparticle) state in the conventional terminology. n_{NT} as settled above is nothing but the number of such basic particles and basic antiparticles in a physical system. According to our picture, a particle corresponds in a sense to the renormalized one in the ordinary picture, and the complication of dynamics in the usual perturbation theoretical treatment is reduced to the complication of the particle structure in our picture. At the present stage one may say that there

is no essential difference between the two particle pictures. But we hope that the difference will be revealed in the further investigations.

Next turning back to the ordinary particle picture, we shall continue our discussions. Another consideration is necessary when we treat, e.g. the two-nucleon system such as a deuteron. Here the contribution of antiparticles is safely neglected, and we have $n_N=n_{NT}=2$. Since this system has two states corresponding to the different values of the isospin ($I=1$ and 0), it is clear that n_{NT} cannot be used any more in place of I. It should be noted here that these two states are respectively symmetric and alternating with respect to the two particles. This suggests that what is important in this case will be the symmetry character among constituent particles. In fact, when the lowest configuration of a composite system consists of either particles or antiparticles we must introduce the *number of alternating pairs* n_{NA} among particles and those among antiparticles in that configuration.* Of course, we may also define this number n_{NA} when both particles and antiparticles are involved in the lowest configuration, but we can prove $n_{NA}=0$ in this case (see the Appendix). The role of n_{NA} can be seen in the nucleus of mass number A, for which there are $A/2+1$ or $(A+1)/2$ ways of assigning isospin according as A is even or odd, and these ways coincide with the possible cases for n_{NA} in number. n_{NA} here defined is characteristic of the physical system and, like n_N, does not change even if a charge independent interaction is switched on.

The above considerations show that an important role is played by n_N, n_{NT} and n_{NA} when we deal with p-n systems. In the Appendix we shall prove that an irreducible representation of $U(2)$ can be specified by three quantum numbers n_N, n_{NT} and n_{NA} which are subject to the condition

$$(n_{NT}-|n_N|)n_{NA}=0, \tag{1}$$

and that the usual isospin I is given by

$$I=(n_{NT}-2n_{NA})/2. \tag{2}$$

According to the condition (1), p-n systems are classified into two types from the standpoint of $U(2)$. One is the type $n_{NT}\neq|n_N|$ and is specified by n_N and n_{NT}. The other is the type $n_{NT}=|n_N|$ and is specified by n_N and n_{NA}. In the former case we have necessarily $n_{NA}=0$.

We are now in a position to illustrate characteristic features of our particle picture. As we have noted above, a basic particle is the totality of the states which are connected through charge independent interactions with the one-particle state in the usual sense. It can never be created nor destroyed by itself. In this picture a physical system involves a definite number of basic particles and antiparticles,

* n_{NA} is defined as follows: Consider a system of n_{NT} particles. If it contains an alternating pair, take off such a pair from the system and examine the remaining system of $n_{NT}-2$ particles. If it still contains alternating pairs, repeat the procedure until the remaining system has no such pairs, namely it is symmetric with respect to the particles. n_{NA} is the number of the repetition of this procedure. Cf. reference 6).

and has a definite symmetry property with respect to its constituent basic particles and antiparticles. Such a definite number and a symmetry character, expressed by the quantum numbers n_{NT} and n_{NA} respectively, do not undergo a change even in the presence of the strong interaction. So, they can be used to specify a physical system regarded as the totality of possible configurations in the usual picture. Throughout the following discussions we make use of the new particle picture unless stated otherwise.

A further feature of our particle picture is revealed when two physical systems are combined as in the case of collisions. The situation is illustrated by an example of pion-nucleon collision. With the notation introduced above, a nucleon and a pion are specified by $(n_N, n_{NT}, n_{NA}) = (1, 1, 0)$ and $(n_N', n_{NT}', n_{NA}') = (0, 2, 0)$ respectively. Fig. 1 shows the processes schematically. The two particles on the left-hand side are partially in the symmetric and partially in the alternating state.

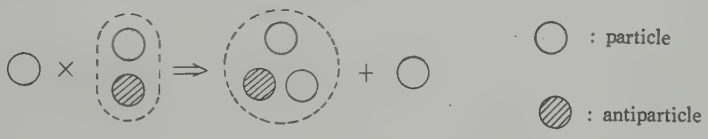


Fig. 1.

The symmetric part makes the transition to the states with the total nucleon number $n_{NT}+n_{NT}'=3$ and with $n_{NT}+n_{NT}'-2=1$, while the alternating part does to a state with the total nucleon number $n_{NT}+n_{NT}'-2=1$. We may understand this situation in the following way: The two states with the total nucleon number 1 may be taken as resulting from a pair annihilation of basic particle and antiparticle. The second term on the right-hand side is just a combination of these states.

The above transitions are written in terms of the new notation:

$$\text{symmetric part} \longrightarrow (1, 3, 0) + (1, 1, 0),$$

$$\text{alternating part} \longrightarrow (1, 1, 0).$$

The reader will easily see that this corresponds to the product rule for the usual isospin: The system (1, 3, 0) has the isospin $I=3/2$, while (1, 1, 0) has $I=1/2$. In fact, we can show that the state composed of one alternating pair of particles and of one antiparticle is derived from the one-particle state by the charge independent interactions and is taken to be a state of the one-particle system from our standpoint. We should notice here that the mechanism of the pair annihilation in our system may be quite different in nature from that in nuclear transitions or in electrodynamics, namely it would operate in an instant or could not be described by the Schrödinger equation.

In concluding this section we summarize the characteristic features of our viewpoint.

(i) What we call a " basic particle " (or " basic antiparticle ") is the totality of states which may be connected with the one-particle (or one-antiparticle) state through charge independent channels in the usual sense. It can never be created nor destroyed by itself. A physical system is characterized by the three quantum numbers n_N, n_{NT} and n_{NA} with the condition (1), all of which should refer to the basic particle and antiparticle inherent in that system.

(ii) In a physical system, if $n_{NT} \neq |n_N|$, then necessarily $n_{NA}=0$. In the case $n_{NT}=|n_N|$, $n_{NA}=0$ is allowed as well as $n_{NA}\neq 0$.

(iii) The quantum numbers n_N, n_{NT} and n_{NA} are conserved by the charge independent interactions.

§ 3. Proton, neutron and Λ-particle system

In the preceding section we have studied a system of protons and neutrons. Let us next discuss a system composed of protons, neutrons and Λ-particles under the assumption of full symmetry, i.e. the invariance for $U(3)$. Its irreducible representation has already been shown to be specified by the three quantum numbers n_B, s_0 and i_0. It is expected that each irreducible representation is realized in nature by a physical system and vice versa.

Now, we can show that s_0 and i_0 may be replaced by other quantities with a more intuitive meaning. Here a physical particle (or system) is understood as the totality of states which are related to each other through the full symmetric interaction in the usual sense. If we define the *total particle number* n_T in the same way as in § 2, the systems are classified into two types according as $n_T \neq |n_B|$ or $n_T=|n_B|$. The case is similar to p-n systems treated in § 2, but now we need somewhat modified discussions because we are treating $U(3)$ instead of $U(2)$. As for the quantum numbers, we must here introduce, besides n_T, the *number of alternating paris among particles* $\hat{n}_A$, the *number of alternating pairs among antiparticles* $\check{n}_A$ and the *number of alternating triplets* n_A (the sum of alternating triplets among particles and those among antiparticles). These can be defined in the same way as n_{NA} in § 2 : Repeat the process of taking off an alternating triplet from the system and define n_A as the number of the repetition of this procedure. Next, examine the remaining system and define $\hat{n}_A(\check{n}_A)$ as the number of the repetition of the process to take off an alternating pair of particles (antiparticles).

We now express these numbers in terms of n_B, s_0 and i_0 (see the Appendix). (A) Type $n_T \neq |n_B|$: This type has often been considered in the applications of the full symmetry theory so far made. In this type there is no alternating triplet, i.e. $n_A=0$, and systems are divided into two cases according as $\hat{n}_A=0$ (Case a) or $\check{n}_A=0$ (Case b).

Case a)
$$n_T=s_0+2i_0,\ \hat{n}_A=0,\qquad \check{n}_A=i_0-(n_B+s_0)/2,\ n_A=0. \tag{3a}$$

Case b) $$n_T=n_B+2s_0,\ \hat{n}_A=-i_0+(n_B+s_0)/2,\ \check{n}_A=0,\ n_A=0. \tag{3b}$$

(B) Type $n_T=|n_B|$: The systems of this type are those composed only of particles (or antiparticles), for example, nuclei and hyperfragments.* There are also two cases which are distinguished by $n_B=n_T$ and $n_B=-n_T$ respectively.

Case c) $$n_T=n_B,\ \hat{n}_A=-i_0+(n_B+3s_0)/2,\ \check{n}_A=0,\ n_A=-s_0. \tag{3c}$$

Case d) $$n_T=-n_B,\ \hat{n}_A=0,\ \check{n}_A=2i_0,\ n_A=-i_0-(n_B+s_0)/2. \tag{3d}$$

We make special mention of the distinctive features of this result: (i) If the number of alternating pairs among particles $\hat{n}_A$ does not vanish, then the number of alternating pairs among antiparticles $\check{n}_A$ must be zero, and vice versa. (ii) The number of alternating triplets n_A does not vanish only when the total particle number n_T coincides with the baryon number n_B apart from the sign.

We can see from the above that an irreducible representation of $U(3)$ is completely specified by the five numbers n_B, n_T, $\hat{n}_A$, $\check{n}_A$ and n_A with the condition

$$(n_T-|n_B|)\,(n_A+\hat{n}_A\check{n}_A)=0. \tag{4}$$

These numbers are conserved by the full symmetric interactions. Only two quantum numbers are essential among n_T, $\hat{n}_A$, $\check{n}_A$ and n_A. This is consistent with the fact that an irreducible representation of $U(3)$ can be characterized by three numbers, e.g. $(n_B,\ s_0,\ i_0)$.

Next we shall make a step forward in the problem of the remaining quantum numbers S, I and I_3. We remark that a configuration of a physical system is a linear combination of "ordered products" of the basic particles and antiparticles.[3] To determine the strangeness quantum number S of the system we have only to pay our attention to the Λ-particles and the anti-Λ-particles in any ordered products and put

$$S=-n_A. \tag{5}$$

In fact, (5) has the value common to all the ordered products in a configuration and gives the strangeness quantum number of the physical system under consideration.

For the study of the isospin we turn our attention to protons, neutrons, antiprotons and antineutrons in the original configuration. Then we can conceive in imagination a p-n system which results from the original p-n-Λ system in the

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* It is not clear whether these examples can be considered on an equal footing as baryons and mesons, which might be composed through a mechanism different from that of a nucleus or hyperfragment. A typical example of the type (B) may be expected in a high energy nucleon-nucleon collision.

following manner: i) In the original configuration all the alternating pairs (triplets) are put in the form $[p, n]$ ($[p, n, \Lambda]$), etc. ii) An alternating pair $[p, \Lambda]$ ($[n, \Lambda]$, $[\bar{p}, \bar{\Lambda}]$, $[\bar{n}, \bar{\Lambda}]$) is replaced by $p(n, \bar{p}, \bar{n})$. iii) An alternating triplet $[p, n, \Lambda]$ ($[\bar{p}, \bar{n}, \bar{\Lambda}]$) is reduced to an alternating pair $[p, n]$ ($[\bar{p}, \bar{n}]$). iv) The remaining Λ or $\bar{\Lambda}$ is taken away. v) The other parts are left unchanged. If we apply the reasoning in § 2 to this imaginary system, we get the value of I for that system, hence for the original p-n-Λ system:

$$I=(n_{NT}-2n_{NA})/2. \tag{2}$$

As to the value of I_3 we can easily see that

$$I_3=(n_p-n_n)/2=n_p-n_N/2. \tag{6}$$

We now summarize the results obtained in this section. In the full symmetry theory a physical system can be characterized by the quantum numbers

$$(n_B,\ n_T,\ \hat{n}_A,\ \check{n}_A,\ n_A;\ n_N,\ n_{NT},\ n_{NA},\ n_p) \tag{7}$$

which are subject to the conditions (1) and (4). The equations (2), (3), (5) and (6) give the relations between these numbers and the old quantum numbers.

§ 4. Characteristics of our particle picture[7]

In the above we have introduced the nine quantum numbers that specify a physical system in the full symmetry theory. To grasp these number intuitively we have merely to make a new particle picture which is similar to the one mentioned in § 2. In this picture, a basic particle is the totality of the states connected through full symmetric interactions with the one-particle state in the usual sense. It should be understood as the fundamental matter in baryon and meson systems. The quantum numbers (7) can be interpreted as the numbers of such basic particles involved in the system considered. A peculiar feature is revealed in the process of composition and decomposition of physical systems. In the following we shall describe a characteristic of our particle picture which is not seen in the case of proton and neutron systems.

The new quantum numbers can be divided into two groups, i.e. the group of $(n_B, n_T, \hat{n}_A, \check{n}_A, n_A)$ and the group of the remaining quantum numbers. As far as the former is concerned, it does not matter whether a constituent basic particle is a proton or a neutron or a Λ-particle. On the contrary, the second group consists of the quantum numbers which are related to the name of constituent basic particles. In our particle picture such a different character between the two groups is of no little significance and brings about some new aspects, as will be seen in the following.

As an example, we first consider the systems $F_3^1(-1, 0, 0)$ and $F_3^1(-1, 1, 0)$, which have the configurations[2),4)]

$$F_3^1(-1,\ 0,\ 0)=\{-(p\Lambda)\bar{p}-(n\Lambda)\bar{n}+2\Lambda\Lambda\bar{\Lambda}\}/2\sqrt{2},$$

$$F_3^1(-1,\ 1,\ 0)=\{(p\Lambda)\bar{p}-(n\Lambda)\bar{n}\}/2.$$

They belong to the same irreducible representation specified by $(n_B,\ s_0,\ i_0)=(1,\ 1,\ 1)$, or by $(n_B,\ n_T,\ \hat{n}_A,\ \check{n}_A,\ n_A)=(1,\ 3,\ 0,\ 0,\ 0)$ in terms of the new quantum numbers. On the other hand, the two systems can be discriminated by n_{NT}, that is, $n_{NT}=0$ for $F_3^1(-1,\ 0,\ 0)$ and $n_{NT}=2$ for $F_3^1(-1,\ 1,\ 0)$. Thus, as far as we are concerned with n_{NT}, the pairs $p\bar{p}$ and $n\bar{n}$ in the former system may be regarded as if they were absent, while their presence is still essential to the latter. This presents a striking contrast to the fact that the pairs $p\bar{p}$, $n\bar{n}$ and $\Lambda\bar{\Lambda}$ are indispensible for $n_T=3$ in both systems. As this example shows, it happens that some nucleon-antinucleon pair plays an essential role in defining $(n_T,\ \hat{n}_A,\ \check{n}_A,\ n_A)$, whereas it does not work in the definition of $(n_{NT},\ n_{NA})$. Such a dual nature of a nucleon-antinucleon pair is a salient feature of the full symmetry theory.

In order to understand this feature more clearly, we shall go ahead by constructing a new model of elementary particles. That is, we first introduce a particle-like entity which results from a basic particle (antiparticle) by shaking off the discrimination among p, n and Λ. Such an entity will be called *basic baryon* (*basic antibaryon*). Next we assume a *p-*, an *n-* and a *Λ-charge*, which attach to the basic baryon (antibaryon) to form the basic particles p, n and Λ ($\bar{p}$, $\bar{n}$ and $\bar{\Lambda}$) respectively. In this model the quantum numbers of the first group can be understood in terms of the basic baryons alone. Of course, the alternation among the basic baryons makes sense only when they can be discriminated between themselves, but the discrimination is assured by the "charges" attaching to the basic baryons. It is obvious that the quantum numbers of the second group should be expressed in terms of the attaching "charges". For the treatment of n_N, n_{NT} and n_{NA}, it is convenient to give a general name of *N-charge* to a p- and an n-charge and to apply the discussions in § 2 to the N-charges. We must further distinguish between a p- and an n-charge for the discussion of n_p.

Now we may understand the dual nature of a nucleon-antinucleon pair as follows: When the system $F_3^1(-1,\ 0,\ 0)$ is composed out of the basic particles, there may occur the annihilation of the N-charges. Throughout this process the number of the basic baryons will not change, because it has already been settled by $n_T=3$, hence some pairs of basic baryon and antibaryon in a physical system may seem as if they were vacant of the N-charges in a certain sense (Fig. 2).

The dual nature in this sense will also be seen in a reaction such as a collision or a decay process, where we must deal with the composition and the decomposition of physical systems. In § 2 it has already been pointed out that these procedures present a peculiarity of our particle picture. Yet it refers to a possible occurrence of pair creations or annihilations of the basic baryons. We must also regard the pair creations and annihilations of the N-charges which may occur in the course of the reaction under consideration.

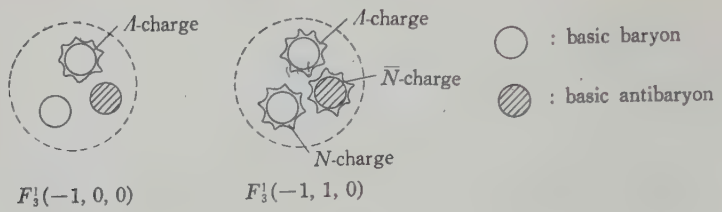


Fig. 2.

It is true that these circumstances are clear from a mathematical point of view. This is obvious about the procedures of composition and decomposition. The dual nature is due to the fact that $U(2)$ is a subgroup of $U(3)$ and an irreducible representation of the latter is decomposed with respect to the former. However, in order to make a step forward in physics, it will be necessary to understand the above circumstances on a realistic ground. This is our future task.

§ 5. Applications

We have shown that a physical system can be specified by the quantities which have a fairly intuitive meaning based on the composite model. The picture of physical systems thus obtained is quite similar to that of the usual nuclear model. A peculiar feature is seen in the rule of composition and decomposition of systems. Here it will be worthwhile to have another look, in the light of our new picture, at the applications of the full symmetry theory so far made.

Two of us (S.S. and M.Y.)[2] have proposed a semi-empirical mass formula for meson and baryon systems. The formula has been derived, after the original one of Matumoto,[8] from a picture which has a close similarity to that of the nuclear model. It has been shown that the mass predicted by the formula has a satisfactory correspondence to experiments and gives a strong support to the full symmetry theory. There, they have treated only the lowest configuration of a system and neglected higher configurations which are connected with the lowest through full symmetric channels. In our picture of physical systems, this means that the formula is to be understood as presenting the ground level of a composite system which consists of a definite number of physical basic particles and antiparticles.

The next problem is the weak decay processes of systems, which have been discussed by three of the authors (M.I., Y.M. and S.O.).[4] For the weak decay processes among baryon and meson systems they have proposed the selection rule

$$\Delta n_B = \Delta s_0 = \Delta i_0 = 0,$$

which can well replace the so-called $|\Delta I| = 1/2$ rule. The new rule is now interpreted as follows: The quantum numbers n_B, n_T, $\hat{n}_A$, $\check{n}_A$ and n_A are all conserved, and what is changed is only the kind of "charges" attached to the basic baryons. The decomposition rule into decay products is referred to in the discussion of § 4.

Now it will also be worthy to examine from our new viewpoint how the full symmetry is destroyed by the interactions realized in nature. We shall discuss them separately.

(i) Λ-nucleon difference: At present we can distinguish Λ-particles from nucleons only by their mass difference and by their behaviours in weak decay processes. A profound meaning behind these phenomena has hardly been understood so far. One should be satisfied at present by saying that the difference induces the non-conservation of the numbers n_T, $\hat{n}_A$, $\check{n}_A$ and n_A.

(ii) Electromagnetic interaction: This violates the conservation of the quantum numbers n_T, $\hat{n}_A$, $\check{n}_A$, n_A, n_{NT} and n_{NA}.

(iii) Weak interactions: Within the framework which we have imposed upon the weak decay processes,[4] the following quantum numbers are not conserved: n_N, n_{NT} and n_{NA}.

§ 6. Concluding remarks

In the present paper we have tried to reveal the implication of the quantum numbers (n_B, s_0, i_0; S, I, I_3) which are characteristic of the full symmetry theory of the composite particle model. It has been shown that these numbers can be replaced by the new quantum numbers whose meaning is far easier to understand intuitively. This result has led us to make a new picture of particle in the composite model. The substance which we have called a system has a similarity to a nucleus in the sense that the substance is composed of a definite number of physical basic particles and antiparticles. But there exist also some new features, which we should like to emphasize once again.

According to our ideas a particle is a substance that represents the totality of the states which are connected by the full symmetric interactions. It can never be created nor destroyed by itself, and should be understood as the fundamental matter in baryon and meson systems. We notice that, contrary to the usual, our particle picture is not based on the concept of free baryons, but involves all the possible states. In our opinion, the quantization of composite systems should be carried out in accordance with this new picture. This will greatly reduce the unnecessary complications in the conventional theory of quantized fields which is based on the image of free baryons. We hope that our particle picture will be of great use in the study of strongly interacting systems.

A peculiarity of our particle picture is, however, brought out in a reaction such as a collision or a decay among physical systems. There an important role is played by the composition and the decomposition rule, which reveals the new feature in terms of the quantum numbers. We know nothing at present as to what causes a composition or a decomposition of physical systems. It may be due to a "quantum jump" as in the case of a quantum mechanical observation, or it may be connected with the concept of "interaction range" among physical systems. This problem requires further investigations.

At first sight it may seem that our scheme is similar in character to the isospin formalism. But we must point out the following peculiar feature of ours: As was shown in a previous paper,[2] we have an empirical mass formula by which each configuration could be associated with a state of some definite energy. This circumstance has never been met with in the case of isospin, and suggests that our scheme contains something more dynamical as compared with the case of isospin which is purely kinematical. As an example, let us consider a collision of a proton and an antiproton. This is schematically expressed by the following in terms of the quantum numbers (7):

$$\begin{aligned}&(1,\,1,\,0,\,0,\,0\,;\;1,\,1,\,0,\,1)\times(-1,\,1,\,0,\,0,\,0\,;\;-1,\,1,\,0,\,-1)\\&=(0,\,2,\,0,\,0,\,0\,;\;0,\,2,\,0,\,0)+(0,\,2,\,0,\,0,\,0\,;\;0,\,0,\,0,\,0)\\&\quad+(0,\,0,\,0,\,0,\,0\,;\;0,\,0,\,0,\,0).\end{aligned}$$

The first term on the right-hand side is the usual neutral pion π^0, and the second is a neutral meson $\pi^{0\prime}$ with isospin 0. The last is a system where neither particle nor antiparticle is present and must be considered as a "vacuum".* This vacuum should have some amount of energy in order that the energy may be conserved by the above process. Thus, if we are to persist in our picture thoroughly, we must assume that the vacuum has some structure with a physical meaning. Such a structure may reflect in a certain sense something dynamical that is inherent in our scheme. This peculiar feature, together with the true meaning of the empirical mass formula, is very important in the realistic aspects of the composite particle model. Further investigations will be necessary in this direction, too.

Acknowledgements

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Appendix

We shall give the mathematical scheme of our present work, which is based on the linear continuous representation of the unitary group $U(n)$ of degree $n=2$ or 3. $U(2)$ refers to the conservation of nucleon number and isospin for p-n systems and $U(3)$ is the group of full symmetry for p-n-Λ systems.

We are interested exclusively in the composite systems. If a state is compos-

* In previous papers[1),2)] we denoted this system by $\pi^{0\prime\prime}$ and treated it as a physical entity like other existing mesons and baryons. It is an interesting problem whether $\pi^{0\prime\prime}$ is a physically realizable matter of a new kind or a vacuum as mentioned above.

ed of $k(\geq 0)$ particles and $l(\geq 0)$ antiparticles, it may be represented by a tensor $T^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ of valence k, l. The case ($n=3$, $k>0$, $l>0$) has already been studied in a previous paper.[3] The other case can be discussed in the same manner as in that paper, hence we give here the results alone.

With respect to the group $U(n)$ the tensor $T^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ can be decomposed into irreducible constituents through the following two steps:

(i) Contraction operation. This operation should be carried out only when both k and l are positive. In the concrete,

$$T^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}=\overset{(0)}{T}{}^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}+\sum_{a,b}\delta^{\kappa_a}_{\lambda_b}\overset{(a,b)}{T}{}^{\kappa_1\cdots\hat{\kappa_a}\cdots\kappa_k}_{\lambda_1\cdots\check{\lambda_b}\cdots\lambda_l}$$
$$+\sum_{ab,cd}\delta^{\kappa_a}_{\lambda_c}\delta^{\kappa_b}_{\lambda_d}\overset{(ab,cd)}{T}{}^{\kappa_1\cdots\hat{\kappa_a}\cdots\hat{\kappa_b}\cdots\kappa_k}_{\lambda_1\cdots\check{\lambda_c}\cdots\check{\lambda_d}\cdots\lambda_l}+\cdots. \qquad (A\cdot 1)$$

On the right-hand side, the indices (0), (a, b), (ab, cd), $\cdots$, designate the tensors T's and the symbol $\wedge$ ($\vee$) tells us to remove the index under (over) itself. These tensors T's are characterized by the property that if we contract a tensor for any pair of a superscript and a subscript the resulting tensor vanishes identically (contraction zero). The summation $\sum_{ab,cd}$ is to be taken for $a, b=1, 2, \cdots, k$ $(a<b)$ and $c, d,=1, 2, \cdots, l$ independently.

(ii) Operation of Young's symmetrizers.[9] When k and l are positive, we apply this operation to the tensors T's on the right-hand side of (A·1) with respect to superscripts and subscripts separately. Therefore the operation may be expressed by a pair of Young's diagrams:

$$\{T(k_1,\cdots,k_n),\ T(l_1,\cdots,l_n)\},$$
$$k_1+\cdots+k_n=k,\ k_1\geq\cdots\geq k_n\geq 0,$$
$$l_1+\cdots+l_n=l,\ l_1\geq\cdots\geq l_n\geq 0. \qquad (A\cdot 2)$$

Here $T(k_1, \cdots, k_n)$ and $T(l_1,\cdots, l_n)$ are Young's diagrams for superscripts and subscripts respectively and it is assumed, without loss of generality, that the tensor in question is of valence k, l. Since the tensor possesses the property of contraction zero, a symmetrizer of some type produces the vanishing tensor. Thus we have only to consider the following cases:

$n=2$ Case a) $\{T(k_1, 0), T(l_1, 0)\}$: $k_1>0$, $l_1>0$.

$n=3$ Case a) $\{T(k_1, 0, 0), T(l_1, l_2, 0)\}$: $k_1>0$, $l_1>0$, $l_2\geq 0$.

Case b) $\{T(k_1, k_2, 0), T(l_1, 0, 0)\}$: $k_1>0$, $k_2\geq 0$, $l_1>0$.

When either k or l is zero, it is sufficient to operate Young's symmetrizers. The corresponding diagram is obtained from (A·2) by putting $l=0$ or $k=0$. That is,

$n=2$ Case b) $\{T(k_1, k_2), T(0, 0)\}$: $k_1>0$, $k_2\geq 0$;

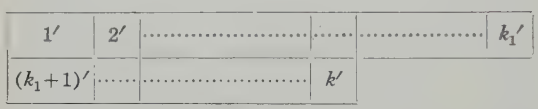
Case c) $\{T(0, 0), T(l_1, l_2)\}$: $l_1>0$, $l_2\geq 0$.

$n=3$ Case c) $\{T(k_1, k_2, k_3), T(0, 0, 0)\}$: $k_1>0,\ k_2>0,\ k_3\geq 0$;

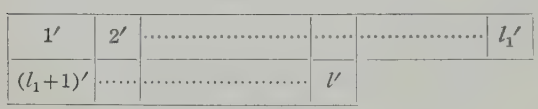
Case d) $\{T(0, 0, 0), T(l_1, l_2, l_3)\}$: $l_1>0,\ l_2>0,\ l_3\geq 0$.

In this way we can decompose each term on the right-hand side of (A·1) to obtain the irreducible constituents of the original tensor $T^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ on the left-hand side. An irreducible part resulting from the first term $\overset{(0)}{T}{}^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ of (A·1) corresponds to the lowest configuration of a physical system. From the second term we attain a higher configuration of a physical system whose lowest configuration is composed of $k-1$ particles and $l-1$ antiparticles, and so on.

An irreducible representation of $U(n)$ can be completely determined by the above k's and l's. Therefore, we can express the quantum numbers in terms of k's and l's. The following is a method of obtaining the expressions of I $(n=2)$, s_0 and i_0 $(n=3)$: Let us first consider an irreducible representation of $U(2)$. The corresponding diagrams can be written as



for superscripts,



for subscripts,

where $(1', 2', \cdots, k')$ and $(1', 2', \cdots, l')$ are permutations of $(1, 2, \cdots, k)$ and $(1, 2, \cdots, l)$ respectively. Let v_{i_0} be the eigenvector of I_3 which belongs to the maximum eigenvalue. Then a non-vanishing component of v_{i_0} is given by $T^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ with

$$\begin{aligned} &\kappa_a=1,\ a=1', 2', \cdots, k_1';\quad \kappa_b=2,\ b=(k_1+1)', \cdots, k';\\ &\lambda_e=2,\ c=1', 2', \cdots, l_1';\quad \lambda_d=1,\ d=(l_1+1)', \cdots, l', \end{aligned} \tag{A·3}$$

and the other non-vanishing components are obtained from this by performing alternations of the κ's and λ's belonging to the same columns. The value of isospin I coincides with that of I_3 for the vector v_{i_0}, hence we have

$$I=(k_1-k_2+l_1-l_2)/2.$$

This reasoning can also be applied to the case $n=3$, in which we have merely to replace v_{i_0} by the vector $v_{s_0i_0}$ introduced in the previous paper[3] and (A·3) by the equations

$$\begin{aligned} &\kappa_a=1,\ a=1', 2', \cdots, k_1';\quad \kappa_b=2,\ b=(k_1+1)', \cdots, (k_1+k_2)';\\ &\kappa_c=3,\ c=(k_1+k_2+1)', \cdots, k';\quad \lambda_d=3,\ d=1', 2', \cdots, l_1';\\ &\lambda_e=2,\ e=(l_1+1)', \cdots, (l_1+l_2)';\quad \lambda_f=1,\ f=(l_1+l_2+1)', \cdots, l'. \end{aligned}$$

The consequence is given by

$$s_0=l_1-k_3,\ i_0=(k_1-k_2+l_2-l_3)/2.$$

The remaining quantum numbers can be easily calculated as follows:

$n=2$ $\quad n_N=k-l,\ n_{NT}=k+l,\ n_{NA}=k_2+l_2$;

$n=3$ $\quad n_B=k-l,\ n_T=k+l,\ \hat{n}_A=k_2-k_3,\ \check{n}_A=l_2-l_3,\ n_A=k_3+l_3$.

Summarizing the above, we have

$n=2$

Case a) $\quad n_N=k-l,\ I=(k+l)/2,\ n_{NT}=k+l,\ n_{NA}=0$;

Case b) $\quad n_N=k,\ I=(k_1-k_2)/2,\ n_{NT}=k,\ n_{NA}=k_2$; $\qquad$ (A·4)

Case c) $\quad n_N=-l,\ I=(l_1-l_2)/2,\ n_{NT}=l,\ n_{NA}=l_2$.

$n=3$

Case a) $\quad n_B=k-l,\ s_0=l_1,\ i_0=(k+l_2)/2,$

$n_T=k+l,\ \hat{n}_A=0,\ \check{n}_A=l_2,\ n_A=0$;

Case b) $\quad n_B=k-l,\ s_0=l,\ i_0=(k_1-k_2)/2$,

$n_T=k+l,\ \hat{n}_A=k_2,\ \check{n}_A=0,\ n_A=0$; $\qquad$ (A·4′)

Case c) $\quad n_B=k,\ s_0=-k_3,\ i_0=(k_1-k_2)/2$,

$n_T=k,\ \hat{n}_A=k_2-k_3,\ \check{n}_A=0,\ n_A=k_3$;

Case d) $\quad n_B=-l,\ s_0=l_1,\ i_0=(l_2-l_3)/2,$

$n_T=l,\ \hat{n}_A=0,\ \check{n}_A=l_2-l_3,\ n_A=l_3$.

In mathematical text-books on the group representations, the results concerning $U(n)$ are often stated in terms of the set of n integers $f_i(i=1,\ 2,\cdots,\ n;\ f_1\geq f_2\geq\cdots\geq f_n)$, which specifies an irreducible representation of $U(n)$. We give the relation between (f_1, f_2) and (n_N, I) and between (f_1, f_2, f_3) and (n_B, s_0, i_0).

$n=2$ $\quad n_N=f_1+f_2,\ I=(f_1-f_2)/2,$ $\qquad$ (A·5)

$f_1=n_N/2+I,\ f_2=n_N/2-I$.

$n=3$ $\quad n_B=f_1+f_2+f_3,\ s_0=-f_3,\ i_0=(f_1-f_2)/2,$ $\qquad$ (A·5′)

$f_1=(n_B+s_0)/2+i_0,\ f_2=(n_B+s_0)/2-i_0,\ f_3=-s_0$.

The proof of (A·5′) is given in the previous paper[3] and (A·5) can be proved in a similar way.

Substituting (A·4) and (A·4′) in these expressions, we get

$n=2$

Case a) $\quad (f_1, f_2)=(k, -l)$;

Case b) $\quad (f_1, f_2)=(k_1, -k_2)$;

Case c) $\quad (f_1, f_2)=(-l_2, -l_1)$.

$n=3$

Case a) $(f_1, f_2, f_3)=(k, -l_2, -l_1)$;

Case b) $(f_1, f_2, f_3)=(k_1, k_2, -l)$;

Case c) $(f_1, f_2, f_3)=(k_1, k_2, k_3)$;

Case d) $(f_1, f_2, f_3)=(-l_3, -l_2, -l_1)$.

This shows that the cases are mutually exclusive and exhaust all the possibilities.

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Electromagnetic Structure of the Nucleon*

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Spectral representations of the electromagnetic form factors of the nucleon for fixed momentum transfer are proved as a function of the square of the four-momentum of the nucleon, based on the general principles of the quantum field theory and, using these representations, two sets of coupled equations for charge and magnetic moment form factors are derived. These equations are solved in an approximation in which only the lowest mass configuration is taken into account which consists of one pion and one nucleon. It is shown that the absorptive parts in this approximation are expressed in terms of the real pion production by virtual photon for $I=J=\frac{1}{2}$ state and pion-nucleon vertex part, from which follows that (3-3) resonant state does not contribute to the form factors as far as the lowest mass configuration is concerned. Inclusion of (3-3) resonance and the second and third resonance requires the calculation of two pions and one nucleon state which is not attempted here.

§ 1. Introduction

Since the pioneer experiments made by Hofstadter et al.[1] electromagnetic properties of the nucleon have revealed itself as one of the most important problems for our understanding the structure of elementary particles and subsequently theoretical calculations have been carried out by various methods: perturbation theory,[2] static theory,[3] dispersion theory[4] and many other methods.[5] In particular, based on the dispersion theoretic approach, which was first formulated by Chew et al.,[4] many efforts have recently been devoted to the calculation of isotopic vector and scalar parts of the nucleon form factors. These calculations clarify that the so-called two-pion state will give the main contribution to the isotopic vector part of form factors and the agreement between theoretical calculations and experiments seems satisfactory for magnetic moment and magnetization mean square radius, though charge density radius appears to be much smaller than experimental value.

As for isotopic scalar part, the " three-pion state " is expected to be the only

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* The result of this work has been reported at the Annual Meeting of the Physical Society of Japan held in Tokyo, April 1960.

state that mainly contributes to the isotopic scalar part. But we have not yet any reliable method for the calculation of this state though several attempts were performed to estimate the contribution by perturbation[6] and static theory.[7]

It has become clear, through these analyses, that we may say that we "understand" semi-quantitatively the "two-pion state" contribution of the isotopic vector parts, while for isotopic scalar part we can at most estimate qualitatively the contribution from "three-pion state". It is our feeling that calculations beyond the lowest mass configuration are not only difficult but also meaningless, since once beyond the lowest mass configuration, the approximation to take one particular state seems to become worse. In this respect, it will be worthwhile to try somewhat different approach to the electromagnetic structure of the nucleon, which at the same time might enable us to derive some new features of the nucleon structure, although we must be careful as to the applicability of the present quantum field theory, in particular charge symmetric pseudoscalar pion theory.

In this paper, therefore, we base our investigation on mass spectral representations for fixed momentum transfer ($q^2 > 0$ space-like), choosing nucleon mass as a variable.* These spectral representations, contrary to the usual spectral representation, are possible to prove rigorously using only general principles: microscopic causality, unitarity, and spectral condition, (§ 2). Next, inspecting the absorptive parts, it is easily seen that in general they contain electromagnetic form factors of the nucleon under consideration and thus we can obtain two sets of coupled algebraic equations for charge and magnetic moment form factors and these equations are formally solved. In particular, if we take the lowest mass configuration, we can express the absorptive part in terms of the photo-pion production matrix element for the $I=J=\frac{1}{2}$ state and the pion-nucleon vertex operator. The matrix element for real pion production by virtual photon was already analysed by Fubini et al.[9] and is known to be expressed in some approximation in terms of nucleon electromagnetic form factors and small $J=I=\frac{1}{2}$ phase shift of pion-nucleon scattering. The vertex part is also expressed by the pion-nucleon phase shift for the $I=J=\frac{1}{2}$ state,[8] (§ 3).

To calculate form factors in lowest mass approximation, we first notice that the 3–3 resonant state never contributes in this approximation and thus we can neglect rescattering corrections in a good approximation. Then, coupled algebraic equations mentioned above are easily set up and solved, and numerical results are given, (§ 4). The final section (§ 5) is devoted to the discussion of the result of the previous sections. In Appendix A an operator equation is given, where the electromagnetic vertex operator is expressed rigorously by the Green's function for photo-pion production process.

§ 2. Derivation of the spectral representation

The quantity with which we shall be concerned is the matrix element of the

* Essentially the same approach has been carried out independently by A. Bincer.[8]

current density operator j_μ taken between one-nucleon states. This matrix element may be expressed as follows:

$$\langle p'|j_\mu|p\rangle=\frac{1}{(2\pi)^3}\left(\frac{m^2}{p_0' p_0}\right)^{1/2}\bar{u}(p')\Gamma_\mu u(p) \qquad (2\cdot1)$$

where p and p' are the initial and the final nucleon momentum.* (Fig. 1)

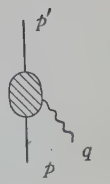


Fig. 1. Electromagnetic form factors.

The most general expression of $\Gamma_\mu(p', p)$ defined by Eq. (2·1) is given in Appendix B and expressed in the following way:

$$\begin{aligned}\Gamma_\mu(p', p)=&i\gamma_\mu G_1+i\sigma_{\mu\nu}q_\nu G_2+q_\mu G_3+i(i\gamma p'+m)\gamma_\mu G_4+i\gamma_\mu(i\gamma p+m)G_7\\&+i(i\gamma p'+m)\sigma_{\mu\nu}q_\nu G_5+i\sigma_{\mu\nu}q_\nu(i\gamma p+m)G_8+(i\gamma p'+m)q_\mu G_6+q_\mu(i\gamma p+m)G_9\\&+(i\gamma p'+m)i\gamma_\mu(i\gamma p+m)G_{10}+(i\gamma p'+m)i\sigma_{\mu\nu}q_\nu(i\gamma p+m)G_{11}\\&+(i\gamma p'+m)q_\mu(i\gamma p+m)G_{12} \qquad (2\cdot2)\end{aligned}$$

where $q_\mu=p'_\mu-p_\mu$ and $G_i(i=1, \cdots, 12)$ are functions of three independent scalar quantities which may be chosen as $p'^2(=-\tilde{\xi})$, p^2 and q^2.

Usual charge and magnetic moment form factors are obtained if we put $p'^2=p^2=-m^2$; i.e. $G_1(p'^2=-m^2,\ p^2=-m^2,\ q^2)=F_1(q^2)$ and $G_2(p'^2=-m^2,\ p^2=-m^2, q^2)=-F_2(q^2)$.

Chew et al.[4] investigated analytic properties of G_1 and G_2 as a function of momentum transfer q^2, fixing the magnitude of p'^2 and p^2 on the mass-shell. Instead, in the following, we shall prove $G_i(i=1, \cdots,6)$ as a function of $\tilde{\xi}(=-p'^2)$, fixing $p^2=-m^2$ and $q^2>0$, has a simple analytic property as a consequence of general principles of quantum field theory: micro-causality, unitarity and spectral condition, which immediately leads to spectral representations under consideration.

Following the formulation of CKGZ,** we consider the quantity (2·1) as an S-matrix element of a transition of the nucleon from the state with momentum p to the state with momentum p' emmiting a virtual quantum q_μ and apply reduction formulas.[10] Then,

$$\begin{aligned}&\langle p'|S|pq_\mu\rangle***\\&=\frac{1}{(2\pi)^3}\left(\frac{m}{2p_0' q_0}\right)^{1/2}\iint d^4x\,d^4y\,e^{iqx-ip'y}\bar{u}(p')\left\langle 0\left|\frac{\delta^2 S}{\delta A_\mu(x)\delta\psi(y)}S^+\right|p\right\rangle\\&=-\frac{1}{(2\pi)^3}\left(\frac{m}{2p_0' q_0}\right)^{1/2}\iint d^4x\,d^4y\,e^{iqx-ip'y}\bar{u}(p')\langle 0|T(j_\mu(x),\eta(y))|p\rangle\\&=-(2\pi)i\left(\frac{m}{2p_0' q_0}\right)^{1/2}\delta(p'-p-q)\bar{u}(p')G_\mu(p'+q), \qquad (2\cdot3)\end{aligned}$$

---

* Spinor $u(p)$ is normalized according to $\bar{u}(p)u(p)=1$ and metric is such that scalar product is defined as $a_\mu b_\mu=\boldsymbol{a}\cdot\boldsymbol{b}-a_0 b_0$.

** The notation used here is the same as that used in CKGZ.[4]

*** Spin indices are suppressed throughout as far as no confusion occurs.

where $j_\mu(x)$ and $\eta(y)$ are defined by

$$j_\mu = i\frac{\delta S}{\delta A_\mu(x)}S^+, \quad \eta(y) = i\frac{\delta S}{\delta \bar{\psi}(y)}S^+, \tag{2·4}$$

and causality conditions imply

$$\begin{aligned} &\frac{\delta j_\mu(x)}{\delta \psi(y)} = 0, \quad x_0 > y_0 \text{ or } (x-y)^2 > 0 \\ &\frac{\delta \eta(y)}{\delta A_\mu(x)} = 0, \quad x_0 < y_0 \text{ or } (x-y)^2 > 0. \end{aligned} \tag{2·5}$$

We have neglected the contribution from equal-time commutator which will be evaluated in Appendix C and $G_\mu(k)$ is defined as a Fourier transform of the causal function $S_\mu(x)$ as follows:

$$G_\mu(p'+q) = \int e^{-(i/2)(p'+q)x} S_\mu(x)\, d^4x \tag{2·6}$$

and

$$\langle 0|T(j_\mu(x), \eta(y))|p\rangle = ie^{(i/2)p(x+y)} S_\mu(x-y). \tag{2·7}$$

In order to prove spectral representation we choose, as usual, a special coordinate system such that

$$\begin{aligned} p' &= (\lambda \boldsymbol{e}, E), \\ q &= (\lambda \boldsymbol{e}, E-m), \\ p &= (0, m), \end{aligned} \tag{2·8}$$

where $\lambda^2 = (E-m)^2 + q^2$ and $\boldsymbol{e}$ is a unit vector, and consider $G_\mu(p'+q)$ as a function of complex E. It should be noticed that we are interested only for the case of $q^2 > 0$ (space-like vector), on account of which $G_\mu(p'+q)$ as a function of complex E is given simply as an analytic continuation of the expression (2·6) and rigorous proof becomes possible, that is, no analytic continuation in terms of $\tau(=-p^2)$ is required and consequently unphysical mass restriction $\mu > (\sqrt{2}-1)m$ does not appear.[4),11)]

Once this point is kept in mind, the procedure of the proof is straightforward and is essentially the same as those given in CKGZ, hence we shall not repeat it here. The result is that $G_\mu^{e,o}(E, q^2)$* as a function of E is regular in the entire complex E plane with a branch point at $E_0(q^2) > 0$ and a cut along the positive real axis.

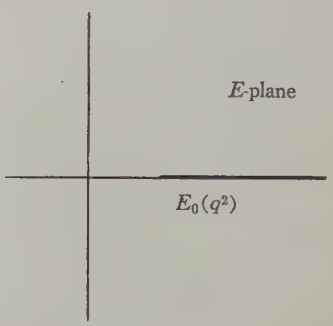


Fig. 2. Complex E plane. $E_0(q^2)$ is a branch point and a cut along the positive real axis.

* $G_\mu^{e,o}(E, q^2)$ means symmetrized and antisymmetrized function in the sense of $\boldsymbol{e}$.

The discontinuity in $G_\mu^{e,o}$ on the cut is known to be given by

$$G_\mu^{+}(E, q^2) = i(2\pi)^4 \sum_n 2E\theta(E)\delta(p'^2+m_n^2)\cdot\langle 0|\eta(0)|n\rangle\cdot\langle n|j_\mu(0)|p\rangle \qquad (2\cdot 9)$$

where m_n indicates the intermediate rest-energy.

In this way the following spectral representation is obtained except a possible subtraction:

$$G_\mu^{e,o}(E, q^2) = \frac{1}{2\pi i}\int_{E_0(q^2)}^{\infty} \frac{G_\mu^{+e,o}(E', q^2)}{E'-E}\, dE', \qquad (2\cdot 10)$$

or we may rewrite Eq. (2·10) as follows, using the relation

$$E = \frac{m^2+q^2+\xi}{2m} \quad \text{and} \quad E_0 = \frac{m^2+q^2+(m+\mu)^2}{2m}$$

$$G_\mu^{e,o}(\xi, q^2) = \frac{1}{2\pi i}\int_{(m+\mu)^2}^{\infty} \frac{G_\mu^{+e,o}(\xi', q^2)}{\xi'-\xi}\, d\xi'. \qquad (2\cdot 11)$$

From Eqs. (2·2), (2·11), we immediately get spectral representations for each form factors:

$$G_i^{S,V}(\xi, q^2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{G_i^{+S,V}(\xi', q^2)}{\xi'-\xi}\, d\xi' \quad (i=1, \cdots, 6) \qquad (2\cdot 12)$$

where superscripts S, V indicate the scalar and vector parts respectively. Explicit relation between $G_\mu(\xi, q^2)$ and $G_i(\xi, q^2)$ is, from Eqs. (2·2), (2·3), (2·6) and (2·7),

$$\begin{aligned} G_\mu(\xi, q^2) = i\ & \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p_0}\right)^{1/2}\Big[i\gamma_\mu G_1(\xi, q^2) + i\sigma_{\mu\nu}q_\nu G_2(\xi, q^2) \\ & + q_\mu G_3(\xi, q^2) + (i\gamma p'+m)i\gamma_\mu G_4(\xi, q^2) \\ & + (i\gamma p'+m)i\sigma_{\mu\nu}q_\nu G_5(\xi, q^2) + (i\gamma p'+m)q_\mu G_6(\xi, q^2)\Big]u(p) \end{aligned} \qquad (2\cdot 13)$$

and $G_i^{+}(\xi, q^2)$ is defined analogous to $G_i(\xi, q^2)$ as follows:

$$\begin{aligned} G_\mu^{+}(\xi, q^2) = \frac{-2}{(2\pi)^{3/2}}\left(\frac{m}{p_0}\right)^{1/2}\Big[& i\gamma_\mu G_1^{+}(\xi, q^2) + i\sigma_{\mu\nu}q_\nu G_2^{+}(\xi, q^2) \\ & + q_\mu G_3^{+}(\xi, q^2) + (i\gamma p'+m)i\gamma_\mu G_4^{+}(\xi, q^2) \\ & + (i\gamma p'+m)i\sigma_{\mu\nu}q_\nu G_5^{+}(\xi, q^2) + (i\gamma p'+m)q_\mu G_6^{+}(\xi, q^2)\Big]u(p). \end{aligned} \qquad (2\cdot 14)$$

Expressions (2·12) are our final form of spectral representations though in practice we should take into account the contribution from equal-time commutator (see § 3). It is to be noticed that G_i^{+} are not linearly independent on account of the conservation of the current which implies

$$q_\mu G_\mu^{+}(\xi, q^2)=0 \tag{2·15}$$

from which the following relations are derived:

$$\begin{aligned} &q^2 G_3^{+S,V}(\xi, q^2)-(\xi-m^2)G_4^{+S,V}(\xi, q^2)=0 \\ &G_1^{+S,V}(\xi, q^2)+2mG_4^{+S,V}(\xi, q^2)-q^2 G_6^{+S,V}(\xi, q^2)=0. \end{aligned} \tag{2·16}$$

In particular, if we set $q^2=0$ or $\xi=m^2$ in Eq. (2·16), we obtain*

$$\begin{aligned} &G_3^{+S,V}(m^2, q^2)=0, \\ &G_1^{+S,V}(\xi, 0)=0, \\ &G_4^{+S,V}(\xi, 0)=0. \end{aligned} \tag{2·17}$$

Eq. (2·16) does not necessarily mean the linear dependence of $G_i(\xi, q^2)$ since Re $G_i(\xi, q^2)$ can contain a possible subtraction term. Re $G_3(\xi, q^2)$, however, also vanishes on the mass shell, $\xi=m^2$.

§ 3. Structure of absorptive parts

In this section we investigate the structure of absorptive parts of charge and magnetic moment form factors. For this purpose, contributions from equal-time commutator are to be taken into account properly since we have assumed no subtractions for charge and magnetic form factors. In Appendix C, we shall prove that the equal-time commutator only contributes to the charge form factor, so we write down spectral representations (2·12) for $G_1^{S,V}(\xi, q^2)$ and $G_2^{S,V}(\xi, q^2)$ in the following form:

$$\begin{aligned} &G_1^{S,V}(\xi, q^2)-G_1^{S,V}(\xi, 0)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{G_1^{+S,V}(\xi', q^2)}{\xi'-\xi}d\xi' \\ &G_2^{S,V}(\xi, q^2)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{G_2^{+S,V}(\xi', q^2)}{\xi'-\xi}d\xi'. \end{aligned} \tag{3·1}$$

Here we have made use of the fact $G_1^{+S,V}(\xi, 0)=0$ (Eq. (2·17)). Putting $\xi=m^2$, we obtain

$$\begin{aligned} &G_1^{S,V}(q^2)=\frac{e}{2}+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{G_1^{+S,V}(\xi', q^2)}{\xi'-m^2}d\xi' \\ &G_2^{S,V}(q^2)=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{G_2^{+S,V}(\xi', q^2)}{\xi'-m^2}d\xi' \end{aligned} \tag{3·2}$$

where $G_1^{S,V}(q^2)=G_1^{S,V}(m^2, q^2)$ and $G_2^{S,V}(q^2)=G_2^{S,V}(m^2, q^2)$ are usual charge and magnetic moment form factors.

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* We assumed $G_i(\xi, q^2)$ are regular at $q^2=0$ which is verified by actual calculations.

In order to investigate the structure of $G_{1,2}^{+S,V}(\xi, q^2)$, we write Eq. (2·9) again:

$$G_\mu^+(\xi, q^2) = i(2\pi)^4 \sum_n 2E\theta(E)\delta(\xi - m_n^2)\langle 0|\eta(0)|n\rangle\cdot\langle n|j_\mu(0)|p\rangle. \quad (2\cdot9)'$$

This form of $G_\mu^+(\xi, q^2)$ indicates that G_μ^+ consists of two processes in which initial nucleon absorbs virtual photon to make a transition to the complex intermediate state and from the intermediate state to the final one-nucleon state. It is then easily seen that in the case of the absorption of virtual photon by the nucleon electromagnetic form factors of the nucleon necessarily appear and general form of $G_{1,2}^{+S,V}(\xi, q^2)$ may be expressed as follows,

$$G_{1,2}^{+S,V}(\xi, q^2) = H_{1,2}^{S,V}(\xi, q^2)G_1^{S,V}(q^2) + I_{1,2}^{S,V}(\xi, q^2)G_2^{S,V}(q^2) + J_{1,2}^{S,V}(\xi, q^2) \quad (3\cdot3)$$

where $H_{1,2}^{S,V}(\xi, q^2)$, $I_{1,2}^{S,V}(\xi, q^2)$ and $J_{1,2}^{S,V}(\xi, q^2)$ represent some analytic functions. Inserting Eq. (3·3) into Eq. (3·2), we get the following two sets of coupled algebraic equations for $G_1^{S,V}(q^2)$ and $G_2^{S,V}(q^2)$:

$$\begin{aligned} G_1^{S,V}(q^2) &= e/2 + \tilde{H}_1^{S,V}(q^2)G_1^{S,V}(q^2) + \tilde{I}_1^{S,V}(q^2)G_2^{S,V}(q^2) + \tilde{J}_1^{S,V}(q^2) \\ G_2^{S,V}(q^2) &= \tilde{H}_2^{S,V}(q^2)G_1^{S,V}(q^2) + \tilde{I}_2^{S,V}(q^2)G_2^{S,V}(q^2) + \tilde{J}_2^{S,V}(q^2) \end{aligned} \quad (3\cdot4)$$

where

$$\tilde{H}_{1,2}^{S,V}(q^2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{H_{1,2}^{S,V}(\xi', q^2)}{\xi' - m^2} d\xi'$$

$$\tilde{I}_{1,2}^{S,V}(q^2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{I_{1,2}^{S,V}(\xi', q^2)}{\xi' - m^2} d\xi'$$

$$\tilde{J}_{1,2}^{S,V}(q^2) = \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{J_{1,2}^{S,V}(\xi', q^2)}{\xi' - m^2} d\xi'.$$

Eq. (3·4) is easily solved and the result is

$$\begin{aligned} G_1^{S,V}(q^2) &= \frac{(e/2 + \tilde{J}_1^{S,V}(q^2))(1 - \tilde{I}_2^{S,V}(q^2)) + \tilde{J}_2^{S,V}(q^2)\tilde{I}_1^{S,V}(q^2)}{(1 - \tilde{H}_1^{S,V}(q^2))(1 - \tilde{I}_2^{S,V}(q^2)) - \tilde{H}_2^{S,V}(q^2)\tilde{I}_1^{S,V}(q^2)} \\ G_2^{S,V}(q^2) &= \frac{(e/2 + \tilde{J}_1^{S,V}(q^2))\tilde{H}_2^{S,V}(q^2) + (1 - \tilde{H}_1^{S,V}(q^2))\tilde{J}_2^{S,V}(q^2)}{(1 - \tilde{H}_1^{S,V}(q^2))(1 - \tilde{I}_2^{S,V}(q^2)) - \tilde{H}_2^{S,V}(q^2)\tilde{I}_1^{S,V}(q^2)}. \end{aligned} \quad (3\cdot5)$$

It should be noticed that the expression (3·5) is exact and contain no approximation. Our final task is to study general properties of $\tilde{H}_{1,2}^{S,V}(q^2)$, $\tilde{I}_{1,2}^{S,V}(q^2)$ and $\tilde{J}_{1,2}^{S,V}(q^2)$. We have not, however, succeeded in finding their general properties except the following one:

$$\lim_{q^2\to 0} \tilde{H}_1^{S,V}(q^2) = \tilde{I}_1^{S,V}(q^2) = \tilde{J}_1^{S,V}(q^2) = 0.$$

In the next section, therefore, we investigate their detailed structure in the case of the lowest mass configuration.

§ 4. Contribution from the lowest mass configuration

It is easily seen from Eq. (2·9)′ that the lowest mass configuration consists of one pion and one nucleon state and in this case we have for the absorptive part the following expression: (Fig. 3)

$$G_\mu^+(\xi, q^2) = i(2\pi)^4 \sum_\sigma \int d^3p_1 d^3q_1 \delta(p' - p_1 - q_1) \times \langle 0|\eta(0)|p_1 q_1 \sigma\rangle \langle \sigma q_1 p_1 | j_\mu | p\rangle \quad (4\cdot1)$$

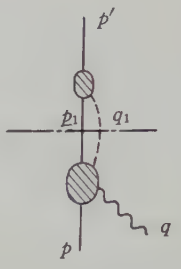


Fig. 3. Contribution to the absorptive parts in the lowest mass configuration; one meson broken line and one nucleon solid line.

where p_1 and q_1 are momentum of the intermediate nucleon and pion, respectively, and $\sum_\sigma$ means sum over the discrete quantum numbers. The factor $\langle 0|\eta(0)|p_1 q_1 \sigma\rangle$ which appeared in Eq. (4·1) is related to the pion-nucleon vertex function and expressed as

$$\langle 0|\eta(0)|p_1 q_1 \sigma\rangle = \frac{1}{(2\pi)^3}\left(\frac{m}{2p_{10}q_{10}}\right)^{1/2} \cdot ig\{K_1(\xi) + (i\gamma p' + m)K_2(\xi)\} \cdot \gamma_5 \tau_\sigma u(p) \quad (4\cdot2)$$

where $K_1(\xi)$ is normalized according to $K_1(m^2)=1$ and g is the renormalized pion-nucleon coupling constant. The quantity $\langle \sigma, q_1 p_1 | j_\mu(0)|p\rangle$ represents the pion production by virtual photon and is investigated in detail by Fubini et al.[9] in connection with the treatment of pion production in electron-nucleon collisions. According to their results, this quantity can be decomposed into six invariant forms and is expressed as:

$$\varepsilon_\mu \langle \sigma q_1 p_1 | j_\mu(0)|p\rangle = -i\left[\frac{1}{(2\pi)^{3/2}}\right]^3 \left(\frac{m^2}{2p_{10}q_{10}p_0}\right)^{1/2} \bar{u}(p_1)\varepsilon_\mu H_\mu^\sigma u(p) \quad (4\cdot3)$$

where ε_μ represents polarization vector of the electromagnetic field and

$$\varepsilon_\mu H_\mu^\sigma = M_A A_\sigma + M_B B_\sigma + M_C C_\sigma + M_D D_\sigma + M_E E_\sigma + M_F F_\sigma \quad (4\cdot4)$$

where

$$\begin{aligned}
M_A &= \frac{1}{2} i\gamma_5 \{\gamma, \gamma\}, && (+)\\
M_B &= 2i\gamma_5 \{P, q\}, && (+)\\
M_C &= \gamma_5 \{\gamma, q_1\}, && (-)\\
M_D &= 2\gamma_5 (\{\gamma, P\} - \frac{1}{2} im\{\gamma, \gamma\}), && (+)\\
M_E &= i\gamma_5 \{q, q_1\}, && (-)\\
M_F &= \gamma_5 \{q, \gamma\}. && (-)
\end{aligned} \quad (4\cdot5)$$

Here $P=(p+p_1)/2$, the $(\pm)$ sign corresponds to the crossing relation and the

following abbreviation is used for the gauge-invariant combination:

$$\{a, b\} \equiv a\varepsilon \cdot bq - aq \cdot b\varepsilon. \tag{4·6}$$

From charge independence, $A_\sigma, \cdots, F_\sigma$ can be further decomposed:

$$A_\sigma = \frac{1}{2}\{\tau_\sigma . \tau_3\} A^+ + \frac{1}{2}[\tau_\sigma, \tau_3] A^- + \tau_\sigma A^0, \text{ etc.} \tag{4·7}$$

The 18 coefficients $A^+, A^-, A^0, \cdots, F^0$ are now functions of three scalars:

$$\nu = -Pq/m, \ \nu_B = q \cdot q_1/2m, \ q^2, \tag{4·8}$$

and assumed to satisfy dispersion relations of the form

$$H_i(\nu, \nu_B, q^2) = C_i(\nu, \nu_B, q^2) + R_i(q^2)\left(\frac{1}{\nu_B - \nu} \pm \frac{1}{\nu_B + \nu}\right)$$
$$+ \frac{1}{\pi}\int_{\nu_0}^{\infty} d\nu' \operatorname{Im} H_i(\nu', \nu_B, q^2)\left(\frac{1}{\nu' - \nu} \pm \frac{1}{\nu' + \nu}\right) \tag{4·9}$$

where $\nu_0 = \nu_B + \mu + \mu^2/2m$, and the sign $(\pm)$ depends on the crossing symmetry and H_i stands for $A^+, \cdots F^0$.

The residue $R_i(q^2)$ of the pole turns out to be

$$\begin{aligned} R[A^{S,V}] &= -ef G_1^{S,V}(q^2), \\ R[B^{S,V}] &= ef G_1^{S,V}(q^2)/2m\nu_B \\ R[C^{S,V}] &= R[D^{S,V}] = f G_2^{S,V}(q^2) \\ R[E^{S,V}] &= R[F^{S,V}] = 0 \end{aligned} \tag{4·10}$$

where $A^V = A^+$ or A^- and $A^S = A^0$, etc. The terms $C_i(\nu, \nu_B, q^2)$ are not determined by this form of dispersion relations. They are determined so as to reproduce the lowest order perturbation result, the procedure of which is justified by dispersion relations for fixed ν, choosing ν_B as a variable. (See the Appendix in reference 9)). They are determined as follows:

$$C_A = C_B = C_C = C_D = C_F = 0,$$
$$D_{E^+} = C_{E^0} = 0,$$
$$C_{E^-} = \frac{2mf}{q^2}\left[\frac{-2G_\pi(q^2)}{2qq_1 - q^2} + \frac{2G_1^V(q^2)}{q_1 \cdot q}\right]$$

where $G_\pi(q^2)$ means the electromagnetic form factor of the pion.

Now, inserting Eq. (4·2) and Eq. (4·3) into Eq. (4·1), we find

$$G_\mu^+(\xi, q^2) = i\frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p_0}\right)^{1/2}\frac{m}{4\pi^2} g \int d^4q_1 d^4p_1 \delta(p' - p_1 - q_1)$$
$$\times \theta(p_{10})\theta(q_{10})\delta(p_1^2 + m^2)\delta(q_1^2 + \mu^2)[\{K_1(\xi) + (i\gamma p' + m)K_2(\xi)\}$$
$$\times \gamma_5 \Lambda^-(p_1)(H_\mu^V \tau_3 + H_\mu^S)]u(p) \tag{4·11}$$

where $\Lambda^-(p_1)=(-ir\not{p}_1+m)/2m$ and $H_\mu{}^V=H_\mu{}^+ +2H_\mu{}^-$ and $H_\mu{}^S=H_\mu{}^0$, thereby

$$\varepsilon_\mu H_\mu^{\pm 0}=M_A A^{\pm 0}+\cdots+M_F F^{\pm 0}. \tag{4·12}$$

From Eqs. (4·9), (4·10) and (4·11) we see that the absorptive part in fact contains electromagnetic form factors of the nucleon as expected in § 3. Furthermore we see that we are only concerned with the matrix element of $I=J=\frac{1}{2}$ state. This is easily seen if we expand $H_\mu^{S,V}$ into partial waves and carry out integration in Eq. (4·11), the result of which agrees those given by A. Bincer.[8] The fact that (3–3) resonant state does not affect the form factors as far as the lowest mass configuration is concerned* enables us to expect that neglecting small rescattering corrections of pion-nucleon $I=J=\frac{1}{2}$ scattering will be a fairly good approximation.

Under this expectation, we neglect rescattering corrections and $H_i(\nu, \nu_B, q^2)$ and $K_i(\xi)$ are approximated as follows,

$$\begin{aligned} H_i^{S,V}(\nu, \nu_B, q^2) &\cong C_i^{S,V}(\nu, \nu_B, q^2)+R_i(q^2)\left(\frac{1}{\nu_B-\nu}\pm\frac{1}{\nu_B+\nu}\right) \\ K_1(\xi) &\cong 1 \\ K_2(\xi) &\cong 0. \end{aligned} \tag{4·13}$$

In the Feynman diagram, this approximation corresponds to Fig. (IV) and calculation is now straightforward. The result is,

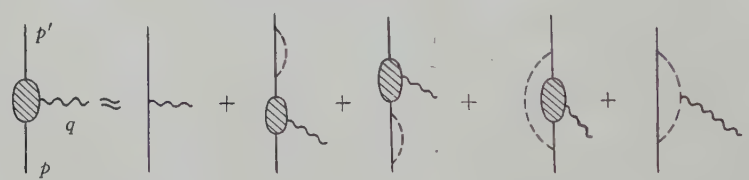


Fig. 4. Feynman graph corresponding to the approximation (4·13).

$$\begin{aligned} G_1^{S,V}(q^2) &= \frac{e}{2}+\widetilde{H}_1^{S,V}(q^2)G_1^{S,V}(q^2)+\tilde{I}_1^{S,V}(q^2)G_2^{S,V}(q^2)+\tilde{J}_1^{S,V}(q^2) \\ G_2^{S,V}(q^2) &= \widetilde{H}_2^{S,V}(q^2)G_1^{S,V}(q^2)+\tilde{I}_2^{S,V}(q^2)G_2^{S,V}(q^2)+\tilde{J}_2^{S,V}(q^2) \end{aligned} \tag{4·14}$$

where

$$\widetilde{H}_1^S(q^2)=\frac{3g^2}{16\pi^2}\{C_1(q^2)-C_1(0)+(m^2-q^2/2)D_1(q^2)+m^2E_1(q^2)\}$$

$$\tilde{I}_1^S(q^2)=-\frac{3g^2}{16\pi^2}m(D_1(q^2)+E_1(q^2))q^2$$

* The so-called second and third resonance in photo-pion production also do not contribute since their resonant states will belong to $D_{3/2}$ and $F_{5/2}$. See § 5.

$$\tilde{J}_1^S(q^2) = -\frac{3eg^2}{32\pi^2} m^2 (D_1(0) + E_1(0))$$

$$\tilde{H}_1^V(q^2) = -\frac{g^2}{16\pi^2} \{C_1(q^2) + 3C_1(0) + (m^2 - q^2/2) D_1(q^2) + m^2 E_1(q^2)\}$$

$$\tilde{I}_1^V(q^2) = +\frac{g^2}{16\pi^2} mq^2 (D_1(q^2) + E_1(q^2))$$

$$\tilde{J}_1^V(q^2) = -\frac{eg^2}{32\pi^2} m^2 (D_1(0) + E_1(0)) + \frac{eg^2}{8\pi^2} \{C_2(q^2) + 4m^2 B_2(q^2)$$
$$- m^2 A_2(q^2) - 2m^2 (D_2(q^2) + E_2(q^2))\}$$

$$\tilde{H}_2^S(q^2) = -\frac{3g^2}{16\pi^2} m (D_1(q^2) + E_1(q^2)) \qquad (4\cdot15)$$

$$\tilde{I}_2^S(q^2) = -\frac{3g^2}{16\pi^2} \{C_1(0) + m^2 (D_1(q^2) + E_1(q^2)) - q^2/2 D_1(q^2)\}$$

$$\tilde{J}_2^S(q^2) = 0$$

$$\tilde{H}_2^V(q^2) = +\frac{g^2}{16\pi^2} m (D_1(q^2) + E_1(q^2))$$

$$\tilde{I}_2^V(q^2) = +\frac{g^2}{16\pi^2} \{m^2 (D_1(q^2) + E_1(q^2)) - q^2/2 D_1(q^2)\}$$

$$\tilde{J}_2^V(q^2) = \frac{eg^2}{16\pi^2} \left\{ \frac{m}{2} A_2(q^2) - 2mB_2(q^2) + (D_2(q^2) + E_2(q^2) m) \right\}$$

and $A_i(q^2)$, etc., are defined as (See reference (12)),

$$A_i(q^2) = 8 \int_0^1 x\,dx \int_0^1 dy \int_0^K \frac{k^3\,dk}{(k^2 + \Delta_i)^3}$$

$$B_i(q^2) = 8 \int_0^1 x^2\,dx \int_0^1 y\,dy \int_0^K \frac{k^3\,dk}{(k^2 + \Delta_i)^3}$$

$$C_i(q^2) = 2 \int_0^1 x\,dx \int_0^1 dy \int_0^K \frac{k^5\,dk}{(k^2 + \Delta_i)^3} \qquad (4\cdot16)$$

$$D_i(q^2) = 8 \int_0^1 x^3\,dx \int_0^1 y(1-y)\,dy \int_0^K \frac{k^3\,dk}{(k^2 + \Delta_i)^3}$$

$$E_i(q^2) = 8 \int_0^1 x^3\,dx \int_0^1 y^2\,dy \int_0^K \frac{k^3\,dk}{(k^2 + \Delta_i)^3}$$

where
$$\Delta_1 = m^2 (x^2 - \eta x + \eta) + x^2 y (1-y) q^2$$
$$\Delta_2 = m^2 \{x^2 + (\eta - 2) x + 1\} + x^2 y (1-y) q^2$$
$$\eta = \mu^2/m^2.$$

From Eq. (4·14), we find that $\widetilde{H}_{1,2}^{S,V}$, ⋯, etc., are the monotonously decreasing function of q^2, and magnetic moment and charge and magnetization mean square radii are directly calculated to become†

$$\mu^S=\frac{\mu^S_{pert}}{1+(3/16\pi^2)g^2\{C_1(0)+m^2(D_1(0)+E_1(0))\}}$$

$$\mu^V=\frac{\mu^V_{pert}}{1-(1/16\pi^2)g^2\{-3C_1(0)+m^2(D_1(0)+E_1(0))\}}$$

$$\langle r_1^2\rangle^S=\frac{\langle r_1^2\rangle^S_{pert}+9/8\pi^2 g^2 m(D_1(0)+E_1(0))\mu^S/e}{1-(3/16\pi^2)g^2m^2(D_1(0)+E_1(0))}$$

$$\langle r_1^2\rangle^V=\frac{\langle r_1^2\rangle^V_{pert}-3/4\pi^2 g^2 m(D_1(0)+E_1(0))\mu^V/e}{1+(1/16\pi^2)g^2\{4C_1(0)+m^2(D_1(0)+E_1(0))\}} \qquad (4\cdot17)$$

$$\langle r_2^2\rangle^S=\frac{\langle r_2^2\rangle^S_{pert}-3/32\pi^2 g^2 m(D_1(0)+E_1(0))\langle r_1^2\rangle^S e/\mu^S}{1+(3/16\pi^2)g^2\{C_1(0)+m^2(D_1(0)+E_1(0))\}}$$
$$+\frac{+9/8\pi^2 g^2\{m^2(D_1'(0)+E_1'(0))-1/2D_1(0)\}}{1+(3/16\pi^2)g^2\{C_1(0)+m^2(D_1(0)+E_1(0))\}}$$

$$\langle r_2^2\rangle^V=\frac{\langle r_2^2\rangle^V_{pert}+1/32\pi^2 g^2 m(D_1(0)+E_1(0))\langle r_1^2\rangle^V e/\mu^V}{1-(1/16\pi^2)g^2\{-3C_1(0)+m^2(D_1(0)+E_1(0))\}}$$
$$-\frac{3/8\pi^2 g^2\{m^2(D_1'(0)+E_1'(0))-1/2D_1(0)\}}{1-(1/16\pi^2)g^2\{-3C_1(0)+m^2(D_1(0)+E_1(0))\}}$$

Table I*

	Experiment	Second Order Perturbation (no cut off)	Second order Perturbation (cut off at $K\sim m$)	Present Approximation
μ^S	-0.06 $(e/2m)$	-1.64	-0.84	-0.33
μ^V	1.8 (〃)	2.16	1.64	1.16
$\langle r_1^2\rangle^S$	~ 0.32 $(1/\mu^2)$	0.11	0.056	0.24** / 0.33
$\langle r_1^2\rangle^V$	~ 0.32 (〃)	0.20	0.16	0.053 / 0.043
$\langle r_2^2\rangle^S$	$\mu^S\langle r_2^2\rangle^S \ll \mu^V\langle r_2^2\rangle^V$	0.57	0.025 / 0.34	0.27 / 1.5
$\langle r_2^2\rangle^V$	~ 0.32 (〃)	0.13	0.15 / 0.098	0.11 / 0.073

* We take $g^2/4\pi=14.5$.

** We have used both calculated and experimental values of μ^S and μ^V to calculate $\langle r_1^2\rangle^S$, etc. Upper and lower figures indicate the case of calculated and experimental $\mu^{S,V}$, respectively.

† Essentially same approximation was once carried out by H. Miyazawa[13] and M. Yamada,[14] though their derivations are quite different.

where D' and E' denote differentiation with respect to q^2, and $\langle r_1^2 \rangle^S_{pert}$, etc., represent the second-order perturbation result. Finally, we calculate Eq. (4·17) with cutoff momentum $K \approx m$, supposing that high frequency part will be properly cut off by the pion nucleon vertex function $K(\tilde{\xi})$. Numerical values are summarized in Table I together with experimental data for convenience.

§ 5. Concluding remarks

In the previous sections, we have investigated charge and magnetic moment form factors, based on spectral representations for fixed momentum transfer, choosing nucleon mass as a variable. These representations are different from the usual ones in the following points;

i) These representations can be proved only by means of general principles;

ii) Thresholds of the absorptive parts for vector parts are the same with those for scalar parts, that is, we can calculate both scalar and vector parts in the same lowest mass configuration;

iii) Absorptive parts contain the electromagnetic form factors of the nucleon and we can get two sets of coupled algebraic equations for vector and scalar parts of form factors.

We have assumed that the subtraction is unnecessary both for charge and magnetic moment form factors, thereby, of course, taking into account the contribution from equal-time commutator. In particular, when only the lowest mass configuration is taken into account, it is possible to express the absorptive parts in terms of the matrix element of pion production by virtual photon and the pion-nucleon vertex part which may be expressed by electromagnetic form factors of the nucleon and small phase shift of the $J=I=\frac{1}{2}$ pion-nucleon scattering. It is to be noticed that the approximation to take the lowest mass configuration in our calculation, at first sight, does not seem to have a direct physical meaning as that used in the usual representation which means to suppose that the lightest mass configuration may dominate the outer range structure of the nucleon. We can interpret, however, in the present case that the inverse of the threshold energy corresponds to a time of the "reaction", forming a complex intermediate state, and the lowest mass configuration corresponds to the longest time reaction which also can be expected to dominate the outer structure of the nucleon.

No contribution of (3–3) resonant state to the form factor is characteristic of the lowest mass approximation, on account of which the approximations to neglect rescattering corrections are possible. We have calculated Eq. (4·17) to see a general features of our approximation and the summarized results in Table I show that the scalar part of the magnetic moment was reduced while the scalar charge density radius was enhanced several times compared to the second order perturbation with cutoff at $K \approx m$. This is due mainly to the property of denominators in Eq. (4·17). It should also be noticed that denominators have the same sensitiveness to the cutoff momentum as magnetic moment. To include the effect

of (3–3) resonant state and moreover the second and third resonances in our formulation we must take into account higher mass configurations, for example, two pion and one nucleon state. In this case the absorptive parts are expressed in term of the pion-nucleon scattering matrix element and double photo-pion production matrix element. In this case, (3–3) resonant state may be expected to contribute to the vector parts, while the second and third resonances to the scalar parts.

Before concluding, a remark should be made on the pion-pion resonance. In our formulation, the contribution of pion-pion resonance, in the lowest mass configuration, is included in the matrix element of pion production by the virtual photon, though we have not touched in this paper on a possible role of pion-pion resonance to the electromagnetic structure of the nucleon.[15]

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Appendix A

Relation between the vertex operator and the Green's function for virtual photo-pion production process

We have shown in § 4 that the matrix element of the electromagnetic vertex operator between free nucleon states are determined by the pion-nucleon vertex and the matrix element for pion production by virtual photon in the approximation where only the states of the lowest total rest mass are taken into account. We will show in the following that not only the above-mentioned matrix elements but also the vertex operator itself may be rigorously represented by the Green's function for photo-pion production.

Following Schwinger,[16] one may define the electromagnetic vertex operator by

$$\Gamma_\mu(x, y; \xi) = i \frac{\partial G^{-1}(x, y)}{\partial e \langle A_\mu(\xi) \rangle} \qquad (A\cdot 1)$$

where $\langle\ \ \rangle$ means vacuum expectation value

$$\langle O(\xi) \rangle = \frac{\langle \mathrm{vac}|O(\xi)|\mathrm{vac} \rangle}{\langle \mathrm{vac}|\mathrm{vac} \rangle}.$$

$G^{-1}(x, y)$ is the inverse of the one-nucleon Green's function interacting with the pseudoscalar and the electromagnetic fields:

$$G^{-1}(x, y)=i\gamma\langle x|p|y\rangle-i\tau_p\gamma\langle x|e\langle A(\xi)\rangle|y\rangle+ig\gamma_5\tau_i\langle x|\langle\phi_i\rangle|y\rangle+M$$
$$=i\gamma\langle x|p|y\rangle-i\tau_p\gamma\langle x|e\langle A(\xi)\rangle|y\rangle$$
$$+(-ig)^2\gamma_5\tau_i G(x, x')\Delta(x, \xi')\Gamma_5^i(x', y; \xi')$$
$$+(-ie)^2\gamma_\mu\tau_p G(x, x')D_{\mu\nu}(x, \xi')\Gamma_\nu(x', y; \xi') \qquad (A\cdot 2)$$

where M is the mass operator of the nucleon, Δ and $D_{\mu\nu}$ are the Green's functions for a pion and a photon respectively, and the sum or integration is understood for repeated suffices or variables respectively.

We discard the last term of Eq. (A·2) in the following, confining our discussions up to the first order of e, although inclusion of the term does not cause any difficulty. Then we have

$$\Gamma_\mu(x, y; \xi)=\gamma_\mu\tau_p\delta(x-\xi)\delta(\xi-y)$$
$$+(-ig)^2\gamma_5(x)\tau_i\Delta(x, \xi')i\frac{\delta}{\delta e\langle A_\mu(\xi)\rangle}\{G(x, x')\Gamma_5^i(x', y; \xi')\}. \qquad (A\cdot 3)$$

Next, let us consider the Green's function for photo-pion production, which is given by[17]

$$G_{i\mu}^{\gamma;\pi}(x, y; \xi\ \eta)=i\left[\langle\phi_i(\xi)\rangle-i\frac{\delta}{\delta J_i(\xi)}\right]\left[A_\mu(\eta)-i\frac{\delta}{\delta J_\mu(\eta)}\right]\cdot G(x, y)\Big|_{J_i=0,\ J_\mu=0}$$
$$=-i\Delta(\xi, \xi')\frac{\delta^2 G(x, y)}{\delta\langle\phi_i(\xi')\rangle\delta\langle A_\nu(\eta')\rangle}\Big|_{J_i=0,\ J_\mu=0}\cdot D_{\nu\mu}(\eta', \eta) \qquad (A\cdot 4)$$

where J_i and J_μ means external sources for the pion field and the electromagnetic field respectively. Making use of the equation

$$\frac{\delta G(x, y)}{\delta\langle\phi_i(\xi')\rangle}=-G(x, x')\frac{\delta G^{-1}(x', y')}{\delta\langle\phi_i(\xi')\rangle}G(y', y)$$
$$=-igG(x, x')\Gamma_5^i(x', y'; \xi')G(y', y), \qquad (A\cdot 5)$$

Eq. (A·4) becomes

$$G_{i\mu}^{\gamma;\pi}(x, y; \xi\ \eta)=(-i)^2\Delta(\xi, \xi')\frac{\delta}{\delta\langle A_\nu(\eta')\rangle}\{gG(x, x')\Gamma_5^i(x', y'; \xi')$$
$$\times G(y', y)\}\Big|_{J_i=J_\mu=0}\cdot D_{\nu\mu}(\eta', \eta)$$
$$=(-i)^2 g\Delta(\xi, \xi')G(x, x')\Gamma_5^i(x', y'; \xi')\frac{\delta G(y', y)}{\delta\langle A_\nu(\eta')\rangle}\Big|_{J_i=J_\mu=0}\cdot D_{\nu\mu}(\eta', \eta)$$
$$+(-i)^2 g\Delta(\xi, \xi')\frac{\delta}{\delta\langle A_\nu(\eta')\rangle}\{G(x, x')\Gamma_5^i(x', y'; \xi')\}\Big|_{J_i=J_\mu=0}\cdot G(y', y)D_{\nu\mu}(\eta', \eta)$$
$$=-(-i)^2 g\Delta(\xi, \xi')G(x, x')\Gamma_5^i(x', y'; \xi')G(y', y'')$$
$$-(-ie)\Gamma_\nu(y'', y'''; \eta')G(y''', y)D_{\nu\mu}(\eta', \eta)$$

$$+(-i)^2 eg\Delta(\xi, \xi')\frac{\delta}{\delta\langle eA_\nu(\eta')\rangle}\{G(x, x')\Gamma_5^i(x', y'; \xi')\}\Big|_{J_i=J_\mu=0}$$
$$\times G(y', y)D_{\nu\mu}(\eta', \eta). \qquad (A\cdot 6)$$

Using Eqs. (A·3) and (A·6), we obtain

$$\Gamma_\nu(x, y'; \xi')G(y', y)D_{\nu\mu}(\xi', \xi) = \gamma_\nu\tau_p\delta(x-\xi')\delta(\xi'-y')G(y', y)D_{\nu\mu}(\xi', \xi)$$
$$+ig/e\,\gamma_5(x)\tau_i G_{i\mu}^{\gamma,\pi}(x, y; x, \xi)$$
$$-\sum^*(x, y')G(y', y'')\Gamma_\nu(y'', y'''; \xi')G(y''', y)D_{\nu\mu}(\xi', \xi) \qquad (A\cdot 7)$$

where

$$\sum^*(x, y') = -(-i)^2 g^2\gamma_5(x)\tau_i G(x, x')\Delta(x, \xi')\Gamma_{5i}(x', y; \xi')$$

gives the proper self-energy operator of the nucleon due to the interaction with the pion field. Eq. (A·7) is the desired formula which shows that the electromagnetic vertex operator is essentially determined by the Green's function for photo-pion production.

Matrix elements between free nucleons of the vertex operator may be derived from Eq. (A·7). Using the following formulas:

$$\lim_{y_0\to-\infty}\int G(x, y)\gamma_4 u(\boldsymbol{p})e^{ipy}(-i)d\boldsymbol{y} = U(\boldsymbol{p})e^{ipx},$$

$$\bar{u}(x)\sum^*(x, y') = 0,$$

where the mass and charge renormalizations are assumed to have been performed already, we obtain

$$\bar{u}_{p'}(x)\Gamma_\nu(x, y'; \xi')u_p(y')D_{\nu\mu}(\xi', \xi)$$
$$= \bar{u}_{p'}(x)\gamma_\nu\tau_p\delta(x-\xi')\delta(\xi'-y')u_p(y')D_{\nu\mu}(\xi', \xi)$$
$$+\lim_{y_0\to-\infty} ig/e\,\bar{u}_{p'}(x)\gamma_5(x)\tau_i\int G_{i\mu}^{\gamma,\pi}(x, y; x, \xi)\gamma_4 u_p(y)\,d\boldsymbol{y}. \qquad (A\cdot 8)$$

This equation shows that if the matrix element for photo-pion production, where all of the participating particles are virtual, is obtained, one can readily evaluate the electromagnetic form factors.

Appendix B

General properties of $\Gamma_\mu(p', p)$

From Lorentz covariance, the most general form of $\Gamma_\mu(p', p)$ may be written in the form given by Eq. (2·2):

$$\Gamma_\mu(p', p) = i\gamma_\mu G_1 + i\sigma_{\mu\nu}q_\nu G_2 + q_\mu G_3 + (i\gamma p' + m)i\gamma_\mu G_4 + i\gamma_\mu(i\gamma p + m)G_7$$
$$+(i\gamma p' + m)i\sigma_{\mu\nu}q_\nu G_5 + i\sigma_{\mu\nu}q_\nu(i\gamma p + m)G_8 + (i\gamma p' + m)q_\mu G_6 + q_\mu(i\gamma p + m)G_9$$

$$+(i\gamma p'+m)i\gamma_\mu(i\gamma p+m)G_{10}+(i\gamma p'+m)i\sigma_{\mu\nu}q_\nu(i\gamma p+m)G_{11}$$
$$+(i\gamma p'+m)q_\mu(i\gamma p+m)G_{12} \qquad (B\cdot 1)$$

where $q_\mu=p_\mu'-p_\mu$ and G_i's are functions of p'^2, p^2 and q^2.

The condition for invariance under charge conjugation,

$$C^{-1}\Gamma_\mu(p', p)C=-[\Gamma_\mu(-p, -p')]^T, \qquad (B\cdot 2)$$

invokes following conditions on G_i's:

$$\begin{aligned}
&G_1(p'^2, p^2, q^2)=G_1(p^2, p'^2, q^2),\\
&G_2(p'^2, p^2, q^2)=G_2(p^2, p'^2, q^2),\\
&G_3(p'^2, p^2, q^2)=-G_3(p^2, p'^2, q^2),\\
&G_4(p'^2, p^2, q^2)=G_7(p^2, p'^2, q^2),\\
&G_5(p'^2, p^2, q^2)=G_8(p^2, p'^2, q^2),\\
&G_6(p'^2, p^2, q^2)=-G_9(p^2, p'^2, q^2),\\
&G_{10}(p'^2, p^2, q^2)=G_{10}(p^2, p'^2, q^2),\\
&G_{11}(p'^2, p^2, q^2)=G_{11}(p^2, p'^2, q^2),\\
&G_{12}(p'^2, p^2, q^2)=-G_{12}(p^2, p'^2, q^2).
\end{aligned} \qquad (B\cdot 3)$$

The third equation in (B·3) shows that $G_3(q^2)$ vanishes when initial and final nucleons are on the mass shell, a property which was noted in § 2.

Appendix C

Evaluation of equal-time commutator

Eq. (2·1) may be expressed as

$$\langle p'|j_\mu(0)|p\rangle=i\frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p_0'}\right)^{1/2}\int d^4x\,\bar{u}(p')e^{-ip'x}\cdot\mathfrak{D}_x\langle 0|T(\psi(x), j_\mu(0))|p\rangle \qquad (C\cdot 1)$$

where $\mathfrak{D}_x=\gamma\partial+m$ and $j_\mu(0)$ defined by Eq. (2·4) is given explicitly when expressed in terms of renormalized Heisenberg operators as follows,

$$j_\mu(0)=\frac{ie}{2}Z_2[\bar{\psi}, \tau_p\gamma_\mu\psi]-\frac{e}{2}Z_3\phi_i T_{ij}\partial_\mu\phi_j \qquad (C\cdot 2)$$

where

$$T_{ij}=\begin{pmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Now, operating the Dirac operator,

$$\mathcal{D}_x T(\psi(x), j_\mu(0)) = T(\eta(x), j_\mu(0)) - i\delta(x_0)[\gamma_4\psi(x), j_\mu(0)]. \qquad (C\cdot 3)$$

The last term of (C·3) is the quantity that we are now going to evaluate which can be easily carried out by using the canonical commutation relation.

$$-i\delta(x_0)[\gamma_4\psi(x), j_\mu(0)] = ed^4(x)\tau_p\gamma_\mu\psi(0). \qquad (C\cdot 4)$$

From (C·4), we obtain the explicit expression of the contribution from equal-time commutator.

$$\frac{1}{(2\pi)^3}\left(\frac{m^2}{p_0' p_0}\right)^{1/2} ie\bar{u}(p')\gamma_\mu\tau_p u(p).$$

Note added in Proof : The following relations can be obtained by making use of the generalized Ward identity,

$$G_4^{S,V}(p'^2, p^2, q^2) - G_7^{S,V}(p'^2, p^2, q^2) - q^2 G_{12}^{S,V}(p'^2, p^2, q^2) = 0,$$
$$G_4^{S,V}(-m^2, -m^2, 0) = G_7^{S,V}(-m^2, -m^2, 0) = 0.$$

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Nuclear Deformation and Nuclear Force. I

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The magnitude of nuclear deformation is determined by the variational method in case of the rotational light nuclei. The Hamiltonian used in this work is essentially the same as that of Brueckner's shell model space, where the reaction matrix is calculated from the Gammel-Thaler potential. The deformed potential model wave function is taken as a trial wave function, where the deformation parameter and the inter-nucleon distance are taken as variational parameters. The central force gives the equilibrium deformation of the same order as the experimental value for some configurations in case of Mg^{24}. The discussion is also given.

§ 1. Introduction

Recently the nuclear phenomena in low energy region have been systematized with some phenomenological models and the cognition of the main features of nuclei has been established. The most important property of nuclei is that they show the independent particle feature in spite of the singular forces between constituents. Secondly, nuclei have some collective behaviours, i.e. the collective rotation and vibration and "superfluidity"—the pairing correlation.

The foundation of the independent particle feature has been given by the Brueckner theory[1] in relation to the actual nuclear force, though it is restricted within the "infinite nuclear matter". It has been thought that the main part of the nuclear forces contributes to the Hartree field and the residual part causes the above collective behaviours. However, at present, the characters of the residual force are assumed rather phenomenologically and they are introduced case by case to explain different problems.

In the case of finite nuclei, some works[2)~4)] have been performed. They have treated the closed shell nuclei because in these cases there is no degenerate state. We are, here, interested in the rotational region where the nuclei have the excitation levels of the intrinsic particle motion and the collective rotational motion. The intrinsic particle motion has been described by the independent particle motion with a deformed potential[5] and the system of independent particles with a largely deformed potential shows the rotational spectra.[6] In the Brueckner theory the potential is determined self-consistently through the reaction matrix between the independent particles. It may be thought that, if there are many nucleons in an incomplete shell, this self-consistent potential will be of a deformed shape.

Our problem is to investigate the deformation of the potential determining the independent particle feature and the rotational spectra in connection with the actual nuclear force. In practice, the self-consistency problem is difficult to carry out, especially in the case of an incomplete shell, so we substitute the problem with a variational method. We perform the Hartree-Fock scheme in the shell model space, where the coordinate space representations of the reaction matrix were given by Brueckner et al.[3]*

Though Brueckner et al. have used the Gammel-Thaler potential as the original force, we have been on the standpoint that we investigate the nuclear problem on the basis of the pion-theoretical potential.[7] However, the numerical results of the reaction matrix based on that potential are not calculated, so we borrow the results based on the Gammel-Thaler potential. Therefore we should understand our results as qualitative one.

As the trial wave function we take the wave function of Nilsson's model which has been shown to be a fairly good model in the rotational region. This wave function contains a parameter which describes the deformation of the one-body potential phenomenologically. The minimum of the total energy is determined by the variation with respect to the deformation and the density which is coupled with the strength of the potential well.

The energy at the equilibrium deformation will be determined by the diagonal parts of the reaction matrix in the representation of the deformed potential model, but they include the non-diagonal parts in the representation of the spherical potential model. Although Elliott[8] introduced the quadrupole-quadrupole interaction as the residual force in the spherical potential model in order to explain the rotational spectra, its strength and the relation to the nuclear force are unknown. Obviously the tensor force seems to contribute to the deformation in an incomplete shell but the quadrupole-quadrupole part in the multipole expansion of the central force will also contribute. In this paper we first take the central force only, and in the succeeding paper we will take into account the tensor and spin-orbit forces. In § 2 we shall discuss the properties of the effective force and the validity of the variational method, and in § 3 the detailed calculation will be presented. We shall show the results in § 4 and also give the concluding discussion.

§ 2. Effective two-body force and variational calculation for nuclear deformation

According to the Brueckner theory the effective two-body force from which the single particle potential is derived by the Hartree-Fock approximation can be given by the reaction matrix of two-body scattering. This effective force depends on the distribution of particles surrounding the interacting nucleon pair. In the

* In the following we shall refer to this paper as BGW.

coordinate space representation it becomes a non-local potential. Brueckner et al.[3] evaluated it, using the Gammel-Thaler potential.

It is very difficult to carry out the self-consistent problem for the finite nuclei in order to calculate the effective two-body force. The main difference between the reaction matrices of the finite nuclei and the infinite nuclear matter appears in the energy denominators of them. This difference will not be decisive, since the energy difference appearing in the energy denominator is quite large and it is supposed that, in comparison with this value, the difference between the energy spectrum of the actual finite nucleus and that of the infinite nuclear matter is very small. Thus it may not introduce a serious error to replace the effective two-body force in the actual nucleus with that of the infinite nuclear matter. Besides, we neglect the one-body transition in the reaction matrix, which vanishes for a large system.

There is another question related to the degenerate configuration. In the Brueckner theory the reaction matrix is evaluated in case of the closed shell configuration. We shall, however, treat the nuclei which have incomplete shell configurations as the ground states. In such cases, Brueckner's method separates the whole problem into two parts, the determination of the reaction matrix and the calculation of the energy change for degenerate states.[2] The former is the same as in the case of the closed shell and the ground state configuration will be determined self-consistently in the whole scheme.* Thus we assume Brueckner's reaction matrix as the effective potential even for the case of the incomplete shell.

According to the numerical calculation by Brueckner et al., the characteristics of this effective force are:

1) There appears non-locality at considerably small distances outside the hard core range $r_c = 0.4 \times 10^{-13}$ cm and it vanishes at distances further than 1×10^{-13} cm.
2) The effective force becomes nearly equal to the original force outside the above distance.
3) The strength of its radial part depends on the relative angular momentum of the interacting nucleon pair and slightly on the density of the nuclear matter.
4) The effective force vanishes inside the hard core range.
5) There is a very high local repulsive core spike at $r = r_c$ and it depends on the density. This dependence is indispensable to the saturation.

Here we take Gammel-Thaler's phenomenological potential for the reason that the result of the calculation is available.[3] In the course of calculation, we assume that

1) The angular momentum dependence of the attractive part is neglected, since the waves with the angular momenta higher than the S-state do not contribute very much to the deformation. This will be discussed in § 4.

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* In our treatment of deformation parameter, the ground state configuration is rather uniquely determined.

2) The radial part which has non-locality is replaced by an effective local function.* Errors in the radial matrix element may be very small since the non-locality appears only in small distance.
3) The density dependence of the attractive part is neglected.
4) For the strength of the repulsive core spike at $r=r_c$, we use the value given in BGW for the S-state of the relative motion. It is noted that the variation with the deformation parameter in the trial function appears under the condition of incompressibility of the nuclear matter, so the spread of the radial wave function varies with the deformation parameter. If we perform the self-consistent problem, this variation is reflected into the reaction matrix and we shall not be able to neglect the part of the core spike. We take into account this effect on the assumption that the inter-nucleon distance r_0 in BGW is replaced by $r_0(\delta)$, whose functional form is determined in (3·6). For the P-state we neglect the hard core spike because it will be reduced to a few percent of that for the S-state.
5) Higher angular momentum components in the effective force than D-state are disregarded since only low angular momentum states contribute to the reaction matrix in the ground state of usual nuclei.

Then the effective potential for the central part is as follows:

$$(\boldsymbol{r}_1, \boldsymbol{r}_2|K|\boldsymbol{r}_1', \boldsymbol{r}_2') = \delta(\boldsymbol{R}-\boldsymbol{R}')\,(\boldsymbol{r}|K|\boldsymbol{r}') \tag{2·1}$$

and

$$(\boldsymbol{r}|K|\boldsymbol{r}') = (\boldsymbol{r}|K|\boldsymbol{r}')_{attractive} + (\boldsymbol{r}|K|\boldsymbol{r}')_{core}, \tag{2·2}$$

where $\boldsymbol{R}=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}$ and $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$. For the attractive part, we write as

$$(\boldsymbol{r}|K(S)|\boldsymbol{r}')_{attractive} = \sum_{l=0}^{2} v_c^{\,S}(r)\cdot\frac{\delta(r-r')}{4\pi r^2}(2l+1)\,P_l(\widehat{\boldsymbol{r},\boldsymbol{r}'}), \tag{2·3}$$

where S denotes eigenstates of two-nucleon system. $v_c^{\,S}(r)$ is an effective local potential, which is given in BGW, and is shown in Fig. 1 for the various states. The core part can be given by

$$(\boldsymbol{r}|K|\boldsymbol{r}')_{core} = \begin{cases} A\dfrac{\delta(r-r_c)\delta(r'-r_c)}{4\pi r_c^{\,2}} f(r_0) & \text{for } S\text{-state} \\ 0 & \text{otherwise} \end{cases} \tag{2·4}$$

and

$$r_c = 0.4\times 10^{-13}\ \text{cm},$$

where $f(r_0)$ is a function of the inter-nucleon distance $r_0(\delta)$, i.e.

---

* The non-locality of the angular part $P_l(\widehat{\boldsymbol{r},\boldsymbol{r}'})$ plays a role of the selection of the diagonal elements with respect to the state of the angular momentum in the relative motion. Therefore, our effective force may be regarded as a local force.

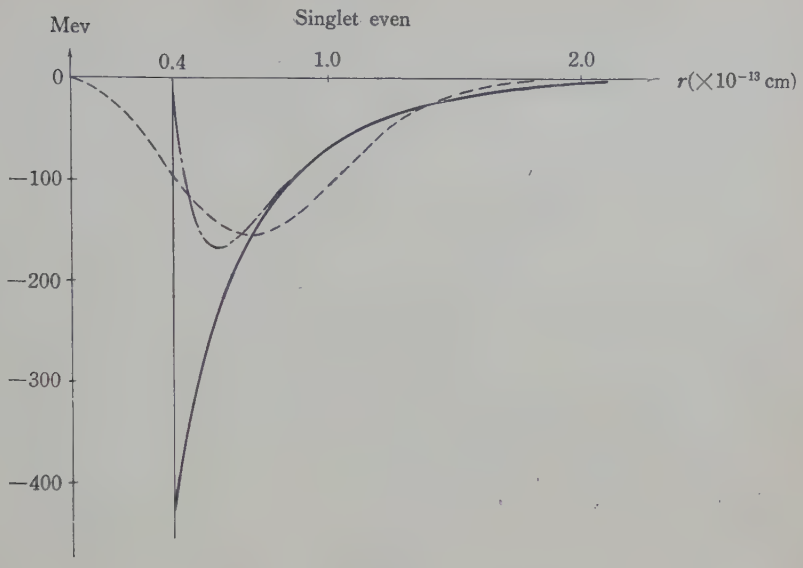


Fig. 1 (a).

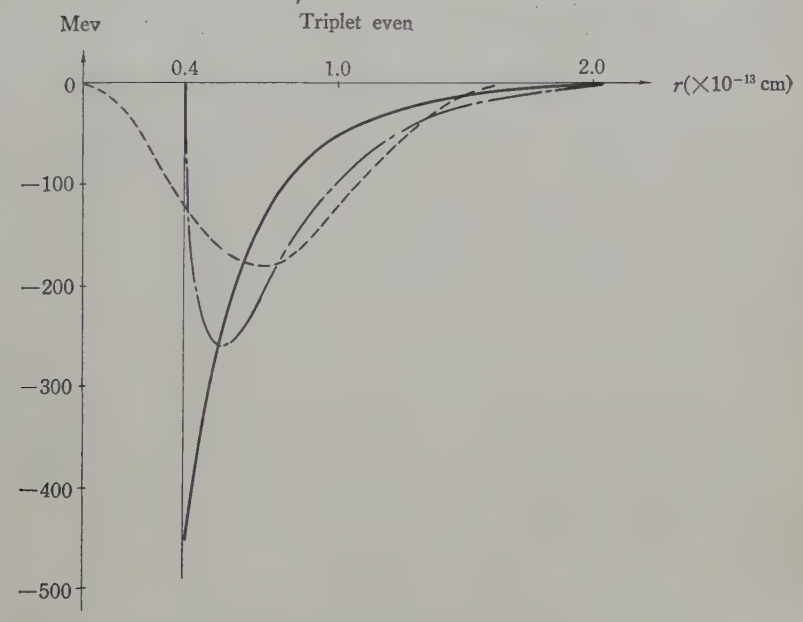


Fig. 1 (b).

Fig. 1. The effective local potential

——— The Gammel-Thaler potential

– – – $\overset{0}{V}{}_c^S(r)$ from BGW. For $r \geqq 0.8\times 10^{-13}$ cm in singlet even and for $r \geqq 2.1\times 10^{-13}$ cm in triplet even the Gammel-Thaler potential is used.

– – – – – The approximation by the form $Vr^2\exp(-\mu r^2)$ for the calculation of $F^S{}_{nl}$ with $n \neq 0$. V and μ are listed in the Table.

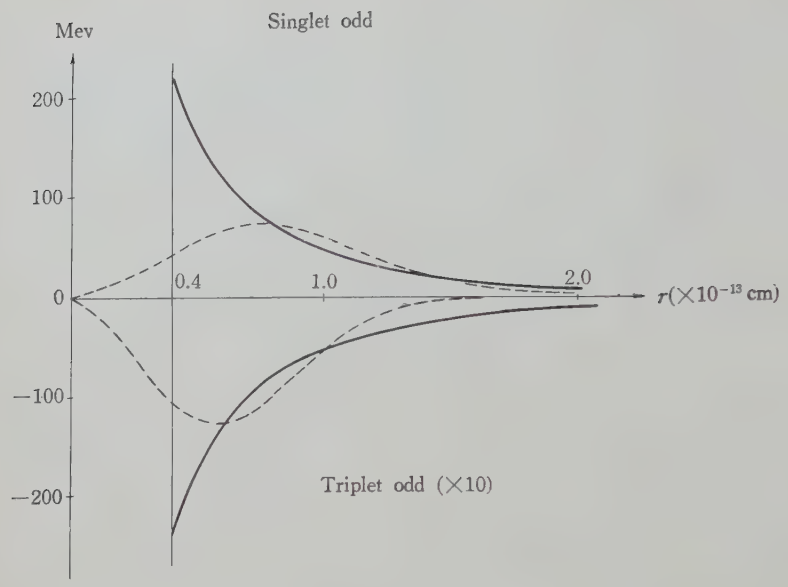


Fig. 1 (c).

Table. V and μ in various states

State	Singlet		Triplet	
	Even	Odd	Even	Odd
V (Mev 10^{26} cm^{-2})	−869.5	344.4	−1074	−105.5
μ (10^{26} cm^{-2})	2.105	1.726	2.162	3.020

$$f(r_0)=\begin{cases}\dfrac{0.544}{1-0.488/r_0(\delta)} & \text{for spin singlet}\\[2ex] \dfrac{0.571^*}{1-0.459/r_0(\delta)} & \text{for spin triplet,}\end{cases} \qquad (2\cdot5)$$

and the constant A is

$$215\ \text{Mev}\times(10^{-13}\ \text{cm}) \quad \text{for spin singlet}$$

$$257\ \text{Mev}\times(10^{-13}\ \text{cm}) \quad \text{for spin triplet.}$$

The Schrödinger equation of the many-body system in the independent particle model space is

$$H\Psi=(H_0+\sum_{\alpha}K_{\alpha}^{c})\Psi=E\Psi, \qquad (2\cdot6)$$

* This value is 0.472 in BGW. It seems to be a misprint.

where H_0 is the total kinetic energy operator, K_α^c is the reaction matrix of the α-th nucleon pair and the superscript c stands for the diagonal part. It was proved by Brueckner et al. that the total energy E is approximately equal to that of actual nucleus. The model wave function is an antisymmetrized product of individual particle wave functions.

Putting $(\Psi, \Psi)=1$, the total energy E can be given by

$$E=(\Psi, H\Psi). \tag{2·7}$$

We determine the expectation value of the total energy E by a variational method. Generally in the self-consistent problem for the nuclei of the rotational region, we should choose a superposition of the shell model wave functions as the "chosen configuration". Then in the variational method, the coefficients of the superposition will be treated as variational parameters in addition to the density. However, since the variational calculation with these many parameters is difficult to carry out, we take a deformed potential model function $\Psi(\delta)$, where δ is a parameter describing the deformation of the potential, so that the superposition is represented as a special functional form of one parameter. In this case, if the equilibrium deformation obtained by the variational method becomes nearly equal to that of the model whose wave function we borrow as a trial function, the self-consistency seems to be satisfied to some extent. Besides, the relation of the deformation to the nuclear force is understood qualitatively. Thus we treat the deformation parameter δ as a variational parameter.

A non-spherical wave function $\Psi(\delta)$ is generally not an eigenstate of total angular momentum I. In order to obtain a better approximation of energy, we must choose a trial function so that its total angular momentum is a good quantum number. For the even-even nuclei we project the state into $I=0$ state in order to evaluate the expectation value of the energy of the ground state. If P^I is the projection operator for the angular momentum I, the ground state energy is

$$E_{tot}(\delta, r_0)=\frac{(\Psi(\delta, r_0), HP^0\Psi(\delta, r_0))}{(\Psi(\delta, r_0), P^0\Psi(\delta, r_0))}, \tag{2·8}$$

where r_0 is another variational parameter corresponding to the density. For the nuclei whose intrinsic and collective motion are well separated or whose deformation are considerably large, Eq. (2·8) can be written as follows:[6]

$$E_{tot}(\delta, r_0)\cong(\Psi, H\Psi)-\frac{\hbar^2}{2\mathcal{J}}(\Psi, \boldsymbol{I}^2\Psi), \tag{2·9a}$$

where

$$\frac{\hbar^2}{2\mathcal{J}}=\frac{(\Psi, H\boldsymbol{I}^2\Psi)-(\Psi, H\Psi)(\Psi, \boldsymbol{I}^2\Psi)}{(\Psi, \boldsymbol{I}^4\Psi)-(\Psi, \boldsymbol{I}^2\Psi)^2}. \tag{2·9b}$$

The minimum point of $E_{tot}(\delta, r_0)$ gives the equilibrium deformation and density, whose values are determined by

$$\frac{\partial E_{tot}(\delta, r_0)}{\partial r_0}=0 \quad \text{and} \quad \frac{\partial E_{tot}(\delta, r_0)}{\partial \delta}=0. \tag{2·10}$$

We discuss in the next section the detailed calculation of Eqs. (2·9a) and (2·10) by using the above effective force. For simplicity of the calculation, we carry out the variational calculation in the case of the light even-even nuclei. As a typical example, the calculation is performed for Mg^{24} which exhibits a large equilibrium deformation ($\delta_{exp}\simeq 0.3$).

§ 3. Calculation

1. *Trial wave function*

We take Nilsson's model wave function $\Psi(\delta, r_0)$[5] as the trial wave function of the intrinsic state. The single particle wave function $\psi_{N\Omega}(\boldsymbol{r}\,;\delta, r_0)$ in Nilsson's model is expanded by the products of the normalized radial wave function $\phi_{Nl}(\nu r)$ for the harmonic oscillator potential, spherical harmonics $Y_{l\Lambda}(\theta, \varphi)$ and spin wave function $\chi_\Sigma(\sigma)$ as follows:

$$\psi_i^m(\boldsymbol{r}_i\,;\delta, r_0)\equiv\psi^m_{N_i\Omega_i}(\boldsymbol{r}_i\,;\delta, r_0)=\sum_{\substack{l_i\Lambda_i\\(\Omega_i=\Lambda_i+\Sigma_i)}} A^m_{N_il_i\Lambda_i}(\delta)\cdot\varphi_{N_il_i\Lambda_i}(\boldsymbol{r}_i\,;\delta, r_0)\cdot\chi_{\Sigma_i}(\sigma_i), \tag{3·1a}$$

where

$$\varphi_{Nl\Lambda}(\boldsymbol{r}\,;\delta, r_0)=\phi_{Nl}(\nu r)\,Y_{l\Lambda}(\theta, \varphi)$$
$$=\sqrt{2\nu^3\frac{(\frac{1}{2}N-\frac{1}{2}l)!}{\{\Gamma(\frac{1}{2}N+\frac{1}{2}l+\frac{3}{2})\}^3}}(\nu r)^l\exp(-\tfrac{1}{2}\nu^2r^2)\,\mathcal{L}^{l+(1/2)}_{(1/2)N-(1/2)l}(\nu^2r^2)\,Y_{l\Lambda}(\theta, \varphi) \tag{3·1b}$$

and

$$\sum_{l\Lambda}|A^m_{Nl\Lambda}(\delta)|^2=1.$$

The quantum numbers N, l, Λ and Σ are the same ones as Nilsson's and $\mathcal{L}^{l+(1/2)}_{(1/2)N-(1/2)l}$ is the associated Laguerre polynomial. A superscript m denotes a level number in Nilsson's diagram and i refers to i-th particle. ν is the inverse radius parameter for harmonic oscillator:

$$\nu=\left(\frac{M\omega}{\hbar}\right)^{1/2}. \tag{3·2}$$

The relation between r_0 and ω will be given in (3·5). The intrinsic wave function of the total system is the antisymmetrized product of the single particle wave functions of the chosen configuration:

$$\Psi(\delta, r_0)=\mathfrak{A}\prod_i^A\psi_i(\boldsymbol{r}_i\,;\delta, r_0), \tag{3·3}$$

where $\mathfrak{A}$ denotes the antisymmetrization operator.

The δ-dependences of the wave function appear in the following two parts:

i) the coefficients $A^{m}{}_{NL\Lambda}(\delta)$

ii) the variable ν in the radial wave function $\phi_{Nl}(\nu r)$, since

$$\omega \equiv \omega(\delta, r_0) = \omega(0, r_0) \cdot \left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{-1/6}, \tag{3·4}$$

which is derived by the condition of constant volume. The relation between $\omega(\delta, r_0)$ and r_0, the inter-nucleon distance, is obtained by putting the average value of r^2 equal to $(3/5)(r_0(\delta)A^{1/3})^2$, i.e.

$$\frac{1}{A}\langle \Psi(\delta) | \sum_i r_i^2 | \Psi(\delta) \rangle = \frac{1}{A} \frac{\hbar}{M\omega(\delta, r_0)} \sum_i^A \left(N_i + \frac{3}{2}\right) = \frac{3}{5}(r_0(\delta)A^{1/3})^2, \tag{3·5}$$

and consequently

$$r_0(\delta) = \left[\frac{\hbar}{M\omega(0, r_0)} \sum_i^A \left(N_i + \frac{3}{2}\right) \Big/ \frac{3}{5}A^{5/3}\right]^{1/2} \cdot \left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{1/12}$$

$$\equiv r_0\left(1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right)^{1/12}. \tag{3·6}$$

We assume this $r_0(\delta)$ to be r_0 in the expression of the hard core spike in BGW in order to take into account the δ-variation of the wave function.

Regarding the configuration in the unfilled shell, it is uniquely determined in the case of Nilsson's model at any value of δ. For example, the configuration of Mg^{24} consists of the states in the closed shell, $N=0,1$, and the No. 6- and No. 7-states in Nilsson's diagram in the unfilled shell, $N=2$. However, Nilsson's shell model has not been ascertained whether it is self-consistent or not. Even if it is self-consistent, the configuration of No. 6 and No. 7 is not necessarily found as the lowest since we do not treat the whole forces but only central one in this paper. Therefore, the variational calculation must be proceeded in the several cases of different configurations. In our case two neutrons and two protons are always filled into a Nilsson's level.

In order to evaluate the matrix elements of the effective forces, we transform the product of the single particle wave functions to the wave function represented in the relative and center-of-mass coordinate space in order to avoid the more complexity of the multipole expansion of the non-local potential. Since the single particle wave function is expressed by a superposition of the wave functions for the spherically symmetric harmonic oscillator potential, we can easily perform the transformation of the spacial part as follows:[9]

$$\varphi_{N_i l_i \Lambda_i}(\boldsymbol{r}_1)\varphi_{N_j l_j \Lambda_j}(\boldsymbol{r}_2) = \sum_{\substack{nlL\lambda \\ \binom{N+n=N_i+N_j}{\Lambda+\lambda=\Lambda_i+\Lambda_j}}} B^{(ij)}_{nlL\lambda}\varphi_{NL\Lambda}(\sqrt{2}\boldsymbol{R}) \cdot \varphi_{nl\lambda}(\boldsymbol{r}/\sqrt{2}). \tag{3·7}$$

$\varphi_{nl\lambda}(\boldsymbol{r}/\sqrt{2})$ and $\varphi_{NL\Lambda}(\sqrt{2}\boldsymbol{R})$ are the same as the harmonic oscillator wave function except their arguments.

2. *Expectation value :* $(\Psi(\delta, r_0),\ H\Psi(\delta, r_0))$

Using the intrinsic wave function (3·3), we calculate the first term in Eq. (2·9a):

$$(\Psi(\delta, r_0),\ H\Psi(\delta, r_0)) = \sum_{i=1}^{A}\int \psi_i^*(\boldsymbol{r}\,;\,\delta,\,r_0)\,T_i\,\psi_i(\boldsymbol{r}\,;\,\delta,\,r_0)\,d\boldsymbol{r}$$
$$+\sum_{i<j}\int \psi_i^*(\boldsymbol{r}_1\,;\,\delta,\,r_0)\,\psi_j^*(\boldsymbol{r}_2\,;\,\delta,\,r_0)\,(\boldsymbol{r}_1\,\boldsymbol{r}_2|K|\,\boldsymbol{r}_1',\,\boldsymbol{r}_2')$$
$$\times(\psi_i(\boldsymbol{r}_1'\,;\,\delta,\,r_0)\,\psi_j(\boldsymbol{r}_2'\,;\,\delta,\,r_0)-\psi_j(\boldsymbol{r}_1'\,;\,\delta,\,r_0)\,\psi_i(\boldsymbol{r}_2'\,;\,\delta,\,r_0))\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_1'\,d\boldsymbol{r}_2', \tag{3·8}$$

where $\boldsymbol{r}$ denotes all the coordinates of one particle.

The kinetic energy can be easily evaluated:

$$E_{kin}(\delta,\,r_0) = \frac{1}{2}\hbar\omega(\delta,\,r_0)\sum_{i=1}^{A}\left(N_i+\frac{3}{2}\right). \tag{3·9}$$

The potential part in (3·8) can be rewritten after carrying out the calculation of the charge state as follows:

$$E_{pot}(\delta,\,r_0) = 2\sum_{\substack{n-n\\p-p}}\{(ij|K(^1E)\,|ij)+(ij|K(^3O)\,|ij)\}$$
$$+\sum_{\substack{n-p\\p-n}}\{(ij|K(^1E)\,|ij)+(ij|K(^3E)\,|ij)+(ij|K(^1O)\,|ij)+(ij|K(^3O)\,|ij)\}, \tag{3·10}$$

where $\sum_{p-p}$, $\sum_{n-n}$ and $\sum_{n-p}$ show the summations over the states of proton-proton pairs, neutron-neutron pairs and neutron-proton pairs, respectively.

After integrating the matrix elements in (3·10) with respect to the center-of-mass coordinates, for a typical example, the matrix element $(ij|K(^1E)\,|ij)$ is reduced by making use of (3·1), (3·7) and (2·3) as follows:*

$$(ij|K(^1E)\,|ij) \equiv \int \psi_i^*(\boldsymbol{r}_1\,;\,\delta,\,r_0)\,\psi_j^*(\boldsymbol{r}_2\,;\,\delta,\,r_0)\,\delta(\boldsymbol{R}-\boldsymbol{R}')\,(\boldsymbol{r}|K(^1E)\,|\boldsymbol{r}')\left(\frac{1-(\sigma_1\sigma_2)}{4}\right)$$
$$\times\psi_i(\boldsymbol{r}_1'\,;\,\delta,\,r_0)\,\psi_j(\boldsymbol{r}_2'\,;\,\delta,\,r_0)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2\,d\boldsymbol{r}_1'\,d\boldsymbol{r}_2'$$
$$=\sum_i\sum_j\sum_{i'}\sum_{j'}A_iA_jA_{i'}A_{j'}\sum_{\substack{nlL\lambda\\(even)}}\sum_{n'l'L\lambda}B_{nlL\lambda}^{(ij)}B_{n'l'L\lambda}^{(ij)}$$
$$\times\int_{spin}\varphi_{nl\lambda}^*\left(\frac{\boldsymbol{r}}{\sqrt{2}}\,;\,\delta,\,r_0\right)\chi_{\Sigma_i}^*\chi_{\Sigma_j}^*\,(\boldsymbol{r}|K(^1E)\,|\boldsymbol{r}')\left(\frac{1-(\sigma_1\sigma_2)}{4}\right)$$
$$\times\varphi_{n'l'\lambda}\left(\frac{\boldsymbol{r}'}{\sqrt{2}}\,;\,\delta,\,r_0\right)\chi_{\Sigma_i}\chi_{\Sigma_j}\frac{d\boldsymbol{r}\,d\boldsymbol{r}'}{(\sqrt{2})^3}$$

---

* $A_{N_il_iA_i}(\delta)$ is abbreviated as A_i and $\sum_i$ denotes $\sum_{l_iA_i}$.

$$= \sum_i \sum_j \sum_{i'} \sum_{j'} A_i A_j A_{i'} A_{j'} \sum_{\substack{nlL\lambda \\ (even)}} \frac{1}{2} |B^{(ij)}_{nlL\lambda}|^2 F^S_{nl}(\delta, r_0) \equiv \sum_{nl} C^{(ij),S}_{nl}(\delta) \cdot F^S_{nl}(\delta, r_0), \tag{3·11}$$

where

$$F^S_{nl}(\delta, r_0) \equiv \int \phi_{nl}\left(\frac{\nu r}{\sqrt{2}}\right) \mathcal{V}_c{}^S(r) \phi_{nl}\left(\frac{\nu r}{\sqrt{2}}\right) \frac{r^2 dr}{(\sqrt{2})^3} \tag{3·12}$$

and here S denotes the singlet even state.

Therefore, we finally obtain the potential energy (3·10) in the following form:

$$E_{pot}(\delta, r_0) = \sum_{ij} \sum_S \sum_{nl} C^{(ij),S}_{nl}(\delta) F^S_{nl}(\delta, r_0). \tag{3·13}$$

$C^{(ij),S}_{nl}(\delta)$'s are composed of the coefficients $A_{NlA}(\delta)$'s and $B^{(ij)}_{nlL\lambda}$'s, and depend on the deformation parameter through $A_{NlA}(\delta)$. $F^S_{nl}(\delta, r_0)$ depends on the deformation and density parameter through $\omega(\delta, r_0)$. The most important contribution to the δ-variation of $E_{pot}(\delta, r_0)$ comes from the $n=l=0$ part and the contributions from other's are less than ten percent in the case of the Gammel-Thaler potential. So we estimate $F_{nl}{}^S$ numerically for $n=l=0$ with an approximation of a few percent, but for $n\neq 0$ $\mathcal{V}_c{}^S(r)$'s are approximated by the form $Vr^2 \exp(-\mu r^2)$. The matrix elements from this form are adjusted at $n=l=0$ to those obtained numerically, and the forms are shown in Fig. 1. The δ-variation of the matrix elements of these forms is slightly larger than that of $\mathcal{V}_c{}^S(r)$ because of the short range of the r^2 times Gauss form. If the configuration of a nucleus is determined, we shall be able to obtain the potential energy $E_{pot}(\delta, r_0)$ at any value of δ and r_0. Our calculation is performed at $\delta=0$, 0.16, 0.32 and 0.48, where $A_{NlA}(\delta)$'s are tabulated in the Appendix of the reference 5).

3. *Expectation value*: $(\hbar^2/2\mathcal{J})(\Psi(\delta, r_0), \boldsymbol{I}^2\Psi(\delta, r_0))$

An analogous calculation has been done by Yoccoz,[6] adopting the wave function for an anisotropic harmonic oscillator potential without one-body spin-orbit potential. The results for some rotational nuclei show good agreement with experiments. Therefore, we take the experimental value of $\mathcal{J}$ instead of that to be calculated according to Eq. (2·9b). The δ-dependence of $\mathcal{J}(\delta)$, which is needed for our variational calculation, is assumed from the experimental data[10] which give the relation between δ_{eq} and $(\mathcal{J}_{experiment}/\mathcal{J}_{rigid})$ for various nuclei.* For Mg^{24}, the curve of $\mathcal{J}(\delta, r_0)/\mathcal{J}_{rig}(\delta, r_0)$ is fitted with the experimental value at $\delta_{eq}=0.3$ and $r_0=1.2\times10^{-13}$ cm.

Since the operator $\boldsymbol{I}^2 \equiv (\sum_i \boldsymbol{j}_i)^2$ consists of two-body operators as the product

* This assumption is not very reasonable since the δ-variation must be performed under the condition of the fixed mass number. However, the last term in Eq. (2·9a) vanishes in the case of $\delta=0$, so this part may not reduce the equilibrium deformation decisively, even if it is calculated by using Eq. (2·9b).

of one-body operators, its expectation value can easily be calculated as follows:

$$(\Psi(\delta, r_0), \boldsymbol{I}^2\Psi(\delta, r_0)) = \sum_{i=1}^{A}\int\psi_i^*(\boldsymbol{r}\,;\delta, r_0)\boldsymbol{j}^2\psi_i(\boldsymbol{r}\,;\delta, r_0)d\boldsymbol{r}$$
$$+2\sum_{i<j}\int\psi_i^*(\boldsymbol{r}_1\,;\delta, r_0)\psi_j^*(\boldsymbol{r}_2\,;\delta, r_0)(\boldsymbol{j}_1\cdot\boldsymbol{j}_2)$$
$$\times(\psi_i(\boldsymbol{r}_1\,;\delta, r_0)\psi_j(\boldsymbol{r}_2\,;\delta, r_0)-\psi_j(\boldsymbol{r}_1\,;\delta, r_0)\psi_i(\boldsymbol{r}_2\,;\delta, r_0))d\boldsymbol{r}_1d\boldsymbol{r}_2. \quad (3\cdot14)$$

The contribution of the closed core to Eq. (3·14) vanishes and only the states in the unfilled shell contribute to it.

Collecting all the above results, i.e. Eqs .(3·9), (3·13) and (3·14), we obtain the expectation value of the total energy $E_{tot}(\delta, r_0)$ as follows:

$$E_{tot}(\delta, r_0)=E_{kin}(\delta, r_0)+E_{pot}(\delta, r_0)+E_J(\delta, r_0),$$

where

$$E_J(\delta, r_0)=-\frac{\hbar^2}{2\mathcal{J}(\delta, r_0)}(\Psi(\delta, r_0), \boldsymbol{I}^2\Psi(\delta, r_0)).$$

§ 4. Results and discussions

In Fig. 2 we show as a typical example the δ-dependence of the total energy $E_{tot}(\delta, r_0)$ of the configuration of No. 6 and No. 7 at $r_0=1.2\times10^{-13}$ cm. The kinetic energy, the potential energy and the second term of Eq. (2·9a) are also shown separately. The kinetic energy increases with increasing δ according to the condition of constant volume. On the other hand the potential energy varies depending on the mixture of the states and on the δ-dependence of the radial integral through the trial wave function satisfying the condition of constant volume. Thus, when the decrease of the potential energy with increasing δ is adequately strong, the total energy has a minimum at a finite value of δ, i.e. $\delta_{eq}\neq0$. The energy $E_J(\delta, r_0)$ contributes to the reduction of the equilibrium deformation slightly.

As for the variation with respect to r_0, we show the energies $(\Psi, H\Psi)$'s of two configurations at several values of r_0 in Fig. 3. The minimum energy is obtained at $r_0\simeq1.2\times10^{-13}$ cm and this value is not sensitive to the configura-

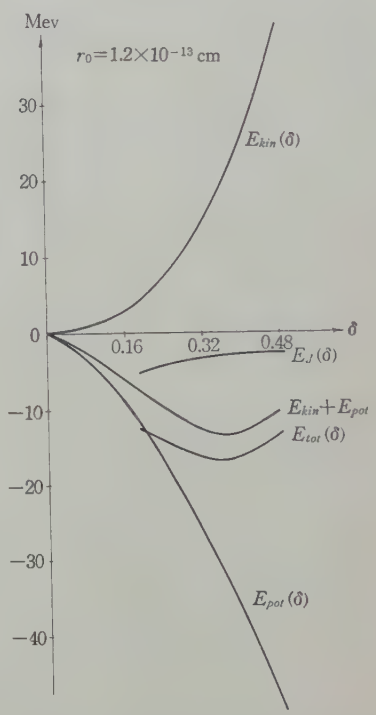


Fig. 2. The total energy $E_{tot}(\delta)$ of the configuration of No. 6 and No. 7 at $r_0=1.2\times10^{-13}$ cm. $E_{kin}(\delta)$, $E_{pot}(\delta)$ and $E_J(\delta)$ are separately shown.

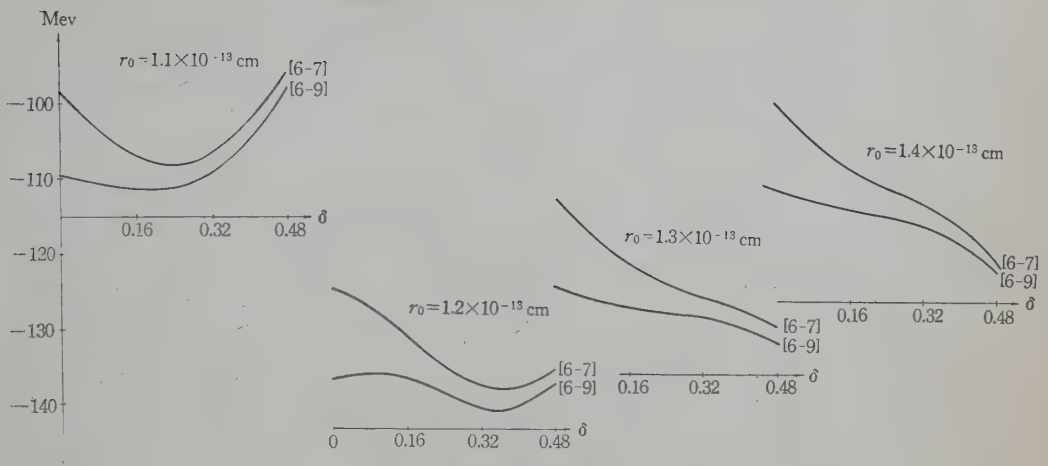


Fig. 3. The energy $E_{kin}(\delta)+E_{pot}(\delta)$ at various r_0's.
Only two configurations are shown here. The relations of other configurations to them are analogous to those in Fig. 4. [·····] shows a configuration in the open shell.

tions. Correspondingly, $\hbar\omega(0, r_0)\cong 15$ Mev, and this is almost equal to Nilsson's phenomenological value. This value of $\hbar\omega$ corresponds to the value $\nu=0.61$ which is somewhat smaller than that by Eden et al.[4] for the Gammel-Thaler potential in case of O^{16}. The minimum value of $E(\delta, r_0)$ is about -140 Mev, which is to be compared with the binding energy 198 Mev[11] of Mg^{24}. Of course, the difference is mainly due to the approximation for the reaction matrix.

The decrease of the potential energy with δ depends on the configuration of the trial wave function and so does the value of the equilibrium deformation also. In Fig. 2 the configuration of the last eight particles is of No. 6 and No. 7, which corresponds to the lowest configuration from the standpoint that Nilsson's shell model is assumed to be self-consistent. However, as we now take only the central part of the reaction matrix, the spin-orbit splitting disappears, so the configuration of No. 6 and No. 7 is not necessarily the lowest. In Fig. 4 the energies of the various configurations at $r_0=1.2\times10^{-13}$ cm are shown. The equilibrium deformation and the minimum energy are obtained for such configurations respectively. If we treat only the central force, we must take the superposition of these various configurations and the equilibrium deforma-

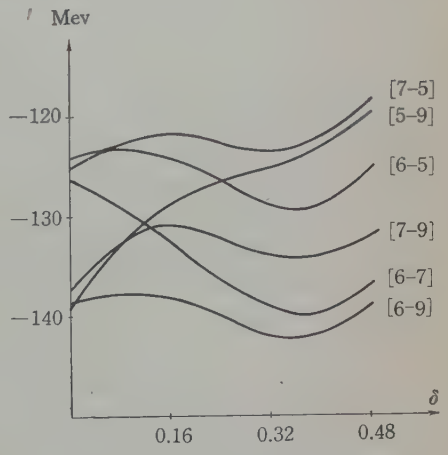


Fig. 4. The energy $E_{kin}(\delta)+E_{pot}(\delta)$ of various configurations at $r_0=1.2\times10^{-13}$ cm. [·····] shows a configuration in the open shell.

tion will be largely reduced. However, with the use of the whole parts of the reaction matrix and under the self-consistent treatment, the particles in such configuration as No. 9 will feel the spin-orbit potential, so the total energy of the configuration of No. 6 and No. 9 will be increased. This circumstance will be shown in the succeeding paper.

It is noted that the small energy differences of the total energy of the various configurations are almost not influenced by the approximation of the reaction matrix because these differences are mainly determined by $C(\delta)$'s and, at any δ and r_0, the matrix elements are the same for each configuration. Thus the relative positions of the energy curves of the various configurations have significance. Here we mainly analyse the case of No. 6 and No. 7, since it is expected that the spin-orbit and the tensor part contribute to lower it. For the configuration of No. 6 and No. 7 the final value of the equilibrium deformation is about $0.3\sim0.4$ at $r_0\cong1.2\times10^{-13}$ cm in case of the central force only. This value seems to be satisfactory in comparison with the experimental value ~0.3.* Anyhow we can say the central part of the reaction matrix plays the main role to obtain a sufficiently large deformation.

The decrease of the potential energy with δ for a configuration is caused by the favourable variations of the kinematical coefficients $C_{nl}(\delta, r_0)$'s and the radial matrix elements $F_{nl}(\delta, r_0)$'s. The contribution from the closed shell to the δ-variation of $C_{nl}(\delta, r_0)$'s vanishes. The large dependences on δ of $C_{nl}(\delta, r_0)$'s come from those which involve, for example, the configuration of No. 6. It is because the single particle state $\varphi_{200}(\boldsymbol{r})$, which is prohibited to be mixed in No. 6 under the spherical potential, is mixed in the deformed region. In Nilsson's formulation, the functions of $A_{N\Lambda}(\delta)$ are almost stationary for $\delta\geqq0.5$, because the asymptotic value corresponding to $\delta=0.75$ is nearly reached there. On the contrary, the variation of $F_{nl}(\delta, r_0)$ with respect to δ is determined through ν, so it is related to the variation with respect to r_0. In the case of a large deformation, it seems that the relation between the density and deformation differs from the above circumstances. Thus we omit the larger value of δ than 0.5.

In our case of $N=2$ shell, the harmonics in the multipole expansion of the force which contribute to the matrix elements are below the 4-th, but in the higher shells the higher harmonics will also contribute. This is because the main contribution comes from the S-state of the relative motion between two nucleons, as discussed below.

It is recognized that the contribution to the deformation from the S-state is about 90% in the Gammel-Thaler potential. This is reduced from the fact that the wave length of the relative motion is of the same order as the nuclear range, so the overlap with the potential is good only for the S-state. However, it is necessary to know the detail of the short-range part in order to obtain a quantita-

* This value is estimated from the experimental value of Na^{23}.

tive result, especially because the S-state is dominant. The above circumstance that the S-state and the central potential mainly contribute to the deformation will remain for other nuclear potentials which give the correct value of the nuclear binding energy, since the central parts of them may have the same order of the strength as Gammel-Thaler's.

There are some problems which we did not consider in this paper. It is noted that, when we treat the deformation of the finite nuclei, it may be thought that the linked cluster terms neglected for the expectation value of energy will be needed. The linked cluster terms are essentially of the many-body feature and related to the collective behaviour. However, the rotational spectra is related to the projection of the intrinsic state in the deformed potential to the eigenstate of the total angular momentum.[6] That is, in the representation of the deformed potential model, this rotational behaviour is thought to be of independent particle feature. Then, even if we take those terms into account, the behaviour concerning the deformation will not be altered.

The pairing correlation in the deformed nucleus is treated by Belyaev[12] and it reduces the energy of the ground state of the even-even nuclei. If we adopt the correlated trial wave function, the total energy will be decreased and the position of the equilibrium will be slightly moved to a smaller value of δ when the stability of the equilibrium is strong as in the case of the configuration of No. 6 and No. 7.

From the above results and discussions we conclude that generally the central part of the reaction matrix in Brueckner's approximation causes to yield the same order of the deformation as the experimental value with some favourable configurations which depend on the number of the nucleons in an open shell.

Acknowledgements

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Nuclear Deformation and Nuclear Force. II

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(August 18, 1960)

In the previous paper,[1] we showed the method to connect the nuclear deformation with the nuclear force and the calculation was carried out as for the central part of the reaction matrix in the case of Mg^{24}. Here we perform the same computation procedure for the tensor and spin-orbit part. Putting the results of the present calculation and the previous one together, we obtain the following conclusions: 1) The nuclear deformation is obtained as $\delta_{eq}=0.35$, which seems to be reasonable in comparison with experimental data. 2) About 150% of the value of the spin-orbit splitting comes from the spin-orbit part of the reaction matrix and it lowers the energy of the configuration of the ground state which has the stable nuclear deformation. 3) The tensor interaction among the particles in the open shell has only small contribution to the deformation. 4) The self-consistency of Nilsson's model is considerably good.

§ 1. Introduction

In the previous paper[1] (to be referred to as I), we showed a method to find the relation between the nuclear two-body force and the nuclear deformation. There we used a variational method in which the nuclear deformation δ and the internucleon distance r_0 are treated as the variational parameters. In order to see a qualitative feature with respect to the deformation we considered this problem in the case of Mg^{24} and obtained the following results: If only the central part of the reaction matrix is taken into account, it produces a stable nuclear deformation comparable with experimental data in the case of a special configuration, but the energy of this configuration is not made the lowest.

Now, we want to treat the two-body spin-orbit force and the tensor force in this paper. Putting the results of I and this paper together, we give an answer to the problem about the nuclear deformation and L-S doublet splitting in relation to the nuclear force.

We treat Mg^{24} in this paper, too. Mg^{24} has a state in which the $N=2$ shell is occupied by eight particles and the energies of all configurations are nearly degenerate if only the central part of the reaction matrix is taken into account. Then the ground state will have a mixed configuration of Nilsson's diagram[2] and will not give so large stable deformation. It is desirable that this degeneracy is removed by taking into account the spin-orbit part and the tensor part of the reaction matrix. By this splitting we can determine the configuration of the trial function which is pure in the meaning of Nilsson's diagram. By ob-

serving the variation of the energy of that configuration with δ and r_0 we can examine the consistency of Nilsson's scheme. Thus we find a picture clarifying the relation between the nuclear force and the independent particle aspect of the ground state of the finite nucleus.

In our present calculation, the tensor interaction between the open shell and the core vanishes, since we do not distort the $N=1$ shell core (or O^{16} core). According to the works by some authors,[3] this interaction can be reduced to the spin-orbit splitting and give about a half of the experimental value of the splitting. Although we are standing on the viewpoint that we investigate the nuclear problem on the basis of the pion-theoretical potential which has the large tensor potential in one-pion exchange region,[4] there are no numerical results of the reaction matrix based on this potential. Therefore we shall carry out the calculation using the results of Brueckner et al. and discuss qualitatively about the pion-theoretical potential.

Under the above consideration we calculate the energy of ground state and the equilibrium deformation by the same method as in I. As the original two-body force we use Gammel-Thaler's phenomenological force.[5] The detailed representation about the force will be given in § 2. In § 3, we show the method of calculation, and in § 4 we summarize the results and discuss some concluding remarks.

§ 2. The effective two-body forces

Standing on the point of view discussed in I, we take Brueckner's reaction matrix obtained in the case of the nuclear matter as the effective two-body force for our variational problem. The coordinate space representation of the reaction matrix was calculated by Brueckner et al.[6] based on the Gammel-Thaler phenomenological two-body potential.

The numerical values and characteristics of these reaction matrix are adopted for our variational problem because, at present, they are only the calculated results, and the properties of nuclear matter are fairly well explained by them. Moreover, it is convenient to use the Gammel-Thaler potential in order to investigate the deformation in relation to the actual two-body force, since it consists of the various types of potentials, i.e. central, tensor and spin-orbit potential. This potential is written by

$$v(r)=v_c(r)+v_T(r)S_{12}+v_{LS}(r)\boldsymbol{l}\cdot\boldsymbol{s}, \qquad (2\cdot1)$$

where the radial parts all have Yukawa form outside a repulsive hard core with radius 0.4×10^{-13} cm.

Using the same notation as that of Brueckner et al., the coordinate space representation of the reaction matrix can be generally represented as follows:

$$(\boldsymbol{r}'|K|\boldsymbol{r})_{singlet}=\frac{4\pi}{(2\pi)^3}\int k^2\,dk\sum_l(2l+1)\,j_l(kr')\,v_{singlet}(r)\,u_{lk}^{lo}(r)\,P_l(\widehat{\boldsymbol{r}\,\boldsymbol{r}'}), \qquad (2\cdot2)$$

and

$$(\boldsymbol{r}'|K|\boldsymbol{r})_{triplet}=A(\boldsymbol{r},\boldsymbol{r}')+B(\boldsymbol{r},\boldsymbol{r}')i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\left(\frac{\boldsymbol{r}'}{r'}\times\frac{\boldsymbol{r}}{r}\right)$$
$$+C(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{\sigma}_1\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\boldsymbol{\sigma}_2\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right) \qquad (2\cdot3)$$
$$+D(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{\sigma}_1\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\times\left(\frac{\boldsymbol{r}'}{r'}\times\frac{\boldsymbol{r}}{r}\right)\boldsymbol{\sigma}_2\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\times\left(\frac{\boldsymbol{r}'}{r'}\times\frac{\boldsymbol{r}}{r}\right),$$

where

$$A(\boldsymbol{r},\boldsymbol{r}')=A_1(\boldsymbol{r},\boldsymbol{r}')+A_2(\boldsymbol{r},\boldsymbol{r}'),$$

$$A_1(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{3}\sum_m(\boldsymbol{r}'|K|\boldsymbol{r})_{m,m},$$

$$A_2(\boldsymbol{r},\boldsymbol{r}')=\frac{\sqrt{2}}{6}\{(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,0}-(\boldsymbol{r}'|K|\boldsymbol{r})_{0,1}\}\cot\theta\,e^{-i\psi}-\frac{2}{3}(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,1}e^{-2i\psi},$$

$$B(\boldsymbol{r},\boldsymbol{r}')=\frac{\sqrt{2}}{4}\{(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,0}+(\boldsymbol{r}'|K|\boldsymbol{r})_{0,1}\}\csc\theta\,e^{-i\psi}, \qquad (2\cdot4)$$

$$C(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4}(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,1}\csc^2\frac{\theta}{2}e^{-2i\psi}+\frac{\sqrt{2}}{8}\{(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,0}-(\boldsymbol{r}'|K|\boldsymbol{r})_{0,1}\}\csc\theta\,e^{-i\psi},$$

$$D(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4}(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,1}\csc^2\frac{\theta}{2}\csc^2\theta\,e^{-2i\psi}$$
$$-\frac{\sqrt{2}}{8}\{(\boldsymbol{r}'|K|\boldsymbol{r})_{-1,0}-(\boldsymbol{r}'|K|\boldsymbol{r})_{0,1}\}\csc^3\theta\cot^2\frac{\theta}{2}e^{-i\psi},$$

and

$$(\boldsymbol{r}'|K|\boldsymbol{r})_{m',m}=\frac{(4\pi)^{3/2}}{(2\pi)^3}\int k^2\,dk\sum_{Jll'l''}(-i)^l i^{l'}(2l'+1)^{1/2}j_l(kr')u^{JS}_{ll'l'}(r)v^{JS}_{l'l''}(r)$$
$$\times(l'\,s0\,m|Jm)\,(lsm-m'\,m'|Jm)\,Y_l^{m-m'}(\widehat{\boldsymbol{r}\,\boldsymbol{r}'}), \qquad (2\cdot5)$$

where $(lsm-m'm'|Jm)$'s are the Clebsch-Gordan coefficients.

In the paper of Brueckner et al., only the central and spin-orbit parts of $(\boldsymbol{r}'|K|\boldsymbol{r})_{triplet}$, which are the terms of $A_1(\boldsymbol{r},\boldsymbol{r}')$ and $B(\boldsymbol{r},\boldsymbol{r}')$ in (2·4), are considered because of closed shell and are explicitly given for the states $l=0$ and 2. In our variational problem, however, we need also tensor part of $(\boldsymbol{r}'|K|\boldsymbol{r})$, i.e. the terms of $A_2(\boldsymbol{r},\boldsymbol{r}')$, $C(\boldsymbol{r},\boldsymbol{r}')$ and $D(\boldsymbol{r},\boldsymbol{r}')$ in (2·4). Using the explicit form for $(\boldsymbol{r}'|K|\boldsymbol{r})_{m',m}$ of (2·5), we obtain the tensor part of $(\boldsymbol{r}'|K|\boldsymbol{r})$ for the states $l=0$, 2 and 1* as follows:

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* We need only to consider the low angular momentum states for the ground state of usual nuclei with normal density.

$$A_2(\boldsymbol{r},\boldsymbol{r}')+C(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{\sigma}_1\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\boldsymbol{\sigma}_2\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)$$

$$+D(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{\sigma}_1\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\times\left(\frac{\boldsymbol{r}'}{r'}\times\frac{\boldsymbol{r}}{r}\right)\boldsymbol{\sigma}_2\cdot\left(\frac{\boldsymbol{r}'}{r'}-\frac{\boldsymbol{r}}{r}\right)\times\left(\frac{\boldsymbol{r}'}{r'}\times\frac{\boldsymbol{r}}{r}\right)$$

$$=T_{20}(r,r')\left\{\sqrt{\frac{6}{5}}\sum_{\mu}S^*_{2\mu}(Y_{00}(\boldsymbol{r}')\,Y_{2\mu}(\boldsymbol{r})+Y_{2\mu}(\boldsymbol{r}')\,Y_{00}(\boldsymbol{r}))\right\} \tag{2·6a}$$

$$+T_{11}(r,r')\left\{\frac{6}{5}\sum_{\mu}S^*_{2\mu}(\sum_{q}(11q\mu-q|2\mu)\,Y_{1q}(\boldsymbol{r}')\,Y_{1\mu-q}(\boldsymbol{r}))\right\} \tag{2·6b}$$

$$+T_{22}(r,r')\left\{-2\sqrt{\frac{3}{7}}\sum_{\mu}S^*_{2\mu}(\sum_{q}(22q\mu-q|2\mu)\,Y_{2q}(\boldsymbol{r}')\,Y_{2\mu-q}(\boldsymbol{r}))\right\}, \tag{2·6c}$$

where

$$T_{20}(r,r')=\frac{2}{\pi}\int k^2\,dk\,j_2(kr')\left\{-u^{11}_{00}(r)\,v_T(r)\right.$$

$$\left.-\frac{1}{\sqrt{8}}u^{11}_{02}(r)\,(v_c(r)-2v_T(r)-3v_{LS}(r))\right\} \tag{2·7a}$$

$$T_{11}(r,r')=\frac{2}{\pi}\int k^2\,dk\,j_1(kr')\left\{-\frac{5}{36}u^{01}_{11}(r)\,(v_c(r)-4v_T(r)-2v_{LS}(r))\right.$$

$$\left.+\frac{5}{24}u^{11}_{11}(r)\,(v_c(r)+2v_T(r)-v_{LS}(r))-\frac{5}{72}u^{21}_{11}(r)\,(v_c(r)-\frac{2}{5}v_T(r)+v_{LS}(r))\right\} \tag{2·7b}$$

$$T_{22}(r,r')=\frac{2}{\pi}\int k^2\,dk\,j_2(kr')\left\{-\frac{7\sqrt{2}}{20}u^{11}_{20}(r)\,v_T(r)\right.$$

$$-\frac{7}{40}u^{11}_{22}(r)\,(v_c(r)-2v_T(r)-3v_{LS}(r))+\frac{7}{24}u^{21}_{22}(r)\,(v_c(r)+2v_T(r)-v_{LS}(r))$$

$$\left.-\frac{7}{60}u^{31}_{22}(r)\,(v_c(r)-\frac{4}{7}v_T(r)+2v_{LS}(r))\right\} \tag{2·7c}$$

and

$$S_{2\pm2}=\frac{1}{2}(\sigma_{1x}\pm i\sigma_{1y})(\sigma_{2x}\pm i\sigma_{2y})$$

$$S_{2\pm1}=\mp\frac{1}{2}\{(\sigma_{1x}\pm i\sigma_{1y})\sigma_{2z}+\sigma_{1z}(\sigma_{2x}\pm i\sigma_{2y})\}$$

$$S_{20}=\sqrt{\frac{1}{6}}\{3\sigma_{1z}\sigma_{2z}-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\}.$$

Now, it is easily shown that the kinematical parts in brackets of (2·6) play the same role as those of the local tensor operator $S_{12}\cdot\delta(\cos\theta-\cos\theta')\cdot\delta(\varphi-\varphi')$ for the states $l=0$, 2 and 1, and that all of the non-local radial part $T(r,r')$'s in (2·7) become $v_T(r)\cdot\delta(r-r')/r^2$ in the Born approximation, as we expect.

According to the above consideration and the numerical calculations of the effective force for the central and spin-orbit part, we can consider that the characteristics of the radial part of the effective force for the tensor part are as follows:

1) It vanishes inside the hard core range.
2) There appears non-locality and angular dependence at considerably small distances outside the hard core range.
3) It becomes nearly equal to the original tensor potential outside $\sim 1 \times 10^{-13}$ cm.
4) It scarcely depends on density.

Under these speculations, we qualitatively assume the suitable effective tensor force based on the Gammel-Thaler potential.

As for the effective central part of potential and the hard core spike, we have

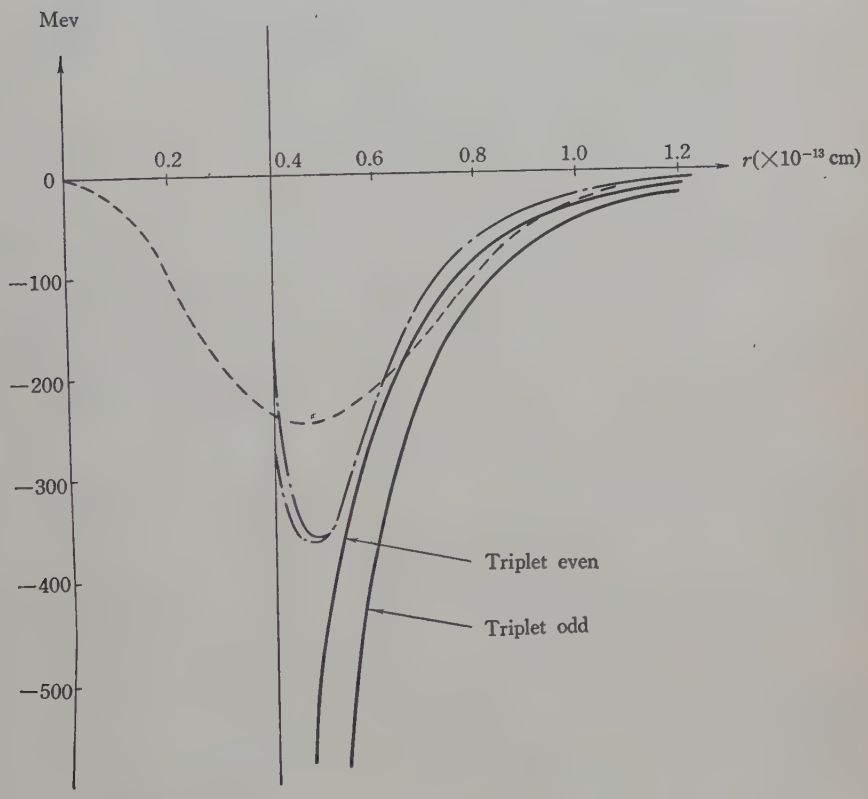


Fig. 1 (a). Spin-Orbit Force.
——— The Gammel-Thaler potential
—·—·— The effective local potential by Brueckner et al.
– – – – – – The effective local potential approximated by $V \cdot r^2 \cdot \exp[-\mu r^2]$

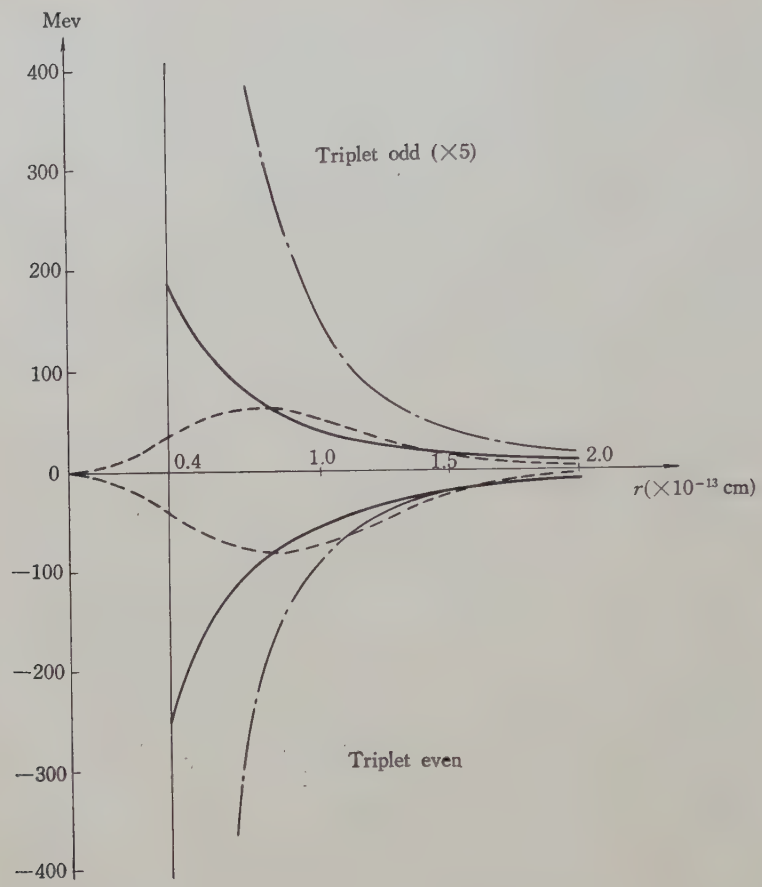


Fig. 1 (b). Tensor Force.

——— The Gammel-Thaler potential

——·—— The pion-theoretical potential

– – – – – – The effective local potential approximated by $V \cdot r^2 \cdot \exp[-\mu r^2]$

already discussed in I. For the spin-orbit and tensor parts, we approximate the radial parts by the local type, since the non-local effect is not large. The form of the radial part of the effective local potential is taken as the form of $V \cdot r^2 \times \exp(-\mu r^2)$ for the convenience of calculation. The magnitude of a matrix element for the spin-orbit part is fitted to that estimated by the numerical calculation for the effective local force of Brueckner et al. These effective forces for the various states are given in Fig. 1.

§ 3. Calculation

Our trial wave function is the same as that used in I, i.e. Nilsson's model

wave functions.[2] In this paper we calculate the contributions from the tensor and the spin-orbit potential. In our approximation of the effective local force, they are analogous to the usual shell model calculation except the radial integral. Following the definition in I, the trial wave function is

$$\Psi(\delta, r_0)=\mathfrak{A}\prod_i^A \phi_i(\boldsymbol{r}_i;\delta, r_0)=\mathfrak{A}\prod_i^A \sum_{l_i\Lambda_i} A_{N_i l_i \Lambda_i}(\delta)\cdot\varphi_{N_i l_i \Lambda_i}(\boldsymbol{r}_i;\delta, r_0)\chi_{\Sigma_i}(\sigma_i), \quad (3\cdot1)$$

and the matrix element which is a part of the potential energy is, for example,

$$\begin{aligned}(ij|K(^3E)|ij)=&\sum_{\substack{l_i\Lambda_i,\ l_j\Lambda_j\\ l_{i'}\Lambda_{i'},\ l_{j'}\Lambda_{j'}}} A_{N_i l_i\Lambda_i}(\delta)A_{N_j l_j\Lambda_j}(\delta)A_{N_{i'} l_{i'}\Lambda_{i'}}(\delta)A_{N_{j'} l_{j'}\Lambda_{j'}}(\delta)\\ &\times\sum_{\substack{nl\lambda\\ L'n'\lambda'}} B^L_{nl\lambda}B^L_{n'l'\lambda'}\left(\frac{1}{2}\ \frac{1}{2}\ \Sigma_i\Sigma_j|1\,m\right)\left(\frac{1}{2}\ \frac{1}{2}\ \Sigma_i'\Sigma_j'|1m'\right)\\ &\times\int\varphi^*_{nl\lambda}\left(\frac{\boldsymbol{r}}{\sqrt{2}};\delta, r_0\right)\chi_1^{m*}\,{}^t\!V_T(r)S_{12}(\text{or }{}^t\!V_{LS}(\boldsymbol{l}\cdot\boldsymbol{s}))\\ &\times\varphi_{nl'\lambda'}\left(\frac{\boldsymbol{r}}{\sqrt{2}};\delta, r_0\right)\chi_1^{m'}\,d\boldsymbol{r}/(\sqrt{2})^3, \end{aligned}\quad (3\cdot2)$$

where χ_1^m is an eigen-function of the total intrinsic spin of two particles. The integral in (3·2) is reduced to the following: For the tensor force

$$\begin{aligned}&\int\varphi^*_{nl\lambda}\left(\frac{\boldsymbol{r}}{\sqrt{2}};\delta, r_0\right)\chi_1^{m*}\,{}^t\!V_T(r)S_{12}\varphi_{nl'\lambda'}\left(\frac{\boldsymbol{r}}{\sqrt{2}};\delta, r_0\right)\chi_1^{m'}\,d\boldsymbol{r}/(\sqrt{2})^3\\ &\equiv\langle\varphi_{nl\lambda}\chi_1^m|\,{}^t\!V_T(r)S_{12}|\varphi_{nl'\lambda'}\chi_1^{m'}\rangle\\ &=F^T_{nl,nl'}(\delta, r_0)\sum_\mu(-1)^\mu\langle Y_{l\lambda}|L_{2\mu}|Y_{l'\lambda'}\rangle\langle\chi_1^m|S_{2,-\mu}|\chi_1^{m'}\rangle\\ &=F^T_{nl,nl'}(\delta, r_0)\cdot(-1)^{m-m'}2\sqrt{10}\sqrt{\frac{2l'+1}{2l+1}}(2l'\,00|l0)(2l'\,m-m'\,\lambda'|l\lambda)\\ &\qquad\times(21m-m'\,m'|1m)\cdot\delta_{m'-m,\lambda-\lambda'}, \end{aligned}\quad (3\cdot3)$$

and for the spin-orbit force

$$\begin{aligned}&\langle\varphi_{nl\lambda}\chi_1^m|\,{}^t\!V_{LS}(r)\cdot(\boldsymbol{l}\cdot\boldsymbol{s})|\varphi_{nl'\lambda'}\chi_1^{m'}\rangle\\ &=\langle\varphi_{nl\lambda}\chi_1^m|\,{}^t\!V_{LS}(r)\left\{\frac{1}{2}(l_+S_-+l_-S_+)+(l_zS_z)\right\}|\varphi_{nl'\lambda'}\chi_1^{m'}\rangle^*\\ &=F^{LS}_{nl,nl'}(\delta, r_0)\cdot\delta_{l,l'}\left\{\frac{1}{2}\sqrt{(l-\lambda+1)(l+\lambda)(m+2)(1-m)}\,\delta_{\lambda,\lambda'+1}\delta_{m,m'-1}\right.\\ &\left.+\frac{1}{2}\sqrt{(l+\lambda+1)(l-\lambda)(2-m)(1+m)}\,\delta_{\lambda,\lambda'-1}\delta_{m,m'+1}+\lambda\cdot m\cdot\delta_{\lambda,\lambda'}\delta_{m,m'}\right\}, \end{aligned}\quad (3\cdot4)$$

* Here the usual notation is used.

where

$$F^{T \text{ or } LS}_{nl,nl'}(\delta, r_0) = \int \phi^*_{nl}\left(\frac{r}{\sqrt{2}}; \delta, r_0\right) v_T(r) \,(\text{or } v_{LS}(r))\, \phi_{nl'}\left(\frac{r}{\sqrt{2}}; \delta, r_0\right) \frac{r^2 dr}{(\sqrt{2})^3}. \tag{3·5}$$

$L_{2\mu}$ is defined as follows:

$$L_{2\mu} = 3\sqrt{\frac{8\pi}{15}}\, Y_{2\mu}.$$

Thus the potential energies for the tensor and the spin-orbit part are written as

$$E_T(\delta, r_0) = \sum_{n,l \geqq l'} C^T_{nl,nl'}(\delta) \cdot F^T_{nl,nl'}(\delta, r_0),$$

and

$$E_{LS}(\delta, r_0) = \sum_{n,l} C^T_{nl,nl}(\delta) \cdot F^{LS}_{nl,nl}(\delta, r_0). \tag{3·6}$$

The factor $C(\delta)$'s represent the behaviour of the kinematical parts of the configurations and $F(\delta, r_0)$'s the dynamical parts. The errors caused by the approximation of the effective local type to the non-locality of the radial part are contained in $F(\delta, r_0)$.

The $C(\delta)$'s of the tensor and the spin-orbit part vanish for the closed shell configuration. The $C^T(\delta)$'s of the tensor part are left only in the open shell and are divided into two groups: One is a group of increasing functions of δ and the other is of decreasing one depending on the configurations of the open shell. If there are a few nucleons in the open shell, either of the two groups is favourable to the large deformation, concerning the sign of the force.

On the other hand, the $C^{LS}(\delta)$'s of the spin-orbit part between the closed and the open shell are left over and carry the main part of the $C^{LS}(\delta)$'s. The values of the above part split into two values with the ratio $(-l)/(l+1)$ according to the total angular momenta of single particles $j=l\pm 1/2$. Comparing with this feature, the δ-variation of $C^{LS}(\delta)$ for the spin-orbit part is small. Thus, it is said that the main effect of the spin-orbit part is to separate the nearly degenerate states so that the pure configuration which has a large δ_{eq} is lowered.

As for $F(\delta, r_0)$, the circumstances are the same as those in I. When the strength of the harmonic oscillator well becomes larger with δ, $F(\delta, r_0)$'s become larger.

Adding the above results (3·6) to the energy obtained in I, we perform the variational calculation

$$\frac{\partial E_{tot}(\delta, r_0)}{\partial r_0} = 0 \quad \text{and} \quad \frac{\partial E_{tot}(\delta, r_0)}{\partial \delta} = 0,$$

where

$$E_{tot}(\delta, r_0) = E_{kin}(\delta, r_0) + E_{pot}(\delta, r_0) + E_T(\delta, r_0) + E_{LS}(\delta, r_0) + E_J(\delta, r_0)$$

and $E_{kin}(\delta, r_0)$, $E_{pot}(\delta, r_0)$ and $E_J(\delta, r_0)$ are given in I.

§ 4. Results and conclusion

We have numerically carried out the computation procedure for Mg^{24} which is shown in § 3. Its results are represented in Figs. 3 and 4. Putting these results and the results of I together, we obtained the total binding energy as a function of parameters δ and r_0 in Figs. 2 and 5.

1. *On the variation with r_0*

At $\delta=0$, the minimum value of the total energy with respect to the variation of r_0 is obtained as -148 Mev at $r_0 \cong 1.2\times10^{-13}$ cm (Fig.2) and this equilibrium value of r_0 differs slightly from the one of I. Therefore, $\hbar\omega(0, r_0)$ is about 15 Mev which is almost equal to Nilsson's value.[2] As for the inverse range parameter ν, the result of Eden et al. in the case of O^{16} is somewhat larger than that derived from our result.[1),7)]

2. *On the spin-orbit splitting*

In the previous paper I, where we treated only the central part of the reaction matrix, what corresponds to the so-called spin-orbit splitting in the shell model did not appear and we could not decide the configuration of the last eight particles. Now, as we take into account the spin-orbit and tensor part, we can obtain

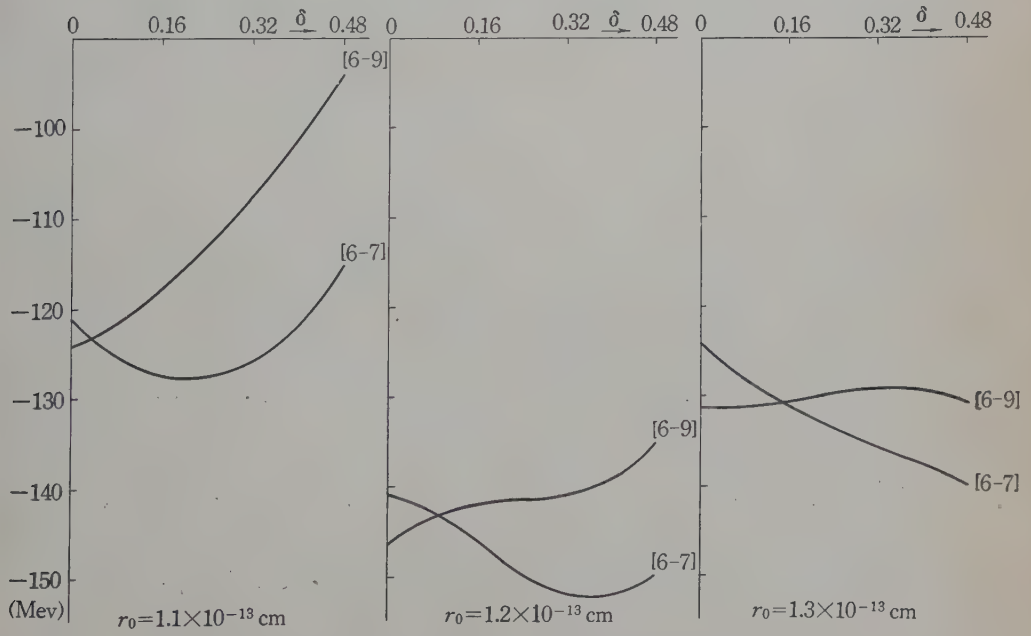


Fig. 2. Total energy of Mg^{24} without $E_J(\delta)$.
The numbers at the ends of each curve indicate the configurations.

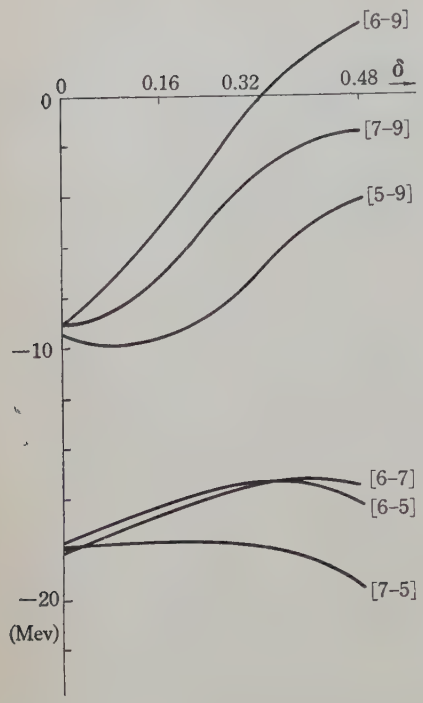


Fig. 3. The energy of the spin-orbit part at $r_0=1.2\times10^{-13}$ cm. The numbers at the ends of each curve indicate the configurations.

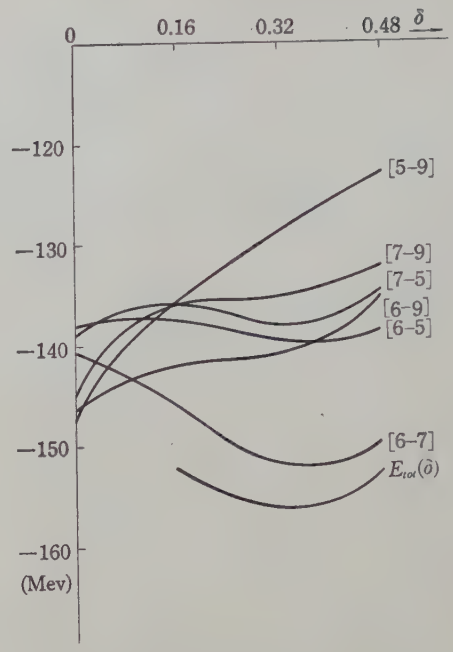


Fig. 5. Total binding energy of Mg^{24} at $r_0=1.2\times10^{-13}$ cm. The numbers at the ends of each curve indicate the configurations. $E_{tot}(\delta)$ represents the energy of ground state of Mg^{24}.

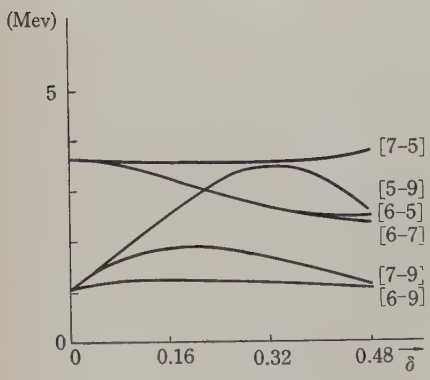


Fig. 4. The energy of the tensor part at $r_0=1.2\times10^{-13}$ cm. The numbers at the ends of each curve indicate the configurations.

the spin-orbit splitting corresponding to the shell model.

The configuration of No. 6 and No. 7 or No. 6 and No. 5 or No. 5 and No. 7* which are the levels splitting from the $d_{5/2}$ level at $\delta=0$ (these configurations are called the 1st group) gives the lowest energy. The energy of the 2nd group (the configuration of No. 6 and No. 9, No. 7 and No. 9, and No. 5 and No. 9) is a half compared with that of the 1st group because the level No. 9 is to be $s_{1/2}$ at $\delta=0$ (Fig. 3). The other configurations give positive energies since they involve such levels as $d_{3/2}$ level at

* The level numbers used here are same as those used by Nilsson.

$\delta=0$. Therefore these configurations can be neglected in the variational procedure.

The difference between the 1st and 2nd group appears as a spin-orbit splitting corresponding to the one-body L-S force of the shell model. Our calculated value of this difference is about 2.2 Mev* per particle at $r_0\cong 1.2\times 10^{-13}$ cm and $\delta=0$, and this value is $\sim 150\%$** of Nilsson's value. This splitting comes mainly from the spin-orbit part of the reaction matrix for the odd state ($\sim 90\%$). The tensor part contributes oppositely in sign but this contribution is small.

Since the decrease of the splitting energies of the 2nd group along with δ is stronger than that of the 1st group, the spin-orbit splitting at the equilibrium deformation becomes larger than the above value.

3. *Contribution from the tensor force*

In order to take the tensor part into account precisely, we should distort the O^{16} core. As the works by several authors[3] have shown, this contributes to two parts; the spin-orbit splitting and the binding energy. As to the former, it has been shown that a half value of experiment is derived in the case of the pion theoretical potential where the tensor part is strong. The Gammel-Thaler potential has a somewhat weak tensor part, so the splitting energy will be smaller. However, in this case, considering the results from the strong two-body spin-orbit potential, we obtain a somewhat larger value with experiment.

Meanwhile, the energies of the tensor interaction among the particles in the open shell are illustrated in Fig. 4. It is noted that the absolute value is so small that the tensor part has only a little influence on the deformation. In the configuration of No.6 and No.7, the decrease of the energy at equilibrium deformation is about 1 Mev, which is a few percent of that of the central part.

4. *On the variation with δ*

Because of the strong δ-dependence of the mixing coefficient $A(\delta)$ which appears in Eq. (3·1), the energy of the configuration of No.6 and No.7 decreases rapidly with increasing δ. Consequently at the point of equilibrium deformation the energy of the configuration of No.6 and No.7 becomes minimum of all configurations and is largely separated from those of the other configurations (Fig. 5). Thus, the configuration of No.6 and No.7 will appear purely in the configuration of the ground state.*** This is caused by the spin-orbit potential and the splitting in this case of the Gammel-Thaler potential is somewhat larger than the experimental value. The spin-orbit part plays a role to determine the single configuration of the deformed potential model as the ground state which has a

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* There are some small fluctuations in accordance with the configuration.

** The reaction matrix calculated by Brueckner et al.[6] is founded on the Gammel-Thaler spin-orbit potential for even states. If we take into account the spin-orbit potential for odd states, this value would be altered to about 200%.

*** Compare this result with Fig. 4 in the previous paper I.

large deformation.*

As pointed out in I, $E_J(\delta)$ has a role to reduce the equilibrium deformation δ_{eq} slightly. We obtain the value $\delta_{eq}=0.35$ and this value is close to the experimental value ~ 0.3.** The minimum energy at this point is about -155 Mev which is comparable with the experimental value -198 Mev.[8] This difference comes from the approximation for the reaction matrix and the non-distortion of O^{16} core.

5. *Discussion on the nuclear force*

We have proceeded with our calculation and discussion using the Gammel-Thaler potential because it has been given phenomenologically over all regions and the calculation of the reaction matrix is available.

Now we discuss the case of the pion-theoretical potential which is confirmed in the outer region and is given qualitatively in the region $r\sim 1.4\times 10^{-13}$ cm and phenomenologically in the inner region.[10] Comparing the pion-theoretical potential with Gammel-Thaler's, the main difference is that the former has the stronger tensor potential and the weaker spin-orbit potential. As to the central force, even if there exist the differences at each state, it is the main part to give the nuclear binding energy, so the deformation will be obtained at the same extent as that from Gammel-Thaler's, as discussed in I. The contribution to the spin-orbit splitting from the two-body spin-orbit potential will be weaker ($\sim 1/5$), but, on the contrary, that from the tensor part is larger and gives about a half of the experimental value. Thus the total spin-orbit splitting will be composed of two parts and be nearly equal to the experimental value. Besides, the deformation will be strengthened by about ten percent because of the stronger tensor force ($\sim 3/2$). Thus even in future when the force is obtained theoretically, the main feature of the relation between the nuclear deformation and the nuclear force will not be altered.

From the above results and discussions, we conclude qualitatively:

1) Through our method of calculation based on Brueckner's reaction matrix, we get the same order of the deformation as the experimental data. Depending on the number of particles in the open shell, this deformation is mainly caused by the central part of the reaction matrix.

2) The magnitude of the spin-orbit splitting is about 150$\sim$200 percent, which is obtained from the spin-orbit part of the reaction matrix using the Gammel-Thaler potential and it connects closely with the deformation.

3) The tensor interaction among the particles in the open shell has only a little contribution to the deformation.

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* It is noted that the interaction energy between different configurations vanishes in our choice of the configuration in which each level is filled up with four particles. Then the configurational mixing does not occur in the first-order perturbation.

** This value is estimated from the neighbouring odd nuclei of Mg^{24}.[9]

4) Nilsson's model seems to be self-consistent in the sense that the configuration of the ground state and the equilibrium deformation are reproduced in a similar fashion to Nilsson's.

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Nuclear Forces in the Momentum Space*

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Several fundamental problems on a two-nucleon system in the momentum space are discussed with the view that they will be useful for treating the two-nucleon problem completely nonstatically, i. e. without making use of the expansion in terms of the inverse of the mass of the nucleon.

General forms for a two-nucleon potential in the momentum space are derived, and the integral equations which are the Fourier transform of the Schrödinger equation and their solutions are briefly discussed. Formulas for matrix elements of the most general types of potentials are evaluated and are applied to the nonstatic one-pion-exchange potential.

§ 1. Introduction

As has been stressed by Taketani and Machida,[1] one of the most important problems in the theory of nuclear forces will be to evaluate the nonstatic effects in a consistent way, especially avoiding the expansion with respect to the inverse of the mass of the nucleon. One way to do so will be to treat the problem in the momentum space throughout. It is the aim of this paper to give fundamental formulas to carry out such a program.

First, we will derive in § 2 the most general expression for a potential in the momentum space, which has been used by Hoshizaki and Machida[2,3] (hereafter referred to as HMI and HMII respectively). In § 3, the integral equations in the momentum space, which are the Fourier transforms of the Schrödinger equations in the configuration space, are discussed, and after separating the angular variables, scattering amplitudes are connected with phase shifts. In § 4, formulas for matrix elements of the most general forms of a potential in the momentum space will be given, and in § 5 we will apply these formulas to the one-pion-exchange potential with full recoil in the case of the pseudoscalar coupling, and in § 6 the phase shifts in the Born approximation will be obtained. Actual solutions of the integral equations derived in § 3 will not be given in this paper, but will be reported in a forthcoming paper.

§ 2. General forms of a potential in the momentum space

We consider the nucleon-nucleon scattering as shown in Fig. 1 in the centre-

* This work has been done as a part of the 1960~61 Annual Research Project on Nuclear Forces organized by the Research Institute for Fundamental Physics, Kyoto University.

of-mass system, and define the following quantities:

$$(1/2)(\boldsymbol{p}+\boldsymbol{p}')\equiv\boldsymbol{q},\quad \boldsymbol{p}-\boldsymbol{p}'\equiv\boldsymbol{k},$$

and

$$\boldsymbol{l}=\frac{\boldsymbol{k}}{|\boldsymbol{k}|},\quad \boldsymbol{m}=\frac{\boldsymbol{q}}{|\boldsymbol{q}|},\quad \text{and}\quad \boldsymbol{n}=\frac{\boldsymbol{k}\times\boldsymbol{q}}{|\boldsymbol{k}\times\boldsymbol{q}|}=\frac{\boldsymbol{p}\times\boldsymbol{p}'}{|\boldsymbol{p}\times\boldsymbol{p}'|}.$$

Then, the most general form of the potential between two particles with spin one half in the momentum space is as follows:

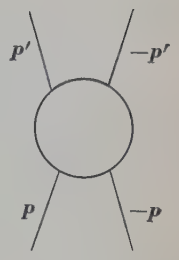


Fig. 1

$$V=V_0+i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot\boldsymbol{n}\cdot V_1+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{l})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{l})\cdot V_2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{m})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{m})\cdot V_3$$
$$+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n})\cdot V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5, \qquad (2\cdot1)$$

where $V_0, \cdots, V_5$ are real functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$. In order to derive Eq. (2·1) we have assumed invariance[4),5),6)] with respect to translation, Galilei transformation, the exchange of two particles, rotation, space reflection and time reversal, and Hermiticity of the potential.

Okubo and Marshak[7)] first derived this general form which corresponds to the Fourier transform of Eq. (2·1) with respect to $\boldsymbol{k}$. Since we will need the relation between x-representation and p-representation which is not necessarily trivial (see the Appendix of HMI), it will be convenient to derive the above formula in our own way.

Consider the matrix element of a potential operator, v, from a two-nucleon state with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ to the one with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$, which may be written as the Fourier transform of a nonlocal potential in x-space in general,

$$\langle\boldsymbol{p}_1',\boldsymbol{p}_2'|v|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle=\frac{1}{(2\pi)^6}\int d\boldsymbol{r}_1'\,d\boldsymbol{r}_2'\,d\boldsymbol{r}_1 d\boldsymbol{r}_2\langle\boldsymbol{r}_1',\boldsymbol{r}_2'|v|\boldsymbol{r}_1,\boldsymbol{r}_2\rangle$$
$$\times\exp i(\boldsymbol{p}_1'\cdot\boldsymbol{r}_1'+\boldsymbol{p}_2'\cdot\boldsymbol{r}_2'-\boldsymbol{p}_1\cdot\boldsymbol{r}_1-\boldsymbol{p}_2\cdot\boldsymbol{r}_2). \qquad (2\cdot2)$$

(I) *Translation invariance*

We have

$$\langle\boldsymbol{r}_1'+\boldsymbol{a},\boldsymbol{r}_2'+\boldsymbol{a}|v|\boldsymbol{r}_1+\boldsymbol{a},\boldsymbol{r}_2+\boldsymbol{a}\rangle=\langle\boldsymbol{r}_1',\boldsymbol{r}_2'|v|\boldsymbol{r}_1,\boldsymbol{r}_2\rangle, \qquad (2\cdot3)$$

where $\boldsymbol{a}$ is an arbitrary vector. Inserting Eq. (2·3) into Eq. (2·2), we obtain

$$\langle\boldsymbol{p}_1',\boldsymbol{p}_2'|v|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle=\langle\boldsymbol{p}_1',\boldsymbol{p}_2'|v|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle\cdot\exp[-i(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2)\boldsymbol{a}]. \qquad (2\cdot4)$$

Therefore, one may write

$$\langle\boldsymbol{p}_1',\boldsymbol{p}_2'|v|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle=\delta(\boldsymbol{p}_1'+\boldsymbol{p}_2'-\boldsymbol{p}_1-\boldsymbol{p}_2)\langle\boldsymbol{p}_1',\boldsymbol{p}_2'|V|\boldsymbol{p}_1,\boldsymbol{p}_2\rangle$$
$$=\delta(\boldsymbol{Q}'-\boldsymbol{Q})\,V(\boldsymbol{Q},\boldsymbol{P}',\boldsymbol{P}), \qquad \text{(i)}$$

where

$$\boldsymbol{Q}=\boldsymbol{p}_1+\boldsymbol{p}_2,\qquad \boldsymbol{Q}'=\boldsymbol{p}_1'+\boldsymbol{p}_2',$$
$$\boldsymbol{P}=(1/2)(\boldsymbol{p}_1-\boldsymbol{p}_2),\quad \boldsymbol{P}'=(1/2)(\boldsymbol{p}_1'-\boldsymbol{p}_2'). \qquad (2\cdot5)$$

(II) *Galilei invariance*

The condition for Galilei invariance gives

$$\langle \boldsymbol{p}_1'+\boldsymbol{q}, \boldsymbol{p}_2'+\boldsymbol{q}|v|\boldsymbol{p}_1+\boldsymbol{q}, \boldsymbol{p}_2+\boldsymbol{q}\rangle=\langle \boldsymbol{p}_1', \boldsymbol{p}_2'|v|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle, \tag{2·6}$$

where $\boldsymbol{q}$ is an arbitrary momentum vector and masses of participating particles are assumed to be equal. Eq. (2·6) gives a condition on $V(\boldsymbol{Q}, \boldsymbol{P}', \boldsymbol{P})$,

$$V(\boldsymbol{Q}+2\boldsymbol{q}, \boldsymbol{P}', \boldsymbol{P})=V(\boldsymbol{Q}, \boldsymbol{P}', \boldsymbol{P}). \tag{2·7}$$

So we have, writing the spin dependence explicitly,

$$V=V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q}), \tag{ii}$$

where

$$\boldsymbol{k}=\boldsymbol{P}-\boldsymbol{P}'=\boldsymbol{p}_1-\boldsymbol{p}_1'=-(\boldsymbol{p}_2-\boldsymbol{p}_2'),$$
$$\boldsymbol{q}=(1/2)(\boldsymbol{P}+\boldsymbol{P}')=(1/4)(\boldsymbol{p}_1+\boldsymbol{p}_1'-\boldsymbol{p}_2-\boldsymbol{p}_2').$$

(III) *Symmetry condition*

V is invariant when particles 1 and 2 are interchanged, i. e.

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}, -\boldsymbol{k}, -\boldsymbol{q}). \tag{iii}$$

(IV) *Rotation invariance*

Invariant functions are functions of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$.

(V) *Space reflection invariance*

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}\ \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, -\boldsymbol{k}, -\boldsymbol{q}). \tag{iv}$$

From (iii) and (iv), we have

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{k}, \boldsymbol{q}). \tag{v}$$

(VI) *Time reversal invariance*

$$V(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, \boldsymbol{q})=V^T(-\boldsymbol{\sigma}^{(1)}, -\boldsymbol{\sigma}^{(2)}, \boldsymbol{k}, -\boldsymbol{q}). \tag{vi}$$

V^T means to reverse the order of operators.

(VII) *Hermiticity*

$$V=V^+.$$

As we assume charge independence, $V_0, \cdots, V_5$ are divided into $\boldsymbol{1}$-terms and $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})$-terms in isotopic space. As for the $\boldsymbol{\sigma}$-dependence, they are separated to the zeroth, the first and the second order terms with respect to $\boldsymbol{\sigma}$. Then, the zeroth order term, V_0, is a function of $\boldsymbol{k}^2$, $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$ from (IV). In the first order term of $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ appear always in a form $\boldsymbol{S}=(1/2)(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})$, i. e. $(\boldsymbol{S}\cdot\boldsymbol{k}\times\boldsymbol{q})$. And the second order terms of $\boldsymbol{\sigma}$ may have the following factors,

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q}),$$
$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\ \text{and}\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q}).$$

Among the above five terms one may be expressed by other four types, so we take the next four terms as independent terms,

$$(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}),\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})\ \text{and}\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q}).$$

Therefore, we have six independent types of potentials, $V_0, \cdots, V_5$, in general, as we see in Eq. (2·1). In the static approximation, only V_0, V_2 and V_5 survive, and if we evaluate the potential up to the first order of the inverse of the mass of the nucleon V_1 appears. V_3 and V_4 appear as the second order correction to the static potential with respect to the inverse of the mass of the nucleon. As our intention is to take the nonstatic effects fully into account, it is necessary to treat all types of potentials on the equal footing.

When we consider the matrix element of a potential, the energy conservation law does not hold in general. However, when we consider the scattering matrix for two nucleons, where the energy must be conserved, we have just five (instead of six) independent types and, furthermore, invariant functions are functions of $\boldsymbol{k}^2$ and $\boldsymbol{q}^2$ only. This is because of the identity,

$$\begin{aligned}&(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\cdot\boldsymbol{q}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})\cdot\boldsymbol{k}^2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})\\&\quad=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})(\boldsymbol{k}\times\boldsymbol{q})^2+(\boldsymbol{k}\cdot\boldsymbol{q})\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})\},\end{aligned}$$

and equations

$$(\boldsymbol{k}\cdot\boldsymbol{q})=0,\ (\boldsymbol{k}\times\boldsymbol{q})^2=\boldsymbol{k}^2\cdot\boldsymbol{q}^2.$$

It should be remarked that the assumption of the conservation of energy in the derivation of a non-static potential introduces serious errors in general (see HMI).

§ 3. Integral equations in the momentum space and phase shifts

In order to compare the potential in § 2 with the experiment, it is necessary to solve the integral equation which corresponds to the Schrödinger equation in the momentum space. We will discuss only the main points, which will be necessary in our later discussions, referring to a paper by Signell[8] for more detailed discussions.

The basic equation is the following,

$$(2E_{p_0}-2E_{p'})\phi(\boldsymbol{p}')=\int d\boldsymbol{p}\,V(\boldsymbol{p},\boldsymbol{p}')\phi(\boldsymbol{p}),\qquad(3\cdot1)$$

where $\boldsymbol{p}$ and $\boldsymbol{p}'$ are momenta as shown in Fig. 1, $E_{p_0}=(1/2)E$ is the incident energy of each particle in the centre-of-mass system, and $E_{p'}=(1/2)H_0$ where H_0 is the free Hamiltonian.

In order to separate the angular variables in Eq. (3·1), we expand the wave function and the potential into the spherical harmonics:

$$\phi(\boldsymbol{p})=\sum_l g_l(p)\cdot Y^M_{J,l,s},\qquad(3\cdot2)$$

where[9]

$$Y^{M}_{J,l,S}=\sum_{m_l,\,m_s} C_{lS}(J,M\,;\,m_l,\,m_s)\,Y^{m_l}_{l}(\Omega_p)\chi^{m_s}_{S}.$$

For simplicity we will treat the uncoupled states and give the relation between scattering amplitudes and phase shifts. The coupled states may also be treated by a slight generalization as discussed by Signell[8]. Using the following matrix elements of the potential,

$$V_l(p,\,p')\equiv\int Y^{M\,*}_{J,l,S}(\Omega_{p'})\,V(\boldsymbol{p},\,\boldsymbol{p}')\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p, \tag{3·3}$$

we can rewrite Eq. (3·1) in one-dimensional equation,

$$(2E_{p_0}-2E_{p'})g_l(p')=\int p^2\,dp\,V_l(p,\,p')g_l(p). \tag{3·4}$$

Thus the scattering solution is given by[10]

$$g_l(p')=\delta(2E_{p_0}-2E_{p'})+\frac{1}{2E_{p_0}-2E_{p'}}\int p^2\,dp\,V_l(p,\,p')g_l(p),$$

and may also be written in the following way:

$$g_l(p')=\delta(2E_{p_0}-2E_{p'})+\mathrm{P}\frac{1}{2E_{p_0}-2E_{p'}}f_l(p'), \tag{3·5}$$

where

$$f_l(p')=\int_0^\infty p^2\,dp\,V_l(p,\,p')g_l(p).$$

Using Eq. (3·5) and

$$\delta(2E_{p_0}-2E_{p'})=\left(\frac{E_{p_0}}{2p_0}\right)\delta(p_0-p'),$$

we obtain an equation satisfied by $f_l(p')$,

$$f_l(p')=f_{l,B}(p')+\mathrm{P}\int_0^\infty\frac{p^2\,dp\,V_l(p,\,p')f_l(p)}{2E_{p_0}-2E_p}, \tag{3·6}$$

where $f_{l,B}(p')$ is the amplitude in Born approximation,

$$f_{l,B}(p')\equiv\frac{p_0\cdot E_{p_0}}{2}V_l(p_0,\,p'). \tag{3·7}$$

Now phase shifts, δ_l, are defined by the asymptotic wave function in the configuration space,

$$\phi_l(r)\sim\sin\left(p_0 r-\frac{l\pi}{2}\right)+\tan\delta_l\cdot\cos\left(p_0 r-\frac{l\pi}{2}\right). \tag{3·8}$$

Transforming this equation into the momentum space in the vicinity of the singularity, we have

$$\phi_l(p') \sim \delta(p_0-p') - \frac{\tan\delta_l}{\pi} \mathrm{P}\cdot\frac{1}{p_0-p'} + R(p'), \tag{3·9}$$

where $R(p')$ is a non-singular function at p_0.

It is seen that Eq. (3·5) is written as follows:

$$g_l(p') \sim \delta(p_0-p') + \mathrm{P}\cdot\frac{1}{p_0-p'} f_l(p_0) + R(p'). \tag{3·10}$$

Comparing Eq. (3·9) with Eq. (3·10), the relation between the scattering solution and phase shifts is given by

$$\tan\delta_l = -\pi f_l(p_0). \tag{3·11}$$

In the Born approximation, we have

$$\tan\delta_{l,B} = -\frac{\pi p_0 E_{p0}}{2} V_l(p_0, p_0). \tag{3·12}$$

So we must solve Eq. (3·6) to obtain scattering phase shifts. Using

$$\frac{1}{2E_{p_0}-2E_p} = \begin{cases} \dfrac{M}{p_0^2-p^2} & \text{for the nonrelativistic case,} \\ \dfrac{E_{p0}+E_p}{2(p_0^2-p^2)} & \text{for the relativistic case,} \end{cases} \tag{3·13}$$

we obtain

$$f_l(p') = f_{l,B}(p') + \mathrm{P}\int_0^\infty \frac{p^2 dp\, U(p', p) f_l(p)}{p_0-p}, \tag{3·14}$$

where $U(p', p)$ is not singular anywhere (we assume $U(p', p)$ is bounded when $p'\to\infty$ or $p\to\infty$), and is defined by

$$\frac{U(p', p)}{p_0-p} \equiv \frac{V(p', p)}{2E_{p_0}-2E_p}. \tag{3·15}$$

Eq. (3·14) is a singular integral equation, the kernel of which has a pole of the first order at a fixed point $p=p_0$ in the range of the integration. The integral equation of this type is generally known to have a solution.[11]

Eq. (3·14) is also easily transformed into a Fredholm integral equation of the third kind with a symmetrical kernel,

$$I(p') F_l(p') = f_{l,B}(p') + \int_0^\infty V_l(p', p) F_l(p) dp, \tag{3·16}$$

where

$$I(p) = \frac{2(E_{p_0}-E_p)}{p^2},$$

$$F_l(p)=\frac{f_l(p)}{I(p)}.$$

Eq. (3·16) may be solved by standard methods used to solve the Fredholm integral equation of the second kind. Evaluation of phase shifts will be treated in a separate paper making use of these integral equations.

§ 4. General formulas for matrix elements of potentials in the momentum space

In this section we calculate the matrix elements of the potentials by operating the most general potentials in § 2 to the eigenfunctions of a two-nucleon system.

If we write

$$V_{l,l'}=\int Y^{M*}_{J,l',S}(\Omega_{p'})\,V\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p,$$

there are four diagonal elements $V_{l,l}$, one for $S=0$ and three for $S=1$ with $l=J$, $J\pm1$. Also, there are two non-diagonal elements $V_{l,l'}$, for $S=1$ between $l'=J\mp1$ and $l=J\pm1$. There are, in all, five independent matrix elements, because $V_{l,l'}=V_{l',l}$. $V_{l,l}$ and $V_{l,l'}$ are real functions and they do not depend on M, the magnetic quantum number, owing to the rotation invariance and hermiticity of the potential. S is also conserved between two states before and after the potential operator because of parity conservation and charge independence. So we may use the following notation,

$$V_{l,l}(p,\,p')\equiv\int Y^{M*}_{J,l,S}(\Omega_{p'})\,V(\boldsymbol{p},\,\boldsymbol{p}')\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p, \tag{4·1}$$

$$V_{l,l'}(p,\,p')\equiv\int Y^{M*}_{J,l',S}(\Omega_{p'})\,V(\boldsymbol{p},\,\boldsymbol{p}')\,Y^{M}_{J,l,S}(\Omega_p)\,d\Omega_{p'}\,d\Omega_p. \tag{4·2}$$

Inserting Eq. (2·1) into Eqs. (4·1) and (4·2), we have carried out the integral of angular variables for each type of $V_0,\cdots,V_5$, and the results are given in the following. Since V_is in Eq. (2·1) contain not only $\boldsymbol{k}^2$ but also $\boldsymbol{q}^2$ and $(\boldsymbol{k}\times\boldsymbol{q})^2$ (which correspond to the most general nonlocal and angular momentum dependent potentials in x space), their calculations are very troublesome though straightforward.

Two states before and after the potential operator may be designated by l, since J and S are conserved between an initial and a final state. For example, a matrix element for $l=J$ in a spin singlet state of V_0 type is written as $\langle j|V_0|j\rangle$.

(1) For spin triplet states:

$$\langle J-1|V_0|J-1\rangle=2\pi A^0_{J-1},$$

$$\langle J|V_0|J\rangle=2\pi A^0_J,$$

$$\langle J+1|V_0|J+1\rangle=2\pi A^0_{J+1},$$

$$\langle J+1|V_0|J-1\rangle\equiv\langle J-1|V_0|J+1\rangle=0.$$

For a spin singlet state:

$$\langle j|V_0|j\rangle=2\pi A_j^0,$$

where

$$A_n^0=\int V_0 P_n(z)\,dz,$$

and V_0 is the same one as V_0 in Eq. (2·1), $P_n(z)$'s are Legendre functions of the first kind, and $z\equiv(\boldsymbol{p}\cdot\boldsymbol{p}')/|\boldsymbol{p}|\cdot|\boldsymbol{p}'|$.

(2) For spin triplet states:

$$\langle J-1|V_1|J-1\rangle=4\pi\frac{(J-1)}{(2J-1)}[A_J^1-A_{J-2}^1],$$

$$\langle J|V_1|J\rangle=4\pi\left[-\frac{(J+1)}{(2J+1)^2}A_{J+1}^1+\frac{1}{(2J+1)}A_{J-1}^1\right],$$

$$\langle J+1|V_1|J+1\rangle=4\pi\left[-\frac{(J+2)}{(2J+3)}A_{J+2}^1+\frac{(J+2)}{(2J+5)}A_J^1\right],$$

$$\langle J+1|V_1|J-1\rangle\equiv\langle J-1|V_1|J+1\rangle=0.$$

For a spin singlet state:

$$\langle j|V_1|j\rangle=0,$$

where

$$A_n^1=\int\frac{V_1 P_n(z)}{\sin(\boldsymbol{p}\cdot\boldsymbol{p}')}dz.$$

(3) For spin triplet states:

$$\langle J-1|V_2|J-1\rangle=2\pi\left[(p^2+p'^2)\frac{A_{J-1}^{2(0)}}{(2J+1)}-2pp'\frac{A_J^{2(0)}}{(2J+1)}\right],$$

$$\langle J|V_2|J\rangle=2\pi\left[(p^2+p'^2)A_J^{2(0)}-\frac{2pp'}{(2J+1)}\{J\cdot A_{J+1}^{2(0)}+(J+1)A_{J-1}^{2(0)}\}\right],$$

$$\langle J+1|V_2|J+1\rangle=2\pi\left[-(p^2+p'^2)\frac{A_{J+1}^{2(0)}}{(2J+1)}+2pp'\frac{A_J^{2(0)}}{(2J+1)}\right],$$

$$\langle J+1|V_2|J-1\rangle\equiv\langle J-1|V_2|J+1\rangle$$
$$=\frac{4\pi\sqrt{J(J+1)}}{(2J+1)}[p'^2\cdot A_{J-1}^{2(0)}+p^2A_{J+1}^{2(0)}-2pp'\cdot A_J^{2(0)}].$$

For a spin singlet state:

$$\langle j|V_2|j\rangle=2\pi[-(p^2+p'^2)A_j^{2(0)}+2pp'A_j^{2(1)}],$$

where

$$A_n^{2(l)}=\int\frac{V_2}{k^2}P_n(z)z^l\,dz.$$

(4) For spin triplet states:

$$\langle J-1|V_3|J-1\rangle=2\pi\left[(p^2+p'^2)\frac{A^{3(0)}_{J-1}}{(2J+1)}+2pp'\frac{A^{3(0)}_{J}}{(2J+1)}\right],$$

$$\langle J|V_3|J\rangle=2\pi\left[(p^2+p'^2)A^{3(0)}_{J}+\frac{2pp'}{(2J+1)}\{J\cdot A^{3(0)}_{J+1}+(J+1)\cdot A^{3(0)}_{J-1}\}\right],$$

$$\langle J+1|V_3|J+1\rangle=-2\pi\left[(p^2+p'^2)\frac{A^{3(0)}_{J+1}}{(2J+1)}+2pp'\frac{A^{3(0)}_{J}}{(2J+1)}\right],$$

$$\langle J+1|V_3|J-1\rangle\equiv\langle J-1|V_3|J+1\rangle$$

$$=\frac{4\pi\sqrt{J(J+1)}}{(2J+1)}[p'^2\cdot A^{3(0)}_{J-1}+p^2A^{3(0)}_{J+1}+2pp'\cdot A^{3(0)}_{J}].$$

For a spin singlet state:

$$\langle j|V_3|j\rangle=-2\pi[(p^2+p'^2)A^{3(0)}_{j}+2pp'\cdot A^{3(1)}_{j}],$$

where

$$A^{3(l)}_{n}=\int\frac{V_3}{\boldsymbol{q}^2}P_n(z)z^l\,dz.$$

(5) For spin triplet states:

$$\langle J-1|V_4|J-1\rangle=2\pi p^2p'^2\left[\frac{2}{(2J+1)}A^{4(1)}_{J}+\frac{(2J-1)}{(2J+1)}A^{4(0)}_{J-1}-A^{4(2)}_{J-1}\right],$$

$$\langle J|V_4|J\rangle=2\pi p^2p'^2\left[-A^{4(0)}_{J}+\frac{(J+2)}{(2J+1)}A^{4(1)}_{J-1}+\frac{(J-1)}{(2J+1)}A^{4(1)}_{J+1}\right],$$

$$\langle J+1|V_4|J+1\rangle=2\pi p^2p'^2\left[-\frac{2}{(2J+1)}A^{4(1)}_{J}-A^{4(2)}_{J+1}+\frac{(2J+3)}{(2J+1)}A^{4(0)}_{J+1}\right],$$

$$\langle J+1|V_4|J-1\rangle\equiv\langle J-1|V_4|J+1\rangle=\frac{4\pi p^2p'^2\sqrt{J(J+1)}}{(2J+1)^2}[A^{4(0)}_{J+1}-A^{4(0)}_{J-1}].$$

For a spin singlet state:

$$\langle j|V_4|j\rangle=2\pi p^2\cdot p'^2[-A^{4(0)}_{j}+A^{4(2)}_{j}],$$

where

$$A^{4(l)}_{n}=\int\frac{V_4\;P_n(z)\cdot z^l}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}dz.$$

(6) For spin triplet states:

$$\langle J-1|V_5|J-1\rangle=2\pi A^{5}_{J-1},$$

$$\langle J|V_5|J\rangle=2\pi A^{5}_{J},$$

$$\langle J+1|V_5|J+1\rangle=2\pi A^{5}_{J+1},$$

$$\langle J+1|V_5|J-1\rangle\equiv\langle J-1|V_5|J+1\rangle=0.$$

For a spin singlet state:

$$\langle j|V_5|j\rangle=-6\pi A_j^5,$$

where

$$A_n^5=\int V_5 P_n(z)\,dz.$$

§ 5. The one-pion-exchange potential (OPEP)

In this paragraph we will consider the one-pion-exchange potential, $V^{(2)}$, assuming the pseudoscalar coupling.* The types V_2, V_3, V_4 and V_5 appear of the six possible types of the potentials in Eq. (2·1),

$$\begin{aligned}V^{(2)}&=(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{l})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{l})V_2+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{m})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{m})V_3\\&\quad+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{n})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{n})V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5\\&=\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{|\boldsymbol{k}^2|}V_2+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{q})}{|\boldsymbol{q}^2|}V_3\\&\quad+\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k}\times\boldsymbol{q})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k}\times\boldsymbol{q})}{|\boldsymbol{k}^2|\cdot|\boldsymbol{q}^2|\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}V_4+(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})\cdot V_5,\end{aligned}\tag{5·1}$$

where

$$\begin{aligned}V_2&=\frac{G}{A+Bz}\boldsymbol{k}^2\left\{\beta+\frac{2\boldsymbol{q}^2}{(\boldsymbol{k}\cdot\boldsymbol{q})}\alpha\right\},\\V_3&=\frac{G}{A+Bz}\boldsymbol{q}^2\left\{\gamma+\frac{\boldsymbol{k}^2\cdot\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})}\right\},\\V_4&=\frac{G}{A+Bz}\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})\frac{2\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})},\\V_5&=\frac{G}{A+Bz}\cdot\frac{-2\alpha}{(\boldsymbol{k}\cdot\boldsymbol{q})}\{\boldsymbol{k}^2\cdot\boldsymbol{q}^2-(\boldsymbol{k}\cdot\boldsymbol{q})^2\},\end{aligned}\tag{5·2}$$

$$\begin{aligned}G&=-\frac{1}{16}g^2\cdot(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{1}{M^2}\frac{(E_p+M)(E_{p'}+M)}{E_p\cdot E_{p'}},\\A&=p^2+p'^2+\mu^2-(E_p-E_{p'})^2,\\B&=-2pp',\\\alpha&=\frac{1}{(1+(E_p/M))^2}-\frac{1}{(1+(E_{p'}/M))^2},\\\beta&=\left(\frac{1}{1+(E_p/M)}+\frac{1}{1+(E_{p'}/M)}\right)^2,\end{aligned}$$

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* Properties of the one-pion-exchange potential, both in the cases of the pseudoscalar and pseudovector couplings, have been investigated in detail by HMI.

$$\gamma=\left(\frac{1}{1+(E_p/M)}-\frac{1}{1+(E_{p'}/M)}\right)^2, \tag{5·3}$$

g is the pseudoscalar coupling constant and μ is the mass of the pion. These potentials contain nonstatic effects completely, i. e. they are not expanded with respect to the inverse of the mass of the nucleon. Only V_2 term remains in the static approximation:

$$\begin{aligned} V^{(2)}_{static}&=-\left(\frac{g}{2M}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{(\boldsymbol{k}^2+\mu^2)}\cdot\frac{1}{(2\pi)^3}\\ &=-\left(\frac{f}{\mu}\right)^2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})(\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})}{(\boldsymbol{k}^2+\mu^2)}\cdot\frac{1}{(2\pi)^3}. \end{aligned} \tag{5·4}$$

When we apply the general formulas in § 4 for the one-pion-exchange potential, we obtain the following matrix elements. From equations given in § 4, we get, for spin singlet states,

$$\begin{aligned} \langle j|V^{(2)}(p,p')|j\rangle=&\int(2\pi)\left[-(p^2+p'^2)\frac{V_2}{\boldsymbol{k}^2}P_J(z)+2pp'\frac{V_2}{\boldsymbol{k}^2}P_J(z)\cdot z\right]dz\\ &-\int(2\pi)\left[(p^2+p'^2)\frac{V_3}{\boldsymbol{q}^2}P_J(z)+2pp'\frac{V_3}{\boldsymbol{q}^2}P_J(z)\,z\right]dz\\ &+\int(2\pi)\,p^2p'^2\left[-\frac{V_4\cdot P_J(z)}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}+\frac{V_4\cdot P_J(z)}{\boldsymbol{k}^2\cdot\boldsymbol{q}^2\cdot\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})}z^2\right]dz\\ &-\int(6\pi)\,V_5\cdot P_J(z)\,dz. \end{aligned} \tag{5·5}$$

Inserting Eq. (5·2) into Eq. (5·5), and carrying out the integral with respect to z, we obtain

$$\langle j|V^{(2)}(p,p')|j\rangle=\langle j|V_2^{(2)}|j\rangle+\langle j|V_3^{(2)}|j\rangle+\langle j|V_4^{(2)}|j\rangle+\langle j|V_5^{(2)}|j\rangle,$$

where

$$\begin{aligned} &\langle j|V_2^{(2)}|j\rangle\\ &=4\pi G\left[\left\{\frac{(p^2+p'^2)\cdot\beta}{B}+\frac{(p^2+p'^2)^2\alpha}{(p^2-p'^2)B}-\frac{2pp'\beta}{B}x_0-\frac{4p^2p'^2\alpha}{(p^2-p'^2)B}x_0^2\right\}Q_J(x_0)\right.\\ &\quad\left.+\left\{\frac{2pp'\beta+(4p^2p'^2\alpha/(p^2-p'^2))x_0}{B}\right\}\delta_{J,0}+\frac{1}{3B}\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\delta_{J,1}\right],\\ &\langle j|V_3^{(2)}|j\rangle\\ &=4\pi G\left[\left\{\frac{(p^2+p'^2)\gamma}{B}+\frac{(p^2+p'^2)^2\alpha}{(p^2-p'^2)B}+\frac{2pp'\gamma}{B}x_0-\frac{4p^2p'^2\alpha}{(p^2-p'^2)B}x_0^2\right\}Q_J(x_0)\right.\\ &\quad\left.+\left\{\frac{-2pp'\gamma+(4p^2p'^2\alpha/(p^2-p'^2))x_0}{B}\right\}\delta_{J,0}+\frac{1}{3B}\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\delta_{J,1}\right], \end{aligned}$$

$$\langle j|V_4^{(2)}|j\rangle$$

$$=16G\frac{\alpha}{B}\left[\frac{p^2p'^2}{(p^2-p'^2)}(1-x_0^2)Q_J(x_0)+\frac{p^2p'^2}{(p^2-p'^2)}\left(x_0\delta_{J,0}+\frac{1}{3}\delta_{J,1}\right)\right],$$

$$\langle j|V_5^{(2)}|j\rangle$$

$$=12\pi G\frac{\alpha}{B}\left[\left\{-\frac{(p^2+p'^2)^2}{(p^2-p'^2)}+(p^2-p'^2)+\frac{4p^2p'^2}{(p^2-p'^2)}x_0^2\right\}Q_J(x_0)\right.$$

$$\left.-\frac{4p^2p'^2}{(p^2-p'^2)}\left(x_0\delta_{J,0}+\frac{1}{3}\delta_{J,1}\right)\right], \qquad (5\cdot6)$$

where $Q_J(x_0)$'s are Legendre functions of the second kind. Then we find the result which agrees with Signell's results from Eq. (5·5) and Eq. (5·6), i.e.

$$\langle j|V^{(2)}(p,p')|j\rangle=-\frac{g^2}{4\pi}\frac{[(x_0-\eta)Q_J(x_0)-\delta_{J,0}]}{2\pi E_p\cdot E_{p'}}, \qquad (5\cdot7)$$

where

$$x_0\equiv\frac{\mu^2+2(E_p\cdot E_{p'}-M^2)}{2pp'},$$

$$\eta\equiv\frac{p(E_{p'}+M)}{2p'(E_p+M)}+\frac{p'(E_p+M)}{2p(E_{p'}+M)}.$$

Other matrix elements for spin triplet states are also evaluated in the same way, and the results are given by

$$\langle J-1|V^{(2)}(p\cdot p')|J-1\rangle$$

$$=\langle J-1|V_2^{(2)}|J-1\rangle+\langle J-1|V_3^{(2)}|J-1\rangle+\langle J-1|V_4^{(2)}|J-1\rangle+\langle J-1|V_5^{(2)}|J-1\rangle,$$

where

$$\langle J-1|V_2^{(2)}|J-1\rangle$$

$$=2\pi G\left[\frac{(p^2+p'^2)}{(2J+1)}\left\{\left(\beta+\frac{(p^2+p'^2)}{(p^2-p'^2)}\alpha+\frac{2pp'\alpha}{(p^2-p'^2)}x_0\right)\frac{(-2)}{B}Q_{J-1}(x_0)\right.\right.$$

$$\left.+\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,1}\right\}-\frac{2pp'}{(2J+1)}\left\{\left(\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}+\frac{2pp'\alpha}{(p^2-p'^2)}x_0\right)\right.$$

$$\left.\left.\times\frac{(-2)}{B}Q_J(x_0)+\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,0}\right\}\right],$$

$$\langle J-1|V_3^{(2)}|J-1\rangle$$

$$=2\pi G\left[\frac{(p^2+p'^2)}{(2J+1)}\left\{\left(\gamma+\frac{(p^2+p'^2)}{(p^2-p'^2)}\alpha-\frac{2pp'\alpha}{(p^2-p'^2)}x_0\right)\frac{(-2)}{B}Q_{J-1}(x_0)\right.\right.$$

$$\left.-\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,1}\right\}+\frac{2pp'}{(2J+1)}\left\{\left(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}-\frac{2pp'\alpha}{(p^2-p'^2)}x_0\right)\right.$$

$$\times\frac{(-2)}{B}Q_J(x_0)-\frac{4pp'\alpha}{(p^2-p'^2)B}\delta_{J,0}\Big\}\Big],$$

$$\langle J-1|V_4^{(2)}|J-1\rangle$$

$$=2\pi G\Big[\frac{8p^2p'^2\alpha}{(p^2-p'^2)(2J+1)}\Big(\frac{2A}{B^2}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)+\frac{4p^2p'^2\alpha(2J-1)}{(p^2-p'^2)(2J+1)}$$

$$\times\Big(-\frac{2}{B}Q_{J-1}(x_0)\Big)-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2x_0^2}{B}Q_{J-1}(x_0)+\frac{2x_0}{B}\delta_{J,1}+\frac{2}{3B}\delta_{J,2}\Big)\Big],$$

$$\langle J-1|V_5^{(2)}|J-1\rangle$$

$$=2\pi G\Big[\frac{8p^2p'^2\alpha}{(p^2-p'^2)B}Q_{J-1}(x_0)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2}{B}Q_{J-1}(x_0)\cdot x_0^2$$

$$+\frac{2x_0}{B}\delta_{J,1}+\frac{2}{3B}\delta_{J,2}\Big)\Big].$$

From this we obtain

$$\langle J-1|V^{(2)}|J-1\rangle=\Big(\frac{g^2}{4\pi}\Big)\frac{[Q_J(x_0)-\eta Q_{J-1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

Also, for $L=J+1$, we obtain

$$\langle J+1|V^{(2)}(p,p')|J+1\rangle$$

$$=\langle J+1|V_2^{(2)}|J+1\rangle+\langle J+1|V_3^{(2)}|J+1\rangle+\langle J+1|V_4^{(2)}|J+1\rangle+\langle J+1|V_5^{(2)}|J+1\rangle,$$

where

$$\langle J+1|V_2^{(2)}|J+1\rangle$$

$$=2\pi G\Big[\frac{-(p^2+p'^2)}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_{J+1}(x_0)-\frac{(p^2+p'^2)}{(2J+1)}\frac{2pp'\alpha}{(p^2-p'^2)}$$

$$\times\Big(\frac{-2x_0}{B}Q_{J+1}(x_0)\Big)+\frac{2pp'}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_J(x_0)$$

$$+\frac{4p^2p'^2\alpha}{(2J+1)(p^2-p'^2)}\Big(\frac{-2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)\Big],$$

$$\langle J+1|V_3^{(2)}|J+1\rangle$$

$$=2\pi G\Big[-\frac{(p^2+p'^2)}{(2J+1)}\Big\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_{J+1}(x_0)+\frac{(p^2+p'^2)}{(p^2-p'^2)}\frac{2pp'\alpha}{(2J+1)}$$

$$\times\Big(\frac{-2x_0}{B}Q_{J+1}(x_0)\Big)-\frac{2pp'}{(2J+1)}\Big\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(\frac{-2}{B}\Big)Q_J(x_0)$$

$$+\frac{4p^2p'^2\alpha}{(2J+1)(p^2-p'^2)}\Big(\frac{-2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)\Big],$$

$$\langle J+1|V_4^{(2)}|J+1\rangle$$

$$=2\pi G\Big[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big\{-\frac{2}{(2J+1)}\Big(-\frac{2x_0}{B}Q_J(x_0)+\frac{2}{B}\delta_{J,0}\Big)$$

$$-\Big(-\frac{2x_0^2}{B}Q_{J+1}(x_0)+\frac{2x_0}{B}\delta_{J,0}+\frac{2}{3B}\delta_{J,1}\Big)+\frac{(2J+3)}{(2J+1)}\Big(-\frac{2}{B}Q_{J+1}(x_0)\Big)\Big\}\Big],$$

$$\langle J+1|V_5^{(2)}|J+1\rangle$$

$$=2\pi G\Big[-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\Big(-\frac{2x_0^2}{B}Q_{J+1}(x_0)$$

$$+\frac{2x_0}{B}\delta_{J,0}+\frac{2}{3B}\delta_{J,1}\Big)\Big],$$

and

$$\langle J+1|V^{(2)}(p,\,p')|J+1\rangle=\Big(\frac{\eta^2}{4\pi}\Big)\frac{[-Q_J(x_0)+\eta Q_{J+1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

And, for $L=J$, we obtain

$$\langle J|V^{(2)}(p,\,p')|J\rangle=\langle J|V_2^{(2)}|J\rangle+\langle J|V_3^{(2)}|J\rangle+\langle J|V_4^{(2)}|J\rangle+\langle J|V_5^{(2)}|J\rangle,$$

where

$$\langle J|V_2^{(2)}|J\rangle$$

$$=2\pi G\Big[(p^2+p'^2)\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}\Big(-\frac{2}{B}Q_J(x_0)\Big)+\frac{2pp'(p^2+p'^2)\alpha}{(p^2-p'^2)}$$

$$\times\Big(-\frac{2}{B}x_0Q_J(x_0)\Big)-\frac{(2pp')}{(2J+1)}\Big\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big\}$$

$$\times\Big\{J\cdot\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+(J+1)\Big(-\frac{2}{B}\Big)Q_{J-1}(x_0)\Big\}$$

$$-\frac{(2pp')}{(2J+1)}\,\frac{(2pp'\alpha)}{(p^2-p'^2)}\Big\{J\Big(-\frac{2}{B}x_0Q_{J+1}(x_0)\Big)$$

$$+(J+1)\Big(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\Big)\Big\}\Big],$$

$$\langle J|V_3^{(2)}|J\rangle$$

$$=2\pi G\Big[(p^2+p'^2)\Big(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big)\Big(-\frac{2}{B}Q_J(x_0)\Big)-\frac{2pp'(p^2+p'^2)\alpha}{(p^2-p'^2)}$$

$$\times\Big(-\frac{2}{B}Q_J(x_0)\,x_0\Big)+\Big(\frac{2pp'}{2J+1}\Big)\Big(\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\Big)$$

$$\times\Big\{J\Big(-\frac{2}{B}\Big)Q_{J+1}(x_0)+(J+1)\Big(-\frac{2}{B}\Big)Q_{J-1}(x_0)\Big\}-\frac{(2pp')}{(2J+1)}$$

$$\times\frac{(2pp'\alpha)}{(p^2-p'^2)}\left\{J\left(-\frac{2}{B}\right)x_0Q_{J+1}(x_0)+(J+1)\left(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)\right\}\Bigg],$$

$$\langle J|V_4^{(2)}|J\rangle$$

$$=2\pi G\Bigg[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left\{-\left(-\frac{2}{B}\right)Q_J(x_0)+\frac{(J+2)}{(2J+1)}\left(-\frac{2x_0}{B}Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)$$

$$+\frac{(J-1)}{(2J+1)}\left(-\frac{2}{B}x_0Q_{J+1}(x_0)\right)\right\}\Bigg],$$

$$\langle J|V_5^{(2)}|J\rangle$$

$$=2\pi G\Bigg[-\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}Q_J(x_0)\right)+\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}x_0^2Q_J(x_0)+\frac{2}{3B}\delta_{J,1}\right)\Bigg],$$

so we obtain

$$\langle J|V^{(2)}(p,p')|J\rangle$$

$$=-\left(\frac{g^2}{4\pi}\right)\frac{[(2J+1)\gamma Q_J(x_0)-JQ_{J+1}(x_0)-(J+1)Q_{J-1}(x_0)]}{2\pi E_pE_{p'}(2J+1)}.$$

For a non-diagonal element, we have

$$\langle J+1|V^{(2)}(p,p')|J-1\rangle\equiv\langle J-1|V^{(2)}(p,p')|J+1\rangle$$

$$=\langle J+1|V_2^{(2)}|J-1\rangle+\langle J+1|V_3^{(2)}|J-1\rangle+\langle J+1|V_4^{(2)}|J-1\rangle,$$

where

$$\langle J+1|V_2^{(2)}|J-1\rangle$$

$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)}\Bigg[\left\{\beta+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\right\}\left\{p'^2\left(-\frac{2}{B}\right)Q_{J-1}(x_0)+p^2\left(-\frac{2}{B}\right)Q_{J+1}(x_0)\right.$$

$$\left.-2pp'\left(-\frac{2}{B}\right)Q_J(x_0)\right\}+\frac{2pp'\alpha}{(p^2-p'^2)}\left\{p'^2\left(-\frac{2}{B}\right)x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right.$$

$$\left.+p^2\left(-\frac{2}{B}\right)x_0Q_{J+1}(x_0)-2pp'\left(\left(-\frac{2}{B}\right)x_0Q_J(x_0)+\frac{2}{B}\delta_{J,0}\right)\right\}\Bigg],$$

$$\langle J+1|V_3^{(2)}|J-1\rangle$$

$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)}\Bigg[\left\{\gamma+\frac{(p^2+p'^2)\alpha}{(p^2-p'^2)}\right\}\left\{p'^2\left(-\frac{2}{B}\right)Q_{J-1}(x_0)+p^2\left(-\frac{2}{B}\right)Q_{J+1}(x_0)\right.$$

$$\left.+2pp'\left(-\frac{2}{B}\right)Q_J(x_0)\right\}-\frac{2pp'\alpha}{(p^2-p'^2)}\left\{p'^2\left(-\frac{2}{B}x_0Q_{J-1}(x_0)+\frac{2}{B}\delta_{J,1}\right)\right.$$

$$\left.+p^2\left(-\frac{2}{B}x_0Q_{J+1}(x_0)\right)-2pp'\left(-\frac{2}{B}x_0Q_J(x_0)+\frac{2}{B}\delta_{J,0}\right)\right\}\Bigg],$$

$$\langle J+1|V_4^{(2)}|J-1\rangle$$
$$=4\pi G\frac{\sqrt{J(J+1)}}{(2J+1)^2}\left[\frac{4p^2p'^2\alpha}{(p^2-p'^2)}\left(-\frac{2}{B}Q_{J+1}(x_0)-\left(-\frac{2}{B}\right)Q_{J-1}(x_0)\right)\right],$$

so we obtain

$$\langle J+1|V^{(2)}(p,p')|J-1\rangle=-\left(\frac{g^2}{4\pi}\right)\frac{\sqrt{J(J+1)}}{2\pi E_pE_{p'}(2J+1)}$$
$$\times\left[\frac{p(E_{p'}+M)}{p'(E_p+M)}Q_{J+1}(x_0)+\frac{p'(E_p+M)}{p(E_{p'}+M)}Q_{J-1}(x_0)-2Q_J(x_0)\right].$$

§ 6. Phase shifts in the Born approximation

Phase shifts for the most general potential in the Born approximation are given without solving the integral equations. Using Eq. (3·12) and Blatt-Biedenharn phase parameters,[12] we obtain the following results, where $|p|=|p'|\equiv p_0$;
for a spin singlet state

$$\tan^1\delta_{j,B}=-\frac{\pi p_0E_{p_0}}{2}\langle j|V(p_0,p_0)|j\rangle;$$

for spin triplet states

$$\tan^3\delta_{J,B}=-\frac{\pi p_0E_{p_0}}{2}\langle J|V(p_0,p_0)|J\rangle,$$

$$x_{J-1,B}=-\frac{\pi p_0E_{p_0}}{2}\langle J-1|V(p_0,p_0)|J-1\rangle,$$

$$x_{J+1,B}=-\frac{\pi p_0E_{p_0}}{2}\langle J+1|V(p_0,p_0)|J+1\rangle,$$

$$y_{J-1,B}=+\frac{\pi p_0E_{p_0}}{2}\langle J+1|V(p_0,p_0)|J-1\rangle,$$

$$y_{J+1,B}=+\frac{\pi p_0E_{p_0}}{2}\langle J-1|V(p_0,p_0)|J+1\rangle,$$

so we have

$$y_{J-1,B}=y_{J+1,B}\equiv y_B.$$

The results may be written in the following way:

(1) *V_0 type*

$$\tan^1\delta_{j,B}=-\pi^2p_0E_{p_0}A_j^0,$$
$$\tan^3\delta_{J,B}=-\pi^2p_0E_{p_0}A_J^0,$$
$$x_{J-1,B}=-\pi^2p_0E_{p_0}A_{J-1}^0,$$
$$x_{J+1,B}=-\pi^2p_0E_{p_0}A_{J+1}^0,$$

$$y_B=0,$$

where

$$A_n^{\,0}=\int V_0(p_0, z)\,P_n(z)\,dz.$$

(2) V_1 *type*

$$\tan^1\delta_{J,B}=0,$$

$$\tan^3\delta_{J,B}=-2\pi^2 p_0 E_{p_0}\left[-\frac{(J+1)}{(2J+1)^2}A^1_{J+1}+\frac{1}{(2J+1)}A^1_{J-1}\right],$$

$$x_{J-1,B}=-2\pi^2 p_0 E_{p_0}\frac{(J-1)}{(2J-1)}[A_J^{\,1}-A^1_{J-2}],$$

$$x_{J+1,B}=-2\pi^2 p_0 E_{p_0}\left[-\frac{(J+2)}{(2J+3)}A^1_{J+2}+\frac{(J+2)}{(2J+5)}A_J^{\,1}\right],$$

$$y_B=0,$$

where

$$A_n^{\,1}=\int\frac{V_1(p_0, z)\,P_n(z)}{\sin(\boldsymbol{p}\cdot\boldsymbol{p}')}\,dz$$

provided that $|p|=|p'|\equiv p_0$.

(3) V_2 *type*

$$\tan^1\delta_{J,B}=-\pi^2 p_0 E_{p_0}[-A_J^{2(0)}+A_J^{2(1)}],$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p_0}\left[A_J^{2(0)}-\frac{J}{(2J+1)}A^{2(0)}_{J+1}-\frac{(J+1)}{(2J+1)}A^{2(0)}_{J-1}\right],$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p_0}\left[\frac{1}{(2J+1)}A^{2(0)}_{J-1}-\frac{1}{(2J+1)}A_J^{2(0)}\right],$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p_0}\left[-\frac{1}{(2J+1)}A^{2(0)}_{J+1}+\frac{1}{(2J+1)}A_J^{2(0)}\right],$$

$$y_B=\pi^2 p_0 E_{p_0}\frac{\sqrt{J(J+1)}}{(2J+1)}[A^{2(0)}_{J-1}+A^{2(0)}_{J+1}-2A_J^{2(0)}],$$

where

$$A_n^{2(l)}=\int\frac{V_2(p_0, z)\,P_n(z)\,z^l}{(1-z)}\,dz.$$

(4) V_3 *type*

$$\tan^1\delta_{J,B}=4\pi^2 p_0 E_{p_0}[A_J^{3(0)}+A_J^{3(1)}],$$

$$\tan^3\delta_{J,B}=-4\pi^2 p_0 E_{p_0}\left[A_J^{3(0)}+\frac{J}{(2J+1)}A^{3(0)}_{J+1}+\frac{(J-1)}{(2J+1)}A^{3(0)}_{J-1}\right],$$

$$x_{J-1,B}=-4\pi^2 p_0 E_{p_0}\left[\frac{1}{(2J+1)}A_{J-1}^{3(0)}+\frac{1}{(2J+1)}A_J^{3(0)}\right],$$

$$x_{J+1,B}=4\pi^2 p_0 E_{p_0}\left[\frac{1}{(2J+1)}A_{J+1}^{3(0)}+\frac{1}{(2J+1)}A_J^{3(0)}\right],$$

$$y_B=4\pi^2 p_0 E_{p_0}\frac{\sqrt{J(J+1)}}{(2J+1)}[A_{J-1}^{3(0)}+A_{J+1}^{3(0)}+2A_J^{3(0)}],$$

where

$$A_n^{3(l)}=\int\frac{V_3(p_0,z)P_n(z)z^l}{(1+z)}dz.$$

(5) *V_4 type*

$$\tan^1\delta_{J,B}=-\pi^2 p_0 E_{p_0}[-A_J^{4(0)}+A_J^{4(2)}],$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p_0}\left[-A_J^{4(0)}+\frac{(J+2)}{(2J+1)}A_{J-1}^{4(1)}+\frac{(J-1)}{(2J+1)}A_{J+1}^{4(1)}\right],$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p_0}\left[\frac{2}{(2J+1)}A_J^{4(1)}+\frac{(2J-1)}{(2J+1)}A_{J-1}^{4(0)}-A_{J-1}^{4(2)}\right],$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p_0}\left[-\frac{2}{(2J+1)}A_J^{4(1)}-A_{J+1}^{4(2)}+\frac{(2J+3)}{(2J+1)}A_{J+1}^{4(0)}\right],$$

$$y_B=2\pi^2 p_0 E_{p_0}\frac{\sqrt{J(J+1)}}{(2J+1)^2}[A_{J+1}^{4(0)}-A_{J-1}^{4(0)}],$$

where

$$A_n^{4(l)}=\int\frac{V_4(p_0,z)P_n(z)z^l}{\sin^2(\boldsymbol{k}\cdot\boldsymbol{q})(1-z^2)}dz$$

provided that

$$|p|=|p'|\equiv p_0.$$

(6) *V_5 type*

$$\tan^1\delta_{J,B}=3\pi^2 p_0 E_{p_0}A_J^5,$$

$$\tan^3\delta_{J,B}=-\pi^2 p_0 E_{p_0}A_J^5,$$

$$x_{J-1,B}=-\pi^2 p_0 E_{p_0}A_{J-1}^5,$$

$$x_{J+1,B}=-\pi^2 p_0 E_{p_0}A_{J+1}^5,$$

$$y_B=0,$$

where

$$A_n^5=\int V_5(p_0,z)p_n(z)dz.$$

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Single Pion Production Process in Pion-Nucleon Collision and the Sakata Model

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Based on the Sakata model, we propose a model for the single pion production process in π-N collision which is essentially an extension of the Lindenbaum-Sternheimer model. It is pointed out that the analyses of sub-Bev single pion production phenomena will give us useful information about the level scheme of the Sakata Model. As a first step of the investigation of this model an analysis of the single pion production process in π^+-p collision at 500 Mev is made. Our model predicts the contribution of the $I=2$ boson isobar as well as the $I=3/2$ fermion isobar (Lindenbaum-Sternheimer model) for this process. Such a prediction is consistent with the present experimental branching ratio $(\pi^+ + p \to \pi^+ + \pi^0 + p)/(\pi^+ + p \to \pi^+ + \pi^+ + n) = 1.5$. We calculate the energy and angular distribution of the pion and nucleon which will be useful for obtaining information about the $I=2$ boson isobar and $I=3/2$ fermion isobar from this process.

§ 1. Introduction

In order to have a realistic understanding of the variety of the elementary particles and the success of the Nakano–Nishijima–Gell-Mann scheme for the strongly interacting particles, Sakata has proposed the idea that all the baryons and mesons are the compound particles composed of the three fundamental particles, proton, neutron and Λ-particle and their antiparticles.[1] Much progress was made along this Sakata's idea on the elementary particle when Ogawa introduced the concept of full symmetry[2] among the fundamental particles in the Sakata model. This full symmetry proposed by Ogawa has been further refined with collaboration of Ikeda and Ohnuki leading to the theory of unitary group $U(3)$ of degree three.[3] In this theory a physical particle state (stable particle or unstable isobar state) is assumed to correspond to a basis vector of an irreducible representation space of $U(3)$.

Applying the semi-empirical mass formula originally proposed by Matumoto[4] and reformulated so as to be adapted for the $U(3)$ theory, the present author and Yonezawa have studied the correspondence between the experimental information and the theoretical levels of the composite system derived from the $U(3)$ theory, and they have found that rather remarkable correspondence seems to exist between the theory and the experiment as far as the reliable experimental data are concerned, and they have stressed that all of the resonant states so far observed in the pion-nucleon and kaon-nucleon reaction can be interpreted as the particle levels

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of Sakata's composite model as well as the ordinary stable particle.[5]

Now we have some accumulation of the experimental information which seems to support the Sakata model and the $U(3)$ theory based on it, but the present experimental data are still insufficient to *establish* a hypothesis of composite model and the symmetry properties inherent in it. Especially we are in poverty of the experimental information about the boson isobar states, partly owing to the experimental difficulty of the direct meson-meson reaction.

It has been emphasized by many authors[6)~9)] that the experimental study of the multiple meson production process will be prospectful for studying such boson resonant states, if they exist, and many theoretical investigations have been performed, through which the evidence of boson resonant state and their structure have been gradually clarified.

The end of this paper is to emphasize that the meson production process in π-N collision will give us a useful information about these boson isobars whose existence is predicted in Sakata's composite particle model, and to propose a scheme that might be able to, in a certain sense, unify the various models that have been proposed to explain the multiple meson production phenomena. This discussion will be given in § 2.

Further, following the scheme given in § 2, we shall give an orientational information of the prediction of the Sakata model for the single pion production process at 500 Mev incident pion energy in π^+-p collision to serve for further experimental and theoretical analyses. Of course it is necessary to perform such an analysis covering all energy region and all channels. In order to obtain the clear evidence for the existence of the levels in our scheme, it is necessary to start the analysis with the phenomena which will involve only few parameters and allow us to confirm each level unambiguously. Then it will be appropriate at first to give the discussion of the phenomena near the threshold, where one has merely to concern with a few lower levels. As such one we choose the π^+-p collision at 500 Mev. In § 3 we shall discuss the theoretical energy spectra of the pion and nucleon and their angular correlations in this process.

§ 2. Some consideration based on the Sakata model

Boson resonant states or boson isobars are not new to the meson physics.[6)~9)] For example, the strong pion-pion interaction[6)] has been proposed to explain the behavior of high energy pion-nucleon reactions, especially with respect to the $I=1/2$ resonant states below 1 Bev laboratory pion kinetic energy and the pion production process at this energy region. In this paper we shall also discuss the boson isobar states that will contribute to the reactions in this energy region. However, our point here is a little different from those previous analyses.

Now let us confine ourselves to the discussion of the single pion production process. There are two main, according to Pontecorvo,[10)] *contradictory* models

for the interpretation of this phenomenon, i.e. the strong pion-pion interaction model and the (3/2, 3/2) nucleon isobar model due to Sternheimer and Lindenbaum.[11]

In our theory[5] based on the Sakata model the second and the third resonances of pion-nucleon reaction are considered due to the composite states which are characterized by the three-body system, $F_3^3(0, 1/2)$ and $F_3^1(0, 1/2)$.* $F_3^3(0, 1/2)$ and $F_3^1(0, 1/2)$ are assigned to the $I=1/2$ resonances at 600 Mev and 900 Mev laboratory pion energies respectively. Since the major part of the features of the total cross section of $I=1/2$ state in the pion-nucleon reaction will be determined by the existence of these levels, we may not necessarily need to introduce either the strong pion-pion interaction model or the nucleon isobar model for the interpretation of the behavior of total cross section below 1 Bev.** However, such circumstance does not deny the existence of the strong pion-pion interaction nor the success of the Lindenbaum-Sternheimer model. Rather the Sakata model will allow us a consistent understanding of these contradictory models.

From the pion energy spectrum of the single pion production process in the pion-nucleon collision around 1 Bev incident pion energy, we know that the (3/2, 3/2) nucleon isobar, or $F_3^1(0, 3/2)$ in our notation, is also produced as well as the stable particles such as nucleons and pions.[10),11] In Sakata's composite particle model this seems to be very natural occurrence, since in this theory all the stable particles and the unstable resonant states bear the same physical significance, and the only matter that makes them appear so different from each other is their life-time.

Accordingly, we can expect that other unstable composite states of the Sakata model will behave similarly as the (3/2, 3/2) isobar state does if the appropriate condition is fulfilled, although now we do not know the condition exactly.

We will shortly see what circumstances are expected when we assume that such a viewpoint holds generally and the single pion production process can be reduced to the two-body cascade decay of the intermediate state such as

$$\begin{array}{l} \pi+N\to B+N \\ \qquad\quad \lfloor\!\to \pi+\pi, \end{array} \tag{1a}$$

$$\begin{array}{l} \pi+N\to F+\pi \\ \qquad\quad \lfloor\!\to \pi+N, \end{array} \tag{1b}$$

where π and N indicate the pion and the nucleon respectively and B and F are the unstable composite states of the Sakata model with the indicated decay modes.***

* For fermion state we use the notation $F_j^i(S, I)$ where subscript j expresses the number of particles plus antiparticles of which the fermion is composed, and superscript i the class to which this state belongs (each class corresponds to each irreducible constituent). S and I in brackets are its strangeness and isotopic spin respectively. For boson state we write similarly as $B_j^i(S, I)$. See reference 5).

** Our present theoretical information of these composite states is limited to their masses, and we do not know how these composite states will appear with definite spin and parity. Its clarification will be an important task for the composite model and will make clear the role of π-π interaction, etc., if they have some contribution in the actually observed resonance phenomena.

*** The interference between (1a) and (1b) does not occur, since in our present model the first step of each reaction is assumed to be a real process.

According to the scheme (1) we give the production threshold energy of the states that may contribute to the single pion production process in Fig. 1. The masses of the levels given in Fig. 1 are those of the experimental data for the fermion states and those of mass formula (3) of the previous article[5] for the boson states. Unfortunately we have not information about the spin and parity of these states at present, but the lack of such information does not prevent one from obtaining the information on mass levels from such a kind of analysis as that in § 3.

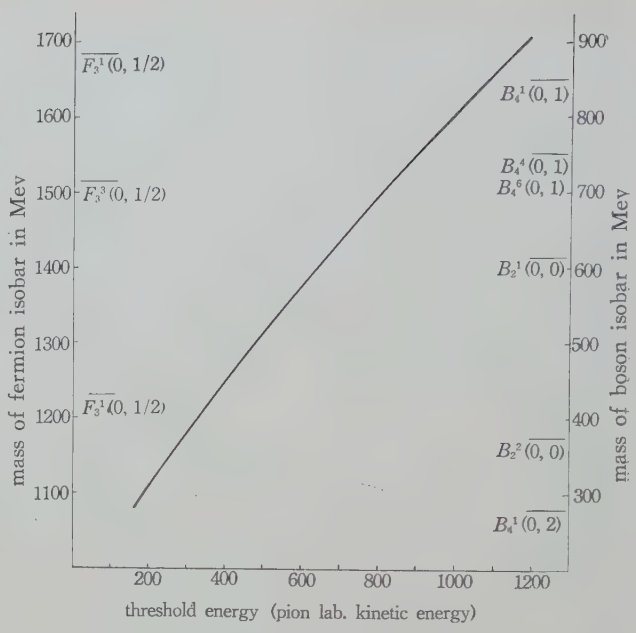


Fig. 1. Threshold energy of the unstable particles according to the scheme (1).

Table I

process	$\pi^+ + p \rightarrow$		$\pi^- + p \rightarrow$		
	$\pi^+ + \pi^0 + p$	$\pi^+ + \pi^+ + n$	$\pi^- + \pi^0 + p$	$\pi^+ + \pi^- + n$	$\pi^0 + \pi^0 + n$
$B(0, 2)$	$\frac{1}{5}\sigma_3$	$\frac{4}{5}\sigma_3$	$\frac{9}{45}\sigma_3$	$\frac{2}{45}\sigma_3$	$\frac{4}{45}\sigma_3$
$B(0, 1)$	σ_3	0	$\frac{1}{9}(4\sigma_1+\sigma_3+4a)$	$\frac{1}{9}(2\sigma_1+2\sigma_3-4a)$	0
$B(0, 0)$	0	0	0	$\frac{4}{9}\sigma_1$	$\frac{2}{9}\sigma_1$
$F(0, 3/2)$	$\frac{4}{15}\sigma_3$(extra π^+) $+\frac{9}{15}\sigma_3$(extra π^0)	$\frac{2}{15}\sigma_3$	$\frac{1}{135}(10\sigma_1+16\sigma_3-40b)$(extra π^-) $+\frac{1}{135}(10\sigma_1+\sigma_3-10b)$(extra π^0)	$\frac{1}{135}(45\sigma_1+18\sigma_3+90b)$(extra π^+) $+\frac{1}{135}(5\sigma_1+8\sigma_3-20b)$(extra π^-)	$\frac{1}{135}(20\sigma_1+2\sigma_3-20b)$
$F(0, 1/2)$	$\frac{1}{3}\sigma_3$(extra π^+)	$\frac{2}{3}\sigma_3$	$\frac{1}{27}(4\sigma_1+\sigma_3+4a)$(extra π^-) $+\frac{1}{27}(4\sigma_1+4\sigma_3-8a)$(extra π^0)	$\frac{1}{27}(8\sigma_1+2\sigma_3+8a)$(extra π^-)	$\frac{1}{27}(2\sigma_1+2\sigma_3-4a)$

From Table I we can easily obtain the charge ratio in each channel. In this Table σ_1 and σ_3 mean the cross section of $I=1/2$ and $I=3/2$ states of each channel where the notation specifying the channel has been omitted, and $a=\sqrt{\sigma_1\sigma_3}\cos\varphi$ and $b=\sqrt{2/5}a$ where φ is the relative phase between $I=1/2$ and $3/2$ states. The term denoted by (extra π) is the contribution from the process in which the designated π is emitted associated with F (cf. reference 11)).

We cannot answer the question what the magnitude of cross section of each process is and how much of the phenomena can be explained by the scheme (1), since we do not yet have the dynamical theory of the composite model. In a rough approximation, Fermi's statistical theory[12] will give us good estimation.

At the low energy near the threshold of the single pion production process we have $F_3^1(0, 3/2)$ as F of the scheme (1). As to B, we have to consider the contribution of $B_4^1(0, 2)$ and $B_2^2(0, 0)$ states. The estimation of the mass by the mass formula (3) of reference 5) gives $2m_\pi$ for $B_4^1(0, 2)$ where m_π is the pion mass. Since no reliable evidence of charge particles with the mass $\sim 2m_\pi$ has been reported, it will be natural to consider that $B_4^1(0, 2)$ may have the mass greater than $2m_\pi$ and decay rapidly into two pions. With respect to $B_2^2(0, 0)$ there are two alternatives, (a) $B_2^2(0, 0)$ is the excited state of the vacuum, and not a new particle with the mass $\sim 2.5m_\pi$, or (b) $B_2^2(0, 0)$ will be new particle and not the excited state of the vacuum. We do not know which alternative corresponds to the reality. In the case (b) there will be two possibilities for the decay. If $B_2^2(0, 0)$ is 0^+ or 2^+, it will decay into two pions very rapidly and contribute to the single pion production process. On the other hand, if it is 0^- or 1^+ and has the mass $< 3m_\pi$, then its dominant decay mode will be $\rightarrow 2\gamma$ or $\rightarrow \pi^0+\gamma$ and will be observed as metastable particle.[13]

As to the experimental data of π^--p reaction near the threshold, the analyses have been performed based on the strong pion-pion interaction model and the Lindenbaum-Sternheimer model respectively.[9),10] It has been reported that some features of the energy spectrum and angular distribution of pions are consistent with the theoretical prediction of the nucleon isobar model and, on the other hand, the large experimental cross section of the single pion production process can be explained by the strong pion-pion interaction. Although the experimental and theoretical investigations are at the very preliminary stage and it is not the purpose of the present article to criticize the analyses so far made, such a situation seems to have some connection with our present scheme that $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ (and possibly $B_2^2(0, 0)$) will contribute to the single pion production process near the threshold. If the mass of $B_4^1(0, 2)$ is $< 3m_\pi$, $B_4^1(0, 2)$ will give larger contribution than $F_3^1(0, 3/2)$ does at the low energy near the threshold. As to the problem of a large cross section[8] compared with the static pion theory, we make no comment, since we have no appropriate way to estimate the cross section. However, a large cross section of the single pion production process will not be unreasonable if we take into account of the phase volume[12] available for

the two-particle process and three-particle process at these energies and there are two (or possibly three) channels opened for the single pion production process.

As the incident pion energy increases, we will see from Fig. 1 that the next higher level $B_2^1(0, 0)$ or $\pi^{0\prime}$ will have influence above 600 Mev incident pion energy. Then, if our scheme is right, we can understand the phenomena of single pion production under 600 Mev incident pion energy in terms of $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ (and $B_2^2(0, 0)$). In this connection there are some interesting experimental data. They are the data on π^+-p reaction at 500 Mev pion energy.[14] It will be concerned only with $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ isobars in our scheme. The data give the branching ratio $\frac{\pi^+ + p \to \pi^+ + \pi^0 + p}{\pi^+ + p \to \pi^+ + \pi^+ + n} = 1.5\ ^{+1.5}_{-0.5}$ for the single pion production process. Although the statistics is still very poor, the interesting point is that this value is far from 6.5 which is the prediction of the Sternheimer-Lindenbaum model. Now, besides $F_3^1(0, 3/2)$, we have $B_4^1(0, 2)$ which gives $1/4$ for this branching ratio and we will have the value which is much smaller than 6.5 of the nucleon isobar model. (It is noted that the contribution of the boson isobar with $I=1$ makes to increase the ratio from 6.5 and $I=0$ isobar does not contribute entirely.) We will discuss the energy spectrum of pions, etc., of this reaction in the next section in view of its interesting features.

Above 600 Mev incident pion energy, the states $B_4^4(0, 1)$, $B_4^6(0, 1)$ and $B_4^1(0, 1)$ will newly appear and contribute to the single pion production, if they decay rapidly into two pions. $B_4^6(0, 1)$ is the antiparticle of $B_4^4(0, 1)$, so that they have the same mass, spin and parity. The estimation of their masses based on the two-body approximation (mass formula (3) of reference 5)) gives the value $\sim$750 Mev. Then they will become influential only above 800 Mev incident pion energy when we assume this value of mass. It is, however, necessary to take into account the theoretical error of order 200 Mev for the estimation by the mass formula, when we apply the formula to the four-body system, and we must consider the possibility that these levels appear at lower energies. The existence of the isobar states having $I=1$ and the corresponding energy have been discussed often.[6),7),16)]

In respect of the opening of the channels of $N+B_4^4(0, 1)$, $N+B_4^6(0, 1)$, $\pi+F_3^3(0, 1/2)$ and $B_4^1(0, 2)+F_3^1(0, 3/2)$, that may happen to concentrate in the energy region 700$\sim$900 Mev incident pion energy, it will be interesting to note that the shoulder behavior of the π^+-p total cross section around 800 Mev pion energy[15]* may have some connection with it. Further information of the masses and spin-parity of $B_4^4(0, 1)$, $B_4^6(0, 1)$ and $B_4^1(0, 2)$ may bring some clarification on this point. Since for the phenomena above 600 Mev we must newly take into account $B_2^1(0, 0)$, $B_4^4(0, 1)$, $B_4^6(0, 1)$, $B_4^1(0, 1)$, $F_3^3(0, 1/2)$ and $F_3^1(0, 1/2)$ besides $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ (and $B_2^2(0, 0)$) successively as the incident energy

* Another possible interpretation from the Sakata model is that it is due to the resonance states $F_5^{6,10}(0, 3/2)$, and this interpretation will be allowed if the fourth broad resonance around 1.3 Bev consists of two resonances, and not three resonances.

increases, the circumstance becomes complicated. But still it will be possible to obtain the information on these levels if we have sufficient accumulation of the experimental data about the branching ratio, energy spectrum of pions and nucleons, angular correlations, etc., in addition to the information obtained from the analyses of the reaction at lower energies. It is evident from the scheme (1) the fermion isobars F's manifest themselves most clearly in the pion energy spectrum, and the boson isobars B's in the nucleon energy spectrum. In regard to this, it will be useful to point out that the pion energy spectrum due to the decay of B is a monotonous function of energy so that the contribution from this channel does not generally wash away the characteristic features of the pion spectrum of the nucleon isobars. For this reason and from the fact that $F_3^1(0, 3/2)$ channel will be dominant in π^--p reaction even at 1 Bev, the qualitative success of the Sternheimer-Lindenbaum model at this energy, which has been shown mainly about the pion energy distribution, will not be unexpectable.

§ 3. Pion production in π^+-p collision at 500 Mev

In the preceding section we have given the prediction that would be expected in the single pion production process initiated by the pion-nucleon collision if we consider the phenomena based on the level scheme of the Sakata model. Our interest in the single pion production phenomena is, first of all, that it will give the information which may serve for establishing the level scheme of composite particle. For this purpose, we need to obtain information about each level. It will be clever to start with the analysis of the events that concern with parameters as few as possible, and then to extend the investigation to the more complicated cases.

As such a simple case, we choose the π^+-p reaction at 500 Mev. As shown in § 2, we will be concerned only with the total isospin $I=3/2$ state and $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ as the unstable isobars in this case. The energy 500 Mev will be high enough to produce these isobars and low enough to avoid the contributions of higher levels, $B_4^1(0, 1)$, etc. We have now no *correct* knowledge about higher levels, $B_4^1(0, 1)$, $B_4^4(0, 1)$ and $B_4^6(0, 1)$. The estimation of two parameter mass formula gives 750 Mev for $B_4^4(0, 1)$ and $B_4^6(0, 1)$ and 850 Mev for $B_4^{4'}(0, 1)$. One may of course expect an error in this estimation. But the maximum mass value of the boson isobar B that can be created at 500 Mev incident pion energy is $\sim$510 Mev, so we can reasonably expect that they will not give a large contribution to the process even if they have relatively small mass. The fact that only $B_4^1(0, 2)$ and $F_3^1(0, 3/2)$ will give large contribution to the process in question is a salient feature of the present model.

Now the cross section of each process is given from Table I by the expression

$$\sigma(\pi^+ + p \to \pi^+ + \pi^0 + p) = \frac{1}{5}\sigma_3(B_4^1(0, 2)) + \frac{13}{15}\sigma_3(F_3^1(0, 3/2)),$$

$$\sigma(\pi^+ + p \to \pi^+ + \pi^+ + n) = \frac{4}{5}\sigma_3(B_4^1(0, 2)) + \frac{2}{15}\sigma_3(F_3^1(0, 3/2)) , \qquad (3)$$

where $\sigma_3(B_4^1(0, 2))$ is the total cross section of the first step of the reaction (1 a) in which $B_4^1(0, 2)$ particle is emitted as B and $\sigma_3(F_3^1(0, 3/2))$ is that of the reaction (1 b). From this we can naturally expect that the branching ratio is in the range

$$0.25 \leqq \frac{\sigma(\pi^+ + p \to \pi^+ + \pi^0 + p)}{\sigma(\pi^+ + p \to \pi^+ + \pi^+ + n)} \leqq 6.5.$$

The value 6.5 corresponds to the case when only $F_3^1(0, 3/2)$ channel exists.

The preliminary experiment at this energy was performed by Willis, which gives $1.5^{+1.5}_{-0.5}$ for this ratio.[14]* This means $\sigma_3(B_4^1(0, 2))/\sigma_3(F_3^1(0, 3/2)) \simeq 2/3$. Although this experiment is a preliminary one and we have not an appropriate way to calculate the magnitude of the cross section, this value may not be unreasonable.**

Unfortunately we have not a reliable experimental information about pion energy spectrum, etc., at this incident energy. However, it is expected that the information about these quantities will increase rapidly so that it will be useful to make clear some features of the spectra of our model.

3-1. *Energy spectrum of pion*

The characteristics of the energy spectrum and angular correlation of the pion and nucleon are essentially determined by kinematical relations in the present model. So if we have full understanding about kinematical relations among the final particles it is rather easy to deduce what the spectra are. Hereafter, discussion is given in the c.m.s. of the total system.

The energy spectrum of the pion consists of three parts. The first is coming from the decay of $F_3^1(0, 3/2)$, $I_1(E_\pi)$, the second is the extra pion which is emitted in company with $F_3^1(0, 3/2)$, $I_2(E_\pi)$, and the third is the pion resulted from the decay of $B_4^1(0, 2)$, $I_3(E_\pi)$.

Assuming the independency of the matrix elements of the isobar momentum and spin state, the energy spectrum in each case can be expressed in c.m.s. as:

$$I_1(E_\pi)\, dE_\pi = N_1 \int_{m_{F\,min}}^{m_{F}{}^{max}} dm_F \rho(m_F) F_1 G_{\pi 1} dE_\pi , \qquad (5)$$

where

$m_{F\,min}^{max} = \left(\dfrac{b_1 \pm \sqrt{b_1^2 - a_1 c_1}}{a_1}\right)^{1/2}$: the available maximum (minimum) mass of $F_3^1(0, 3/2)$ for a fixed energy E_π of decay pion,

---

* The experiment at 300~750 Mev gives 1.0 for this ratio.[14]

** From the argument of the phase volume and weight of spin state, we have $\sigma_3(B_4^1(0, 2))/\sigma_3(F_3^1(0, 3/2)) \cong 0.77$ for spinless $B_4^1(0, 2)$ with mass 350 Mev and $F_3^1(0, 3/2)$ with mass 1230 Mev.

$$a_1=W^2+2m_\pi^2-2WE_\pi,$$

$$b_1=m_N^2W^2+m_\pi^4+(W^2-m_N^2)WE_\pi-2W^2E_\pi^2,$$

$$c_1=(m_N^4-2m_N^2m_\pi^2-m_\pi^4+m_\pi^2W^2)W^2+m_\pi^6+2(m_N^2-m_\pi^2)(W^2-m_\pi^2)WE_\pi,$$

$$G_{\pi 1}=\frac{m_F}{2p_Fp_{\pi F}},$$

$$F_1=E_Fp_F,$$

$E_F=\dfrac{W^2+m_F^2-m_\pi^2}{2W}$: the energy of $F_3^1(0, 3/2)$ in c.m.s.,

p_F : the momentum of $F_3^1(0, 3/2)$ in c.m.s.,

$E_{\pi F}=\dfrac{m_F^2+m_\pi^2-m_N^2}{2m_F}$: the energy of decay pion in the rest system of $F_3^1(0, 3/2)$,

$p_{\pi F}$: the momentum of decay pion in the rest system of $F_3^1(0, 3/2)$,

W : the total energy of the system in c.m.s..

$$I_2(E_\pi)\,dE_\pi=N_2\rho(m_F)F_1G_{\pi 2}\,dE_\pi, \tag{6}$$

where

$$G_{\pi 2}=\frac{W}{m_F}.$$

$$I_3(E_\pi)\,dE_\pi=N_3\int_{m_{B\min}}^{m_B{}^{max}}dm_B\rho(m_B)F_3G_{\pi 3}\,dE_\pi, \tag{7}$$

where

$m_{B\,min}^{\;max}=\left(\dfrac{b_3\pm\sqrt{b_3^2-a_3c_3}}{a_3}\right)^{1/2}$: the available maximum (minimum) mass of $B_4^1(0, 2)$ for a fixed energy E_π of decay pion,

$$a_3=W^2+m_\pi^2-2WE_\pi,$$

$$b_3=m_\pi^2(W^2+m_N^2)+(W^2-m_N^2)WE_\pi-2W^2E_\pi^2,$$

$$c_3=m_\pi^2(W^2-m_N^2)^2,$$

$$G_{\pi 3}=\frac{m_B}{2p_Bp_{\pi B}},$$

$$F_3=E_Np_N,$$

$E_N=\dfrac{W^2+m_N^2-m_B^2}{2W}$: the energy of nucleon in c.m.s.,

p_N : the momentum of nucleon in c.m.s.,

$E_B=W-E_N$: the energy of $B_4^1(0, 2)$ in c.m.s.,

p_B : the momentum of $B_4^1(0, 2)$ in c.m.s.;

$E_{\pi B}=\frac{m_B}{2}$: the energy of decay pion in the rest system of $B_4^1(0, 2)$,

$p_{\pi B}$: the momentum of decay pion in the rest system of $B_4^1(0, 2)$.

In Eqs. (5)$\sim$(7) N_1, N_2 and N_3 are the normalization constants. $\rho(m_F)$ is the mass spectrum of $F_3^1(0, 3/2)$ particle and $\rho(m_B)$ is the one of $B_4^1(0, 2)$ particle. F_1 and F_3 are the two-body phase space factors of the first steps of the reactions (1b) and (1a) divided by the boson (pion or $B_4^1(0, 2)$) energy. G_π's are the factors which give the energy spectrum of the pion from the decay of the isobars of masses m_F and m_B. Our discussion is essentially the same with the L-S model except for a few minor simplifications.

We have not correct information on the mass spectrum of each level, and it is very dependent on the unknown properties of the system (spin, parity, formation interaction, decay interaction, etc.). We assume the form

$$\rho(m_F)\sim(m_F-m_N-m_\pi)\times\exp\left[-\frac{(m_F-m_F^0)^2}{\Gamma_F^2}\right]. \tag{8}$$

However, this assumption will not be so essential for our present purpose. The parameters of $\rho(m_F)$ are chosen so as to approximately give the energy dependence of the total cross section of π^+-p reaction at low energy ($m_F^0=1230$ Mev, $\Gamma_F=66$ Mev).

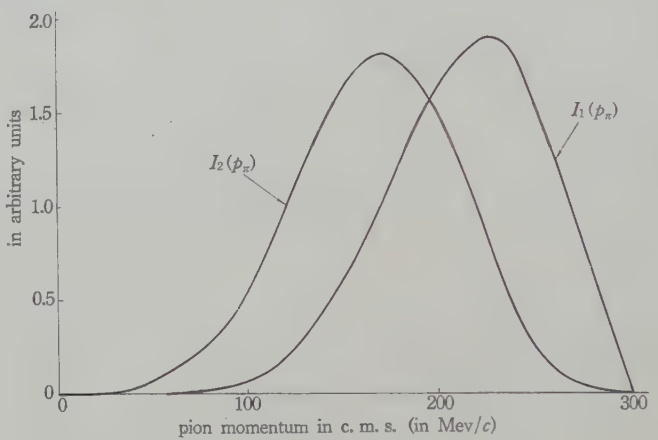


Fig. 2. Calculated momentum spectrum of pion in the reaction (1b).

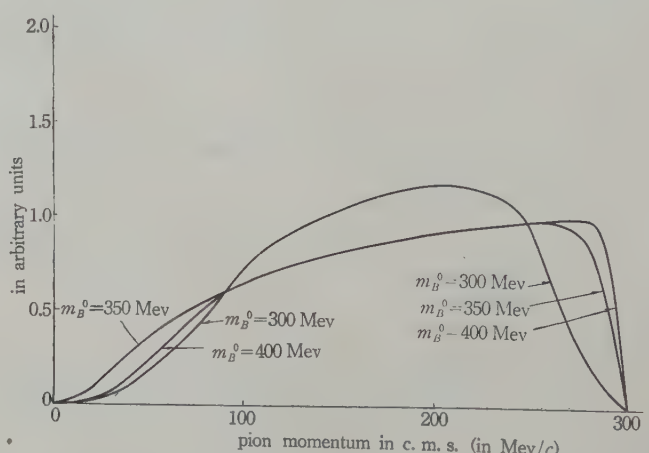


Fig. 3. Calculated momentum spectrum of pion in the reaction (1a).

For the $B_4^1(0, 2)$ particle the parameters are more arbitrary. The theoretical estimation of mass of $B_4^1(0, 2)$ gives $2m_\pi$, but owing to the situation that no evidence of charged boson with mass $\sim 2m_\pi$, it will be natural to consider $m_B > 2m_\pi$ and $B_4^1(0, 2)$ will decay rapidly. Even in this case $\rho(m_B)$ is still too arbitrary. Similar to the case of $\rho(m_F)$, we choose

$$\rho(m_B) \sim (m_B - 2m_\pi) \exp[-(m_B - m_B^0)^2/\Gamma_B^2], \tag{9}$$

with $m_B^0 = 300$, 350, 400 Mev and $\Gamma_B = 30$ Mev. The feature of the spectrum with the different parameters can be guessed from these data to some extent. The momentum spectra $I_i(p_\pi)$ are given from the corresponding energy spectrum $I_i(E_\pi)$ by the relation $I_i(p_\pi) = I_i(E_\pi)\, dE_\pi/dp_\pi$.

In Fig. 2 we give the momentum distribution of the pion calculated from $I_1(E_\pi)$ and $I_2(E_\pi)$. In Fig. 3 we give the distribution due to $I_3(E_\pi)$.

The energy spectrum in each process can be expressed in terms of $I_1(E_\pi)$, $I_2(E_\pi)$ and $I_3(E_\pi)$ as

π^+ spectrum in $\pi^+ + p \to \pi^+ + \pi^0 + p$

$$I(E_\pi)\, dE_\pi = \left[\frac{1}{15}\sigma_3(F_3^1(0, 3/2))\{9I_1(E_\pi) + 4I_2(E_\pi)\} + \frac{1}{5}\sigma_3(B_4^1(0, 2))\, I_3(E_\pi)\right] dE_\pi, \tag{10}$$

π^0 spectrum in $\pi^+ + p \to \pi^+ + \pi^0 + p$

$$I(E_\pi)\, dE_\pi = \left[\frac{1}{15}\sigma_3(F_3^1(0, 3/2))\{4I_1(E_\pi) + 9I_2(E_\pi)\} + \frac{1}{5}\sigma_3(B_4^1(0, 2))\, I_3(E_\pi)\right] dE_\pi, \tag{11}$$

π^+ spectrum in $\pi^+ + p \to \pi^+ + \pi^+ + n$

$$I(E_\pi)\, dE_\pi = \left[\frac{2}{15}\sigma_3(F_3^1(0, 3/2))\{I_1(E_\pi) + I_2(E_\pi)\} + \frac{4}{5}\sigma_3(B_4^1(0, 2))\, I_3(E_\pi)\right] dE_\pi. \tag{12}$$

For the interest of the reader, the theoretical momentum spectrum of the pion in each process is given in Figs. 4 and 5 for the case of $m_B^0 = 350$ Mev and $\sigma_3(B_4^1(0, 2))/\sigma_3(F_3^1(0, 3/2)) = 2/3$. The experimental data of Willis are also given in Figs. 4 and 5.

Although it will be too early to discuss these experimental data, Willis has noted that in the $\pi^+ + p \to \pi^+ + \pi^+ + n$ process the pions are emitted with very high or low energies. A similar observation has also been made for the experimental data of Blevins et al.[14] Apparently the characteristic feature of the spectrum of $F_3^1(0, 3/2)$ channel does not show such a tendency. The existence of $B_4^1(0, 2)$ will reduce such a discrepancy, if not satisfactory.

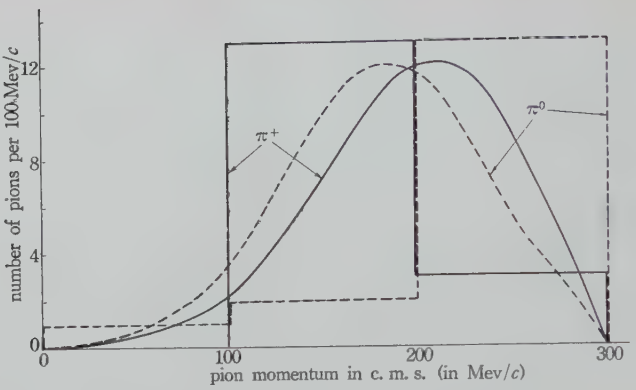


Fig. 4. Calculated and experimental momentum spectra of π^+ in $\pi^+ + p \to \pi^+ + \pi^0 + p$ (solid curve) and π^0 in $\pi^+ + p \to \pi^+ + \pi^0 + p$ (dashed curve).

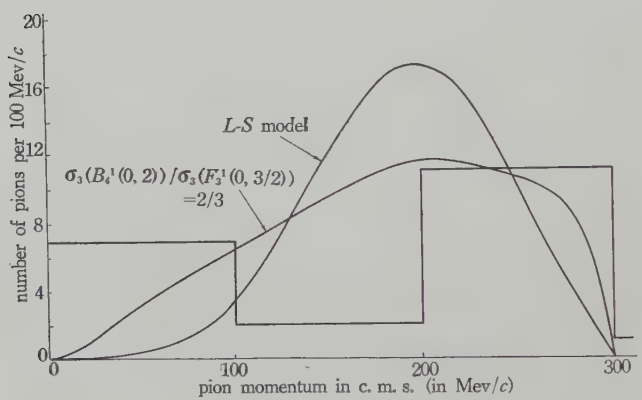


Fig. 5. Calculated and experimental momentum spectrum of π^+ in $\pi^+ + p \to \pi^+ + \pi^+ + n$.

3-2. *Energy spectrum of nucleon*

The energy spectrum of nucleon consists of two parts. One is the decay product of $F_3^1(0, 3/2)$, $J_1(E_N)$, and the other is the extra nucleon which is emitted in company of $B_4^1(0, 2)$, $J_2(E_N)$.

$J_1(E_N)$ and $J_2(E_N)$ are expressed as:

$$J_1(E_N)\,dE_N = N_1' \int_{m_{F\,min}}^{m_{F}{}^{max}} dm_F \rho(m_F) F_1 G_{N1}\,dE_N\,, \tag{13}$$

where

$m_{F\,min}^{\;\;max} = \left(\dfrac{a \pm \sqrt{b^2 - ac}}{a}\right)^{1/2}$: the available maximum (minimum) mass of $F_3^1(0, 3/2)$ for a fixed energy E_N of decay nucleon,

$$a=W^2+m_N^2-2WE_N,$$

$$b=m_\pi^2(W^2+m_N^2)+(W^2+m_N^2-2m_\pi^2)\,WE_N-2W^2E_N^2,$$

$$c=m_N^2(W^2-m_\pi^2)^2+(m_N^2-m_\pi^2)^2W^2-2(m_N^2-m_\pi^2)\,(W^2-m_\pi^2)\,WE_N,$$

$$G_{N1}=\frac{m_F}{2p_Fp_{NF}}=G_{\pi 1}.$$

$$J_2(E_N)\,dE_N=N_2'\rho(m_B)\,F_3G_{N2}\,dE_N, \tag{14}$$

where

$$G_{N2}=\frac{W}{m_B}.$$

The notation N_1', N_2', G_{N1} and G_{N2} should be understood analogously as in the case of the pion energy spectrum. In Fig. 6 we give the momentum spectrum of nucleon calculated from $J_1(E_N)$ and $J_2(E_N)$.

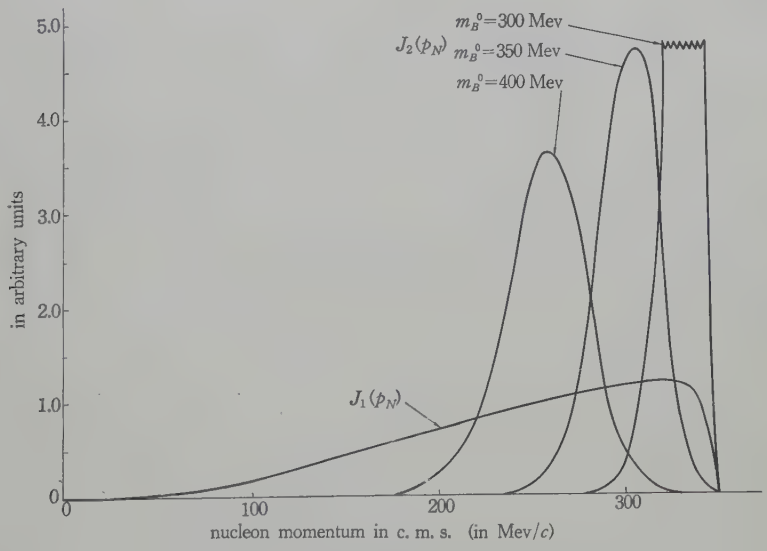


Fig. 6. Calculated momentum spectrum of nucleon.

The energy spectrum of nucleon in each process can be expressed in terms of $J_1(E_N)$ and $J_2(E_N)$ as

p spectrum in $\pi^++p\rightarrow\pi^++\pi^0+p$

$$J(E_N)\,dE_N=\left\{\frac{13}{15}\sigma_3(F_3^1(0,3/2))\,J_1(E_N)+\frac{1}{5}\sigma_3(B_4^1(0,2))\,J_2(E_N)\right\}dE_N, \tag{15}$$

n spectrum in $\pi^+ + p \to \pi^+ + \pi^+ + n$

$$J(E_N)\,dE_N = \left\{ \frac{2}{15}\,\sigma_3(F_3^1(0, 3/2))\,J_1(E_N) + \frac{4}{5}\,\sigma_3(B_4^1(0, 2))\,J_2(E_N) \right\} dE_N . \tag{16}$$

Fixing the $\sigma_3(B_4^1(0, 2))/\sigma_3(F_3^1(0, 3/2))$ ratio by the experimental data of Willis, and assuming $m_B{}^0 = 350$ and 400 Mev, we obtain the spectra in Figs. 7 and 8.

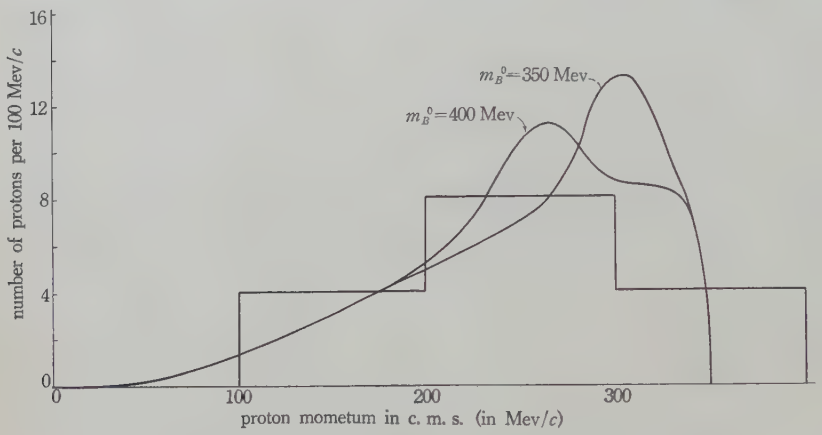


Fig. 7. Calculated and experimental momentum spectrum of p in $\pi^+ + p \to \pi^+ + \pi^0 + p$.

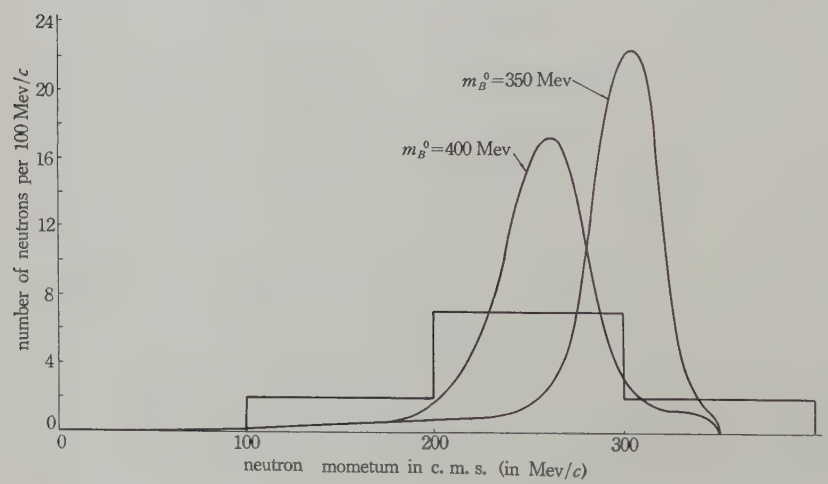


Fig. 8. Calculated and experimental momentum spectrum of n in $\pi^+ + p \to \pi^+ + \pi^+ + n$.

3-3. *Angular correlations*

In the preceding discussions we have given information on the momentum spectrum of the pion and nucleon. The obtained spectra have characteristic features in each

case, so that even with the momentum spectrum one can make a certain detailed discussion for our present scheme, if sufficient experimental data are supplied.

In this section we will give some crude information about the angular correlation between final products. It will also be useful in comparing the present scheme with experiment.

Different from the case of mometum spectrum, the feature of the angular correlation is rather strongly dependent on the mass of unstable isobars.

So, for the angular correlation, we give the theoretical spectrum when $\rho(m_F)\sim\delta(m_F-m_F{}^0)$ and $\rho(m_B)\sim\delta(m_B-m_B{}^0)$. Such a discussion may be helpful to understand the mutual correlation of the mass spectrum and the angular correlation.

a) Angular correlation between pions

The angular correlation between pions consists of two parts. One is the correlation between the extra pion and the decay pion in the reaction (1b), $K_1(\theta_{\pi\pi})$, and the other is the correlation between the pions which are the decay products of the reaction (1a), $K_2(\theta_{\pi\pi})$. They can be written as

$$K_1(\theta_{\pi\pi})\,d(\cos\theta_{\pi\pi})=\int dm_F\rho(m_F)\,\frac{\gamma_F{}^2(x_\pi-\delta_\pi)^3}{2(1-\delta_\pi x_\pi)\cos^3\theta_{\pi\pi}}\,d(\cos\theta_{\pi\pi}),\qquad(17)$$

where x_π is given by the relation $\dfrac{\sqrt{1-x_\pi{}^2}}{\gamma_F(x_\pi-\delta_\pi)}=\tan\theta_{\pi\pi}$, and $\gamma_F=\dfrac{E_F}{m_F}$ and $\delta_\pi=p_F E_{\pi F}/E_F p_{\pi F}$.

$$K_2(\theta_{\pi\pi})\,d(\cos\theta_{\pi\pi})=\int dm_B\rho(m_B)\,\frac{\{(\gamma_B{}^2-1)y^2+(1-\gamma_B{}^2\delta_B{}^2)\}^3}{4\gamma_B{}^2\delta_B{}^2y[(\gamma_B{}^2-1)(1-y^2)+\gamma_B{}^2(1-\delta_B{}^2)]\cos^3\theta_{\pi\pi}}\,d(\cos\theta_{\pi\pi}),\qquad(18)$$

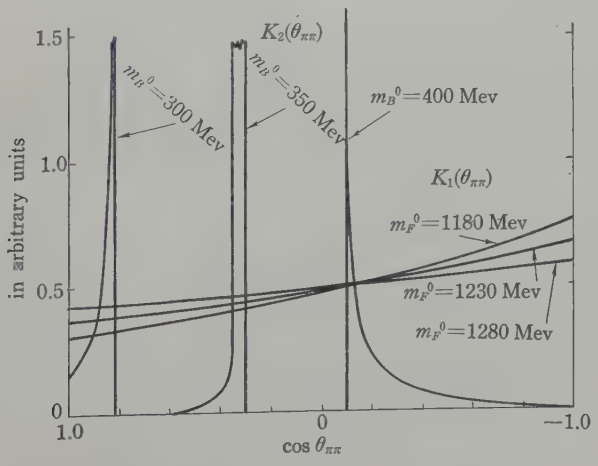


Fig. 9. The graphs of K_1 and K_2.

where y is given by the relation $\dfrac{2\gamma_B\delta_B\sqrt{1-y^2}}{(\gamma_B^2-1)y+(1-\gamma_B^2\delta_B^2)}=\tan\theta_{\pi\pi}$, and $\gamma_B=\dfrac{E_B}{m_B}$ and $\delta_B=p_BE_{\pi B}/E_Bp_{\pi B}$. These $K_1(\theta_{\pi\pi})$ and $K_2(\theta_{\pi\pi})$ are given in Fig. 9 with $m_F{}^0=1180$, 1230, 1280 Mev and $m_B{}^0=300$, 350, 400 Mev.

b) Angular correlation between pion and nucleon

The angular correlation between nucleon and pion consists of three parts, the correlation between the extra pion and the nucleon resulted from the decay of isobar $F_3{}^1(0, 3/2)$, $L_1(\theta_{\pi N})$, the correlation between the pion and the nucleon which are the decay products of $F_3{}^1(0, 3/2)$, $L_2(\theta_{\pi N})$ and the correlation between the extra nucleon and the pion from the decay of $B_4{}^1(0, 2)$, $L_3(\theta_{\pi N})$. They are expressed respectively as

$$L_1(\theta_{\pi N})\,d(\cos\theta_{\pi N})=\int dm_F\rho(m_F)\,\frac{\gamma_F{}^2(x_N-\delta_N)^3}{2(1-\delta_Nx_N)\cos^3\theta_{\pi N}}\,d(\cos\theta_{\pi N}), \tag{19}$$

where x_N is given by the relation $\dfrac{\sqrt{1-x_N{}^2}}{\gamma_F(x_N-\delta_N)}=\tan\theta_{\pi N}$, and $\delta_N=\dfrac{p_FE_{NF}}{E_Fp_{NF}}$;

$$L_2(\theta_{\pi N})\,d(\cos\theta_{\pi N})=\int dm_F\rho(m_F)$$
$$\times\frac{[(\gamma_F{}^2-1)z^2+\gamma_F{}^2(\delta_N-\delta_\pi)z+(1-\gamma_F{}^2\delta_N\delta_\pi)]^3\,d(\cos\theta_{\pi N})}{2\gamma_F{}^2(\delta_N+\delta_\pi)^2\{-(\gamma_F{}^2-1)z^3+(2\gamma_F{}^2-\gamma_F{}^2\delta_F\delta_\pi-1)z+\gamma_F{}^2(\delta_N-\delta_\pi)\}\cos^3\theta_{\pi N}}, \tag{20}$$

where z is given by the relation $\dfrac{\gamma_F(\delta_N-\delta_\pi)\sqrt{1-z^2}}{\gamma_F{}^2(z-\delta_\pi)(z+\delta_N)+(1-z^2)}=\tan\theta_{\pi N}$;

$$L_3(\theta_{\pi N})\,d(\cos\theta_{\pi N})=\int dm_B\rho(m_B)\,\frac{\gamma_B{}^2(x_B-\delta_B)}{2(1-\delta_Bx_B)\cos^3\theta_{\pi N}}\,d(\cos\theta_{\pi N}), \tag{21}$$

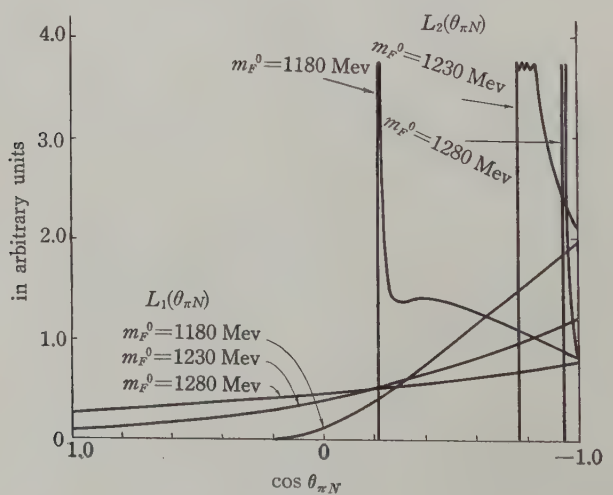


Fig. 10. The graphs of L_1 and L_2.

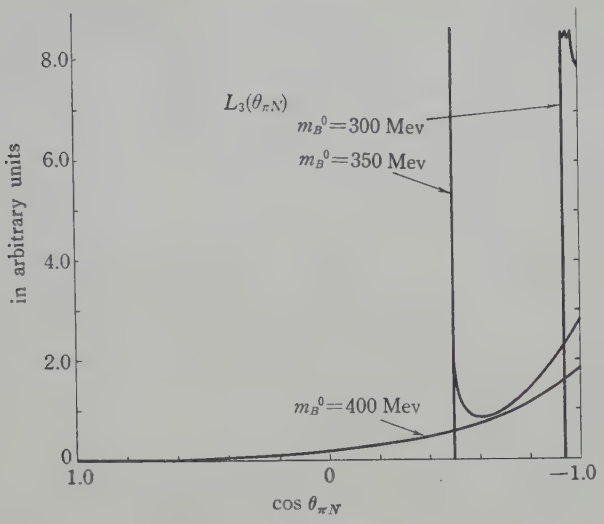


Fig. 11. The graphs of L_3.

where x_B is given by the relation $\frac{\sqrt{1-x_B{}^2}}{\gamma_B(x_B-\delta_B)}=\tan\theta_{\pi N}$. $L_1(\theta_{\pi N})$, $L_2(\theta_{\pi N})$ and $L_3(\theta_{\pi N})$ are given in Figs. 10 and 11.

In our opinion, the investigation of the pion-pion angular correlation seems to be more prospective than the pion-nucleon correlation to make clear the mutual correspondence between our model and the experiment.

Besides the investigation of the angular correlations between final products, the analysis of the correlations between the incident particle and the secondary particles is also important. For example, the Adair type analysis[17] will be interesting for the determination of the spin and parity of the isobar.

§ 4. Some remarks

In the preceding sections we have given an analysis of the single pion production process in π-N collision. Our analysis is essentially an extension of the Lindenbaum-Sternheimer model to the case where the boson isobar exists besides the fermion isobar. Our aim in this paper is to give rough information in order to help further theoretical and experimental investigation.

For the present stage of the composite theory, it is one of the important problems to find out what a correspondence exists between the theoretical levels (such as $U(3)$ theory) and the experimental levels. Although we believe the composite levels manifest themselves as stable particles or as resonant states in nature, it is rather difficult at present to show theoretically *how these unstable composite levels show themselves in the interaction.* But this will not give an objection to the necessity of such a kind of phenomenological analysis so far made. Our feeling for the necessary step to the future advance is to confirm a corre-

spondence between the theoretical levels and the experimental levels and to study the various processes such as the transitions between levels of composite system as far as possible, even if it were based on a rather naive picture. For such a purpose, we have proposed a systematic investigation of the pion production process, starting from the low energy phenomena which may involve few unknown parameters.

The theoretical analysis of the pion production process has so far been performed along two main different approaches (the L-S model and the strong pion-pion interaction model) rather independently, and the analyses in each model have shown some consistency with the experimental results. Our standpoint here proposed may give a logical basis for these models and show that both, if we use old terminology, the L-S model and the strong pion-pion interaction model are necessary to explain the phenomena.

Finally, it is also noted that the unstable composite states may appear as the reaction products and play important roles, not only in the multiple meson production process initiated by the pion collision, but also in other processes such as the nucleon-nucleon collision and nucleon-antinucleon annihilation.[16] For example, we can increase the multiplicity of pion without taking a large interaction volume, if we take account of the boson isobars predicted in the Sakata model, although it will require further extensive work to find out what is really going on in these phenomena.

We hope that experimental and theoretical analyses will be performed so as to make clear the scheme proposed in this paper.

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Energy Spectrum of Lattices with Defects. I

—— *General Theory* ——

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(August 29, 1960)

An attempt is presented for calculating the spectrum of the vibrational frequencies as well as that of electronic energy levels of crystal lattices with defects. It is shown that the Fourier or the Laplace transform of the spectrum can be obtained from the trace of density matrices which satisfy a set of simultaneous differential-difference equations.

Formal solutions of these equations are given, which are analogous to the integral expressions for additive functions of normal mode frequencies originally used by Montroll and Potts. As another approach to the solution of the equations, the perturbation expansion method is employed, which yields a cluster expansion or an expansion in terms of the concentration of defects. Finally the moment method and its correspondence to the random walk problem are discussed.

§ 1. Introduction

In recent years there have been a number of papers on the problem of the vibration frequency spectrum as well as the density of electronic energy levels for crystalline solids containing lattice defects. Montroll[1] invented a computational method for the evaluation of the frequency spectrum of periodic lattices by means of calculating the trace of a matrix of a secular determinant by which the eigenfrequencies were determined. Montroll's method, the moment trace method has been applied by his collaborators[2)~4)] and Hori[5] to determine the vibrational spectrum of one-dimensional disordered lattices, and energetic works have been presented in a series of papers. Ford[6] used the same method to study the density of electronic states of one-dimensional liquid metals. Mahanty et al.[7] gave another method which is a simple application of the method of evaluating additive functions of the normal mode frequencies[8] originally used by Montroll and Potts.[9] While for the same problem, a powerful method of computation was presented by Dean,[10] which, however, can be applied only to one-dimensional lattices. Although these various methods yield several results of physical interests, they do not seem to provide a proper basis for practical calculations of spectra for three-dimensional lattices.

Apart from a series of works as mentioned above, there are papers of Dyson,[11] Lifshitz and Stepanova,[12] and Schmidt.[13] They have features of considerable mathematical elegance. However, it seems to be very difficult to apply them to practical calculations. Parmenter[14] studied the density of electronic energy levels of three-dimensional disordered lattices by using Nordheim-Muto's virtual crystal

approximation.[15),16)] The most striking result of this theory is the prediction of a tailing-off of the density of states curve into a forbidden band. His theory, however, fails to give a complete description of the spectra, because the broadening of each in-band level is not taken into account.

The purpose of the present paper is to present a new approach, which seems to provide with a more powerful technique than the previous ones for evaluating the frequency spectrum as well as the spectrum (the density of states) of electronic levels for solids containing impurities. We start with the Fourier or the Laplace transform of the spectrum. In § 2 it is shown that both of them can be obtained by introducing density matrices. Formally the equations satisfied by the density matrices are analogous to the Schrödinger equation or the Bloch equation. This permits us to apply the various methods employed in quantum mechanics. The present method seems to have advantages over previous ones in that the calculation of the spectrum can be reduced to solve these equations, and then to evaluate the trace. Here it should be emphasized that the present method enables us to obtain the spectrum without knowing each energy eigenvalue and corresponding eigenfunction and therefore without considering the secular determinant.

In § 3 we present a fundamental formulation of the problem. Our basic equations which determine the energy eigenvalues are a set of difference equations, and so the density matrices satisfy differential-difference equations. Our procedure for solving these equations is to use the method of the Green's function similar to that employed in scattering theory.[17)~19)] It is shown that they are reduced to simultaneous integral equations of the convolution type.

In § 4 we study the Laplace transforms of the density matrices, which are adequate for obtaining formal solution. They satisfy a set of difference equations which are closely connected with the ones encountered in the time-independent impurity problem;[20)~22),9)] exact solutions of these equations are obtained by solving secular equations whose orders are equal to the number of lattice sites over which the impurity potential extends. Perturbation calculation for the difference equations is carried out in § 5, from which an expansion in terms of the concentration of impurities is obtained. This perturbation expansion is intimately connected with the multiple scattering problem in quantum mechanics.[23),24)] It is shown that each expansion term corresponds to graphs which include various types of clusters of impurities. In § 6 the moment method is discussed together with its connection with the random walk problem.

In this paper we present a general formulation of the problem, still confining ourselves to study the formal properties of the energy spectrum without specifying simple models. The application of the result to the construction of the spectrum will be left in a forthcoming paper.

§ 2. Density matrices

We consider a crystal which contains N atoms arranged in a Bravais lattice.

Here N is assumed to be infinitely large. We assume that there are identical impurities introduced substitutionally into this lattice. The distribution of the squares of the normal mode frequencies of the vibration as well as the density of one-electron energy states are both designated by the same notation $N(E)$, where E is the energy eigenvalue in the latter case, and is the square of the eigenfrequency ω^2 in the former case. The Fourier transform of $N(E)$ is given by

$$M(\alpha)=\int_{-\infty}^{\infty}\exp(i\alpha E)\,N(E)\,dE. \tag{2·1}$$

It is rewritten as

$$M(\alpha)=\frac{1}{N}\sum_{n}\exp(i\alpha E_n). \tag{2·2}$$

Then $M(\alpha)$, the characteristic function of $N(E)$, is given by

$$M(\alpha)=\frac{1}{N}\,\mathrm{Tr}\{\rho(\alpha)\}, \tag{2·3}$$

where

$$\rho(\alpha)=\exp(i\alpha L) \tag{2·4}$$

is a density matrix, and where L is an operator corresponding to the energy eigenvalue E, but it is not yet specified. $\rho(\alpha)$ satisfies the differential equation

$$\frac{1}{i}\frac{\partial\rho}{\partial\alpha}=L\rho,\quad \rho(0)=1, \tag{2·5}$$

which is formally equivalent to the Schrödinger equation if α is replaced by time t. The operator L can ordinarily be decomposed into the sum of an unperturbed term L_0 and a perturbational term V:

$$L=L_0+V. \tag{2·6}$$

We study the Laplace transform of Eq. (2·5): Let

$$\sigma(s)=\int_{0}^{\infty}\exp(-s\alpha)\rho(\alpha)\,d\alpha, \tag{2·7}$$

and then

$$\sigma(s)=\frac{1}{s-iL_0}+\frac{i}{s-iL_0}V\sigma(s). \tag{2·8}$$

The variable s must be understood as a complex variable with positive or zero real part.

Similarly, we can define the bilateral Laplace transform of $N(E)$:

$$K(\alpha)=\int_{-\infty}^{\infty}\exp(-\alpha E)\,N(E)\,dE. \tag{2·9}$$

This transform always exists, since $N(E)$ vanishes except in limited regions of E. As before, $K(\alpha)$ is given by

$$K(\alpha)=\frac{1}{N}\mathrm{Tr}\phi(\alpha), \tag{2·10}$$

where

$$\phi(\alpha)=\exp(-\alpha L). \tag{2·11}$$

It satisfies the equation

$$\frac{\partial\phi}{\partial\alpha}=-L\phi,\quad \phi(0)=1, \tag{2·12}$$

which is precisely the familiar density matrix encountered in statistical mechanics. Applying the Laplace transformation to the above equation, we have

$$\tau(s)=-\frac{1}{s-L_0}-\frac{1}{s-L_0}V\tau(s). \tag{2·13}$$

Similar equations were obtained by Watson[25] in his theory on the evaluation of the partition function in a quantum-mechanical system.

§ 3. Difference equations and integral equations

So far, we have not specified the operators L_0 and V. In the case of lattice vibrations E is determined by the time-independent equation of motion for atoms. Here we assume that each component of the displacement of one atom is independent of each other.* The equations of motion of the harmonic vibration for one component of the displacement may be written in the form:

$$\sum_{\boldsymbol{R}_j}\{E(\boldsymbol{R}_i-\boldsymbol{R}_j)+V(\boldsymbol{R}_i,\boldsymbol{R}_j)\}u(\boldsymbol{R}_j)=Eu(\boldsymbol{R}_i), \tag{3·1}$$

$$(i=1,2,\cdots N)$$

where

$$E(\boldsymbol{R}_i-\boldsymbol{R}_j)=\kappa(\boldsymbol{R}_i-\boldsymbol{R}_j)/m \tag{3·2}$$

is the term due to the unperturbed lattice, $\kappa(\boldsymbol{R}_i-\boldsymbol{R}_j)$ being the force constant between the atom at $\boldsymbol{R}_i$ and the atom at $\boldsymbol{R}_j$, and m being the mass of one atom. $V(\boldsymbol{R}_i,\boldsymbol{R}_j)$ is the corresponding term due to the impurity potential, the explicit form of which, however, is determined by the type of the impurities.

Similar difference equations arise when we make use of a tight-binding approximation for the electronic states; the one-electron crystal wave function is approximated by the Bloch sum, constructed by taking a linear combination of orthogonal atomic orbitals, i. e. for example, the Wannier functions or the orthogonalized atomic orbitals. In the general case, of course, we must approximate the

* This is true, for example, for a simple cubic lattice with nearest neighbour interaction.

crystal wave function by several Bloch sums, one for each of the interacting atomic levels, but for the present we consider only a single sum:

$$\Psi(\boldsymbol{r}) = \sum_{\boldsymbol{R}_j} u(\boldsymbol{R}_j)\varphi(\boldsymbol{r}-\boldsymbol{R}_j), \tag{3·3}$$

where $\boldsymbol{r}$ denotes the space coordinate of the electron, $\Psi(\boldsymbol{r})$ is the crystal wave function, and $\varphi(\boldsymbol{r}-\boldsymbol{R}_j)$ is the orthogonal atomic orbitals centered at $\boldsymbol{R}_j$, and where the sum extends over the lattice sites making up the crystal. The one-electron Hamiltonian of a system is given by

$$H = H_0 + V(\boldsymbol{r}), \tag{3·4}$$

where H_0 is the one-electron operator for the unperturbed lattice, the sum of a kinetic energy, a Hartree field of the electron cloud, and a potential energy of the ion cores which reflect the periodicity of the perfect lattice, and $V(\boldsymbol{r})$ is the perturbation due to impurities. The energy eigenvalue is determined by the equation:

$$(H_0 + V(\boldsymbol{r}))\Psi = E\Psi. \tag{3·5}$$

If we substitute (3·3) in the above equation, multiply the resulting equation by $\varphi^*(\boldsymbol{r}-\boldsymbol{R}_i)$, and, finally, integrate over the space coordinate, we obtain a set of difference equations for the u's which have exactly the same form as Eq. (3·1). Here $E(\boldsymbol{R}_i-\boldsymbol{R}_j)$ and $V(\boldsymbol{R}_i, \boldsymbol{R}_j)$ are given by

$$E(\boldsymbol{R}_i-\boldsymbol{R}_j) = \int \varphi^*(\boldsymbol{r}-\boldsymbol{R}_i)\, H_0 \varphi(\boldsymbol{r}-\boldsymbol{R}_j)\, d\boldsymbol{r}, \tag{3·6}$$

$$V(\boldsymbol{R}_i,\ \boldsymbol{R}_j) = \int \varphi^*(\boldsymbol{r}-\boldsymbol{R}_i)\, V(\boldsymbol{r}) \varphi(\boldsymbol{r}-\boldsymbol{R}_j)\, d\boldsymbol{r}, \tag{3·7}$$

where $\varphi^*(\boldsymbol{r}-\boldsymbol{R}_i)$ is the complex conjugate of $\varphi(\boldsymbol{r}-\boldsymbol{R}_i)$, and $d\boldsymbol{r}$ designates the volume element of $\boldsymbol{r}$.

Eqs. (3·1) define the operators L_0 and V explicitly; namely it can be written as the following operator equation:

$$(L_0 + V)u(\boldsymbol{R}) = Eu(\boldsymbol{R}), \tag{3·8}$$

when we define L_0 by

$$L_0 = E(-i\nabla), \tag{3·9}$$

where $E(-i\nabla)$ is given by replacing $\boldsymbol{k}$ by $-i\nabla$ in the following equation,

$$E(\boldsymbol{k}) = \sum_{\boldsymbol{R}} \exp(i\boldsymbol{k}\cdot\boldsymbol{R}) E(\boldsymbol{R}), \tag{3·10}$$

and $Vu(\boldsymbol{R})$ by

$$Vu(\boldsymbol{R}) = \sum_{\boldsymbol{R}_i, \boldsymbol{R}_j} \Delta(\boldsymbol{R}-\boldsymbol{R}_i)\, V(\boldsymbol{R}_i, \boldsymbol{R}_j) u(\boldsymbol{R}_j), \tag{3·11}$$

where $\Delta(\boldsymbol{R}-\boldsymbol{R}_i)$ is the three-dimensional Kronecker's delta, and the sum over $\boldsymbol{R}_i$ extends over all lattice sites at which the perturbation has nonvanishing matrix element.

Using the coordinate representation, Eq. (2·8) is expressed as follows:

$$\sigma(\boldsymbol{R}_l, \boldsymbol{R}_m; s) = \left\langle \boldsymbol{R}_l \left| \frac{1}{s - iL_0} \right| \boldsymbol{R}_m \right\rangle + i \sum_{\boldsymbol{R}_j, \boldsymbol{R}_k} \left\langle \boldsymbol{R}_l \left| \frac{1}{s - iL_0} \right| \boldsymbol{R}_j \right\rangle \times V(\boldsymbol{R}_j, \boldsymbol{R}_k) \sigma(\boldsymbol{R}_k, \boldsymbol{R}_m; s), \tag{3·12}$$

or more simply

$$\sigma_{lm}(s) = g_{lm}(s) + i \sum_{j,k} g_{lj}(s) V_{jk} \sigma_{km}(s). \tag{3·13}$$

$$(l = 1, 2, 3, \cdots; m = 1, 2, 3, \cdots).$$

Here we have used a simplified notation:

$$\sigma_{lm}(s) \equiv \sigma(\boldsymbol{R}_l, \boldsymbol{R}_m; s), \quad V_{jk} \equiv V(\boldsymbol{R}_j, \boldsymbol{R}_k), \quad \sum_{j,k} \to \sum_{\boldsymbol{R}_j, \boldsymbol{R}_k}$$

and the matrix elements g_{lm} is given by

$$g_{lm}(s) = -\frac{\Omega}{(2\pi)^3} \int \frac{\exp\{i\boldsymbol{k} \cdot (\boldsymbol{R}_l - \boldsymbol{R}_m)\}}{s - iE(\boldsymbol{k})} d\boldsymbol{k}, \tag{3·14}$$

where Ω is the volume of the unit cell, and the integration is carried out over the first Brillouin zone in $\boldsymbol{k}$-space.* Eqs. (3·13) constitute a set of simultaneous equations. We notice that there are as many equations as the squares of the number of lattice sites in the crystal. Transforming back to the original matrix $\rho(\alpha)$, we obtain from Eqs. (3·13) a set of simultaneous integral equations:

$$\rho_{lm}(\alpha) = h_{lm}(\alpha) + i \sum_{j,k} \int_0^\alpha h_{lj}(\gamma) V_{jk} \rho_{km}(\alpha - \gamma) d\gamma, \tag{3·15}$$

$$(l = 1, 2, 3, \cdots; m = 1, 2, 3, \cdots)$$

where

$$h_{lm}(\alpha) = \frac{\Omega}{(2\pi)^3} \int \exp\{i\alpha E(\boldsymbol{k}) + i\boldsymbol{k} \cdot (\boldsymbol{R}_l - \boldsymbol{R}_m)\} d\boldsymbol{k}. \tag{3·16}$$

This shows that $h_{ll}(\alpha)$ is independent of l. Thus we may write as

$$h_{ll}(\alpha) = h_0(\alpha). \tag{3·17}$$

As is easily seen from Eq. (2·1) and Eq. (3·16), $h_0(\alpha)$ equals the value of $M(\alpha)$ for the unperturbed lattice

$$h_0(\alpha) = \frac{\Omega}{(2\pi)^3} \int \exp\{i\alpha E(\boldsymbol{k})\} d\boldsymbol{k} = M_0(\alpha). \tag{3·18}$$

In exactly the same way we get a set of simultaneous equations for the τ's:

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* Hereafter the integration over $\boldsymbol{k}$ always means the one that is taken over the first Brillouin zone.

$$\tau_{lm}(s)=q_{lm}(s)-\sum_{j,k}q_{lj}(s)\,V_{jk}\tau_{km}(s), \tag{3·19}$$

$$(l=1,\,2,\,\cdots;\ m=1,\,2,\,\cdots)$$

where

$$q_{lm}(s)=\frac{\Omega}{(2\pi)^3}\int\frac{\exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_l-\boldsymbol{R}_m)\}}{s-E(\boldsymbol{k})}\,d\boldsymbol{k}. \tag{3·20}$$

The corresponding simultaneous integral equations are given by

$$\phi_{lm}(\alpha)=p_{lm}(\alpha)-\sum_{j,k}\int_0^{\alpha}p_{lj}(\gamma)\,V_{jk}\phi_{km}(\alpha-\gamma)\,d\gamma, \tag{3·21}$$

$$(l=1,\,2,\,\cdots;\ m=1,\,2,\,\cdots)$$

where

$$p_{lm}(\alpha)=\frac{\Omega}{(2\pi)^3}\int\exp\{-\alpha E(\boldsymbol{k})+i\boldsymbol{k}\cdot(\boldsymbol{R}_l-\boldsymbol{R}_m)\}\,d\boldsymbol{k}. \tag{3·22}$$

As before for the unperturbed lattice, $K(\alpha)$ is in accordance with

$$p_0(\alpha)=\frac{\Omega}{(2\pi)^3}\int\exp\{-\alpha E(\boldsymbol{k})\}\,d\boldsymbol{k}=K_0(\alpha). \tag{3·23}$$

Hereafter we do not consider Eqs. (3·19) and Eqs. (3·21), since formally they are quite analogous to Eqs. (3·13) and Eqs. (3·15) respectively.

§ 4. Formal solutions of fundamental equations

In Eqs. (3·13) the matrix of the coefficients of the σ's has a simplifying feature. Let us change the numbering l and arrange such σ_{lm} that corresponds to the nonvanishing V_{lm} into the upper left part of the matrix. If this is done, the above-mentioned matrix has the form in Fig. 1., where the shaded region is a region of nonvanishing elements and the unshaded region is the region which has vanishing elements except for the indicated 1's. The matrix of the upper left corner which is enclosed by the heavy lines and the corresponding determinant are designated by $A(s)$ and $D(s)$ respectively. The pq element is given by

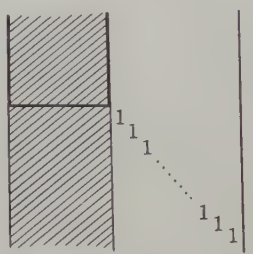


Fig. 1

$$(A(s))_{pq}=\delta_{pq}-i\sum_j g_{pj}(s)\,V_{jq}. \tag{4·1}$$

Thanks to this we may first solve our simultaneous equations (3·13) by limiting l only for the lattice sites over which the impurity potential has nonvanishing matrix elements, and obtain the corresponding σ's. Then inserting these σ's into the right-hand side of Eqs. (3·13), we have the remaining matrix elements of σ:

$$\sigma_{lm}(s)=g_{lm}(s)+\frac{i}{ND(s)}\sum_{jpq}g_{lj}(s)V_{jp}B_{pq}(s)g_{qm}(s)\,, \tag{4·2}$$

where $B_{pq}(s)$ is the cofactor of $(A(s))_{pq}$. The trace of the σ's is given by

$$\frac{1}{N}\operatorname{Tr}\sigma=g_0+\frac{i}{N^2D(s)}\sum_{l}\sum_{jpq}g_{lj}V_{jp}B_{pq}g_{ql}\,, \tag{4·3}$$

where

$$g_0\equiv g_{ll}=\frac{\Omega}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{s-iE(\boldsymbol{k})}\,, \tag{4·4}$$

and the sum over l extends over all lattice sites in the crystal. Using the periodic boundary condition, this sum is carried out as follows:

$$\sum_{l}g_{lj}g_{ql}=-Ng'_{qj}\,, \tag{4·5}$$

where the dash indicates the differentiation with respect to s. From this relation (4·3) is reduced to

$$\frac{1}{N}\operatorname{Tr}\sigma=g_0-\frac{i}{ND(s)}\sum_{jpq}g'_{qj}V_{jp}B_{pq}\,. \tag{4·6}$$

Considering (4·1), we have

$$\frac{1}{N}\operatorname{Tr}\sigma=g_0+\frac{1}{N}\frac{D'(s)}{D(s)}\,. \tag{4·7}$$

The inverse Laplace transformation of the above equation is given by

$$M(\alpha)=h_0(\alpha)+\frac{1}{N}\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\exp(\alpha s)\,d\{\log D(s)\}\,, \tag{4·8}$$

where C is real and positive. This is the formal solution of simultaneous integral equations (3·15). Eq. (4·7) as well as Eq. (4·8) can be further simplified when V_{jk} depends only on the relative distance between $\boldsymbol{R}_j$ and $\boldsymbol{R}_k$: in Eq. (4·1) the sum over j is done to give the following result,

$$(A)_{pq}=\delta_{pq}-f_{pq}\,, \tag{4·9}$$

where f_{pq} is given by

$$f_{pq}(s)=\frac{\Omega}{(2\pi)^3}\int\frac{iV(\boldsymbol{k})\exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_p-\boldsymbol{R}_q)\}}{s-iE(\boldsymbol{k})}\,d\boldsymbol{k}\,, \tag{4·10}$$

$V(\boldsymbol{k})$ being the Fourier coefficient of the impurity potential:

$$V(\boldsymbol{k})=\sum_{R}V(\boldsymbol{R})\exp(-i\boldsymbol{k}\cdot\boldsymbol{R})\,. \tag{4·11}$$

Taking into account the relations

$$D(s)=0,\quad\text{as}\quad|s|\to\infty\,, \tag{4·12}$$

we see that the Bromwitch integral in Eq. (4·8) is transformed to a contour integral along a closed, counterclockwise curve which encloses all zeros and all poles of the D's. Therefore, formally Eq. (4·8) has close resemblance to the contour integral formula for evaluating additive functions of the normal mode frequencies originally obtained by Montroll and Potts.

Let us consider Eq. (4·7) and Eq. (4·9) for some simplified cases. For the unperturbed lattice, we see that σ_{ll} is independent of l. Let σ_0 be the value of $(1/N)\mathrm{Tr}\,\sigma$ for the unperturbed lattice, we have

$$\sigma_0 = g_0. \tag{4·13}$$

Hereafter we assume that $V(\boldsymbol{k})$ is the Fourier coefficient of the perturbative potential due to one impurity. Then for the one-impurity problem, the correction term to σ_0, σ_1 is given by

$$\sigma_1 = -\frac{1}{N}\frac{f_0'}{1-f_0'}, \tag{4·14}$$

where

$$f_0 = \frac{\Omega}{(2\pi)^3}\int \frac{iV(\boldsymbol{k})}{s-iE(\boldsymbol{k})}d\boldsymbol{k}. \tag{4·15}$$

σ_1 is apparently independent of the location of the impurity. This is due to the periodic boundary condition. Next we consider the case when two impurities are located with relative distance $\boldsymbol{R}$. Then the correction term to σ_0, $\sigma_2(\boldsymbol{R})$ is

$$\sigma_2(\boldsymbol{R}) = -\frac{1}{N}\left\{\frac{f_0+f_{\boldsymbol{R}}}{1-f_0-f_{\boldsymbol{R}}} + \frac{f_0-f_{\boldsymbol{R}}}{1-f_0+f_{\boldsymbol{R}}}\right\}, \tag{4·16}$$

where

$$f_{\boldsymbol{R}} = \frac{\Omega}{(2\pi)^3}\int \frac{iV(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{R})}{s-iE(\boldsymbol{k})}d\boldsymbol{k}. \tag{4·17}$$

The " interaction-σ " of a pair of two impurities is defined as the difference between the σ of a system of two interacting impurities and that of a pair of isolated impurities and is given by

$$\sigma_2(\boldsymbol{R}) - 2\sigma_1 = -\frac{2}{N}\frac{(1-f_0)f_{\boldsymbol{R}}f_{\boldsymbol{R}}' + f_0'f_{\boldsymbol{R}}^2}{(1-f_0)\{(1-f_0)^2 - f_{\boldsymbol{R}}^2\}}. \tag{4·18}$$

The inverse Laplace transformation which yields the impurity levels is easily carried out. The zero of the determinant $D(s)$ may exist for a purely imaginary value of s, $s=i\varepsilon$ (ε being real) such that

$$D(i\varepsilon) = 0 \qquad \begin{matrix} \text{for } \varepsilon > E_1, \\ \text{or for } \varepsilon < E_2, \end{matrix} \tag{4·19}$$

where E_1 and E_2 denote respectively the maximum and the minimum value of $E(\boldsymbol{k})$ of the unperturbed lattice. The former corresponds to the case in which

the impurity potential is positive, while the latter to the case in which the impurity potential is negative. Remembering Eq. (4·1) or Eq. (4·9), we can rewrite Eq. (4·19) explicitly as

$$\det\left|\delta_{pq}-\sum_{j}V_{jq}\frac{\Omega}{(2\pi)^3}\int\frac{\exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_p-\boldsymbol{R}_j)\}}{\varepsilon-E(\boldsymbol{k})}\,d\boldsymbol{k}\right|=0, \tag{4·20}$$

or

$$\det\left|\delta_{pq}-\frac{\Omega}{(2\pi)^3}\int\frac{V(\boldsymbol{k})\exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_p-\boldsymbol{R}_q)\}}{\varepsilon-E(\boldsymbol{k})}\,d\boldsymbol{k}\right|=0. \tag{4·21}$$

These are just the secular equations for determining the impurity levels which are displaced out of the main continuum of the unperturbed energy band. If the solutions of these equations are designated by $E_j\,(j=1, 2, 3, \cdots)$, the contour integration in Eq. (4·8) gives the following contribution to $M(\alpha)$:

$$\frac{1}{N}\sum_{j}\exp(i\alpha E_j). \tag{4·22}$$

Therefore the spectrum of the impurity levels is given by

$$\frac{1}{N}\sum_{j}\delta(E-E_j). \tag{4·23}$$

In order to study the one- and two-impurity problem, let us consider Eq. (4·7) or Eqs. (4·14) and (4·16). For the one-impurity problem the impurity level ε_0 is determined by the equation

$$\frac{\Omega}{(2\pi)^3}\int\frac{V(\boldsymbol{k})}{\varepsilon_0-E(\boldsymbol{k})}\,d\boldsymbol{k}=1. \tag{4·24}$$

When two impurities are located with relative distance $\boldsymbol{R}$, the eigenvalues are given by

$$\frac{\Omega}{(2\pi)^3}\int\frac{V(\boldsymbol{k})}{\varepsilon-E(\boldsymbol{k})}\{1\pm\exp(i\boldsymbol{k}\cdot\boldsymbol{R})\}=1. \tag{4·25}$$

The calculation of the contour integration which gives the shifts of in-band levels is not so simple, because in this case the zeros as well as the poles of the D's have nearly continuous distribution on the imaginary axis. For this reason, we here consider only the one-impurity problem. The denominator of Eq. (4·14) is rewritten as follows:

$$1-\int\frac{iN_1(E)}{s-iE}\,dE, \tag{4·26}$$

where

$$N_1(E)=\frac{\Omega}{(2\pi)^3}\int_{E(\boldsymbol{k})=E}\frac{V(\boldsymbol{k})}{|\nabla_{\boldsymbol{k}}E(\boldsymbol{k})|}\,dS_k \tag{4·27}$$

where dS_k is the surface element of the constant energy surface $E(\boldsymbol{k})=E$ in $\boldsymbol{k}$-space. The zeros as well as the poles of (4·26) may exist for a purely imaginary value of $s=i\varepsilon$, ε being real. Replacing the integral by the sum, we have

$$1-\frac{W}{N}\sum_n \frac{N_1(E_{n0})}{\varepsilon-E_{n0}} \tag{4·28}$$

where $E_{n0}(n=0, 1, 2, 3, \cdots)$ is the energy levels of the unperturbed lattice, and W is the width of the energy band $E(\boldsymbol{k})$. The equation obtained by letting (4·28) be zero is the familiar equation giving the shifts of in-band levels as well as impurity levels. The inverse Laplace transformation can be achieved through the application of the theorem on the contour integration :[26] Let C be a contour in the z-plane, and let $f(z)$ be a function analytic inside and on C. Let $\phi(z)$ be another function which is analytic inside and on C except at a finite number of poles ; let the zeros of $\phi(z)$ in the interior of C be $a_1, a_2, \cdots$, and let their degrees of multiplicity be $r_1, r_2, \cdots$; let its poles in the interior of C be $b_1, b_2, \cdots$, and let their degrees of multiplicity be $s_1, s_2 \cdots$. Then

$$\frac{1}{2\pi i}\int_C f(z)\frac{\phi'(z)}{\phi(z)}dz=\sum_i r_i f(a_i)-\sum_j s_j f(b_j), \tag{4·29}$$

the summations being extended over all the zeros and poles of $\phi(z)$. Then the inverse Laplace transformation of Eq. (4·14) becomes

$$\frac{1}{N}\sum_n \{\exp(iE_n\alpha)-\exp(iE_{n0}\alpha)\}, \tag{4·30}$$

where E_n is the energy eigenvalue determined by letting (4·28) be zero. Therefore the spectrum is given by

$$\frac{1}{N}\{\sum{}' \delta(E-E_n)+\delta(E-\varepsilon_0)\}-N_0(E), \tag{4·31}$$

where the prime on the summation indicates that the impurity level which is obtained by (4·28) must be excluded, and where $N_0(E)$ is the spectrum of the unperturbed lattice, which is obtained from Eq. (3·17) and is assumed to be known.

§ 5. Random lattices and perturbation calculation

By a random lattice we mean here the one in which each atom in a regular lattice is replaced with a definite probability by an impurity atom of fixed kind. We shall develop a perturbation method for solving Eqs. (3·13), which will allow us to obtain another expression for $(1/N)\mathrm{Tr}\,\sigma$. With the use of the method of iteration, Eqs. (3·13) are solved as follows :

$$\sigma_{lm}=g_{lm}+i\sum_{a\alpha} g_{la}V_{a\alpha}g_{\alpha m}+i^2\sum_{a\alpha}\sum_{b\beta} g_{la}V_{a\alpha}g_{ab}V_{b\beta}g_{\beta m}+\cdots. \tag{5·1}$$

$$(l=1, 2, \cdots ;\ m=1, 2, \cdots)$$

When $V_{a\alpha}$ depends only on the relative distance between $\boldsymbol{R}_a$ and $\boldsymbol{R}_\alpha$, following the same procedure used to derive Eq. (4·9), from Eq. (4·1) we can carry out the sum over $\alpha, \beta, \cdots$. The result is

$$\sigma_{lm}=g_{lm}+\sum_a g_{la}f_{am}+\sum_{ab} g_{la}f_{ab}f_{bm}+\sum_{abc} g_{la}f_{ab}f_{bc}f_{cm}+\cdots. \tag{5·2}$$

From Eq. (5·1) we obtain

$$\langle (1/N)\ \mathrm{Tr}\sigma\rangle=g_0-i\langle\sum_{a\alpha} g'_{\alpha a}V_{a\alpha}\rangle-i^2\langle\sum_{a\alpha}\sum_{b\beta} g_{\beta a}V_{a\alpha}g'_{\alpha b}V_{b\beta}\rangle-\cdots, \tag{5·3}$$

where the angular brackets denote an average over all possible arrangements of the impurities. Using the following relation analogous to Eq. (4·5) :

$$\sum_l g_{la}f_{bl}=-Nf'_{ba}, \tag{5·4}$$

we get from Eq. (5·2)

$$\langle (1/N)\ \mathrm{Tr}\sigma\rangle=g_0-cf_0'-\langle\sum_{ab} f_{ab}f'_{ba}\rangle-\langle\sum_{abc} f_{ab}f_{bc}f'_{ca}\rangle-\cdots, \tag{5·5}$$

where c is the concentration of the impurities. Applying to Eq. (5·3) the inverse Laplace transformation, we obtain the perturbation expansion for $\langle M(\alpha)\rangle$:

$$\langle M(\alpha)\rangle=M_0(\alpha)+\langle M_1(\alpha)\rangle+\langle M_2(\alpha)\rangle+\cdots, \tag{5·6}$$

where

$$\langle M_1(\alpha)\rangle=i\alpha\langle\sum_{a\alpha} V_{a\alpha}h_{a\alpha}(\alpha)\rangle, \tag{5·7}$$

$$\langle M_n(\alpha)\rangle=i^n\int_0^{\alpha}\int_0^{\alpha-\alpha_1}\cdots\int_0^{\alpha-\alpha_1-\cdots-\alpha_{n-1}}\langle\sum_{abc\cdots d}\sum_{\alpha\beta\cdots\gamma} V_{a\alpha}h_{ab}(\alpha_2)V_{b\beta}h_{\beta c}(\alpha_3)\times\cdots$$
$$\times h_{\gamma d}(\alpha_n)V_{d\delta}h_{\delta a}(\alpha-\alpha_2-\alpha_3-\cdots-\alpha_n)\rangle\, d\alpha_1 d\alpha_2\cdots d\alpha_n. \tag{5·8}$$
$$(n=2, 3, 4, \cdots)$$

Following the same procedure, from Eq. (5·5) we have the expansion

$$\langle M_1(\alpha)\rangle=ic\alpha b_0(\alpha), \tag{5·9}$$

$$\langle M_n(\alpha)\rangle=i^n\int_0^{\alpha}\int_0^{\alpha-\alpha_1}\cdots\int_0^{\alpha-\alpha_1-\cdots-\alpha_{n-1}}\langle\sum_{abc\cdots de} b_{ab}(\alpha_2)b_{bc}(\alpha_3)\times\cdots\times b_{de}(\alpha_n)$$
$$\times b_{ea}(\alpha-\alpha_2-\alpha_3-\cdots-\alpha_n)\rangle\, d\alpha_1 d\alpha_2\cdots d\alpha_n, \tag{5·10}$$
$$(n=2, 3, 4, \cdots)$$

where

$$b_{ab}(\alpha)=\frac{\Omega}{(2\pi)^3}\int V(\boldsymbol{k})\exp\{i\alpha E(\boldsymbol{k})+i\boldsymbol{k}\cdot(\boldsymbol{R}_a-\boldsymbol{R}_b)\}\,d\boldsymbol{k},$$
$$b_{aa}(\alpha)\equiv b_0(\alpha). \tag{5·11}$$

Next we shall develop a perturbation expansion in a slightly different way,

which is similar to the methods employed in multiple scattering theory. Our procedure is to rewrite (5·2) such that in each expansion term no two adjacent indices may be alike. For this purpose we rewrite them as

$$\sigma_{lm}=g_{lm}+\sum_a g_{la}\Omega_{am}, \tag{5·12}$$

where Ω_{am} satisfies the equation

$$\Omega_{am}=f_{am}+\sum_b f_{ab}\Omega_{bm}. \tag{5·13}$$

From this equation we get

$$\Omega_{am}=\frac{1}{d}f_{am}+\frac{1}{d}\sum_b{}' f_{ab}\Omega_{bm}, \tag{5·14}$$

where

$$d=1-f_0. \tag{5·15}$$

In the above the prime on the summation indicates that $a\neq b$. Then Eq. (5·14) gives by iteration

$$\Omega_{am}=\frac{1}{d}f_{am}+\frac{1}{d^2}\sum_b{}' f_{ab}f_{bm}+\frac{1}{d^3}\sum_b{}'\sum_c{}' f_{ab}f_{bc}f_{cm}+\cdots. \tag{5·16}$$

Inserting this solution into (5·12), we have the result

$$\left\langle\frac{1}{N}\mathrm{Tr}\,\sigma\right\rangle=g_0-c\frac{f_0'}{d}-\frac{1}{d^2}\left\langle\sum_a\sum_b{}' f_{ab}f_{ba}'\right\rangle-\frac{1}{d^3}\left\langle\sum_a\sum_b{}'\sum_c{}' f_{ab}f_{bc}f_{ca}'\right\rangle-\cdots. \tag{5·17}$$

This is the expansion in terms of the exact solution of the one-impurity problem. Such an expansion has frequently been used in the multiple scattering theory. We can represent the terms of our series by closed diagrams through the various sites of the impurities. These are shown in Figs. 2, 3, and 4.

The series (5·17) is rearranged as follows :



Fig. 2. Graphical representation of the third term of (5·17).



Fig. 3. Graphical representation of the fourth term of (5·17).



Fig. 4. Graphical representation of the fifth term of (5·17).

$$\left\langle \frac{1}{N}\operatorname{Tr}\sigma \right\rangle = g_0 - c\frac{f_0'}{d} - \left\langle \sum_{ab}'' \frac{1}{d^2} f_{ab} f_{ba}' + \frac{1}{d^3} f_0' f_{ab} f_{ba} + \frac{1}{d^4} f_{ab} f_{ba} f_{ab} f_{ba}' + \frac{1}{d^5} f_0' (f_{ab} f_{ba})^2 + \cdots \right\rangle + \left\langle \sum_{abc}'' \frac{1}{d^3} f_{ab} f_{bc} f_{ca}' + \cdots \right\rangle + \cdots, \tag{5·18}$$

where the double prime on the summation indicates that all indices must be different. In this expression we have thus separated the perturbed term of $(1/N)\operatorname{Tr}\sigma$ into scattering by single impurities, by pairs, by triplets, etc. This procedure, for example, corresponds to picking out such diagrams as shown in Fig. 5 from Figs. 2, 3, and 4 for Eq. (5·17). It is useful when we consider the effect of the correlation among the impurities.



Fig. 5. Graphical illustration of the interaction between a pair of the impurities.

The second term of (5·17) which is proportional to the concentration of the impurities is related to the solution of the one-impurity problem and is in fact equal to

$$Nc\sigma_1. \tag{5·19}$$

The effect of the correlation between a pair of the impurities is involved in "ladders" which repeatedly runs back and forth between a and b. Physically, we can think of them as resulting from a pair of closely coupled impurities. It is represented in the first sum of Eq. (5·18), which is reduced to

$$-\sum_{ab}'' \frac{d f_{ab} f_{ba}' + f_0' f_{ab} f_{ba}}{d(d^2 - f_{ab} f_{ba})}. \tag{5·20}$$

Let $w(\boldsymbol{R})/N$ be the probability that two impurities are located with relative distance $\boldsymbol{R}$; the normalization condition is

$$\sum_{\boldsymbol{R}\neq 0} w(\boldsymbol{R})/N = 1. \tag{5·21}$$

By considering (4·20), the above equation is written as

$$N\frac{c^2}{2}\sum_{\boldsymbol{R}\neq 0} w(\boldsymbol{R})\{\sigma_2(\boldsymbol{R}) - 2\sigma_1\}. \tag{5·22}$$

From Eqs. (4·15), (5·19), and (5·22), we have the series in powers of c:

$$\left\langle \frac{1}{N}\operatorname{Tr}\sigma \right\rangle = \sigma_0 + Nc\sigma_1 + N\frac{c^2}{2}\sum_{\boldsymbol{R}\neq 0}\{\sigma_2(\boldsymbol{R}) - 2\sigma_1\} + \cdots. \tag{5·23}$$

A similar expansion formula was obtained by Lifshitz and Stepanova. However, the explicit forms of σ_1 and $\sigma_2(\boldsymbol{R})$ obtained by them were not fit for practical calculations.

§ 6. The moment method and the random walk problem

The moment of the spectrum is obtained from Eq. (2·1); the m-th moment is given by

$$\mu_m = i^{-m}[d^m M(\alpha)/d\alpha^m]_{\alpha=0}. \tag{6·1}$$

Using Eq. (5·6), we obtain the expansion for μ_m:

$$\mu_m = \mu_m(0) + \mu_m(1) + \mu_m(2) + \cdots + \mu_m(m). \tag{6·2}$$

where

$$\mu_m(n) = i^{-m}[d^m M_n(\alpha)/d\alpha^m]_{\alpha=0}. \tag{6·3}$$

$$(n=0, 1, 2, 3, \cdots)$$

Here we have used the following relation,

$$\mu_m(n) = 0 \quad \text{unless } n < m, \tag{6·4}$$

which is easily derived from Eq. (5·8) or Eq. (5·10). The expansion Eq. (6·2) is exact in a sense that the n-th order perturbation does not contribute to the moments whose orders are less than n.

The m-th moment of the unperturbed lattice is given by

$$\mu_m(0) = \frac{\Omega}{(2\pi)^3}\int E(\boldsymbol{k})^m d\boldsymbol{k}. \tag{6·5}$$

From Eq. (5·7) we see that the contribution from the first order perturbation is

$$\langle \mu_m(1) \rangle = m\langle \sum_{\alpha\boldsymbol{\alpha}} V_{\alpha\alpha} H_{\alpha\alpha}(m-1) \rangle, \tag{6·6}$$

where

$$H_{\alpha\alpha}(m) = \frac{\Omega}{(2\pi)^3}\int E(\boldsymbol{k})^m \exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_\alpha - \boldsymbol{R}_\alpha)\} d\boldsymbol{k}. \tag{6·7}$$

When $m=0$, it becomes

$$H_{\alpha\alpha}(0) = \delta(\alpha - a). \tag{6·8}$$

If we use Eq. (5·9), we have

$$\langle \mu_m(1) \rangle = mc\frac{\Omega}{(2\pi)^3}\int V(\boldsymbol{k}) E(\boldsymbol{k})^{m-1} d\boldsymbol{k}. \tag{6·9}$$

The calculation of the higher moments are made in a straightforward way. After some manipulations, from Eq. (5·8) we have the results:

$$\langle \mu_m(n) \rangle = \Big\langle \sum_{ab\cdots cd} \sum_{\alpha\beta\gamma\cdots\delta} \sum_{p=n}^{m} H_{\delta a}(m-p+p_{n-1}-1)\, V_{a\alpha} \sum_{p_1=n-1}^{p-1} H_{\alpha b}(p-p_1-1) \times V_{b\beta} \times \cdots \times V_{c\gamma} \sum_{p_{n-1}=1}^{p_{n-2}-1} H_{\gamma d}(p_{n-2}-p_{n-1}-1)\, V_{d\delta} \Big\rangle. \tag{6·10}$$

$$(n=2,\ 3,\ 4,\cdots m)$$

We present here the first five moments:

$$\mu_0 = 1, \tag{6·11}$$

$$\langle \mu_1 \rangle = \mu_1(0) + \Big\langle \sum_{a} V_{a\alpha} H_{a\alpha}(0) \Big\rangle, \tag{6·12}$$

$$\langle \mu_2 \rangle = \mu_2(0) + 2\Big\langle \sum_{a\alpha} V_{a\alpha} H_{a\alpha}(1) \Big\rangle + \Big\langle \sum_{ab} V_{ab} V_{ba} \Big\rangle, \tag{6·13}$$

$$\langle \mu_3 \rangle = \langle \mu_3(0) \rangle + 3\Big\langle \sum_{a\alpha} V_{a\alpha} H_{a\alpha}(2) \Big\rangle + 3\Big\langle \sum_{ab\alpha} V_{a\alpha} H_{ab}(1)\, V_{b\alpha} \Big\rangle, \tag{6·14}$$

$$\begin{aligned} \langle \mu_4 \rangle = {} & \langle \mu_4(0) \rangle + 4\Big\langle \sum_{a\alpha} V_{a\alpha} H_{a\alpha}(3) \Big\rangle + 2\Big\langle \sum_{ab\alpha\beta} V_{a\alpha} H_{ab}(1)\, V_{b\beta} H_{\beta a}(1) \Big\rangle \\ & + 4\Big\langle \sum_{ab\alpha} V_{a\alpha} H_{ab}(2)\, V_{b\alpha} \Big\rangle + 4\Big\langle \sum_{abc\alpha} V_{a\alpha} H_{ab}\, V_{bc}\, V_{c\alpha} \Big\rangle \\ & + \Big\langle \sum_{abcd} V_{ab}\, V_{bc}\, V_{cd}\, V_{da} \Big\rangle. \end{aligned} \tag{6·15}$$

In the above we have used Eq. (6·8).

If we consider Eq. (5·10), we get

$$\langle \mu_m(n) \rangle = \Big\langle \sum_{ab\cdots c} \sum_{p=n}^{m} S_{ca}(m-p+p_{n-1}-1) \sum_{p_1=n-1}^{p-1} S_{ab}(p-p_1-1) \times \cdots \times \sum_{p_{n-1}=1}^{p_{n-2}-1} S_{bc}(p_{n-2}-p_{n-1}-1) \Big\rangle. \tag{6·16}$$

$$(n=2,\ 3,\ 4,\cdots)$$

The first five moments are then given by

$$\mu_0 = 1, \tag{6·17}$$

$$\langle \mu_1 \rangle = \mu_1(0) + cS_0(0), \tag{6·18}$$

$$\langle \mu_2 \rangle = \mu_2(0) + 2cS_0(1) + \Big\langle \sum_{ab} S_{ab}(0)\, S_{ba}(0) \Big\rangle, \tag{6·19}$$

$$\langle \mu_3 \rangle = \mu_3(0) + 3cS_0(2) + 3\Big\langle \sum_{ab} S_{ab}(1)\, S_{ba}(0) \Big\rangle + \Big\langle \sum_{abc} S_{ab}(0)\, S_{bc}(0)\, S_{ca}(0) \Big\rangle, \tag{6·20}$$

$$\begin{aligned} \langle \mu_4 \rangle = {} & \mu_4(0) + 4cS_0(3) + \Big\langle \sum_{ab} 2S_{ab}(1)\, S_{ba}(1) + 4S_{ab}(2)\, S_{ba}(0) \Big\rangle \\ & + 4\Big\langle \sum_{abc} S_{ab}(1)\, S_{bc}(0)\, S_{ca}(0) \Big\rangle + \Big\langle \sum_{abcd} S_{ab}(0)\, S_{bc}(0)\, S_{cd}(0)\, S_{da}(0), \end{aligned} \tag{6·21}$$

where

$$S_{ab}(n) = \frac{\Omega}{(2\pi)^3} \int V(\boldsymbol{k})\, E(\boldsymbol{k})^n \exp\{i\boldsymbol{k}\cdot(\boldsymbol{R}_a - \boldsymbol{R}_b)\}\, d\boldsymbol{k}. \tag{6·22}$$

$$(n=0,\ 1,\ 2,\cdots)$$

Here we have put

$$S_0(n)=S_{aa}(n), \tag{6·23}$$

since it is independent of a. Eqs. (6·11) and (6·17) show that the area of the spectrum remains unchanged. From Eqs. (6·12) and (6·18), we see that only the diagonal part of the impurity potential contributes to the first moments.

The moments are intimately connected with the random walk problem. First we consider the moments of the unperturbed lattice. Inserting (3·10) into (6·5), we have another expression for $\mu_m(0)$:

$$\mu_m(0)=\sum_{\boldsymbol{R}_1\boldsymbol{R}_2\cdots\boldsymbol{R}_m} E(\boldsymbol{R}_1)E(\boldsymbol{R}_2)\cdots E(\boldsymbol{R}_m)\,\Delta(\boldsymbol{R}_1+\boldsymbol{R}_2+\cdots+\boldsymbol{R}_m). \tag{6·24}$$

This equation represents a kind of random walk in which the walker walks on the lattice sites and then returns to the starting point after m steps; the length of each step is determined by the $\boldsymbol{R}$-dependence of $E(\boldsymbol{R})$. In this walk, however, the walker is allowed to stay at some lattice sites. In order to prohibit this staying, the origin of the energy which have so far been unspecified must be chosen as

$$E(0)=0. \tag{6·25}$$

Let λ_m be the m-th moment with this specification of the energy origin. Then we have

$$\lambda_m(0)=\sum_{\boldsymbol{R}_1\boldsymbol{R}_2\cdots\boldsymbol{R}_m}{}' E(\boldsymbol{R}_1)E(\boldsymbol{R}_2)\cdots E(\boldsymbol{R}_m)\,\Delta(\boldsymbol{R}_1+\boldsymbol{R}_2+\cdots+\boldsymbol{R}_m), \tag{6·26}$$

where the prime on the summation indicates that $\boldsymbol{R}_1\neq 0$, $\boldsymbol{R}_2\neq 0$, $\cdots$, $\boldsymbol{R}_m\neq 0$. The schematic feature of the above equation is shown in Fig. 6. If $E(\boldsymbol{k})$ is characterized by the nearest neighbour interaction, the walker cannot return to the starting point after odd-number steps, so that

$$\lambda_{2m+1}(0)=0. \tag{6·27}$$

$$(m=0,\ 1,\ 2,\cdots)$$

This shows that the energy band of the unperturbed lattice is symmetric about the origin of the energy axis.

Next we consider the correction terms due to the perturbation in the expression for each moment. Using Eq. (3·10), $H_{aa}(m)$ can be rewritten as

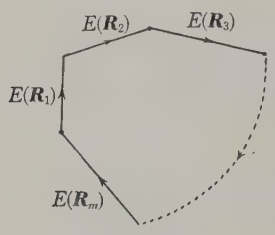


Fig. 6

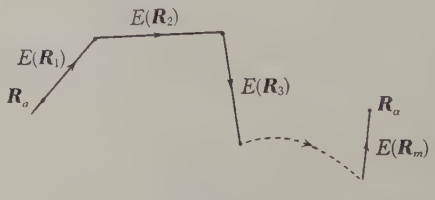


Fig. 7

$$H_{\alpha\alpha}(m)=\sum_{R_1 R_2\cdots R_m} E(\boldsymbol{R}_1)E(\boldsymbol{R}_2)\cdots E(\boldsymbol{R}_m)\,\Delta(\boldsymbol{R}_1+\boldsymbol{R}_2+\cdots+\boldsymbol{R}_m+\boldsymbol{R}_\alpha-\boldsymbol{R}_\alpha). \quad (6\cdot28)$$

This equation represents the random walk in which the walker walks from $\boldsymbol{R}_\alpha$ to $\boldsymbol{R}_\alpha$ after m steps. This is illustrated in Fig. 7. In the same way $S_{\alpha b}(m)$ can be written as

$$S_{\alpha b}(m)=\sum_{R_1 R_2\cdots R_{m+1}} V(\boldsymbol{R}_1)E(\boldsymbol{R}_2)\cdots E(\boldsymbol{R}_{m+1})\,\Delta(\boldsymbol{R}_1+\boldsymbol{R}_2+\cdots+\boldsymbol{R}_{m+1}+\boldsymbol{R}_\alpha-\boldsymbol{R}_b). \quad (6\cdot29)$$

This represents the walk of $(m+1)$ steps, one of which is made by the impurity potential.

From Eqs. (6·26), (6·28), and (6·29), it is shown that the moments are described by the random walks among the lattice sites. Therefore the evaluation of the moments can be reduced to study the number of possible paths for the given configuration of the distribution of the impurities.

§ 7. Summary

In this paper the effect of a perturbation due to impurity potentials on the energy spectrum of periodic lattices was studied. The theory differs from the previous ones in that the evaluation of the Fourier or the Laplace transform of the spectrum is reduced to solving a set of differential-difference equations or equivalent simultaneous integral equations satisfied by density matrices. We saw in § 2 that the calculation of the density matrices is logically connected with the solution of the Schrödinger equation in the scattering theory; if L is the Hamiltonian operator, Eq. (2·8) or Eq. (2·13) becomes an integral equation. In our formulation of the problem in § 3, our particular intention was to construct the theory on a basis of the theory of perturbed periodic lattices first developed by Koster and Slater, in which the basic eigenvalue equation for the impurity problem was a set of difference equations. It was shown in § 4 that this enables us to solve Eq. (2·8) exactly. The calculation of Eq. (4·8) or Eq. (4·13) was shown to be equivalent to solve the eigenvalue problem directly, that is to say, to solve Eq. (3·1). Therefore such a procedure is rather inadequate for practical calculations when the number of the impurities is large. In § 5, we showed a perturbation expansion method to solve Eqs. (3·13), which yields also a formal expansion formula in a series of powers of the concentration of the impurities. From (5·17), we see that the calculation of $(1/N)\,\mathrm{Tr}\,\sigma$ is reduced to studying the problem how we can select various expansion terms belonging to a particular type of graphs in a compact form which allows us to carry out actual calculations. Perhaps this is one of the most important tasks for our problem. Here it is to be noted that apart from the spectrum of in-band levels, the ordinary perturbation method fails to give the spectrum of impurity levels or of an impurity band. In § 6, the general expressions of the moments were obtained. A correspondence to

the random walk problem was shown to elucidate the problem from a different standpoint. Also it was pointed out that the moment is obtained by explicitly calculating the number of possible paths of the walks for a given configuration of the impurities.

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Symmetry in Sakata's Model and Weak Interactions. II

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In the previous paper we have proposed a scheme for non-leptonic decays of hyperons and K-mesons in the full symmetry theory of Sakata's composite model. We shall make a further investigation of our decay scheme. First it will be shown that the relations between various decay amplitudes are independent of particular configurations assigned to the particles which participate in the reactions. Next the K-3π decay will be studied to show that our decay scheme is rather successful as in the previous paper. Finally we shall make some discussions concerning anomalous decay modes in which there appear one or more $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ in place of π^0.

§ 1. Introduction

In the theory of full symmetry,[1),2),3)] we assume that there exists a certain symmetry (full symmetry) among the basic particles p, n and Λ in Sakata's composite model.[4)] This assumption has achieved a fair success in the study of strong interactions. Particularly, a mass formula[5)] has been found for the strongly interacting particles on the analogy of the nuclear model. And as a result it has been made clear that there is a remarkable correspondence between the states predicted by the full symmetry theory and the resonance levels established in scattering experiments. In view of these circumstances we anticipate a bright future before the full symmetry theory of Sakata's model, although there still remain many unsettled problems, especially the problem of dynamics.

In a previous paper[6)] (hereafter referred to as I) we have studied the weak interactions in the scheme of the full symmetry theory. Following the same line of investigation, we shall discuss in the present paper some problems left untouched in I.

To begin with, we shall outline our view on the non-leptonic decays of hyperons and K-mesons. In the full symmetry theory a physical system is considered to be the totality of the configurations associated with the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. The first three of these numbers specify an irreducible representation of the full symmetry group, i. e. the unitary group $U(3)$ of degree three. Thus a hyperon or a K-meson is expressed by a vector of an irreducible representation space. We assume that this vector remains in the same repre-

sentation space when the particle is subject to a non-leptonic weak decay. This means that such a decay reaction is governed by the selection rule

$$\Delta n_B = \Delta s_0 = \Delta i_0 = 0, \tag{1·1}$$

which is embodied in that the interaction causing a decay process can be expressed in terms of the operators of the group $U(3)$. We have the following possibilities of the primary weak interactions $\boldsymbol{O}$:

$$\begin{aligned} &\text{For} \quad \Delta S=1, \quad \boldsymbol{O}=N_{23},\ N_{13}I_-\ \text{and}\ I_-N_{13}, \\ &\text{For} \quad \Delta S=-1, \quad \boldsymbol{O}=N_{32},\ N_{31}I_+\ \text{and}\ I_+N_{31}. \end{aligned} \tag{1·2}$$

Along this line of thought we have analysed in I the hyperon decays and the K-$2\pi^*$ decays, and have found that the selection rule (1·1) may well replace the so-called $|\Delta I|=1/2$ rule. There, as the initial states we have taken* the $(2+1)$-body configurations for hyperons and $(2+2)$-body configurations for K-mesons. For instance, the decays $\Lambda\to\pi^0+n$ and $\Lambda\to\pi^-+p$ have been considered as the transition

$$F_3(\Lambda)\to F_1(N)\cdot B_2(\pi). \tag{1·3}$$

However, we have a great many ways of treatment, that is,

$$\begin{aligned} F_5(\Lambda)&\to F_1(N)\cdot B_4(\pi), \\ &\to F_3(N)\cdot B_2(\pi), \\ F_7(\Lambda)&\to F_1(N)\cdot B_6(\pi), \\ &\to F_3(N)\cdot B_4(\pi), \\ &\to F_5(N)\cdot B_2(\pi), \end{aligned} \tag{1·4}$$

and so on. Here $F_n(\alpha)$ $(B_n(\beta))$ stands for an n-body configuration of a fermion α (boson β). According to our ideas, a physical particle should be understood as the totality of all possible configurations. Then the transition schemes in (1·3) and (1·4) are all equivalent and should lead to the same conclusion. This, however, is not so apparent at first sight because of the complexity of higher configurations.

The first purpose of the present paper is to clarify this point and to show that our conjecture is actually true. In the above example, the relation between the amplitudes for $\Lambda\to\pi^-+p$ and $\Lambda\to\pi^0+n$ is independent of whether we regard the reactions as (1·3) or as any of (1·4). Such a fact can be proved in general by considering the structure of higher configurations and fully utilizing the characteristics of our decay scheme. This result gives support not only to our view on the weak interactions as outlined above but also to our new particle picture which has been fully discussed in other place.[7]

The second aim of this paper is to investigate some problems which have not

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* A $(k+l)$-body configuration consists of k basic particles and l basic antiparticles.

been touched in I. In § 3 we shall deal with the K-3π decays to show that our decay scheme gives the same results as the conventional $|\Delta I|=1/2$ rule does, as was the case with the other decay processes of hyperons and K-mesons. In § 4 a further investigation will be made concerning the anomalous decay modes of hyperons and K-mesons in which there appears as a decay product $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ predicted by the full symmetry theory. The final section will be devoted to the concluding remarks.

§ 2. Structure of higher configurations

In our theory a physical system may be represented as the totality of the configurations associated with the same quantum numbers (n_B, s_0, i_0; S, I, I_3). From this standpoint, in considering the mutual relation among some reactions, what is important should be the quantum numbers of the systems and not the concrete configurations thereof. We shall show below that this is actually the case so far as the interaction which induces the reactions is expressible in terms of the infinitesimal operators of the group $U(3)$. For this purpose we begin with the discussion of the structure of higher configurations.

Let us consider a physical system specified by the quantum numbers (n_B, s_0, i_0; S, I, I_3). To this system there correspond many configurations, which transform in the same manner with respect to $U(3)$. The lowest of these configurations is a tensor $\overset{(0)}{T}{}^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ that possesses the following properties: (1) The tensor is reduced to zero when we contract it for any pair of a superscript and a subscript. (2) The tensor has a certain symmetry character which arises from the corresponding Young's diagram.

Now, multiply $\overset{(0)}{T}{}^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$ by m Kronecker deltas $\delta^{\kappa_{k+1}}_{\lambda_{l+1}}\cdots\delta^{\kappa_{k+m}}_{\lambda_{l+m}}$ and permute the indices $(1, 2, \cdots, k+m)$ and $(1, 2, \cdots, l+m)$ separately. Then we get

$$\overset{(0)}{T}{}^{\kappa_\alpha\cdots\kappa_\eta}_{\lambda_\rho\cdots\lambda_\sigma}\delta^{\kappa_\sigma}_{\lambda_\tau}\cdots\delta^{\kappa_\delta}_{\lambda_s}. \tag{2·1}$$

For brevity's sake we call (2·1) simply a product of $\overset{(0)}{T}$ and m δ's, and denote it by $\overset{(0)}{T}\delta\cdots\delta$. For example, $\pi^-\delta$ stands for the following configurations:*

$$(\pi^-)^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}=(np\bar{p}\bar{p}+nn\bar{p}\bar{n}+n\Lambda\bar{p}\bar{\Lambda})/\sqrt{3},$$
$$(\pi^-)^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}=(np\bar{p}\bar{p}+nn\bar{n}\bar{p}+n\Lambda\bar{\Lambda}\bar{p})/\sqrt{3},$$
$$(\pi^-)^{\kappa_2}_{\lambda_1}\delta^{\kappa_1}_{\lambda_2}=(pn\bar{p}\bar{p}+nn\bar{p}\bar{n}+\Lambda n\bar{p}\bar{\Lambda})/\sqrt{3},$$
$$(\pi^-)^{\kappa_2}_{\lambda_2}\delta^{\kappa_1}_{\lambda_1}=(pn\bar{p}\bar{p}+nn\bar{n}\bar{p}+\Lambda n\bar{\Lambda}\bar{p})/\sqrt{3}.$$

The configurations (2·1) are neither always linearly independent nor orthogonal

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* In ref. 3) we put $\delta_\lambda{}^\kappa=p\bar{p}+n\bar{n}+\Lambda\bar{\Lambda}$, but it is convenient to normalize it as $\delta_\lambda{}^\kappa=(p\bar{p}+n\bar{n}+\Lambda\bar{\Lambda})/\sqrt{3}$. Then (2·1) has the same norm as $\overset{(0)}{T}{}^{\kappa_1\cdots\kappa_k}_{\lambda_1\cdots\lambda_l}$.

to each other. It is easy to make of them a complete set of higher configurations which are linearly independent and orthogonal. Any higher configuration $T^{\kappa_1\cdots\kappa_{k+m}}_{\lambda_1\cdots\lambda_{l+m}}$ of the system can be expressed as a linear combination of the tensors (2·1), that is,

$$T^{\kappa_1\cdots\kappa_{k+m}}_{\lambda_1\cdots\lambda_{l+m}}=L(\overset{(0)}{T}{}^{\kappa_a\cdots\kappa_b}_{\lambda_p\cdots\lambda_q}\delta^{\kappa_c}_{\lambda_r}\cdots\delta^{\kappa_d}_{\lambda_s}). \tag{2·2}$$

This tensor does not satisfy the above condition (1).

A transformation belonging to $U(3)$ leaves each irreducible representation space invariant. In particular, when we operate an infinitesimal operator U of $U(3)$ to (2·2), we have

$$(UT)^{\kappa_1\cdots\kappa_{k+m}}_{\lambda_1\cdots\lambda_{l+m}}=L((\overset{(0)}{UT})^{\kappa_a\cdots\kappa_b}_{\lambda_p\cdots\lambda_q}\delta^{\kappa_c}_{\lambda_r}\cdots\delta^{\kappa_d}_{\lambda_s}), \tag{2·3}$$

because $\delta_\lambda{}^\kappa$ is invariant with respect to $U(3)$. The coefficients of the linear combination in (2·3) are the same as those in (2·2). This feature of higher configurations plays a very important role in the subsequent discussions.

Now, we shall consider a reaction

$$A+B+\cdots\rightarrow C+D+\cdots, \tag{2·4}$$

where A, B, C, D, etc., denote the relevant particles. If the configurations T_α, T_β, $\cdots$ (indices suppressed) are assigned to the particles A, B, $\cdots$, the configuration of the initial state becomes a product of T_α, T_β, $\cdots$ (cf. the paragraph below (2·1).), that is,

$$T_\alpha\cdot T_\beta\cdots. \tag{2·5}$$

Similarly, the configuration of the final state is

$$T_\gamma\cdot T_\delta\cdots. \tag{2·6}$$

To obtain the amplitude of the reaction (2·4) it is convenient to expand (2·5) and (2·6) in terms of the basis vectors for the representation of $U(3)$. If T_α, T_β, $\cdots$ are of $(k_\alpha+l_\alpha)$-, $(k_\beta+l_\beta)$-, $\cdots$ bodies, then (2·5) is a $(k+l)$-body configuration with

$$k=k_\alpha+k_\beta+\cdots,\quad l=l_\alpha+l_\beta+\cdots.$$

We have already given a method of constructing the basis vectors for $(k+l)$-body composite systems.[3] We make them orthonormal and denote the resulting basis vectors by $T_i(q)$, where q denotes the set of quantum numbers and i discriminates one $(k+l)$-body configuration from another with the same set of quantum numbers q. Then we can easily expand (2·5) as follows:

$$T_\alpha\cdot T_\beta\cdots=\sum_{i,q}C^i(q)T_i(q). \tag{2·7}$$

The final state should be a $(k+l)$-body configuration as well as the initial one. Therefore, in like manner we have

$$T_\gamma \cdot T_\delta \cdots = \sum_{i,q}{}' C^i(q) T_i(q). \tag{2·8}$$

In (2·7) and (2·8) the coefficients on the right-hand side correspond to those of Clebsch-Gordan for angular momenta. We here assume, as is the case in our decay scheme, that the reaction (2·4) is induced by the interaction $\boldsymbol{O}$ which can be expressed in terms of the infinitesimal operators of $U(3)$. Then the initial state makes a transition to

$$\boldsymbol{O}(T_\alpha \cdot T_\beta \cdots) = \sum_{i,q} C^i(q) \boldsymbol{O} T_i(q) = \sum_{i,q,q'} C^i(q)\,(q|q')\,T_i(q') \equiv \sum_{i,q}{}'' C^i(q) T_i(q).$$

The reaction (2·4) can take place through various channels, which may be specified by the quantum numbers and the possible configurations. Thus the amplitude desired can be given by

$$\begin{aligned} A &= \sum_{i,j,q,q'}{}' C^i(q)\,{}'' C^j(q') \langle T_i(q), T_j(q') \rangle a_{ij}(q, q') \\ &= \sum_{i,j,q,q'}{}' C^i(q)\,{}'' C^j(q') \delta_{ij} \delta_{qq'} a_{ij}(q, q') = \sum_{i,q}{}' C^i(q)\,{}'' C^i(q) a_{ii}(q, q), \end{aligned} \tag{2·9}$$

where $\langle T_i(q), T_j(q') \rangle$ denotes the inner product and $a_{ij}(q, q')$ is the amplitude for the channel specified by ij and qq'.

Next we analyse the reaction (2·4) in terms of higher configurations. We can do this by multiplying (2·5) and (2·6) by one $\delta_\lambda{}^\kappa$ with the same permutation of indices (cf. (2·1)). Correspondingly, $T_i(q)$ in (2·7) and (2·8) must be multiplied by $\delta_\lambda{}^\kappa$ with the same permutation. That is, the initial and the final state can be expanded as

$$T_\alpha \cdot T_\beta \cdots \delta = \sum_{i,q} C^i(q) T_i(q) \delta, \tag{2·7′}$$

$$T_\gamma \cdot T_\delta \cdots \delta = \sum_{i,q}{}' C^i(q) T_i(q) \delta, \tag{2·8′}$$

with the same coefficients as those in (2·7) and (2·8) respectively. When the interaction $\boldsymbol{O}$ is switched on, (2·7′) makes a transition to

$$\boldsymbol{O}(T_\alpha \cdot T_\beta \cdots \delta) = \boldsymbol{O}(T_\alpha \cdot T_\beta \cdots)\delta = \sum_{i,q}{}'' C^i(q) T_i(q) \delta.$$

Here use is made of the fact that the interaction $\boldsymbol{O}$ which induces the reaction can be expressed in terms of the infinitesimal operators of $U(3)$. Accordingly, we have the following expression for the amplitude of the reaction (2·4) analysed in terms of higher configurations:

$$\begin{aligned} \bar{A} &= \sum_{i,j,q,q'}{}' C^i(q)\,{}'' C^j(q') \langle T_i(q)\delta,\ T_j(q')\delta \rangle \bar{a}_{ij}(q, q') \\ &= \sum_{i,j,q,q'}{}' C^i(q)\,{}'' C^j(q') \langle T_i(q), T_j(q') \rangle \bar{a}_{ij}(q, q') = \sum_{i,q}{}' C^i(q)\,{}'' C^i(q) \bar{a}_{ii}(q, q). \end{aligned} \tag{2·9′}$$

The amplitude $\bar{a}_{ii}$ shows that the corresponding channel is specified by the configuration $T_i\delta$ and not by T_i. (2·9′) differs from (2·9) only in this $\bar{a}_{ii}$. Therefore, if we consider a second reaction, say,

$$A+B+\cdots\rightarrow C'+D'+\cdots, \tag{2·10}$$

the relation between the amplitudes of the reactions (2·4) and (2·10) is the same whether they are analysed by the scheme (2·9) or by (2·9′). This consequence is consistent with our view that a physical system is the totality of possible configurations. Furthermore, we may say that it gives support to the new particle picture developed in a previous paper.[7]

In the above discussions it is essential that the interaction $\boldsymbol{O}$ is expressible in terms of the infinitesimal operators of $U(3)$. Accordingly, in our scheme of weak interactions, we can safely analyse a decay process by making use of a particular configuration for each particle. In particular, the relations obtained in I among various decay amplitudes are valid irrespective of the configurations adopted there. In the next section, on the basis of the above result, we shall study the K-3π decay left untreated in the previous paper I.

§ 3. Analysis of the K-3π decay

In this section we shall analyse the K-3π decay processes. According to our scheme, the initial state of $K\rightarrow 3\pi$ must be a configuration of $(3+3)$-bodies or more, because the lowest configuration of a pion is a $(1+1)$-body state. On the other hand, as was shown in the preceding section, the relations among amplitudes of various reactions are irrespective of particular configurations of the initial or final system. Hence, when we deal with the $K\rightarrow 3\pi$ decay, it is sufficient to take a $(3+3)$-body configuration of a K-meson as the initial state.

Such a configuration, denoted by $K^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}$, can be expressed linearly in terms of the cofigurations $\delta^{\kappa_i}_{\lambda_p}\delta^{\kappa_j}_{\lambda_q}K^{\kappa_k}_{\lambda_r}$, thus

$$K^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}=L(\delta^{\kappa_i}_{\lambda_p}\delta^{\kappa_j}_{\lambda_q}K^{\kappa_k}_{\lambda_r}). \tag{3·1}$$

Here the notation is the same as in (2·1), $K_\lambda{}^\kappa$ being the lowest configuration of K. If we write the configurations in parentheses of (3·1) concretely, we have

$$\begin{aligned}
&\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}K^{\kappa_3}_{\lambda_3},\quad \delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}K^{\kappa_2}_{\lambda_3},\quad \delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}K^{\kappa_1}_{\lambda_3},\\
&\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_3}K^{\kappa_3}_{\lambda_2},\quad \delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}K^{\kappa_2}_{\lambda_2},\quad \delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\kappa_3}K^{\kappa_1}_{\lambda_2},\\
&\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}K^{\kappa_3}_{\lambda_3},\quad \delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}K^{\kappa_2}_{\lambda_3},\quad \delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}K^{\kappa_1}_{\lambda_3},\\
&\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}K^{\kappa_3}_{\lambda_1},\quad \delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}K^{\kappa_2}_{\lambda_1},\quad \delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}K^{\kappa_1}_{\lambda_1},\\
&\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}K^{\kappa_3}_{\lambda_2},\quad \delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}K^{\kappa_2}_{\lambda_2},\quad \delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}K^{\kappa_1}_{\lambda_2},\\
&\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}K^{\kappa_3}_{\lambda_1},\quad \delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}K^{\kappa_2}_{\lambda_1},\quad \delta^{\kappa_3}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}K^{\kappa_1}_{\lambda_1}.
\end{aligned} \tag{3·1′}$$

We can easily see that these configurations are linearly independent of each other.

When the decay interaction $\boldsymbol{O}$ is switched on, $K^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}$ makes a transition to $(\boldsymbol{O}K)^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}$, which has the expression

$$(\boldsymbol{O}K)^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}=L(\delta^{\kappa_i}_{\lambda_p}\,\delta^{\kappa_j}_{\lambda_q}(\boldsymbol{O}K)^{\kappa_k}_{\lambda_r}), \qquad (3\cdot2)$$

with the same coefficients as those in (3·1). $(\boldsymbol{O}K)_\lambda{}^\kappa$, the final states of $K_\lambda{}^\kappa$, are given in Table I, where we have assumed that the $K^0\to3\pi$ decay occurs through $K_2{}^{0*}$ and that the final state must be of the form

Table I

final states of \ $\boldsymbol{O}$	N_{32}	I_+N_{31}	$N_{31}I_+$
$(K^+)^\kappa_\lambda$	$(\pi^+)^\kappa_\lambda$	$-(\pi^+)^\kappa_\lambda$	0
$(K_2{}^0)^\kappa_\lambda$	0	$\left(\frac{\sqrt{3}}{2}\pi^{0\prime}-\frac{3}{2}\pi^0\right)^\kappa_\lambda$	$\left(\frac{\sqrt{3}}{2}\pi^{0\prime}-\frac{3}{2}\pi^0\right)^\kappa_\lambda$

$$(\boldsymbol{O}|K^0\rangle-\boldsymbol{O}^*|\overline{K}{}^0\rangle)/\sqrt{2}.$$

Here $\boldsymbol{O}$ stands for N_{32}, $N_{31}I_-$ and I_+N_{31}, and the corresponding $\boldsymbol{O}^*$ does for N_{23}, I_-N_{13} and $N_{13}I_-$ respectively. Table I tells us the following: To the $K_2{}^0$-3π decay, the interaction N_{32} makes no contribution and I_+N_{31} and $N_{31}I_+$ give the same result. As to the K^+-3π decay, N_{32} and I_+N_{31} bring about the same consequence, while $N_{31}I_+$ does not cause the decay.

Now we must express $(\boldsymbol{O}K)^{\kappa_1\kappa_2\kappa_3}_{\lambda_1\lambda_2\lambda_3}$ in terms of various final states. The calculation is straightforward but rather tedious, and the result is given in Appendix 1. From this and Table I we can easily calculate the decay amplitudes of K-meson to various charge states of 3π. Appendix 2 shows the coefficients of these amplitudes with respect to the $I=1$ and 0 channels.

In general, an $I=1$ state of 3π system is expressed by a superposition of three independent (orthonormal) wave functions X, Y and Z. X is totally symmetric with respect to all pions, and Y (or Z) is symmetric (or alternative) only for the first two pions. For practical purposes it is convenient to use the decay amplitude to each of these states. We shall discuss each decay mode separately in what follows.

1) $K^+\to3\pi$ *decay*

In this case there are two possible decay modes $K^+\to\pi^++\pi^++\pi^-$ and $K^+\to\pi^++\pi^0+\pi^0$. Table I shows that both the final 3π states must have $I=1$. The decay interactions N_{32} and I_+N_{31} give the same result, while $N_{31}I_+$ makes no contribution.

1a) $K^+\to\pi^++\pi^++\pi^-$

* On account of the CP invariance, the $3\pi^0$ system is a possible decay product of $K_2{}^0$ but not of $K_1{}^0$. As to $\pi^++\pi^-+\pi^0$ we cannot conclude that it is a decay product of $K_2{}^0$ without some knowledge of the wave function. But the conclusion does hold to the extent that the matrix element is symmetric in π^+ and π^-.

The isospin wave functions for the final 3π states have the following expressions:

$$\begin{aligned} X&=(\pi^+\pi^+\pi^-+\pi^+\pi^-\pi^++\pi^-\pi^+\pi^+)/\sqrt{3}\,,\\ Y&=(2\pi^+\pi^+\pi^--\pi^+\pi^-\pi^+-\pi^-\pi^+\pi^+)/\sqrt{6}\,,\\ Z&=(\pi^+\pi^-\pi^+-\pi^-\pi^+\pi^+)/\sqrt{2}\,. \end{aligned} \tag{3·3}$$

The decay amplitudes to these states are

$$\begin{aligned} A_X&=\{-(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9)/3-2(\alpha_2+\alpha_{13}+\alpha_{14})/3\}/\sqrt{3}\,,\\ A_Y&=\{-(\alpha_2-\alpha_4-\alpha_5-\alpha_7-\alpha_9+\alpha_{14})/3-2(\alpha_6+\alpha_8-\alpha_{13})/3\}/\sqrt{6}\,,\\ A_Z&=(\alpha_2-\alpha_4+\alpha_5+\alpha_7-\alpha_9-\alpha_{14})/3\sqrt{2}\,. \end{aligned} \tag{3·4}$$

In terms of these amplitudes we obtain the decay probability of $K^+\to\pi^++\pi^++\pi^-$ as follows:

$$P(K^+\to\pi^++\pi^++\pi^-)=|A_X|^2+|A_Y|^2+|A_Z|^2.$$

1b) $K^+\to\pi^++\pi^0+\pi^0$

In this case we have the following expressions corresponding to (3·3) and (3·4):

$$\begin{aligned} X&=(\pi^+\pi^0\pi^0+\pi^0\pi^+\pi^0+\pi^0\pi^0\pi^+)/\sqrt{3}\,,\\ Y&=(2\pi^0\pi^0\pi^+-\pi^+\pi^0\pi^0-\pi^0\pi^+\pi^0)/\sqrt{6}\,,\\ Z&=(\pi^+\pi^0\pi^0-\pi^0\pi^+\pi^0)/\sqrt{2}\,; \end{aligned} \tag{3·5}$$

$$\begin{aligned} B_X&=\{(\alpha_2+\alpha_{13}+\alpha_{14})/3+(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9)/6\}/\sqrt{3}\,,\\ B_Y&=\{-(\alpha_2-\alpha_4-\alpha_5-\alpha_7-\alpha_9+\alpha_{14})/3-2(\alpha_6+\alpha_8-\alpha_{13})/3\}/\sqrt{6}\,,\\ B_Z&=-(\alpha_2-\alpha_4+\alpha_5+\alpha_7-\alpha_9-\alpha_{14})/3\sqrt{2}\,. \end{aligned} \tag{3·6}$$

Thus the decay probability of $K^+\to\pi^++\pi^0+\pi^0$ becomes

$$P(K^+\to\pi^++\pi^0+\pi^0)=|B_X|^2+B_Y|^2+|B_Z|^2.$$

It follows from (3·4) and (3·6) that

$$-A_X=2B_X,\quad A_Y=B_Y,\quad -A_Z=B_Z. \tag{3·7}$$

Hence the branching ratio of the K^+-3π decay is given by

$$\frac{P(K^+\to\pi^++\pi^0+\pi^0)}{P(K^+\to\pi^++\pi^++\pi^-)}=\frac{|B_X|^2+|B_Y|^2+|B_Z|^2}{|A_X|^2+|A_Y|^2+|A_Z|^2}=\frac{1+x}{4+x}, \tag{3·8}$$

where

$$x=4\{|A_Y|^2+|A_Z|^2\}/|A_X|^2\geq 0.$$

From (3·8) we can easily get

$$\frac{1}{4}\leq\frac{P(K^+\to\pi^++\pi^0+\pi^0)}{P(K^+\to\pi^++\pi^++\pi^-)}\leq 1. \tag{3·9}$$

If we assume, as is usually done, that the totally symmetric state is predominant, the ratio (3·8) has the minimum value, i. e.

$$\frac{P(K^+\to\pi^+ +\pi^0+\pi^0)}{P(K^+\to\pi^+ +\pi^+ +\pi^-)}=\frac{1}{4}. \tag{3·10}$$

2) $K_2^0\to 3\pi$ *decay*

In this case we have two decay modes, that is, $K_2^0\to 3\pi^0$ and $K_2^0\to\pi^+ +\pi^- +\pi^0$. Three pions in the final state may be in a state of $I=1$ and of $I=0$. The decay interaction N_{32} does not contribute and $I_+ N_{31}$ and $N_{31} I_+$ produce the same result.

2a) $K_2^0\to\pi^+ +\pi^- +\pi^0$

The isospin wave functions for the final states are as follows:

For $I=1$

$$\begin{aligned}
X&=(\pi^+\pi^-\pi^0+\pi^+\pi^0\pi^- +\pi^0\pi^+\pi^- +\pi^0\pi^-\pi^+ +\pi^-\pi^+\pi^0+\pi^-\pi^0\pi^+)/\sqrt{6},\\
Y&=(\pi^+\pi^0\pi^- +\pi^0\pi^+\pi^- +\pi^0\pi^-\pi^+ +\pi^-\pi^0\pi^+ -2\pi^+\pi^-\pi^0-2\pi^-\pi^+\pi^0)/\sqrt{12},\\
Z&=(\pi^+\pi^0\pi^- -\pi^0\pi^+\pi^- -\pi^0\pi^-\pi^+ +\pi^-\pi^0\pi^+)/2.
\end{aligned} \tag{3·11}$$

For $I=0$

$$W=(\pi^+\pi^0\pi^- -\pi^0\pi^+\pi^- +\pi^0\pi^-\pi^+ -\pi^-\pi^0\pi^+ +\pi^-\pi^+\pi^0-\pi^+\pi^-\pi^0)/\sqrt{6}.$$

The corresponding decay amplitudes are

$$\begin{aligned}
A_X'&=\{-2(\alpha_2+\alpha_{13}+\alpha_{14})/3-(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9)/3\}/\sqrt{6},\\
A_Y'&=\{2(-\alpha_2+\alpha_4+\alpha_5+\alpha_7+\alpha_9-\alpha_{14})/3-4(\alpha_6+\alpha_8-\alpha_{13})/3\}/\sqrt{12},\\
A_Z'&=(-\alpha_2+\alpha_4-\alpha_5-\alpha_7+\alpha_9+\alpha_{14})/3,\\
A_W'&=(\beta_4+\beta_5+\beta_6-\beta_7-\beta_8-\beta_9)/3\sqrt{2}.
\end{aligned} \tag{3·12}$$

2b) $K_2^0\to 3\pi^0$

Since the final three pions are totally symmetric, we have

$$X=\pi^0\pi^0\pi^0,\quad Y=Z=W=0. \tag{3·13}$$

The decay amplitudes are given by

$$\begin{aligned}
B_X'&=(\alpha_2+\alpha_{13}+\alpha_{14})/3+(\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9)/6,\\
B_Y'&=B_Z'=B_W'=0.
\end{aligned} \tag{3·14}$$

Because of the relation

$$-\sqrt{6}\,A_X'=2B_X', \tag{3·15}$$

we have the branching ratio

$$\frac{P(K_2^0\to 3\pi^0)}{P(K_2^0\to\pi^+ +\pi^- +\pi^0)}=\frac{|B_X'|^2}{|A_X'|^2+|A_Y'|^2+|A_Z'|^2+|A_W'|^2/3}=\frac{3}{2}\,\frac{1}{1+x}, \tag{3·16}$$

where

$$x=\{|A_Y'|^2+|A_Z'|^2+|A_W'|^2/3\}/|A_X'|^2\geq 0.$$

From this we get

$$0\leq\frac{P(K_2^0\to 3\pi^0)}{P(K_2^0\to\pi^++\pi^-+\pi^0)}\leq\frac{3}{2}.$$

In particular, if we assume that the symmetric state X is predominant, the branching ratio becomes maximum:

$$\frac{P(K_2^0\to 3\pi^0)}{P(K_2^0\to\pi^++\pi^-+\pi^0)}=\frac{3}{2}.$$

Finally we search for the ratio of the decay rates $K^+\to 3\pi$ and $K_2^0\to 3\pi$. It follows from (3·4) and (3·12) that

$$A_X=\sqrt{2}\,A_X',\quad \sqrt{2}\,A_Y=A_Y',\quad \sqrt{2}\,A_Z=-A_Z'.$$

Making use of these relations together with (3·7) and (3·15), we have

$$\frac{P(K^+\to 3\pi)}{P(K_2^0\to 3\pi)}=\frac{|A_X|^2+|A_Y|^2+|A_Z|^2+|B_X|^2+|B_Y|^2+|B_Z|^2}{|A_X'|^2+|A_Y'|^2+|A_Z'|^2+|B_X'|^2+|A_W'|^2/3}=\frac{1}{1+x},$$

where

$$x=\frac{1}{3}\,\frac{|A_W'|^2}{5|A_X|^2/4+2|A_Y|^2+2|A_Z|^2}\geq 0.$$

Therefore, we get

$$0\leq\frac{P(K^+\to 3\pi)}{P(K_2^0\to 3\pi)}\leq 1.$$

This ratio reaches its maximum value when the totally symmetric state X is predominant:

$$\frac{P(K^+\to 3\pi)}{P(K_2^0\to 3\pi)}=1.$$

The foregoing results coincide with those obtained from the $|\Delta I|=1/2$ rule in the usual theory.[8] This shows that our selection rule $\Delta n_B=\Delta s_0=\Delta i_0=0$ in the weak interactions is extremely effective for all the decays detected up to now.

§ 4. Decay processes involving $\pi^{0\prime}$ and $\pi^{0\prime\prime}$

We have so far considered the observed decay modes of hyperons and K-mesons and have found that our selection rule

$$\Delta n_B=\Delta s_0=\Delta i_0=0 \qquad (1\cdot 1)$$

is very effective as a substitute for the conventional $|\Delta I|=1/2$ rule.

On the other hand, in the scheme of the full symmetry theory, there are some states predicted but not observed as yet. A low energy state among them will not be so difficult to detect, if it does exist in nature. In this respect we are greatly interested in the two possible states $\pi^{0\prime}$ and $\pi^{0\prime\prime}$ discussed in ref. 2) and 5). The former state $\pi^{0\prime}$ belongs to the same irreducible representation of $U(3)$ as a pion does, and will be similar in character to a pion except the isospin value. Yamaguchi[9] examined the possibility of the existence of such a particle. Sawada and Yonezawa[10] pointed out that the existence of $\pi^{0\prime}$ is favourable to explain the electron energy spectra of the K_{e3}-decay in terms of the V-A theory. As to $\pi^{0\prime\prime}$ we have no knowledge except that $S=I=0$. It may not be a bound state, or it may be more pertinent to call it the vacuum rather than a particle in a sense,[7] because $\pi^{0\prime\prime}$ is a scalar with respect to $U(3)$. Though these points are beyond our present knowledge, we shall tentatively treat $\pi^{0\prime}$ and $\pi^{0\prime\prime}$ as unknown particles.

It is possible that hyperons and K-mesons have new decay modes involving $\pi^{0\prime}$ and $\pi^{0\prime\prime}$ besides the observed modes. This point will be studied in this section by presuming the selection rule (1·1). Of course, for such a new decay mode to be possible, the mass of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ should be small to some extent. Sawada and Yonezawa obtained $m_{\pi^{0\prime}} \cong 615$ Mev and $m_{\pi^{0\prime\prime}} \cong 377$ Mev* on the basis of their mass formula.[5] But, without sticking to these values, we here only assume that $m_{\pi^{0\prime}}$ and $m_{\pi^{0\prime\prime}}$ are not smaller than m_{π^0}.

Now, the following are the possible new decay modes which we are going to discuss:

For hyperons

$$\Lambda \to \pi^{0\prime} + n, \quad \Lambda \to \pi^{0\prime\prime} + n, \tag{4·1}$$

$$\Sigma_1^+ \to \pi^{0\prime} + p, \quad \Sigma_1^+ \to \pi^{0\prime\prime} + p, \tag{4·2}$$

$$\Xi^0 \to \pi^{0\prime} + \Lambda, \quad \Xi^0 \to \pi^{0\prime\prime} + \Lambda. \tag{4·3}$$

For K-mesons

$$\left.\begin{aligned} &K^+ \to \pi^+ + \pi^{0\prime}, \quad K^+ \to \pi^+ + \pi^{0\prime\prime}, \\ &K_1^0 \to \pi^0 + \pi^{0\prime}, \quad K_1^0 \to \pi^0 + \pi^{0\prime\prime}, \\ &K_1^0 \to \pi^{0\prime} + \pi^{0\prime}, \quad K_1^0 \to \pi^{0\prime\prime} + \pi^{0\prime\prime}, \quad K_1^0 \to \pi^{0\prime} + \pi^{0\prime\prime}. \end{aligned}\right\} \tag{4·4}$$

$$\left.\begin{aligned} &K^+ \to \pi^+ + \pi^0 + \pi^{0\prime}, \quad K^+ \to \pi^+ + \pi^0 + \pi^{0\prime\prime}, \\ &K^+ \to \pi^+ + 2\pi^{0\prime}, \quad K^+ \to \pi^+ + 2\pi^{0\prime\prime}, \quad K^+ \to \pi^+ + \pi^{0\prime} + \pi^{0\prime\prime}, \\ &K_2^0 \to \pi^+ + \pi^- + \pi^{0\prime}, \quad K_2^0 \to \pi^+ + \pi^- + \pi^{0\prime\prime}, \quad K_2^0 \to 2\pi^0 + \pi^{0\prime}, \end{aligned}\right\} \tag{4·5}$$

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* There is some doubt about $m_{\pi^{0\prime\prime}}$, because the above value is obtained by applying the mass formula to the configuration $B_2^2(0, 0)$ that is not the lowest. It is possible that $\pi^{0\prime\prime}$ may be a vacuum[7] and its ground level may be of zero energy.

$K_2^0 \to 2\pi^0 + \pi^{0\prime\prime}$, $K_2^0 \to \pi^0 + 2\pi^{0\prime}$, $K_2^0 \to \pi^0 + 2\pi^{0\prime\prime}$,
$K_2^0 \to \pi^0 + \pi^{0\prime} + \pi^{0\prime\prime}$, $K_2^0 \to 3\pi^{0\prime}$, $K_2^0 \to 3\pi^{0\prime\prime}$,
$K_2^0 \to 2\pi^{0\prime} + \pi^{0\prime\prime}$, $K_2^0 \to \pi^{0\prime} + 2\pi^{0\prime\prime}$.

We next obtain the decay amplitude for each process. For this purpose we need the product of configurations of the two particles in the final state. This

Table II. Decay amplitudes for hyperons

Decay mode \ Interaction	N_{23}	$I_- N_{13}$	$N_{13} I_-$
$\Lambda \to \pi^- + p$ (A_-)	$\frac{\sqrt{2}}{4} a_3^2\left(0, \frac{1}{2}\right) - \frac{1}{2} a_3^4\left(0, \frac{1}{2}\right)$	$\frac{\sqrt{2}}{4} a_3^2\left(0, \frac{1}{2}\right) - \frac{1}{2} a_3^4\left(0, \frac{1}{2}\right)$	0
$\Lambda \to \pi^0 + n$ (A_0)	$-\frac{1}{4} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{2}}{4} a_3^4\left(0, \frac{1}{2}\right)$	$-\frac{1}{4} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{2}}{4} a_3^4\left(0, \frac{1}{2}\right)$	0
$\Lambda \to \pi^{0\prime} + n$ (A_0')	$-\frac{\sqrt{3}}{12} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{6}}{12} a_3^4\left(0, \frac{1}{2}\right)$	$-\frac{\sqrt{3}}{12} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{6}}{12} a_3^4\left(0, \frac{1}{2}\right)$	0
$\Lambda \to \pi^{0\prime\prime} + n$ (A_0'')	$\frac{\sqrt{6}}{3} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{3}}{3} a_3^4\left(0, \frac{1}{2}\right)$	$\frac{\sqrt{6}}{3} a_3^2\left(0, \frac{1}{2}\right) + \frac{\sqrt{3}}{3} a_3^4\left(0, \frac{1}{2}\right)$	0
$\Sigma_1^+ \to \pi^0 + p$ (A_1^0)	$-\frac{1}{6} a_3^1\left(0, \frac{1}{2}\right) + \frac{2}{3} a_3^1\left(0, \frac{3}{2}\right)$	$2 a_3^1\left(0, \frac{3}{2}\right)$	$\frac{1}{6} a_3^1\left(0, \frac{1}{2}\right) + \frac{4}{3} a_3^1\left(0, \frac{3}{2}\right)$
$\Sigma_1^+ \to \pi^+ + n$ (A_1^+)	$\frac{\sqrt{2}}{6} a_3^1\left(0, \frac{1}{2}\right) + \frac{\sqrt{2}}{3} a_3^1\left(0, \frac{3}{2}\right)$	$\sqrt{2} a_3^1\left(0, \frac{3}{2}\right)$	$-\frac{\sqrt{2}}{6} a_3^1\left(0, \frac{1}{2}\right) + \frac{2\sqrt{2}}{3} a_3^1\left(0, \frac{3}{2}\right)$
$\Sigma_1^+ \to \pi^{0\prime} + p$ $(A_1^{0\prime})$	$\frac{\sqrt{3}}{2} a_3^1\left(0, \frac{1}{2}\right)$	0	$-\frac{\sqrt{3}}{2} a_3^1\left(0, \frac{1}{2}\right)$
$\Sigma_1^+ \to \pi^{0\prime\prime} + p$ $(A_1^{0\prime\prime})$	0	0	0
$\Sigma_1^- \to \pi^- + n$ (A_1^-)	$\sqrt{2} a_3^1\left(0, \frac{3}{2}\right)$	$\sqrt{2} a_3^1\left(0, \frac{3}{2}\right)$	0
$\Xi^0 \to \pi^0 + \Lambda$	$\frac{\sqrt{2}}{2} a_3^1(-1, 1)$	$\sqrt{2} a_3^1(-1, 1)$	$\frac{\sqrt{2}}{2} a_3^1(-1, 1)$
$\Xi^0 \to \pi^{0\prime} + \Lambda$	$\frac{\sqrt{6}}{2} a_3^1(-1, 0)$	0	$-\frac{\sqrt{6}}{2} a_3^1(-1, 0)$
$\Xi^0 \to \pi^{0\prime\prime} + \Lambda$	0	0	0
$\Xi^- \to \pi^- + \Lambda$	$a_3^1(-1, 1)$	$a_3^1(-1, 1)$	0

product can directly be calculated from the one-body configurations of (p, n, Λ) and $(1+1)$-body configurations of $(\pi, \pi^{0\prime}, \pi^{0\prime\prime})$ which are given in ref. 5). The results are listed in Appendix 3. The decay amplitude for each process can be evaluated in much the same way as in I and the results are given in Table II, where the amplitudes for the normal decay modes are also listed for the sake of comparison.

In what follows we shall discuss each process separately.*

1) *Λ-decay*

A new mode does not occur unless the mass of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ is smaller than about 175 Mev. From Table II we easily see the following:

(i) The interaction $N_{13}I_-$ does not induce (4·1) any more than it induces the normal decay modes.

(ii) The interactions N_{23} and I_-N_{13} lead to the same results. Denoting the decay amplitudes to $(p\pi^-)$, $(n\pi^0)$, $(n\pi^{0\prime})$ and $(n\pi^{0\prime\prime})$ by A_-, A_0, A_0' and A_0'' respectively, we have in general the following relations:

$$A_-=-\sqrt{2}\,A_0,\quad A_0'=A_0/\sqrt{3}. \tag{4·6}$$

In the special case of $a_3^4=0$ or $a_3^2=0$, a further relation exists among them:

$$\begin{aligned} &A_0''=-4\sqrt{6}\,A_0/3 \quad \text{for } a_3^2\neq 0,\ a_3^4=0,\\ &A_0''=2\sqrt{6}\,A_0/3 \quad \text{for } a_3^2=0,\ a_3^4\neq 0. \end{aligned} \tag{4·7}$$

In any case the decay amplitudes of the new modes (4·1) are comparable to those of the normal modes.** Thus, in order to be compatible with experiments it is necessary to make the frequency of (4·1) low by assuming the mass of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ to be very near $m_\Lambda-m_N$ or higher.

2) *Σ-decay*

Among the two possible states $\Sigma_1=F_3^1(-1, 1)$ and $\Sigma_2=F_3^3(-1, 1)$, the latter is forbidden to decay through the primary decay interactions $\boldsymbol{O}$. Therefore we here consider only the former. For (4·2) to be realized the upper limit of $m_{\pi^{0\prime}}$ or $m_{\pi^{0\prime\prime}}$ is about 250 Mev.

(i) $\Sigma_1^+\to\pi^{0\prime\prime}+p$ is strictly forbidden for any decay interaction. This is because $\pi^{0\prime\prime}p$ belongs to the same representation as p does, while Σ_1^+ belongs to an inequivalent representation.

(ii) The decay interactions I_-N_{13} and $N_{13}I_-$ forbid the decay $\Sigma_1^+\to\pi^{0\prime}+p$ and $\Sigma_1^-\to\pi^-+n$ respectively and are inconsistent with experiments.

(iii) Denote the amplitudes for $\Sigma_1^+\to\pi^0+p$, $\Sigma_1^+\to\pi^{0\prime}+p$, $\Sigma_1^+\to\pi^++n$ and $\Sigma_1^-\to\pi^-+n$ by A_1^0, $A_1^{0\prime}$, A_1^+ and A_1^- respectively. Then for the interaction N_{23} we have in general

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* Tables IV, V and VI in the paper I give the results of operating various decay interactions $\boldsymbol{O}$ to the initial states.

** A_0'' may be reduced to zero by a fortuitous cancellation between $a_3^2(0, 1/2)$ and $a_3^4(0, 1/2)$, but it is not the case with A_0' on account of (4·6).

$$\sqrt{2}A_1^0+A_1^+=A_1^-,\quad \sqrt{2}A_1^+-A_1^0=A_1^{0\prime}/\sqrt{3}. \tag{4·8}$$

In the special case of $a_3^1(0, 1/2)=0$, $A_1^{0\prime}$ vanishes but the branching ratio of the normal modes becomes $|A_1^0|^2:|A_1^+|^2:|A_1^-|^2=2:1:9$, which is contradictory to experiments. If $a_3^1(0, 3/2)=0$, $\Sigma_1^-\to\pi^-+n$ is forbidden. If $\sqrt{2}A_1^0$, A_1^+ and A_1^- form a right-angled isosceles triangle as is consistent with the behaviour of the branching ratio and the asymmetry factor of the normal modes,[8] $A_1^{0\prime}$ is never smaller than the others and even larger than any of them. Thus we must consider that the slowness of the decay mode (4·2) is only due to a possible high mass value of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$.

3) *Ξ-decay*

The new mode (4·3) is possible only when the mass of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ is smaller than about 205 Mev.

(i) The $\pi^{0\prime\prime}$ mode is always forbidden. This is due to the same reason as in the case of $\pi^{0\prime\prime}$ mode in Σ_1^+-decay.

(ii) The interaction $N_{13}I_-$ forbids the decay $\Xi^-\to\pi^-+\Lambda$.

(iii) The interaction I_-N_{13} forbids the $\pi^{0\prime}$ mode. In this case the ratio of the two normal decay modes is

$$\frac{P(\Xi^0\to\pi^0+\Lambda)}{P(\Xi^-\to\pi^-+\Lambda)}=2. \tag{4·9}$$

(iv) For the interaction N_{23}, the decay amplitudes of the normal modes are of $I=1$, while that of the $\pi^{0\prime}$ mode is of $I=0$. Accordingly, both may occur independently of each other. In this case the ratio between the normal modes are

$$\frac{P(\Xi^0\to\pi^0+\Lambda)}{P(\Xi^-\to\pi^-+\Lambda)}=\frac{1}{2}, \tag{4·10}$$

which coincides with the $|\Delta I|=1/2$ rule.

4) *Two-body decay of K-mesons*

The anomalous modes (4·4) can be classified into the following two groups according as the decay amplitudes are of $I=1$ or 0.

$I=1$ channel

$$\left.\begin{aligned} &K^+\to\pi^++\pi^{0\prime},\\ &K^+\to\pi^++\pi^{0\prime\prime},\\ &K_1^0\to\pi^0+\pi^{0\prime},\\ &K_1^0\to\pi^0+\pi^{0\prime\prime},\end{aligned}\right\}\quad (m_{\pi^{0\prime}}\text{ or }m_{\pi^{0\prime\prime}}<m_K-m_\pi\sim 360\text{ Mev}). \tag{4·11}$$

$I=0$ channel

$$\begin{aligned} &K_1^0\to\pi^{0\prime}+\pi^{0\prime\prime}, && (m_{\pi^{0\prime}}+m_{\pi^{0\prime\prime}}\leq m_{K_1^0}),\\ &K_1^0\to 2\pi^{0\prime}, && (m_{\pi^{0\prime}}\leq m_{K_1^0}/2\sim 250\text{ Mev}),\\ &K_1^0\to 2\pi^{0\prime\prime}, && (m_{\pi^{0\prime\prime}}\leq m_{K_1^0}/2\sim 250\text{ Mev}).\end{aligned} \tag{4·12}$$

Note that the normal modes $K_1^0 \to 2\pi^0$ and $K_1^0 \to \pi^+ + \pi^-$ also belong to the $I=0$ channel and that $K^+ \to \pi^+ + \pi^0$ is forbidden by our selection rule[6] as well as by the conventional $|\Delta I| = 1/2$ rule. Therefore, only $(\pi^+\pi^{0\prime})$ or $(\pi^+\pi^{0\prime\prime})$ in (4·11) is possible as the two-body decay of K^+. From Table III we find the following.

Table III. Amplitude for the two-body decay of K-mesons

Decay mode \ Interaction	N_{32}	$I_+ N_{31}$	$N_{31} I_+$
$K^+ \to \pi^+ + \pi^0$	0		
$K^+ \to \pi^+ + \pi^{0\prime}$	$-\frac{\sqrt{30}}{30} a_4^2(0,1) + \frac{\sqrt{6}}{6} a_4^8(0,1)$		
$K^+ \to \pi^+ + \pi^{0\prime\prime}$	$\sqrt{\frac{15}{6}} a_4^2(0,1) - \frac{1}{2} a_4^5(0,1) - \frac{1}{2} a_4^7(0,1) + \sqrt{\frac{3}{6}} a_4^8(0,1)$	$-\frac{1}{2} \times$ (results of N_{32})	$\frac{1}{2} \times$ (results of N_{32})
$K_1^0 \to \pi^+ + \pi^-$	$\frac{\sqrt{5}}{5} a_4^2(0,0) - a_4^8(0,0)$		
$K_1^0 \to \pi^0 + \pi^0$	$-\frac{\sqrt{10}}{10} a_4^2(0,0) + \frac{\sqrt{2}}{2} a_4^8(0,0)$		
$K_1^0 \to \pi^0 + \pi^{0\prime}$	$-\frac{\sqrt{30}}{15} a_4^2(0,1) + \frac{\sqrt{6}}{3} a_4^8(0,1)$		
$K_1^0 \to \pi^0 + \pi^{0\prime\prime}$	$\frac{\sqrt{15}}{3} a_4^2(0,1) - a_4^5(0,1) - a_4^7(0,1) + \frac{\sqrt{3}}{3} a_4^8(0,1)$		
$K_1^0 \to \pi^{0\prime} + \pi^{0\prime\prime}$	$\frac{\sqrt{5}}{2} a_4^2(0,0) - \frac{\sqrt{3}}{2} a_4^5(0,0) - \frac{\sqrt{3}}{2} a_4^7(0,0) + \frac{1}{2} a_4^8(0,0)$		
$K_1^0 \to \pi^{0\prime} + \pi^{0\prime}$	$\frac{\sqrt{10}}{10} a_4^2(0,0) - \frac{\sqrt{2}}{2} a_4^8(0,0)$		
$K_1^0 \to \pi^{0\prime\prime} + \pi^{0\prime\prime}$	0		

(i) $K_1^0 \to 2\pi^{0\prime\prime}$ is absolutely forbidden.

(ii) The decay interaction $I_+ N_{31}$ (or $N_{31} I_+$) gives the amplitudes different from those for N_{32} only by a factor $-1/2$ (or $1/2$). Thus the relations among various amplitudes are the same for all interactions.

(iii) For the $I=1$ channel the following relations hold in general among the decay amplitudes of (4·11):

$$2A(\pi^+\pi^{0\prime}) = A(\pi^0\pi^{0\prime}), \quad 2A(\pi^+\pi^{0\prime\prime}) = A(\pi^0\pi^{0\prime\prime}). \tag{4·13}$$

If any of the possible four classes of K-meson (class 2, 5, 7, 8) is predominant over the others, we have further relations as follows:

$$\begin{aligned} &A(\pi^+\pi^{0\prime}) = A(\pi^0\pi^{0\prime}) = 0 && \text{if classes 5 and 7 are predominant,} \\ &A(\pi^+\pi^{0\prime}) = \sqrt{2}\, A(\pi^+\pi^{0\prime\prime}) && \text{if class 8 is predominant,} \\ &A(\pi^+\pi^{0\prime}) = -\sqrt{2}\, A(\pi^+\pi^{0\prime\prime})/5 && \text{if class 2 is predominant.} \end{aligned} \tag{4·14}$$

(iv) For the $I=0$ channel we have in general the following relations among

the various amplitudes involving the normal decay modes:

$$A(\pi^+\pi^-) = -\sqrt{2}\,A(2\pi^0) = \sqrt{2}\,A(2\pi^{0\prime}). \tag{4·15}$$

In the special case in which any of the possible four classes of K-meson is predominant over the others, we have further the following relations:

$$\begin{aligned} &A(\pi^{0\prime}\pi^{0\prime\prime}) = -5A(2\pi^0)/\sqrt{2} && \text{if class 2 is predominant,} \\ &A(\pi^{0\prime}\pi^{0\prime\prime}) = A(2\pi^0)/\sqrt{2} && \text{if class 8 is predominant,} \\ &A(\pi^+\pi^-) = A(2\pi^0) = A(2\pi^{0\prime}) = 0 && \text{if classes 5 and 7 are predominant.} \end{aligned} \tag{4·16}$$

Since $A(\pi^+\pi^-) \neq 0$, we can conclude that the anomalous modes should be realized appreciably unless $m_{\pi^{0\prime}}$ or $m_{\pi^{0\prime\prime}}$ is far larger than m_π and that they can be forbidden only by the energy conservation law.

5) *Three-body decay of K-mesons*

For the processes (4·5) to be realized the mass of $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ should be almost equal to m_π or smaller. Even if these modes can be realized, the question of the two-body anomalous decay precedes that of the decay under codsideration. Therefore, we do not go into these modes any further and only point out that the $K_2^0 \to 3\pi^{0\prime\prime}$ process is absolutely forbidden.

In conclusion we summarize the results in this section.

a) Among the anomalous decay modes (4·1) to (4·5), the following reactions never occur by any interaction:

$$\begin{aligned} &\Sigma_1^+ \to \pi^{0\prime\prime} + p, \quad \Xi^0 \to \pi^{0\prime\prime} + \Lambda, \\ &K_1^0 \to 2\pi^{0\prime\prime}, \qquad K_2^0 \to 3\pi^{0\prime\prime}. \end{aligned} \tag{4·17}$$

b) If $m_{\pi^{0\prime}}$ or $m_{\pi^{0\prime\prime}}$ is of the same order of magnitude as m_π, the other anomalous decay modes will be realized to a considerable extent unless the matrix elements are cancelled by chance. In particular, possible non-leptonic decay modes of K^+-meson in our scheme are only anomalous ones.

c) In order to explain the slowness of these anomalous processes, it is necessary to assume that $m_{\pi^{0\prime}}$ or $m_{\pi^{0\prime\prime}}$ is very close to or even larger than $m_K - m_\pi$. The mass value predicted by Sawada and Yonezawa[5] for $\pi^{0\prime}$ or $\pi^{0\prime\prime}$ is actually of such an order of magnitude.

§ 5. Concluding remarks

In conclusion we shall make some remarks about the results which have been obtained in the previous paper I and the present one.

For the discussions of the non-leptonic decay of hyperons and mesons, we have assumed that the interaction causing a reaction can be expressed in terms of the infinitesimal operators of the group $U(3)$. Under this assumption, as shown in § 2, the relations among the amplitudes of various reactions have no connection with particular configurations of the particles which participate in the reactions.

On account of this fact all the conclusions which we have derived from those relations are valid regardless of the configurations employed in deduction. That fact also gives support to our view that a physical system is represented by the totality of configurations with the same quantum numbers.

We have found the following results concerning the observed non-leptonic decays of hyperons and K-mesons. For Λ- and K-decays our selection rule (1·1) is equivalent to the conventional $|\Delta I|=1/2$ rule, while it is less restrictive for Σ- and Ξ-decays and is consistent with experiments. Hence we may conclude that the full symmetry among the basic particles p, n and Λ is well reflected not only in the strong interactions but in the phenomena induced by the weak interactions. The $|\Delta I|=1/2$ rule may be interpreted as a manifestation of such a fact. Of course, it is a question that all the characteristics of the weak interactions can be understood in the scheme of full symmetry. For instance, in our decay scheme, the $|\Delta I|=3/2$ or $5/2$ mixture cannot be treated so easily as the $|\Delta I|=1/2$ rule. We leave the deviation from the $|\Delta I|=1/2$ rule as an open question. Anyhow, one should bear in mind a possible breakdown of the full symmetry among the basic particles.

We have analysed in the last section various anomalous decay processes involving $\pi^{0\prime}$ and $\pi^{0\prime\prime}$. Of these decay modes only (4·18) can be forbidden by our selection rule (1·1). In particular, we must resort to the energy conservation law in order to forbid the processes in which one or more $\pi^{0\prime}$ appear in place of π^0. It is an interesting problem whether such a particle $\pi^{0\prime}$ will be discovered or not. If it does really exist, it will be a pseudo-scalar particle with $I=0$. We are looking forward to a further experimental study on this point.

We have considered that a physical system is represented by the totality of the configurations specified by the same quantum numbers $(n_B, s_0, i_0; S, I, I_3)$. In this respect we should notice a significant result which was derived from the mass formula of Sawada and Yonezawa. That is, all the particle states or the resonance levels observed in scattering experiments are associated with the lowest configurations, and there have never been observed the levels corresponding to higher configurations. This fact seems to indicate that a higher configuration does not correspond to any physical reality by itself. The situation is closely connected with the new particle picture which has been developed in a previous paper.[7] According to this view, a system is composed of a definite number of physical basic particles and antiparticles, and any higher configuration can automatically be included in the lowest one if we only regard the constituents of the latter as physical particles or antiparticles. Thus it is the energy level of such a physical system that has been predicted by the mass formula of Sawada and Yonezawa.[5] As was noted in the last section of ref. 7), the above problem is connected with the one whether $\pi^{0\prime\prime}$ is a particle like the existing mesons or it is a vacuum in the sense of our new particle picture. These problems require a further investigation.

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Appendix 1

Decomposition of $\delta\cdot\delta\cdot(\boldsymbol{O}K)$ into three-boson states

a) $I=1$ channel of K^+-decay

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^+)^{\kappa_3}_{\lambda_1}=\frac{1}{3}(\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+)+\frac{\sqrt{6}}{6}(\pi^+\pi^{0\prime\prime}\pi^0-\pi^0\pi^{0\prime\prime}\pi^+)$$
$$-\frac{\sqrt{2}}{6}(\pi^+\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^+)+\frac{\sqrt{3}}{3}\bar{K}^0\pi^{0\prime\prime}K^+$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_2}_{\lambda_2}=-\frac{1}{3}(\pi^+\pi^+\pi^-+\pi^-\pi^+\pi^+-\pi^0\pi^+\pi^0-\pi^{0\prime}\pi^+\pi^{0\prime}-\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}$$
$$-K^+\pi^+K^--K^-\pi^+K^++K^0\pi^+\bar{K}^0+\bar{K}^0\pi^+K^0)$$

$$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_1}_{\lambda_3}=\frac{1}{3}(\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+)-\frac{\sqrt{6}}{6}(\pi^+\pi^{0\prime\prime}\pi^0-\pi^0\pi^{0\prime\prime}\pi^+)$$
$$-\frac{\sqrt{2}}{6}(\pi^+\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^+)+\frac{\sqrt{3}}{3}K^+\pi^{0\prime\prime}\bar{K}^0$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}(\pi^+)^{\kappa_3}_{\lambda_1}=\frac{1}{6}(\pi^+\pi^0\pi^0-\pi^0\pi^+\pi^0+\pi^0\pi^0\pi^+)+\frac{1}{18}(\pi^+\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^+)$$
$$+\frac{1}{9}(\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+)-\frac{\sqrt{3}}{18}(\pi^+\pi^0\pi^{0\prime}+\pi^+\pi^{0\prime}\pi^0-\pi^0\pi^+\pi^{0\prime}$$
$$+\pi^{0\prime}\pi^+\pi^0-\pi^0\pi^{0\prime}\pi^+-\pi^{0\prime}\pi^0\pi^+)+\frac{\sqrt{6}}{18}(\pi^+\pi^0\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^0-\pi^0\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^0$$
$$-\pi^0\pi^{0\prime\prime}\pi^+-\pi^{0\prime\prime}\pi^0\pi^+)-\frac{\sqrt{2}}{18}(\pi^+\pi^{0\prime}\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^{0\prime}$$
$$+\pi^{0\prime}\pi^{0\prime\prime}\pi^++\pi^{0\prime\prime}\pi^{0\prime}\pi^+)-\frac{1}{3}(\pi^+\pi^-\pi^+-\pi^+K^-K^++\bar{K}^0K^0\pi^+)$$
$$-\frac{\sqrt{2}}{6}(\pi^0\bar{K}^0K^+-\bar{K}^0K^+\pi^0)+\frac{\sqrt{6}}{9}\bar{K}^0\pi^{0\prime}K^+-\frac{\sqrt{6}}{18}(\pi^{0\prime}\bar{K}^0K^++\bar{K}^0K^+\pi^{0\prime})$$
$$+\frac{\sqrt{3}}{9}(\pi^{0\prime\prime}\bar{K}^0K^++\bar{K}^0K^+\pi^{0\prime\prime}+\bar{K}^0\pi^{0\prime\prime}K^+)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_2}_{\lambda_2}=\frac{1}{6}(\pi^0\pi^0\pi^+-\pi^+\pi^0\pi^0+\pi^0\pi^+\pi^0)+\frac{1}{18}(\pi^{0\prime}\pi^{0\prime}\pi^++\pi^+\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime})$$

$$+\frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime\prime}\pi^{+}+\pi^{0\prime\prime}\pi^{+}\pi^{0\prime\prime\prime}+\pi^{+}\pi^{0\prime\prime}\pi^{0\prime\prime\prime})-\frac{\sqrt{3}}{18}(\pi^{0}\pi^{0\prime}\pi^{+}+\pi^{0\prime}\pi^{0}\pi^{+}-\pi^{+}\pi^{0\prime}\pi^{0}$$

$$+\pi^{+}\pi^{0}\pi^{0\prime}-\pi^{0\prime}\pi^{+}\pi^{0}-\pi^{0}\pi^{+}\pi^{0\prime})+\frac{\sqrt{6}}{18}(\pi^{0}\pi^{0\prime\prime}\pi^{+}+\pi^{0\prime\prime}\pi^{0}\pi^{+}-\pi^{+}\pi^{0\prime\prime}\pi^{0}+\pi^{+}\pi^{0}\pi^{0\prime\prime}$$

$$-\pi^{0\prime\prime}\pi^{+}\pi^{0}-\pi^{0}\pi^{+}\pi^{0\prime\prime})-\frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{+}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{+}+\pi^{+}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{+}\pi^{0\prime}\pi^{0\prime\prime}$$

$$+\pi^{0\prime\prime}\pi^{+}\pi^{0\prime}+\pi^{0\prime}\pi^{+}\pi^{0\prime\prime})-\frac{1}{3}(\pi^{-}\pi^{+}\pi^{+}-K^{-}K^{+}\pi^{+}+K^{0}\pi^{+}\bar{K}^{0})$$

$$-\frac{\sqrt{2}}{6}(\bar{K}^{0}K^{+}\pi^{0}-K^{+}\pi^{0}\bar{K}^{0})+\frac{\sqrt{6}}{9}\pi^{0\prime}K^{+}\bar{K}^{0}-\frac{\sqrt{6}}{18}(\bar{K}^{0}K^{+}\pi^{0\prime}+K^{+}\pi^{0\prime}\bar{K}^{0})$$

$$+\frac{\sqrt{3}}{9}(\bar{K}^{0}K^{+}\pi^{0\prime\prime}+K^{+}\pi^{0\prime\prime}\bar{K}^{0}+\pi^{0\prime\prime}K^{+}\bar{K}^{0})$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^{+})^{\kappa_1}_{\lambda_2}=\frac{1}{6}(\pi^{0}\pi^{+}\pi^{0}-\pi^{0}\pi^{0}\pi^{+}+\pi^{+}\pi^{0}\pi^{0})+\frac{1}{18}(\pi^{0\prime}\pi^{+}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^{+}+\pi^{+}\pi^{0\prime}\pi^{0\prime})$$

$$+\frac{1}{9}(\pi^{0\prime\prime}\pi^{+}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{+}+\pi^{+}\pi^{0\prime\prime}\pi^{0\prime\prime})-\frac{\sqrt{3}}{18}(\pi^{0\prime}\pi^{+}\pi^{0}+\pi^{0}\pi^{+}\pi^{0\prime}-\pi^{0\prime}\pi^{0}\pi^{+}$$

$$+\pi^{0}\pi^{0\prime}\pi^{+}-\pi^{+}\pi^{0}\pi^{0\prime}-\pi^{+}\pi^{0\prime}\pi^{0})+\frac{\sqrt{6}}{18}(\pi^{0\prime\prime}\pi^{+}\pi^{0}+\pi^{0}\pi^{+}\pi^{0\prime\prime}-\pi^{0\prime\prime}\pi^{0}\pi^{+}+\pi^{0}\pi^{0\prime\prime}\pi^{+}$$

$$-\pi^{+}\pi^{0}\pi^{0\prime\prime}-\pi^{+}\pi^{0\prime\prime}\pi^{0})-\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^{+}\pi^{0\prime}+\pi^{0\prime}\pi^{+}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{+}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{+}$$

$$+\pi^{+}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{+}\pi^{0\prime\prime}\pi^{0\prime})-\frac{1}{3}(\pi^{+}\pi^{-}\pi^{+}-K^{+}\pi^{+}K^{-}+\pi^{+}\bar{K}^{0}K^{0})$$

$$-\frac{\sqrt{2}}{6}(K^{+}\pi^{0}\bar{K}^{0}-\pi^{0}\bar{K}^{0}K^{+})+\frac{\sqrt{6}}{9}K^{+}\bar{K}^{0}\pi^{0\prime}-\frac{\sqrt{6}}{18}(K^{+}\pi^{0\prime}\bar{K}^{0}+\pi^{0\prime}\bar{K}^{0}K^{+})$$

$$+\frac{\sqrt{3}}{9}(K^{+}\pi^{0\prime\prime}\bar{K}^{0}+\pi^{0\prime\prime}\bar{K}^{0}K^{+}+K^{+}\bar{K}^{0}\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}(\pi^{+})^{\kappa_3}_{\lambda_2}=-\frac{1}{6}(\pi^{+}\pi^{0}\pi^{0}-\pi^{0}\pi^{+}\pi^{0}-\pi^{0}\pi^{0}\pi^{+})+\frac{1}{18}(\pi^{+}\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{+}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^{+})$$

$$+\frac{1}{9}(\pi^{+}\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{+}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{+})-\frac{\sqrt{3}}{18}(-\pi^{+}\pi^{0}\pi^{0\prime}+\pi^{+}\pi^{0\prime}\pi^{0}+\pi^{0}\pi^{+}\pi^{0\prime}$$

$$+\pi^{0\prime}\pi^{+}\pi^{0}-\pi^{0}\pi^{0\prime}\pi^{+}-\pi^{0\prime}\pi^{0}\pi^{+})-\frac{\sqrt{6}}{18}(\pi^{+}\pi^{0}\pi^{0\prime\prime}-\pi^{+}\pi^{0\prime\prime}\pi^{0}-\pi^{0}\pi^{+}\pi^{0\prime\prime}-\pi^{0\prime\prime}\pi^{+}\pi^{0}$$

$$+\pi^{0}\pi^{0\prime\prime}\pi^{+}+\pi^{0\prime\prime}\pi^{0}\pi^{+})-\frac{\sqrt{2}}{18}(\pi^{+}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{+}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{+}\pi^{0\prime}$$

$$+\pi^{0\prime}\pi^{0\prime\prime}\pi^{+}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{+})-\frac{1}{3}(\pi^{-}\pi^{+}\pi^{+}-K^{-}\pi^{+}K^{+}+K^{0}\bar{K}^{0}\pi^{+})$$

$$-\frac{\sqrt{2}}{6}(-K^{+}\bar{K}^{0}\pi^{0}+\bar{K}^{0}\pi^{0}K^{+})-\frac{\sqrt{6}}{18}(K^{+}\bar{K}^{0}\pi^{0\prime}+\bar{K}^{0}\pi^{0\prime}K^{+}-2\pi^{0\prime}\bar{K}^{0}K^{+})$$

$$+\frac{\sqrt{3}}{9}(K^+\bar{K}^0\pi^{0\prime\prime}+\bar{K}^0\pi^{0\prime\prime}K^++\pi^{0\prime\prime}\bar{K}^0K^+)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_2}_{\lambda_1}=-\frac{1}{6}(\pi^0\pi^0\pi^+-\pi^+\pi^0\pi^0-\pi^0\pi^+\pi^0)+\frac{1}{18}(\pi^{0\prime}\pi^{0\prime}\pi^++\pi^+\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime})$$

$$+\frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime})-\frac{\sqrt{3}}{18}(-\pi^0\pi^{0\prime}\pi^++\pi^{0\prime}\pi^0\pi^++\pi^0\pi^{0\prime}\pi^0$$

$$+\pi^+\pi^0\pi^{0\prime}-\pi^{0\prime}\pi^+\pi^0-\pi^0\pi^+\pi^{0\prime})-\frac{\sqrt{6}}{18}(\pi^0\pi^{0\prime\prime}\pi^+-\pi^{0\prime\prime}\pi^0\pi^+-\pi^+\pi^{0\prime\prime}\pi^0-\pi^+\pi^0\pi^{0\prime\prime}$$

$$+\pi^{0\prime\prime}\pi^+\pi^0+\pi^0\pi^+\pi^{0\prime\prime})-\frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^{0\prime\prime}\pi^++\pi^{0\prime\prime}\pi^{0\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^{0\prime}+\pi^+\pi^{0\prime}\pi^{0\prime\prime}$$

$$+\pi^{0\prime\prime}\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime\prime})-\frac{1}{3}(\pi^+\pi^+\pi^--\pi^+K^+K^-+\bar{K}^0\pi^+K^0)$$

$$-\frac{\sqrt{2}}{6}(-\bar{K}^0\pi^0K^++\pi^0K^+\bar{K}^0)-\frac{\sqrt{6}}{18}(\bar{K}^0\pi^{0\prime}K^++\pi^{0\prime}K^+\bar{K}^0-2\bar{K}^0K^+\pi^{0\prime})$$

$$+\frac{\sqrt{3}}{9}(\bar{K}^0\pi^{0\prime\prime}K^++\pi^{0\prime\prime}K^+\bar{K}^0+\bar{K}^0K^+\pi^{0\prime\prime})$$

$$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_1}_{\lambda_3}=-\frac{1}{6}(\pi^0\pi^+\pi^0-\pi^0\pi^0\pi^+-\pi^+\pi^0\pi^0)+\frac{1}{18}(\pi^{0\prime}\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^++\pi^+\pi^{0\prime}\pi^{0\prime})$$

$$+\frac{1}{9}(\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime})-\frac{\sqrt{3}}{18}(-\pi^{0\prime}\pi^+\pi^0+\pi^0\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^0\pi^+$$

$$+\pi^0\pi^{0\prime}\pi^+-\pi^+\pi^0\pi^{0\prime}-\pi^+\pi^{0\prime}\pi^0)-\frac{\sqrt{6}}{18}(\pi^{0\prime\prime}\pi^+\pi^0-\pi^0\pi^+\pi^{0\prime\prime}-\pi^{0\prime\prime}\pi^0\pi^+-\pi^0\pi^{0\prime\prime}\pi^+$$

$$+\pi^+\pi^0\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^0)-\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^++\pi^{0\prime}\pi^{0\prime\prime}\pi^+$$

$$+\pi^+\pi^{0\prime}\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^{0\prime})-\frac{1}{3}(\pi^+\pi^-\pi^+-K^+K^-\pi^++\pi^+K^0\bar{K}^0)$$

$$-\frac{\sqrt{2}}{6}(-\pi^0K^+\bar{K}^0+K^+\bar{K}^0\pi^0)-\frac{\sqrt{6}}{18}(\pi^{0\prime}K^+\bar{K}^0+K^+\bar{K}^0\pi^{0\prime}-2K^+\pi^{0\prime}\bar{K}^0)$$

$$+\frac{\sqrt{3}}{9}(\pi^{0\prime\prime}K^+\bar{K}^0+K^+\bar{K}^0\pi^{0\prime\prime}+K^+\pi^{0\prime\prime}\bar{K}^0)$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^+)^{\kappa_3}_{\lambda_3}=\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_2}_{\lambda_2}=\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}$$

$$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_1}_{\lambda_1}=\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^+)^{\kappa_3}_{\lambda_3}=\frac{1}{3}(\pi^0\pi^0\pi^++\pi^{0\prime}\pi^{0\prime}\pi^++\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+-\pi^+\pi^-\pi^+-\pi^-\pi^+\pi^++K^+K^-\pi^+$$

$$+K^-K^+\pi^+-K^0\bar{K}^0\pi^+-\bar{K}^0K^0\pi^+)$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_1}_{\lambda_1}=\frac{1}{3}(\pi^+\pi^0\pi^0+\pi^+\pi^{0\prime}\pi^{0\prime}+\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}-\pi^+\pi^+\pi^--\pi^+\pi^-\pi^++\pi^+K^+K^-$$

$$+\pi^+K^-K^+-\pi^+K^0\bar{K}^0-\pi^+\bar{K}^0K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_2}_{\lambda_1}=-\frac{\sqrt{6}}{6}(-\pi^+\pi^0\pi^{0\prime\prime}+\pi^0\pi^+\pi^{0\prime\prime})-\frac{\sqrt{2}}{6}(\pi^+\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^+\pi^{0\prime\prime})$$
$$+\frac{1}{3}(\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime})+\frac{\sqrt{3}}{3}\bar{K}^0K^+\pi^{0\prime\prime}$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_3}(\pi^+)^{\kappa_3}_{\lambda_2}=-\frac{\sqrt{6}}{6}(-\pi^{0\prime\prime}\pi^+\pi^0+\pi^{0\prime\prime}\pi^0\pi^+)-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^+\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^+)$$
$$+\frac{1}{3}(\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^+)+\frac{\sqrt{3}}{3}\pi^{0\prime\prime}\bar{K}^0K^+$$

$$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_1}_{\lambda_2}=-\frac{\sqrt{6}}{6}(-\pi^0\pi^+\pi^{0\prime\prime}+\pi^+\pi^0\pi^{0\prime\prime})-\frac{\sqrt{2}}{6}(\pi^{0\prime}\pi^+\pi^{0\prime\prime}+\pi^+\pi^{0\prime}\pi^{0\prime\prime})$$
$$+\frac{1}{3}(\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^{0\prime\prime})+\frac{\sqrt{3}}{3}K^+\bar{K}^0\pi^{0\prime\prime}$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_2}_{\lambda_3}=-\frac{\sqrt{6}}{6}(-\pi^{0\prime\prime}\pi^0\pi^++\pi^{0\prime\prime}\pi^+\pi^0)-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^{0\prime}\pi^++\pi^{0\prime\prime}\pi^+\pi^{0\prime})$$
$$+\frac{1}{3}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^++\pi^{0\prime\prime}\pi^+\pi^{0\prime\prime})+\frac{\sqrt{3}}{3}\pi^{0\prime\prime}K^+\bar{K}^0$$

b) $I=1$ channel of K_2^0-decay

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_3}_{\lambda_1}=-\frac{\sqrt{2}}{6}(\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^0)+\frac{1}{3}(\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0)$$
$$+\frac{\sqrt{6}}{6}(-\pi^-\pi^{0\prime\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^-+K^-\pi^{0\prime\prime}K^++\bar{K}^0\pi^{0\prime\prime}K^0)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_2}_{\lambda_2}=\frac{1}{3}(\pi^0\pi^0\pi^0+\pi^{0\prime}\pi^0\pi^{0\prime}+\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}-\pi^+\pi^0\pi^--\pi^-\pi^0\pi^++K^+\pi^0K^-$$
$$+K^-\pi^0K^+-K^0\pi^0\bar{K}^0-\bar{K}^0\pi^0K^0)$$

$$\delta^{\kappa_3}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_3}=-\frac{\sqrt{2}}{6}(\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^0)+\frac{1}{3}(\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0)$$
$$+\frac{\sqrt{6}}{6}(-\pi^+\pi^{0\prime\prime}\pi^-+\pi^-\pi^{0\prime\prime}\pi^++K^-\pi^{0\prime\prime}K^-+K^0\pi^{0\prime\prime}\bar{K}^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}(\pi^0)^{\kappa_3}_{\lambda_1}=\frac{1}{18}(\pi^{0\prime}\pi^{0\prime}\pi^0+\pi^{0\prime}\pi^0\pi^{0\prime}+\pi^0\pi^{0\prime}\pi^{0\prime})+\frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}+\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{18}(\pi^0\pi^{0\prime}\pi^{0\prime\prime}+\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^0)$$
$$+\frac{1}{6}\pi^0\pi^0\pi^0+\frac{1}{6}(-\pi^-\pi^+\pi^0-\pi^0\pi^-\pi^++\pi^-\pi^0\pi^+-\pi^+\pi^-\pi^0+\pi^+\pi^0\pi^--\pi^0\pi^+\pi^-$$
$$+K^-K^+\pi^0+\pi^0K^-K^+-\pi^0\bar{K}^0K^0-\bar{K}^0K^0\pi^0)+\frac{\sqrt{2}}{6}(-K^-K^0\pi^+-\pi^-\bar{K}^0K^+$$
$$+\bar{K}^0K^+\pi^-+\pi^+K^-K^0)+\frac{\sqrt{3}}{18}(\pi^-\pi^+\pi^{0\prime}+\pi^{0\prime}\pi^-\pi^++\pi^-\pi^{0\prime}\pi^+-\pi^+\pi^-\pi^{0\prime}$$

$$-\pi^+\pi^{0\prime}\pi^- - \pi^{0\prime}\pi^+\pi^- - K^-K^+\pi^{0\prime} - \pi^{0\prime}K^-K^+ - \pi^{0\prime}\bar{K}^0K^0 - \bar{K}^0K^0\pi^{0\prime}$$
$$+2K^-\pi^{0\prime}K^+ + 2\bar{K}^0\pi^{0\prime}K^0) + \frac{\sqrt{6}}{18}(-\pi^-\pi^+\pi^{0\prime\prime} - \pi^{0\prime\prime}\pi^-\pi^+ - \pi^-\pi^{0\prime\prime}\pi^+$$
$$+\pi^+\pi^-\pi^{0\prime\prime} + \pi^+\pi^{0\prime\prime}\pi^- + \pi^{0\prime\prime}\pi^+\pi^- + K^-K^+\pi^{0\prime\prime} + \pi^{0\prime\prime}K^-K^+ + \pi^{0\prime\prime}\bar{K}^0K^0$$
$$+\bar{K}^0K^0\pi^{0\prime\prime} + K^-\pi^{0\prime\prime}K^+ + \bar{K}^0\pi^{0\prime\prime}K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_2}_{\lambda_3} = \frac{1}{18}(\pi^{0\prime}\pi^0\pi^{0\prime} + \pi^0\pi^{0\prime}\pi^{0\prime} + \pi^{0\prime}\pi^{0\prime}\pi^0) + \frac{1}{9}(\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0 + \pi^0\pi^{0\prime\prime}\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^{0\prime\prime}\pi^0 + \pi^{0\prime}\pi^{0\prime}\pi^0 + \pi^0\pi^{0\prime\prime}\pi^{0\prime} + \pi^{0\prime\prime}\pi^0\pi^{0\prime} + \pi^0\pi^{0\prime}\pi^{0\prime\prime} + \pi^{0\prime}\pi^0\pi^{0\prime\prime})$$
$$+\frac{1}{6}\pi^0\pi^0\pi^0 + \frac{1}{6}(-\pi^+\pi^0\pi^- - \pi^-\pi^+\pi^0 + \pi^0\pi^+\pi^- - \pi^-\pi^0\pi^+ + \pi^0\pi^-\pi^+ - \pi^+\pi^-\pi^0$$
$$+K^+\pi^0K^- + K^-K^+\pi^0 - \bar{K}^0K^0\pi^0 - K^0\pi^0\bar{K}^0) + \frac{\sqrt{2}}{6}(-K^0\pi^+K^- - \bar{K}^0K^+\pi^-$$
$$+K^+\pi^-\bar{K}^0 + K^-K^0\pi^+) + \frac{\sqrt{3}}{18}(\pi^+\pi^{0\prime}\pi^- + \pi^-\pi^+\pi^{0\prime} + \pi^{0\prime}\pi^+\pi^- - \pi^-\pi^{0\prime}\pi^+$$
$$-\pi^{0\prime}\pi^-\pi^+ - \pi^+\pi^-\pi^{0\prime} - K^+\pi^{0\prime}K^- - K^-K^+\pi^{0\prime} - \bar{K}^0K^0\pi^{0\prime} - K^0\pi^{0\prime}\bar{K}^0$$
$$+2\pi^{0\prime}K^+K^- + 2\pi^{0\prime}K^0\bar{K}^0) + \frac{\sqrt{6}}{18}(-\pi^+\pi^{0\prime\prime}\pi^- - \pi^-\pi^+\pi^{0\prime\prime} - \pi^{0\prime\prime}\pi^+\pi^- + \pi^-\pi^{0\prime\prime}\pi^+$$
$$+\pi^{0\prime\prime}\pi^-\pi^+ + \pi^+\pi^-\pi^{0\prime\prime} + K^+\pi^{0\prime\prime}K^- + K^-K^+\pi^{0\prime\prime} + \bar{K}^0K^0\pi^{0\prime\prime} + K^0\pi^{0\prime\prime}\bar{K}^0$$
$$+\pi^{0\prime\prime}K^+K^- + \pi^{0\prime\prime}K^0\bar{K}^0)$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_1}_{\lambda_2} = \frac{1}{18}(\pi^{0\prime}\pi^{0\prime}\pi^0 + \pi^0\pi^{0\prime}\pi^{0\prime} + \pi^{0\prime}\pi^0\pi^{0\prime}) + \frac{1}{9}(\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0 + \pi^{0\prime\prime}\pi^0\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^0\pi^{0\prime} + \pi^{0\prime}\pi^0\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^{0\prime}\pi^0 + \pi^0\pi^{0\prime}\pi^{0\prime\prime} + \pi^{0\prime}\pi^{0\prime\prime}\pi^0 + \pi^0\pi^{0\prime\prime}\pi^{0\prime})$$
$$+\frac{1}{6}\pi^0\pi^0\pi^0 + \frac{1}{6}(-\pi^0\pi^-\pi^+ - \pi^+\pi^0\pi^- + \pi^+\pi^-\pi^0 - \pi^0\pi^+\pi^- + \pi^-\pi^+\pi^0 - \pi^-\pi^0\pi^+$$
$$+\pi^0K^-K^+ + K^+\pi^0K^- - K^0\pi^0\bar{K}^0 - \pi^0\bar{K}^0K^0) + \frac{\sqrt{2}}{6}(-\pi^+K^-K^0 - K^+\pi^-\bar{K}^0$$
$$+\pi^-\bar{K}^0K^+ + K^0\pi^+K^-) + \frac{\sqrt{3}}{18}(\pi^{0\prime}\pi^-\pi^+ + \pi^+\pi^{0\prime}\pi^- + \pi^+\pi^-\pi^{0\prime} - \pi^{0\prime}\pi^+\pi^-$$
$$-\pi^-\pi^+\pi^{0\prime} - \pi^-\pi^{0\prime}\pi^+ - \pi^{0\prime}K^-K^+ - K^+\pi^{0\prime}K^- - K^0\pi^{0\prime}\bar{K}^0 - \pi^{0\prime}\bar{K}^0K^0$$
$$+2K^+K^-\pi^{0\prime} + 2K^0\bar{K}^0\pi^{0\prime}) + \frac{\sqrt{6}}{18}(-\pi^{0\prime\prime}\pi^-\pi^+ - \pi^+\pi^{0\prime\prime}\pi^- - \pi^+\pi^-\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^+\pi^- + \pi^-\pi^+\pi^{0\prime\prime} + \pi^-\pi^{0\prime\prime}\pi^+ + \pi^{0\prime\prime}K^-K^+ + K^+\pi^{0\prime\prime}K^- + K^0\pi^{0\prime\prime}\bar{K}^0$$
$$+\pi^{0\prime\prime}\bar{K}^0K^0 + K^+K^-\pi^{0\prime\prime} + K^0\bar{K}^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_3}_{\lambda_2} = \frac{1}{18}(\pi^{0\prime}\pi^{0\prime}\pi^0 + \pi^0\pi^{0\prime}\pi^{0\prime} + \pi^{0\prime}\pi^0\pi^{0\prime}) + \frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0 + \pi^0\pi^{0\prime\prime}\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^0\pi^{0\prime\prime})$$

$$-\frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^0\pi^{0\prime}+\pi^0\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^0+\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^0)$$

$$+\frac{1}{6}\pi^0\pi^0\pi^0+\frac{1}{6}(-\pi^+\pi^-\pi^0-\pi^-\pi^0\pi^++\pi^0\pi^-\pi^+-\pi^-\pi^+\pi^0+\pi^0\pi^+\pi^--\pi^+\pi^0\pi^-$$

$$+K^+K^-\pi^0+K^-\pi^0K^+-\bar{K}^0\pi^0K^0-K^0\bar{K}^0\pi^0)+\frac{\sqrt{2}}{6}(-K^0K^-\pi^+-\bar{K}^0\pi^-K^+$$

$$+K^+\bar{K}^0\pi^-+K^-\pi^+K^0)+\frac{\sqrt{3}}{18}(\pi^+\pi^-\pi^{0\prime}+\pi^-\pi^{0\prime}\pi^++\pi^{0\prime}\pi^-\pi^+-\pi^-\pi^+\pi^{0\prime}$$

$$-\pi^{0\prime}\pi^+\pi^--\pi^+\pi^{0\prime}\pi^--K^+K^-\pi^{0\prime}-K^-\pi^{0\prime}K^+-\bar{K}^0\pi^{0\prime}K^0-K^0\bar{K}^0\pi^{0\prime}$$

$$+2\pi^{0\prime}K^-K^++2\pi^{0\prime}\bar{K}^0K^0)+\frac{\sqrt{6}}{18}(-\pi^+\pi^-\pi^{0\prime\prime}-\pi^-\pi^{0\prime\prime}\pi^+-\pi^{0\prime\prime}\pi^-\pi^+$$

$$+\pi^-\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^-+\pi^+\pi^{0\prime\prime}\pi^-+K^+K^-\pi^{0\prime\prime}+K^-\pi^{0\prime\prime}K^++\bar{K}^0\pi^{0\prime\prime}K^0$$

$$+K^0\bar{K}^0\pi^{0\prime\prime}+\pi^{0\prime\prime}K^-K^++\pi^{0\prime\prime}\bar{K}^0K^0)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_2}_{\lambda_1}=\frac{1}{18}(\pi^{0\prime}\pi^0\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^0+\pi^0\pi^{0\prime}\pi^{0\prime})+\frac{1}{9}(\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0+\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime})$$

$$-\frac{\sqrt{2}}{18}(\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^0\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^0+\pi^{0\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime})$$

$$+\frac{1}{6}\pi^0\pi^0\pi^0+\frac{1}{6}(-\pi^-\pi^0\pi^+-\pi^0\pi^+\pi^-+\pi^-\pi^+\pi^0-\pi^+\pi^0\pi^-+\pi^+\pi^-\pi^0-\pi^0\pi^-\pi^+$$

$$+K^-\pi^0K^++\pi^0K^+K^--\pi^0K^0\bar{K}^0-\bar{K}^0\pi^0K^0)+\frac{\sqrt{2}}{6}(-K^-\pi^+K^0-\pi^-K^+\bar{K}^0$$

$$+\bar{K}^0\pi^-K^++\pi^+K^0K^-)+\frac{\sqrt{3}}{18}(\pi^-\pi^{0\prime}\pi^++\pi^{0\prime}\pi^+\pi^-+\pi^-\pi^+\pi^{0\prime}-\pi^+\pi^{0\prime}\pi^-$$

$$-\pi^+\pi^-\pi^{0\prime}-\pi^{0\prime}\pi^-\pi^+-K^-\pi^{0\prime}K^+-\pi^{0\prime}K^+K^--\pi^{0\prime}K^0\bar{K}^0-\bar{K}^0\pi^{0\prime}K^0$$

$$+2K^-K^+\pi^{0\prime}+2\bar{K}^0K^0\pi^{0\prime})+\frac{\sqrt{6}}{18}(-\pi^-\pi^{0\prime\prime}\pi^+-\pi^{0\prime\prime}\pi^+\pi^--\pi^-\pi^+\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^-$$

$$+\pi^+\pi^-\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^-\pi^++K^-\pi^{0\prime\prime}K^++\pi^{0\prime\prime}K^+K^-+\pi^{0\prime\prime}K^0\bar{K}^0+\bar{K}^0\pi^{0\prime\prime}K^0$$

$$+K^-K^+\pi^{0\prime\prime}+\bar{K}^0K^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_3}=\frac{1}{18}(\pi^0\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^0\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^0)+\frac{1}{9}(\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0)$$

$$-\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^{0\prime}\pi^0+\pi^{0\prime}\pi^{0\prime\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime}+\pi^0\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^0\pi^{0\prime\prime}+\pi^0\pi^{0\prime}\pi^{0\prime\prime})$$

$$+\frac{1}{6}\pi^0\pi^0\pi^0+\frac{1}{6}(-\pi^0\pi^+\pi^--\pi^+\pi^-\pi^0+\pi^+\pi^0\pi^--\pi^0\pi^-\pi^++\pi^-\pi^0\pi^+-\pi^-\pi^+\pi^0$$

$$+\pi^0K^+K^-+K^+K^-\pi^0-K^0\bar{K}^0\pi^0-\pi^0K^0\bar{K}^0)+\frac{\sqrt{2}}{6}(-\pi^+K^0K^--K^+\bar{K}^0\pi^-$$

$$+\pi^-K^+\bar{K}^0+K^0K^-\pi^+)+\frac{\sqrt{3}}{18}(\pi^{0\prime}\pi^+\pi^-+\pi^+\pi^-\pi^{0\prime}+\pi^+\pi^{0\prime}\pi^--\pi^{0\prime}\pi^-\pi^+$$

$$-\pi^-\pi^{0\prime}\pi^+-\pi^-\pi^+\pi^{0\prime}-\pi^{0\prime}K^+K^--K^+K^-\pi^{0\prime}-K^0\bar{K}^0\pi^{0\prime}-\pi^{0\prime}K^0\bar{K}^0$$
$$+2K^+\pi^{0\prime}K^-+2K^0\pi^{0\prime}\bar{K}^0)+\frac{\sqrt{6}}{18}(-\pi^{0\prime\prime}\pi^+\pi^--\pi^+\pi^-\pi^{0\prime\prime}-\pi^+\pi^{0\prime\prime}\pi^-$$
$$+\pi^{0\prime\prime}\pi^-\pi^++\pi^-\pi^{0\prime\prime}\pi^++\pi^-\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}K^+K^-+K^+K^-\pi^{0\prime\prime}+K^0\bar{K}^0\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}K^0\bar{K}^0+K^+\pi^{0\prime\prime}K^-+K^0\pi^{0\prime\prime}\bar{K}^0)$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_3}_{\lambda_3}=\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_2}_{\lambda_2}=\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}$$

$$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_1}_{\lambda_1}=\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^0)^{\kappa_3}_{\lambda_3}=-\frac{1}{3}(-\pi^0\pi^0\pi^0-\pi^{0\prime}\pi^{0\prime}\pi^0-\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0+\pi^+\pi^-\pi^0+\pi^-\pi^+\pi^0-K^+K^-\pi^0$$
$$-K^-K^+\pi^0+K^0\bar{K}^0\pi^0+\bar{K}^0K^0\pi^0)$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_1}=-\frac{1}{3}(-\pi^0\pi^0\pi^0-\pi^0\pi^{0\prime}\pi^{0\prime}-\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^0\pi^+\pi^-+\pi^0\pi^-\pi^+-\pi^0K^+K^-$$
$$-\pi^0K^-K^++\pi^0K^0\bar{K}^0+\pi^0\bar{K}^0K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_2}_{\lambda_1}=-\frac{\sqrt{2}}{6}(\pi^0\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^0\pi^{0\prime\prime})+\frac{1}{3}(\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime})$$
$$+\frac{\sqrt{6}}{6}(-\pi^-\pi^+\pi^{0\prime\prime}+\pi^+\pi^-\pi^{0\prime\prime}+K^-K^+\pi^{0\prime\prime}+\bar{K}^0K^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_3}_{\lambda_2}=-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^0\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^0)+\frac{1}{3}(\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0)$$
$$+\frac{\sqrt{6}}{6}(-\pi^{0\prime\prime}\pi^-\pi^++\pi^{0\prime\prime}\pi^+\pi^-+\pi^{0\prime\prime}K^-K^++\pi^{0\prime\prime}\bar{K}^0K^0)$$

$$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_1}_{\lambda_2}=-\frac{\sqrt{2}}{6}(\pi^{0\prime}\pi^0\pi^{0\prime\prime}+\pi^0\pi^{0\prime}\pi^{0\prime\prime})+\frac{1}{3}(\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime}+\pi^0\pi^{0\prime\prime}\pi^{0\prime\prime})$$
$$+\frac{\sqrt{6}}{6}(-\pi^+\pi^-\pi^{0\prime\prime}+\pi^-\pi^+\pi^{0\prime\prime}+K^+K^-\pi^{0\prime\prime}+K^0\bar{K}^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_2}_{\lambda_3}=-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^{0\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime})+\frac{1}{3}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^{0\prime\prime})$$
$$+\frac{\sqrt{6}}{6}(-\pi^{0\prime\prime}\pi^+\pi^-+\pi^{0\prime\prime}\pi^-\pi^++\pi^{0\prime\prime}K^+K^-+\pi^{0\prime\prime}K^0\bar{K}^0)$$

c) $I=0$ channel of K_2^0-decay

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_3}_{\lambda_1}=\frac{\sqrt{2}}{6}(-\pi^0\pi^{0\prime\prime}\pi^0+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime})+\frac{1}{3}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime})$$
$$+\frac{\sqrt{2}}{6}(\pi^-\pi^{0\prime\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^--K^-\pi^{0\prime\prime}K^++\bar{K}^0\pi^{0\prime\prime}K^0$$
$$+2K^+\pi^{0\prime\prime}K^--2K^0\pi^{0\prime\prime}\bar{K}^0)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_2}_{\lambda_2}=\frac{1}{3}\,(\pi^0\pi^{0\prime}\pi^0+\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}-\pi^+\pi^{0\prime}\pi^--\pi^-\pi^{0\prime}\pi^++K^+\pi^{0\prime}K^-$$
$$+K^-\pi^{0\prime}K^+-K^0\pi^{0\prime}\bar{K}^0-\bar{K}^0\pi^{0\prime}K^0)$$

$$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_1}_{\lambda_3}=\frac{\sqrt{2}}{6}\,(-\pi^0\pi^{0\prime\prime}\pi^0+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime})+\frac{1}{3}\,(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime})$$
$$+\frac{\sqrt{2}}{6}\,(\pi^+\pi^{0\prime\prime}\pi^-+\pi^-\pi^{0\prime\prime}\pi^+-K^+\pi^{0\prime\prime}K^-+K^0\pi^{0\prime\prime}\bar{K}^0$$
$$+2K^-\pi^{0\prime\prime}K^+-2\bar{K}^0\pi^{0\prime\prime}K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}(\pi^{0\prime})^{\kappa_3}_{\lambda_1}=\frac{1}{6}\,\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}+\frac{1}{18}\,(\pi^0\pi^0\pi^{0\prime}+\pi^0\pi^{0\prime}\pi^0+\pi^{0\prime}\pi^0\pi^0)-\frac{\sqrt{2}}{18}\,(\pi^0\pi^0\pi^{0\prime\prime}+\pi^0\pi^{0\prime\prime}\pi^0$$
$$+\pi^{0\prime\prime}\pi^0\pi^0)+\frac{\sqrt{2}}{18}\,(\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime})+\frac{1}{9}\,(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}$$
$$+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime})+\frac{\sqrt{3}}{18}\,(\pi^0\pi^-\pi^++\pi^-\pi^+\pi^0+\pi^+\pi^0\pi^--\pi^-\pi^0\pi^+$$
$$-\pi^0\pi^+\pi^--\pi^+\pi^-\pi^0-K^-K^+\pi^0-\pi^0K^-K^+-\bar{K}^0K^0\pi^0-\pi^0\bar{K}^0K^0+2K^+\pi^0K^-$$
$$+2K^0\pi^0\bar{K}^0)+\frac{\sqrt{6}}{18}\,(K^-K^0\pi^++\pi^-\bar{K}^0K^++\bar{K}^0K^+\pi^-+\pi^+K^-K^0$$
$$-2K^0\pi^+K^--2K^+\pi^-\bar{K}^0)+\frac{1}{18}\,(-\pi^{0\prime}\pi^-\pi^+-\pi^-\pi^+\pi^{0\prime}-\pi^+\pi^{0\prime}\pi^--\pi^-\pi^{0\prime}\pi^+$$
$$-\pi^{0\prime}\pi^+\pi^--\pi^+\pi^-\pi^{0\prime}+K^-K^+\pi^{0\prime}+\pi^{0\prime}K^-K^+-\bar{K}^0K^0\pi^{0\prime}-\pi^{0\prime}\bar{K}^0K^0$$
$$-2K^+\pi^{0\prime}K^-+2K^0\pi^{0\prime}\bar{K}^0-4\pi^{0\prime}K^0\bar{K}^0+4\pi^{0\prime}K^+K^-+4K^+K^-\pi^{0\prime}-4K^0\bar{K}^0\pi^{0\prime}$$
$$-2K^-\pi^{0\prime}K^++2\bar{K}^0\pi^{0\prime}K^0)+\frac{\sqrt{2}}{18}\,(\pi^{0\prime\prime}\pi^-\pi^++\pi^-\pi^+\pi^{0\prime\prime}+\pi^+\pi^{0\prime\prime}\pi^-+\pi^-\pi^{0\prime\prime}\pi^+$$
$$+\pi^{0\prime\prime}\pi^+\pi^-+\pi^+\pi^-\pi^{0\prime\prime}-K^-K^+\pi^{0\prime\prime}-\pi^{0\prime\prime}K^-K^++\bar{K}^0K^0\pi^{0\prime\prime}+\pi^{0\prime\prime}\bar{K}^0K^0$$
$$+2K^+\pi^{0\prime\prime}K^--2K^0\pi^{0\prime\prime}\bar{K}^0-2\pi^{0\prime\prime}K^0\bar{K}^0+2\pi^{0\prime\prime}K^+K^-+2K^+K^-\pi^{0\prime\prime}$$
$$-2K^0\bar{K}^0\pi^{0\prime\prime}-K^-\pi^{0\prime\prime}K^++\bar{K}^0\pi^{0\prime\prime}K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}=\frac{1}{6}\,\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}+\frac{1}{18}\,(\pi^0\pi^{0\prime}\pi^0+\pi^{0\prime}\pi^0\pi^0+\pi^0\pi^0\pi^{0\prime})-\frac{\sqrt{2}}{18}\,(\pi^0\pi^{0\prime\prime}\pi^0+\pi^{0\prime\prime}\pi^0\pi^0$$
$$+\pi^0\pi^0\pi^{0\prime\prime})+\frac{\sqrt{2}}{18}\,(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime})+\frac{1}{9}\,(\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime})+\frac{\sqrt{3}}{18}\,(\pi^-\pi^+\pi^0+\pi^+\pi^0\pi^-+\pi^0\pi^-\pi^+-\pi^0\pi^+\pi^--\pi^+\pi^-\pi^0-\pi^-\pi^0\pi^+$$
$$-K^+\pi^0K^--K^-K^+\pi^0-K^0\pi^0\bar{K}^0-\bar{K}^0K^0\pi^0+2\pi^0K^-K^++2\pi^0\bar{K}^0K^0)$$
$$+\frac{\sqrt{6}}{18}\,(K^0\pi^+K^-+\bar{K}^0K^+\pi^-+K^+\pi^-\bar{K}^0+K^-K^0\pi^+-2\pi^+K^-K^0$$
$$-2\pi^-\bar{K}^0K^+)+\frac{1}{18}\,(-\pi^-\pi^+\pi^{0\prime}-\pi^+\pi^{0\prime}\pi^--\pi^{0\prime}\pi^-\pi^+-\pi^{0\prime}\pi^+\pi^--\pi^+\pi^-\pi^{0\prime}$$

$$-\pi^-\pi^{0\prime}\pi^+ + K^+\pi^{0\prime}K^- + K^-K^+\pi^{0\prime} - K^0\pi^{0\prime}\bar{K}^0 - \bar{K}^0K^0\pi^{0\prime} - 2\pi^{0\prime}K^-K^+$$
$$+2\pi^{0\prime}\bar{K}^0K^0 - 4K^0\bar{K}^0\pi^{0\prime} + 4K^+K^-\pi^{0\prime} + 4K^-\pi^{0\prime}K^+ - 4\bar{K}^0\pi^{0\prime}K^0 - 2\pi^{0\prime}K^+K^-$$
$$+2\pi^{0\prime}K^0\bar{K}^0) + \frac{\sqrt{2}}{18}(\pi^-\pi^+\pi^{0\prime\prime} + \pi^+\pi^{0\prime\prime}\pi^- + \pi^{0\prime\prime}\pi^-\pi^+ + \pi^{0\prime\prime}\pi^+\pi^- + \pi^+\pi^-\pi^{0\prime\prime}$$
$$+\pi^-\pi^{0\prime\prime}\pi^+ - K^+\pi^{0\prime\prime}K^- - K^-K^+\pi^{0\prime\prime} + K^0\pi^{0\prime\prime}\bar{K}^0 + \bar{K}^0K^0\pi^{0\prime\prime} + 2\pi^{0\prime\prime}K^-K^+$$
$$-2\pi^{0\prime\prime}\bar{K}^0K^0 - 2K^0\bar{K}^0\pi^{0\prime\prime} + 2K^+K^-\pi^{0\prime\prime} + 2K^-\pi^{0\prime\prime}K^+ - 2\bar{K}^0\pi^{0\prime\prime}K^0$$
$$-\pi^{0\prime\prime}K^+K^- + \pi^{0\prime\prime}K^0\bar{K}^0)$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_1}_{\lambda_2} = \frac{1}{6}\pi^{0\prime}\pi^{0\prime}\pi^{0\prime} + \frac{1}{18}(\pi^{0\prime}\pi^0\pi^0 + \pi^0\pi^{0\prime}\pi^0 + \pi^0\pi^0\pi^{0\prime}) - \frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^0\pi^0 + \pi^0\pi^0\pi^{0\prime\prime}$$
$$+\pi^0\pi^{0\prime\prime}\pi^0) + \frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime} + \pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime} + \pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}) + \frac{1}{9}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime} + \pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}) + \frac{\sqrt{3}}{18}(\pi^+\pi^0\pi^- + \pi^0\pi^-\pi^+ + \pi^-\pi^+\pi^0 - \pi^+\pi^-\pi^0 - \pi^-\pi^0\pi^+$$
$$-\pi^0\pi^+\pi^- - \pi^0K^-K^+ - K^+\pi^0K^- - \pi^0\bar{K}^0K^0 - K^0\pi^0\bar{K}^0 + 2K^-K^+\pi^0$$
$$+2\bar{K}^0K^0\pi^0) + \frac{\sqrt{6}}{18}(\pi^+K^-K^0 + K^+\pi^-\bar{K}^0 + \pi^-\bar{K}^0K^+ + K^0\pi^+K^-$$
$$-2K^-K^0\pi^+ - 2\bar{K}^0K^+\pi^-) + \frac{1}{18}(-\pi^+\pi^{0\prime}\pi^- - \pi^{0\prime}\pi^-\pi^+ - \pi^-\pi^+\pi^{0\prime} - \pi^+\pi^-\pi^{0\prime}$$
$$-\pi^-\pi^{0\prime}\pi^+ - \pi^{0\prime}\pi^+\pi^- + \pi^{0\prime}K^-K^+ + K^+\pi^{0\prime}K^- - \pi^{0\prime}\bar{K}^0K^0 - K^0\pi^{0\prime}\bar{K}^0$$
$$-2K^-K^+\pi^{0\prime} + 2\bar{K}^0K^0\pi^{0\prime} - 4\bar{K}^0\pi^{0\prime}K^0 + 4K^-\pi^{0\prime}K^+ + 4\pi^{0\prime}K^+K^- - 4\pi^{0\prime}K^0\bar{K}^0$$
$$-2K^+K^-\pi^{0\prime} + 2K^0\bar{K}^0\pi^{0\prime}) + \frac{\sqrt{2}}{18}(\pi^+\pi^{0\prime\prime}\pi^- + \pi^{0\prime\prime}\pi^-\pi^+ + \pi^-\pi^+\pi^{0\prime\prime} + \pi^+\pi^-\pi^{0\prime\prime}$$
$$+\pi^-\pi^{0\prime\prime}\pi^+ + \pi^{0\prime\prime}\pi^+\pi^- - \pi^{0\prime\prime}K^-K^+ - K^+\pi^{0\prime\prime}K^- + \pi^{0\prime\prime}\bar{K}^0K^0 + K^0\pi^{0\prime\prime}\bar{K}^0$$
$$+2K^-K^+\pi^{0\prime\prime} - 2\bar{K}^0K^0\pi^{0\prime\prime} - 2\bar{K}^0\pi^{0\prime\prime}K^0 + 2K^-\pi^{0\prime\prime}K^+ + 2\pi^{0\prime\prime}K^+K^-$$
$$-2\pi^{0\prime\prime}K^0\bar{K}^0 - K^+K^-\pi^{0\prime\prime} + K^0\bar{K}^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}(\pi^{0\prime})^{\kappa_3}_{\lambda_2} = \frac{1}{6}\pi^{0\prime}\pi^{0\prime}\pi^{0\prime} + \frac{1}{18}(\pi^0\pi^0\pi^{0\prime} + \pi^{0\prime}\pi^0\pi^0 + \pi^0\pi^{0\prime}\pi^0) - \frac{\sqrt{2}}{18}(\pi^0\pi^{0\prime\prime}\pi^0 + \pi^0\pi^0\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^0\pi^0) + \frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime} + \pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}) + \frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}$$
$$+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime} + \pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}) + \frac{\sqrt{3}}{18}(\pi^-\pi^0\pi^+ + \pi^+\pi^-\pi^0 + \pi^0\pi^+\pi^- - \pi^0\pi^-\pi^+ - \pi^+\pi^0\pi^-$$
$$-\pi^-\pi^+\pi^0 - K^+K^-\pi^0 - K^-\pi^0K^+ - K^0\bar{K}^0\pi^0 - \bar{K}^0\pi^0K^0 + 2\pi^0K^+K^-$$
$$+2\pi^0K^0\bar{K}^0) + \frac{\sqrt{6}}{18}(K^0K^-\pi^+ + \bar{K}^0\pi^-K^+ + K^+\bar{K}^0\pi^- + K^-\pi^+K^0 - 2\pi^+K^0K^-$$
$$-2\pi^-K^+\bar{K}^0) + \frac{1}{18}(-\pi^-\pi^{0\prime}\pi^+ - \pi^+\pi^-\pi^{0\prime} - \pi^{0\prime}\pi^+\pi^- - \pi^{0\prime}\pi^-\pi^+ - \pi^+\pi^{0\prime}\pi^-$$

$$-\pi^-\pi^+\pi^{0\prime}+K^+K^-\pi^{0\prime}+K^-\pi^{0\prime}K^+-K^0\bar{K}^0\pi^{0\prime}-\bar{K}^0\pi^{0\prime}K^0-2\pi^{0\prime}K^+K^-$$
$$+2\pi^{0\prime}K^0\bar{K}^0-4K^0\pi^{0\prime}\bar{K}^0+4K^+\pi^{0\prime}K^-+4K^-K^+\pi^{0\prime}-4\bar{K}^0K^0\pi^{0\prime}-2\pi^{0\prime}K^-K^+$$
$$+2\pi^{0\prime}\bar{K}^0K^0)+\frac{\sqrt{2}}{18}(\pi^-\pi^{0\prime\prime}\pi^++\pi^+\pi^-\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^-+\pi^{0\prime\prime}\pi^-\pi^++\pi^+\pi^{0\prime\prime}\pi^-$$
$$+\pi^-\pi^+\pi^{0\prime\prime}-K^+K^-\pi^{0\prime\prime}-K^-\pi^{0\prime\prime}K^++\bar{K}^0\pi^{0\prime\prime}K^0+K^0\bar{K}^0\pi^{0\prime\prime}+2\pi^{0\prime\prime}K^+K^-$$
$$-2\pi^{0\prime\prime}K^0\bar{K}^0-2K^0\pi^{0\prime\prime}\bar{K}^0+2K^+\pi^{0\prime\prime}K^-+2K^-K^+\pi^{0\prime\prime}-2\bar{K}^0K^0\pi^{0\prime\prime}$$
$$-\pi^{0\prime\prime}K^-K^++\pi^{0\prime\prime}\bar{K}^0K^0)$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_2}_{\lambda_1}=\frac{1}{6}\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}+\frac{1}{18}(\pi^0\pi^{0\prime}\pi^0+\pi^0\pi^0\pi^{0\prime}+\pi^{0\prime}\pi^0\pi^0)-\frac{\sqrt{2}}{18}(\pi^0\pi^{0\prime\prime}\pi^0+\pi^0\pi^0\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^0\pi^0)+\frac{\sqrt{2}}{18}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime})+\frac{1}{9}(\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}$$
$$+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime})+\frac{\sqrt{3}}{18}(\pi^-\pi^0\pi^++\pi^+\pi^-\pi^0+\pi^0\pi^+\pi^--\pi^0\pi^-\pi^+-\pi^+\pi^0\pi^-$$
$$-\pi^-\pi^+\pi^0-K^-\pi^0K^+-\pi^0K^+K^--\bar{K}^0\pi^0K^0-\pi^0K^0\bar{K}^0+2K^+K^-\pi^0$$
$$+2K^0\bar{K}^0\pi^0)+\frac{\sqrt{6}}{18}(K^-\pi^+K^0+\pi^-K^+K^0+K^0\pi^-K^++\pi^+K^0K^-$$
$$-2K^0K^-\pi^+-2K^+\bar{K}^0\pi^-)+\frac{1}{18}(-\pi^{0\prime}\pi^+\pi^--\pi^-\pi^{0\prime}\pi^+-\pi^+\pi^-\pi^{0\prime}-\pi^-\pi^+\pi^{0\prime}$$
$$-\pi^{0\prime}\pi^-\pi^+-\pi^+\pi^{0\prime}\pi^-+K^-\pi^{0\prime}K^++\pi^{0\prime}K^+K^--\bar{K}^0\pi^{0\prime}K^0-\pi^{0\prime}K^0\bar{K}^0$$
$$-2K^+K^-\pi^{0\prime}+2K^0\bar{K}^0\pi^{0\prime}-4\pi^{0\prime}\bar{K}^0K^0+4\pi^{0\prime}K^-K^++4K^+\pi^{0\prime}K^--4K^0\pi^{0\prime}\bar{K}^0$$
$$-2K^-K^+\pi^{0\prime}+2\bar{K}^0K^0\pi^{0\prime})+\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^+\pi^-+\pi^-\pi^{0\prime\prime}\pi^++\pi^+\pi^-\pi^{0\prime\prime}+\pi^-\pi^+\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^-\pi^++\pi^+\pi^{0\prime\prime}\pi^--K^-\pi^{0\prime\prime}K^+-\pi^{0\prime\prime}K^+K^-+\bar{K}^0\pi^{0\prime\prime}K^0+\pi^{0\prime\prime}K^0\bar{K}^0$$
$$+2K^+K^-\pi^{0\prime\prime}-2K^0\bar{K}^0\pi^{0\prime\prime}-2\pi^{0\prime\prime}\bar{K}^0K^0+2\pi^{0\prime\prime}K^-K^++2K^+\pi^{0\prime\prime}K^-$$
$$-2K^0\pi^{0\prime\prime}\bar{K}^0-K^-K^+\pi^{0\prime\prime}+\bar{K}^0K^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_3}_{\lambda_1}\delta^{\kappa_1}_{\lambda_2}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}=\frac{1}{6}\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}+\frac{1}{18}(\pi^{0\prime}\pi^0\pi^0+\pi^0\pi^{0\prime}\pi^0+\pi^0\pi^0\pi^{0\prime})-\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^0\pi^0+\pi^0\pi^{0\prime\prime}\pi^0$$
$$+\pi^0\pi^0\pi^{0\prime\prime})+\frac{\sqrt{2}}{18}(\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime})+\frac{1}{9}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}$$
$$+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime})+\frac{\sqrt{3}}{18}(\pi^+\pi^-\pi^0+\pi^0\pi^+\pi^-+\pi^-\pi^0\pi^+-\pi^+\pi^0\pi^--\pi^-\pi^+\pi^0$$
$$-\pi^0\pi^-\pi^+-\pi^0K^+K^--K^+K^-\pi^0-\pi^0K^0\bar{K}^0-K^0\bar{K}^0\pi^0+2K^-\pi^0K^+$$
$$+2\bar{K}^0\pi^0K^0)+\frac{\sqrt{6}}{18}(\pi^+K^0K^-+K^+\bar{K}^0\pi^-+\pi^-K^+\bar{K}^0+K^0K^-\pi^+$$
$$-2K^-\pi^+K^0-2\bar{K}^0\pi^-K^+)+\frac{1}{18}(-\pi^+\pi^-\pi^{0\prime}-\pi^{0\prime}\pi^+\pi^--\pi^-\pi^{0\prime}\pi^+-\pi^+\pi^{0\prime}\pi^-$$

$$-\pi^-\pi^+\pi^{0\prime}-\pi^{0\prime}\pi^-\pi^++\pi^{0\prime}K^+K^-+K^+K^-\pi^{0\prime}-\pi^{0\prime}K^0\bar{K}^0-K^0\bar{K}^0\pi^{0\prime}$$
$$-2K^-\pi^{0\prime}K^++2\bar{K}^0\pi^{0\prime}K^0-4\bar{K}^0K^0\pi^{0\prime}+4K^-K^+\pi^{0\prime}+4\pi^{0\prime}K^-K^+-4\pi^{0\prime}\bar{K}^0K^0$$
$$-2K^+\pi^{0\prime}K^-+2K^0\pi^{0\prime}\bar{K}^0)+\frac{\sqrt{2}}{18}(\pi^+\pi^-\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^+\pi^-+\pi^-\pi^{0\prime\prime}\pi^++\pi^+\pi^{0\prime\prime}\pi^-$$
$$+\pi^-\pi^+\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^-\pi^+-\pi^{0\prime\prime}K^+K^--K^+K^-\pi^{0\prime\prime}+\pi^{0\prime\prime}K^0\bar{K}^0+K^0\bar{K}^0\pi^{0\prime\prime}$$
$$+2K^-\pi^{0\prime\prime}K^+-2\bar{K}^0\pi^{0\prime\prime}K^0-2\bar{K}^0K^0\pi^{0\prime\prime}+2K^-K^+\pi^{0\prime\prime}+2\pi^{0\prime\prime}K^-K^+$$
$$-2\pi^{0\prime\prime}\bar{K}^0K^0-K^+\pi^{0\prime\prime}K^-+K^0\pi^{0\prime\prime}\bar{K}^0)$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^{0\prime})^{\kappa_3}_{\lambda_3}=\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_2}_{\lambda_2}=\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}$$

$$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_1}_{\lambda_1}=\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^{0\prime})^{\kappa_3}_{\lambda_3}=-\frac{1}{3}(-\pi^0\pi^0\pi^{0\prime}-\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}-\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime}+\pi^+\pi^-\pi^{0\prime}+\pi^-\pi^+\pi^{0\prime}-K^+K^-\pi^{0\prime}$$
$$-K^-K^+\pi^{0\prime}+K^0\bar{K}^0\pi^{0\prime}+\bar{K}^0K^0\pi^{0\prime})$$

$$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_1}_{\lambda_1}=-\frac{1}{3}(-\pi^{0\prime}\pi^0\pi^0-\pi^{0\prime}\pi^{0\prime}\pi^{0\prime}-\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^+\pi^-+\pi^{0\prime}\pi^-\pi^+-\pi^{0\prime}K^+K^-$$
$$-\pi^{0\prime}K^-K^++\pi^{0\prime}K^0\bar{K}^0+\pi^{0\prime}\bar{K}^0K^0)$$

$$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}=-\frac{\sqrt{2}}{6}(\pi^0\pi^0\pi^{0\prime\prime}-\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime})+\frac{1}{3}(\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{6}(-\pi^+\pi^-\pi^{0\prime\prime}-\pi^-\pi^+\pi^{0\prime\prime}+K^-K^+\pi^{0\prime\prime}-\bar{K}^0K^0\pi^{0\prime\prime}-2K^+K^-\pi^{0\prime\prime}$$
$$+2K^0\bar{K}^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^{0\prime})^{\kappa_3}_{\lambda_1}=-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^0\pi^0-\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime})+\frac{1}{3}(\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime\prime}\pi^{0\prime\prime}\pi^{0\prime})$$
$$-\frac{\sqrt{2}}{6}(-\pi^{0\prime\prime}\pi^+\pi^--\pi^{0\prime\prime}\pi^-\pi^++\pi^{0\prime\prime}K^-K^+-\pi^{0\prime\prime}\bar{K}^0K^0-2\pi^{0\prime\prime}K^+K^-$$
$$+2\pi^{0\prime\prime}K^0\bar{K}^0)$$

$$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_1}_{\lambda_2}=-\frac{\sqrt{2}}{6}(\pi^0\pi^0\pi^{0\prime\prime}-\pi^{0\prime}\pi^{0\prime}\pi^{0\prime\prime})+\frac{1}{3}(\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime}+\pi^{0\prime}\pi^{0\prime\prime}\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{6}(-\pi^-\pi^+\pi^{0\prime\prime}-\pi^+\pi^-\pi^{0\prime\prime}+K^+K^-\pi^{0\prime\prime}-K^0\bar{K}^0\pi^{0\prime\prime}-2K^-K^+\pi^{0\prime\prime}$$
$$+2\bar{K}^0K^0\pi^{0\prime\prime})$$

$$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}=-\frac{\sqrt{2}}{6}(\pi^{0\prime\prime}\pi^0\pi^0-\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime})+\frac{1}{3}(\pi^{0\prime\prime}\pi^{\prime 0}\pi^{0\prime}+\pi^{0\prime\prime}\pi^{0\prime}\pi^{0\prime\prime})$$
$$-\frac{\sqrt{2}}{6}(-\pi^{0\prime\prime}\pi^-\pi^+-\pi^{0\prime\prime}\pi^+\pi^-+\pi^{0\prime\prime}K^+K^--\pi^{0\prime\prime}K^0\bar{K}^0-2\pi^{0\prime\prime}K^-K^+$$
$$+2\pi^{0\prime\prime}\bar{K}^0K^0)$$

Appendix 2·1

$K^+ \to 3\pi$ decay, $I=1$ channel

final 3π / $I=1$ $\delta\delta\pi^+$	$\pi^+\pi^+\pi^-$	$\pi^+\pi^-\pi^+$	$\pi^-\pi^+\pi^+$	$\pi^+\pi^0\pi^0$	$\pi^0\pi^+\pi^0$	$\pi^0\pi^0\pi^+$	Amplitude
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^+)^{\kappa_3}_{\lambda_1}$	0	0	0	0	0	0	α_1
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_2}_{\lambda_2}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	α_2
$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_1}_{\lambda_3}$	0	0	0	0	0	0	α_3
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}(\pi^+)^{\kappa_3}_{\lambda_1}$	0	$-\frac{1}{3}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_4
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_2}_{\lambda_3}$	0	0	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	α_5
$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^+)^{\kappa_1}_{\lambda_2}$	$-\frac{1}{3}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	α_6
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}(\pi^+)^{\kappa_3}_{\lambda_2}$	0	0	$-\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	α_7
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_2}_{\lambda_1}$	$-\frac{1}{3}$	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	α_8
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_1}_{\lambda_3}$	0	$-\frac{1}{3}$	0	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_9
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^+)^{\kappa_3}_{\lambda_3}$	0	0	0	0	0	0	α_{10}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_2}_{\lambda_2}$	0	0	0	0	0	0	α_{11}
$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_1}_{\lambda_1}$	0	0	0	0	0	0	α_{12}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^+)^{\kappa_3}_{\lambda_3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	α_{13}
$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_1}_{\lambda_1}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	0	α_{14}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^+)^{\kappa_2}_{\lambda_1}$	0	0	0	0	0	0	α_{15}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_3}(\pi^+)^{\kappa_3}_{\lambda_2}$	0	0	0	0	0	0	α_{16}
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_1}_{\lambda_2}$	0	0	0	0	0	0	α_{17}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^+)^{\kappa_2}_{\lambda_3}$	0	0	0	0	0	0	α_{18}

Appendix 2·2

$K_2^0 \to 3\pi$ decay, $I=1$ channel

final 3π / $I=1$ $\delta\delta\pi^0$	$\pi^+\pi^-\pi^0$	$\pi^+\pi^0\pi^-$	$\pi^-\pi^+\pi^0$	$\pi^-\pi^0\pi^+$	$\pi^0\pi^-\pi^+$	$\pi^0\pi^+\pi^-$	$\pi^0\pi^0\pi^0$	Amplitude
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_3}_{\lambda_1}$	0	0	0	0	0	0	0	α_1
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_2}_{\lambda_2}$	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	α_2

$\delta^{\kappa_3}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_3}$	0	0	0	0	0	0	0	α_3
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_3}_{\lambda_1}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_4
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_2}_{\lambda_3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	α_5
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^0)^{\kappa_1}_{\lambda_2}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_6
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}(\pi^0)^{\kappa_3}_{\lambda_2}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	α_7
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_2}_{\lambda_1}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_8
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_3}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	α_9
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^0)^{\kappa_3}_{\lambda_3}$	0	0	0	0	0	0	0	α_{10}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_2}_{\lambda_2}$	0	0	0	0	0	0	0	α_{11}
$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_1}_{\lambda_1}$	0	0	0	0	0	0	0	α_{12}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^0)^{\kappa_3}_{\lambda_3}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$	0	0	0	$\frac{1}{3}$	α_{13}
$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_1}_{\lambda_1}$	0	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	α_{14}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_2}_{\lambda_1}$	0	0	0	0	0	0	0	α_{15}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_3}(\pi^0)^{\kappa_3}_{\lambda_2}$	0	0	0	0	0	0	0	α_{16}
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^0)^{\kappa_1}_{\lambda_2}$	0	0	0	0	0	0	0	α_{17}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^0)^{\kappa_2}_{\lambda_3}$	0	0	0	0	0	0	0	α_{18}

Appendix 2·3

$K_2^0 \to 3\pi$ decay, $I=0$ channel

$I=0$ $\delta\delta\pi^{0\prime}$ \ final 3π	$\pi^+\pi^-\pi^0$	$\pi^+\pi^0\pi^-$	$\pi^-\pi^+\pi^0$	$\pi^-\pi^0\pi^+$	$\pi^0\pi^-\pi^+$	$\pi^0\pi^+\pi^-$	$\pi^0\pi^0\pi^0$	Amplitude
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_2}(\pi^{0\prime})^{\kappa_3}_{\lambda_1}$	0	0	0	0	0	0	0	β_1
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_2}_{\lambda_2}$	0	0	0	0	0	0	0	β_2
$\delta^{\kappa_3}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^{0\prime})^{\kappa_1}_{\lambda_3}$	0	0	0	0	0	0	0	β_3
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_3}(\pi^{0\prime})^{\kappa_3}_{\lambda_1}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	0	β_4
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	0	β_5
$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_1}(\pi^{0\prime})^{\kappa_1}_{\lambda_2}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	0	β_6
$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_2}_{\lambda_1}(\pi^{0\prime})^{\kappa_3}_{\lambda_2}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	0	β_7

$\delta^{\kappa_1}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_2}_{\lambda_1}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	0	β_8
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_1}_{\lambda_3}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	$-\frac{\sqrt{3}}{18}$	$\frac{\sqrt{3}}{18}$	0	β_9
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_2}(\pi^{0\prime})^{\kappa_3}_{\lambda_3}$	0	0	0	0	0	0	0	β_{10}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_2}_{\lambda_2}$	0	0	0	0	0	0	0	β_{11}
$\delta^{\kappa_2}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_1}_{\lambda_1}$	0	0	0	0	0	0	0	β_{12}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_2}_{\lambda_1}(\pi^{0\prime})^{\kappa_3}_{\lambda_3}$	0	0	0	0	0	0	0	β_{13}
$\delta^{\kappa_2}_{\lambda_3}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_1}_{\lambda_1}$	0	0	0	0	0	0	0	β_{14}
$\delta^{\kappa_1}_{\lambda_2}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_2}_{\lambda_1}$	0	0	0	0	0	0	0	β_{15}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_2}_{\lambda_3}(\pi^{0\prime})^{\kappa_3}_{\lambda_2}$	0	0	0	0	0	0	0	β_{16}
$\delta^{\kappa_2}_{\lambda_1}\delta^{\kappa_3}_{\lambda_3}(\pi^{0\prime})^{\kappa_1}_{\lambda_2}$	0	0	0	0	0	0	0	β_{17}
$\delta^{\kappa_1}_{\lambda_1}\delta^{\kappa_3}_{\lambda_2}(\pi^{0\prime})^{\kappa_2}_{\lambda_3}$	0	0	0	0	0	0	0	β_{18}

Appendix 3

Product of two configurations

$$\pi^{0\prime}n \equiv B_2^1(0\,;\,0,\,0)n = -\frac{3}{4}F_3^1\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)-\frac{\sqrt{3}}{12}F_3^2\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)$$

$$-\frac{\sqrt{6}}{4}F_3^3\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)-\frac{\sqrt{6}}{12}F_3^4\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)$$

$$\pi^{0\prime\prime}n \equiv B_2^2(0\,;\,0,\,0)n = \frac{\sqrt{6}}{3}F_3^2\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)-\frac{\sqrt{3}}{3}F_3^4\left(0\,;\,\frac{1}{2},\,-\frac{1}{2}\right)$$

$$\pi^{0\prime}p \equiv B_2^1(0\,;\,0,\,0)p = -\frac{3}{4}F_3^1\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)-\frac{\sqrt{3}}{12}F_3^2\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)$$

$$-\frac{\sqrt{6}}{4}F_3^3\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)-\frac{\sqrt{6}}{12}F_3^4\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)$$

$$\pi^{0\prime\prime}p \equiv B_2^2(0\,;\,0,\,0)p = \frac{\sqrt{6}}{3}F_3^2\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)-\frac{\sqrt{3}}{3}F_3^4\left(0\,;\,\frac{1}{2},\,\frac{1}{2}\right)$$

$$\pi^{0\prime}\Lambda \equiv B_2^1(0\,;\,0,\,0)\Lambda = \frac{\sqrt{3}}{2}F_3^1(-1\,;\,0,\,0)+\frac{\sqrt{3}}{6}F_3^2(-1\,;\,0,\,0)$$

$$-\frac{\sqrt{6}}{6}F_3^4(-1\,;\,0,\,0)$$

$$\pi^{0\prime\prime}\Lambda \equiv B_2^2(0\,;\,0,\,0)\Lambda = \frac{\sqrt{6}}{3}F_3^2(-1\,;\,0,\,0)+\frac{\sqrt{3}}{3}F_3^4(-1\,;\,0,\,0)$$

$$\pi^+\pi^{0\prime}=B_2^1(0\,;\,1,\,1)\,B_2^1(0\,;\,0,\,0)=\frac{\sqrt{30}}{10}B_4^1(0\,;\,1,\,1)-\frac{\sqrt{30}}{30}B_4^2(0\,;\,1,\,1)$$

$$-\frac{1}{2}B_4^4(0\,;\,1,\,1)+\frac{1}{2}B_4^6(0\,;\,1,\,1)+\frac{\sqrt{6}}{6}B_4^8(0\,;\,1,\,1)$$

$$\pi^+\pi^{0\prime\prime}=B_2^1(0\,;\,1,\,1)\,B_2^2(0\,;\,0,\,0)=\frac{\sqrt{15}}{6}B_4^2(0\,;\,1,\,1)-\frac{1}{2}B_4^5(0\,;\,1,\,1)$$

$$-\frac{1}{2}B_4^7(0\,;\,1,\,1)+\frac{\sqrt{3}}{6}B_4^8(0\,;\,1,\,1)$$

$$\pi^0\pi^{0\prime}=B_2^1(0\,;\,1,\,0)\,B_2^1(0\,;\,0,\,0)=\frac{\sqrt{30}}{10}B_4^1(0\,;\,1,\,0)-\frac{\sqrt{30}}{30}B_4^2(0\,;\,1,\,0)$$

$$-\frac{1}{2}B_4^4(0\,;\,1,\,0)+\frac{1}{2}B_4^6(0\,;\,1,\,0)+\frac{\sqrt{6}}{6}B_4^8(0\,;\,1,\,0)$$

$$\pi^0\pi^{0\prime\prime}=B_2^1(0\,;\,1,\,0)\,B_2^2(0\,;\,0,\,0)=\frac{\sqrt{15}}{6}B_4^2(0\,;\,1,\,0)-\frac{1}{2}B_4^5(0\,;\,1,\,0)$$

$$-\frac{1}{2}B_4^7(0\,;\,1,\,0)+\frac{\sqrt{3}}{6}B_4^8(0\,;\,1,\,0)$$

$$\pi^{0\prime}\pi^{0\prime}=B_2^1(0\,;\;\,0,\;\,0)\,B_2^1(0\,;\,0,\,0)=\frac{3\sqrt{30}}{20}B_4^1(0\,;\,0,\,0)+\frac{\sqrt{30}}{30}B_4^2(0\,;\,0,\,0)$$

$$+\frac{\sqrt{6}}{12}B_4^3(0\,;\,0,\,0)-\frac{\sqrt{6}}{6}B_4^8(0\,;\,0,\,0)-\frac{\sqrt{3}}{6}B_4^9(0\,;\,0,\,0)$$

$$\pi^{0\prime}\pi^{0\prime\prime}=B_2^1(0\,;\,0,\,0)\,B_2^2(0\,;\,0,\,0)=\frac{\sqrt{15}}{6}B_4^2(0\,;\,0,\,0)-\frac{1}{2}B_4^5(0\,;\,0,\,0)$$

$$-\frac{1}{2}B_4^7(0\,;\,0,\,0)+\frac{\sqrt{3}}{6}B_4^8(0\,;\,0,\,0)$$

$$\pi^{0\prime\prime}\pi^{0\prime\prime}=B_2^2(0\,;\,0,\,0)\,B_2^2(0\,;\,0,\,0)=\frac{\sqrt{6}}{3}B_4^3(0\,;\,0,\,0)+\frac{\sqrt{3}}{3}B_4^9(0\,;\,0,\,0)$$

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Letters to the Editor

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A Calculation of the Electrical Conductivity

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October 14, 1960

Constantinov and Perel[1] have recently obtained the kinetic equation, or the integral equation that is obeyed by the "thermodynamical" two-particle Green function, based upon the method of summation of graphs. The same technique can be applied to the calculation of the static electrical conductivity.* In this case the calculation is simple and straightforward, and moreover it can be shown that the result obtained is exact in the weak coupling limit.

We start from the formula[3]

$$\sigma=\beta\int_0^\infty dt\, F_{\mu\mu}(t) \tag{1}$$

$$F_{\mu\nu}(t)=\langle j_\mu(t) j_\nu\rangle_0 = (e/m)^2 \sum_{kk'} k_\mu k_\nu' f_{kk'}(t) \tag{2}$$

where $f_{kk'}$ is the two-particle Green function

$$f_{kk'}(t)=\langle T_c(a_k^+ a_k)_t\,(a_{k'}^+ a_{k'})_0 \times\exp[-i\int_C dt V(t)]\rangle_0. \tag{3}$$

The integration in the last factor is to be taken over t along the path C, which is shown in Fig. 1. T_c is the chronological ordering operator along this path. $\langle\ \rangle_0$ means the average over the unperturbed grand canonical ensemble and $(\ )_t$ denotes the interaction representation of the operator. V is the interaction part of the Hamiltonian H, which is expressed as $H=H_0+V$.



Fig. 1

If we introduce the Laplace transform of $f_{kk'}(t)$:

$$f_{kk'}(s)=\int_0^\infty dt\, e^{-st} f_{kk'}(t), \tag{4}$$

then it can be proved that $f_{kk'}(s)$ satisfies the following integral equation :

$$s f_{kk'}(s)=\delta_{kk'} f_k^+ f_k^- + \sum_q f_{qk'}(s)\, W_{qk}, \tag{5}$$

where f_k^- is the Fermi distribution function, $f_k^+=1-f_k^-$, $W_{qk}=W_{qk}^0-\delta_{qk}\cdot\sum_p W_{kp}^0$ and W_{qk}^0 is the transition probability from the state $\boldsymbol{q}$ to the state $\boldsymbol{k}$.

If we define

$$g_k^{\nu}(s)=\sum_{k'} k_\nu' f_{kk'}(s), \tag{6}$$

* The same idea has recently been applied by Abe[2] to the problem of the mobility of ion in a dilute hard-sphere Bose gas.

the formula (1) can be written as

$$\sigma=\beta(e/m)^2\sum_k k_\mu g_k{}^\mu(0), \qquad (7)$$

and we can get in the limit $s\to 0$ the integral equation for $g_k=g_k(0)$ from (5), viz.,

$$\sum_k k_\mu \Gamma(k) g_k{}^\nu=\sum_k k_\mu k_\nu f_k^+ f_k^-, \qquad (8)$$

where

$$k_\mu \Gamma(k)=\sum_p p_\mu W^0_{k,k-p}. \qquad (9)$$

The solution of (8) is evidently

$$g_k{}^\nu=[k_\nu f_k^+ f_k^-/\Gamma(k)]+h_k{}^\nu, \qquad (10)$$

where h_k satisfies the homogeneous equation,

$$\sum_k k_\mu \Gamma(k) h_k{}^\nu=0.$$

Inserting this expression into (7) we finally obtain

$$\sigma=\beta(e/m)^2\sum_k k_\mu k_\mu f_k^+ f_k^-/\Gamma(k), \qquad (11)$$

where h_k gives no contribution to the result.

Particularly for isotropic case,

$$\sigma=(ne^2/m)\tau(k_0)\ ;\ \tau(k)^{-1}=\Gamma(k). \qquad (11')$$

In (11′), n is the number density of electrons and k_0 denotes the Fermi momentum.

As typical examples we take the two cases of the impurity scattering and phonon scattering. In the first case we get

$$V=\sum_{kk'} v(\boldsymbol{k}-\boldsymbol{k}')\, a_k^+ a_{k'} \sum_{j=1}^{N} e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{r}_j}$$

where r_j is the position of the impurity center and $v(k)$ is the Fourier component of the impurity potential. By averaging over the random distribution of impurity centers, we get

$$W^0_{qk}=2\pi N|v(\boldsymbol{q}-\boldsymbol{k})|^2\delta(E_q-E_k). \qquad (12)$$

Performing the integration in (9) we have

$$\Gamma(k)=(2\pi)^2 N\rho(E_k)\int_{-1}^{1} d(\cos\vartheta) \times(1-\cos\vartheta)\,|v(k,\vartheta)|^2. \qquad (13)$$

Thus (11′) becomes the standard formula for the case of impurity scattering.[4]

In the case of phonon scattering, we have

$$V=\sum_{p,q}\alpha_q a^+_{p+q} a_p (b_q+b^+_{-q}),$$

where we get in the lowest approximation

$$W^0_{k,k-q}=2\pi\alpha_q{}^2(f^-_{k-q}/f^-_k)[N_q\delta(E_k -E_{k-q}-\omega_q)+(1+N_q) \times\delta(E_k-E_{k-q}+\omega_q)]. \qquad (14)$$

Then it is easily seen that

$$\Gamma(k)=(2\pi/k^2)\sum_q \alpha_q{}^2(\boldsymbol{k}\cdot\boldsymbol{q})\,(f^-_{k-q}/f^-_k) \times[N_q\delta(E_k-E_{k-q}-\omega_q) +(1+N_q)\delta(E_k-E_{k-q}+\omega_q)]. \qquad (15)$$

In (14) and (15) N_q is the phonon distribution function, $\omega_q=cq$ and $E_k=k^2/2m$. A little transformation from (15) shows that

$$\Gamma(k_0)=Am\,J_5(\theta/T)/2\pi k_0{}^3(\beta c)^5 \qquad (16)$$

where we have put $\alpha_q{}^2=Aq$, or in the notation of Sommerfeld and Bethe[5] $A=2C^2/9MNc$. θ is the Debye temperature and J_5 is the well-known function in the theory of conductivity.[5] The expression (15) shows that σ is identi-

cal with the well-known Grüneisen formula.

Details will be published in near future.

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Remarks on Eden's "Proof" of the Mandelstam Representation

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October 22, 1960

Mandelstam[1] conjectured the possibility of a double dispersion representation for the two-particle scattering amplitude. Recently, Eden[2] has proposed a "proof" of the Mandelstam representation in every order of perturbation theory in the case of no anomalous threshold. His sufficient conditions for the possibility of the Mandelstam representation are essentially as follows:

(1) There is a complex neighbourhood of real axis where the scattering amplitude is analytic with respect to all energy variables.

(2) A single dispersion relation holds for each energy variable.

In this note, we shall present a very simple counter-example against the above-mentioned Eden's "proof".

In our previous work on the validity of multiple dispersion representation,[3] we have found that a double dispersion representation is not always possible in the case of no anomalous threshold. Such a typical example (see (4·1) of reference 3)) is

$$M(s,t) = \int_0^1\int_0^1 \frac{\delta(1-x_1-x_2)\,dx_1\,dx_2}{(x_1^2+x_2^2)[x_1(m^2-s)+x_2(m^2-t)]},$$

which of course satisfies Eden's sufficient conditions with respect to the energy variables s and t. But the absorptive part for $s'>m^2$ and $t<m^2$ is

$$\frac{1}{\pi}\operatorname{Im}M(s'+i\varepsilon,t) = \frac{s'-t}{(m^2-s')^2+(m^2-t)^2},$$

which has complex poles in the case of the analytic continuation with respect to t. Thus, for real s' and complex t, $M(s'\pm i\varepsilon, t)$ is generally non-analytic in spite of the analyticity of $M(s', t)$ on the cut-plane. This singularity comes out from the values other than the (real) critical ones of x in the case of the parametric integration. It seems to the author that the above point is the most crucial problem in the proof of the Mandelstam representation.

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Note on the Pion Propagator in the Composite Theory

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November 1, 1960

Recently, many authors[1] investigated the phenomena related to the nucleon core, by using the assumption[2)-4)] that the pion is a particle which is composed of nucleons and antinucleons. In these approaches, the direct nucleon-nucleon interaction is considered as a fundamental interaction because the contribution from the dissociated state of a pion to nucleon-antinucleon pairs may become dominant in the region of nucleon core. From this standpoint, they calculated only the contribution from the first few terms of the perturbation expansion concerning the direct nucleon-nucleon interaction. The pion propagator will be given by a chain of closed loops of nucleon pairs, which is constructed from the higher order terms of the fundamental interaction. However, it is not clear even for the case of interaction between two nucleons of which separation is very small whether the contribution from the chain may be small or not. Then, it is very interesting to study the pion propagator in the composite theory. This is the purpose of this short note.

Let us consider the neutron-proton scattering as an example. The fundamental interaction is given as[5]

$$(2\pi)^{-3}\cdot f\cdot\int dp_1\,dp_2\,dp_3\,dp_4\,\delta^3(p_1+p_2-p_3-p_4):(\bar{\psi}^p(p_4)\gamma_5\psi^n(p_2))\times(\bar{\psi}^n(p_3)\gamma_5\psi^p(p_1)):C(p_1)\,C(p_4), \tag{1}$$

where $C(p)$ is a cutoff factor and we assume such a form as

$$C(p)=\left[\frac{-M^2+\Lambda^2}{p^2+\Lambda^2}\right] \tag{2}$$

where M is the nucleon mass.



Fig. 1

The interaction of these particles which corresponds to the exchange of the one positive-pion is schematically represented by Fig. 1 and the total contribution of those diagrams can be written as

$$(2\pi)^{-2}\cdot f\cdot\int dp_1\,dp_2\,dp_3\,dp_4\,\delta^4(p_1+p_2-p_3-p_4)\times[:(\bar{\psi}^p(p_4)\gamma_5\psi^n(p_2))\times D(p_4-p_2)\,(\bar{\psi}^n(p_3)\gamma_5\psi^p(p_1)):]\times C(p_1)\,C(p_4), \tag{3}$$

where $D(p_4-p_2)$ corresponds to the pion propagator and has a form

$$D(p)=\Pi(p^2)+\Pi(p^2)^2+\Pi(p^2)^3+\cdots=\frac{\Pi(p^2)}{1-\Pi(p^2)}. \tag{4}$$

$\Pi(p^2)$ is the contribution from a closed loop and is given, in the lowest approximation, by

$$\Pi(p^2)=if(2\pi)^{-4}\int \mathbf{Sp}\left[S_F\left(\frac{p}{2}+k\right)\gamma_5 S_F\left(-\frac{p}{2}+k\right)\right]C^2(k)\,d^4k, \tag{5}$$

$$=\left(\frac{f\Lambda^2}{16\pi^2}\right)\cdot\int_{4M^2}^{\infty}\frac{\rho(\kappa^2)}{p^2+\kappa^2}\,d\kappa^2, \qquad (6)$$

where

$$\rho(\kappa^2)=\theta(\kappa^2-4M^2)\left[\frac{4\kappa^2}{\Lambda^2}\right]$$
$$\times\left\{\left[\frac{1}{4}-\frac{M^2}{\kappa^2}\right]^{1/2}+\theta(\kappa^2-4\Lambda^2)\right.$$
$$\times\left[1-\frac{2(\Lambda^2-M^2)}{\kappa^2}\right]\left[\frac{1}{4}-\frac{\Lambda^2}{\kappa^2}\right]^{1/2}$$
$$-2\theta(\kappa^2-(\Lambda+M)^2)\left[1-\frac{\Lambda^2+M^2}{\kappa^2}\right]$$
$$\left.\times\left[\frac{(\Lambda^2+M^2-\kappa^2)^2-4\Lambda^2M^2}{4\kappa^2}\right]^{1/2}\right\}. \qquad (7)$$

Next we consider the Bethe-Salpeter equation for the one positive-pion amplitude "$\varphi(x|y)$":

$$\varphi(x|y)=if\int S_F(x-\xi)\gamma_5\cdot\boldsymbol{Sp}[\gamma_5\varphi(\xi|\xi)]$$
$$\times S_F(\xi-y)\,d^4\xi, \qquad (8)$$

which is reduced to the following eigenvalue equation,

$$\Pi(p^2;f\Lambda^2)|_{p^2=-\mu^2}=1, \qquad (9)$$

where μ is the mass of the pion and (9) gives the relation between f and Λ^2. From Eqs. (9) and (6), we get

$$1-\Pi(p^2)=\left(\frac{f\Lambda^2}{16\pi^2}\right)\cdot\left[(p^2+\mu^2)\right.$$
$$\left.\times\int\frac{\rho(\kappa^2)}{(p^2+\kappa^2)(-\mu^2+\kappa^2)}\,d\kappa^2\right], \qquad (10)$$

and by inserting (10) and (6) into (4), $D(p)$ becomes

$$D(p)=\left[\frac{1}{p^2+\mu^2}\right]\cdot F(p^2), \qquad (11)$$

where

$$F(p^2)=\frac{\int\frac{\rho(\kappa^2)}{p^2+\kappa^2}\,d\kappa^2}{\int\frac{\rho(\kappa^2)}{(p^2+\kappa^2)(-\mu^2+\kappa^2)}\,d\kappa^2}. \qquad (12)$$

The first factor in the right-hand side of Eq. (11) just agrees with the free pion propagator in the Yukawa theory and the second factor $F(p^2)$ corresponds to the correction for the free propagator. The behaviour of the function $F(p^2)$ is figured in Fig. 2 for $\Lambda^2=10M^2$ and is nearly constant because of the cancellation of the numerator and denominator, i.e.

$$F(p^2)=[11.2-0.123(p^2/M^2)]\cdot M^2,$$
$$\text{for } |p^2|\ll 4M^2,$$
$$F(p^2)\simeq 12.53\,M^2, \quad \text{for } |p^2|\gg 4\Lambda^2.$$

Therefore we may put as

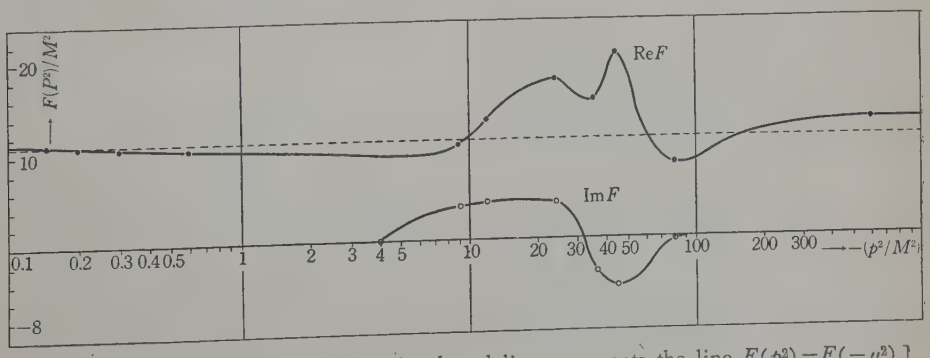


Fig. 2. The behaviour of $F(p^2)$ [The dotted line represents the line $F(p^2)=F(-\mu^2)$.]

$$D(p) \simeq \text{const.} \times \frac{1}{p^2+\mu^2}, \qquad (13)$$

for all p^2. Now we can choose this constant as $F(p^2)|_{p^2=-\mu^2}$. Consequently, we get the Yukawa coupling constant

$$(G_\pi^2/2) = f \cdot F(p^2)|_{p^2=-\mu^2},$$

which agrees with that determined by Maki in connection with the *real* meson.[3]

In conclusion, the contribution from the pion propagator can not be neglected even for the case of the small separation of two nucleons.

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On the Small-Angle Scattering of High-Energy Protons

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November 7, 1960

It is well known that the small-angle polarization of high-energy protons elastically scattered from nuclei can be calculated by using nucleon-nucleon scattering phase shifts.[1] Kerman, McManus and Thaler[2] (KMT), for example, evaluated P/θ_L (P=polarization, θ_L=laboratory scattering angle) at 90, 156 and 310 Mev and obtained a good agreement with proton-carbon polarization data. This note is to point out that, in such an analysis, contributions due to small nucleon-nucleon phase shifts of the higher partial waves are often appreciable, for all Legendre polynomials $P_L(\theta)$ in the expression of nucleon-nucleon forward scattering amplitudes are almost equal to one.

In their calculation of (P/θ_L) at 310 Mev, KMT used phase shifts obtained by Stapp et al.[3] (Solution 1) for T (isotopic spin) $=1$ states but did not include $\delta(^3H_6)$ to evaluate the scattering amplitudes in their Tables II and III. Inclusion of these phase shifts, which is 1.3°, changes the value of (P/θ_L) from 7.4 percent/degree given in KMT to 8.6 percent/degree. Since their amplitudes for $T=0$ contain only $L \leq 5$ phase shifts which were calculated from the Gammel-Thaler potential,[4] contributions from $L \geq 6$ phase shifts to these amplitudes must also be considered. Although Gammel-Thaler potential is quite different from one-pion exchange potential, it is expected that contributions from the tail part of the potential could be approximately estimated by the method employed in the modified phase-shift analysis.[5),6] Assuming g^2 (pion-nucleon coupling constant) $=12$ and 20, one gets $(P/\theta_L)=9.4$ and 8.7 percent/degree, respectively. Here, Solution 1 of the modified analysis[6]

has been taken as the phase shifts for $T=1$ states.

The importance of higher partial-wave phase shifts is also apparent in the analysis of n-p scattering. Indeed, scattering amplitudes in Table II of KMT give $\sigma_{np}(\theta=0°)=10.2$, 6.8 and 5.3 mb/sterad at 90, 156 and 310 Mev, respectively, whereas the corresponding values given in reference 4), which presumably contain all contributions from higher partial waves, are 11.1, 7.7 and 5.9 mb/sterad.

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Electromagnetic Mass Shift of Σ Hyperons

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The recent experimental data of the masses of Σ hyperons[1] indicate that the mass of Σ^0 is rather close to the arithmetic mean of the masses of Σ^+ and Σ^-,

$$(1/2)\,(M_{\Sigma^+}+M_{\Sigma^-})-M_{\Sigma^0} = 1.2\pm 0.4 \text{ Mev}, \qquad (1)$$

while the Σ^- is heavier than the Σ^+ by a fairly large amount, $M_{\Sigma^-}-M_{\Sigma^+}=6.6\pm 0.2$ Mev. The purpose of the present report is to show that the left-hand side of (1) approximately vanishes, if the e^2-order self-energy of hyperons is calculated by using the several vertex relations[2] deduced as a consequence of the proposed symmetries of strong interactions, and making an assumption that the direct pion current contribution dominates the isovector part of the vertex.

It is convenient to write the electromagnetic vertex Γ_μ in terms of two invariant functions F_1 and F_2 called the charge and magnetic moment form functions, respectively, $\Gamma_\mu(p', p)=\gamma_\mu F_1[(p'-p)^2]-i\sigma_{\mu\nu}(p'-p)_\nu F_2[(p'-p)^2]$. The vertex relations can now be written as

$$\left.\begin{aligned} F_i^{(\Sigma)} &= A_i+T_3(-B_i+C_i) \\ F_i^{(\Sigma^0\Lambda^0)} &= B_i+C_i \end{aligned}\right\} (i=1, 2), \qquad (2)$$

which have been obtained[2] by assuming the charge independent strong interactions of baryons with π and K mesons represented by the doublet form of Gell-Mann and Pais,[3] and using the minimal electromagnetic interactions of baryons and mesons written in three parts, $H_{em}=H_a+H_b+H_c$, following Tanaka's argument,[4] where H_a concerns the isoscalar part of nucleon, Ξ and K currents, H_b the isoscalar part of Σ and Λ currents and isovector part of K current, and H_c the isovector part of all baryons and π currents. It is

assumed that all baryons are spin 1/2 particles and the K meson is spin 0 particle. In Eq. (2), A_i denotes the contribution from H_a, B_i and C_i are those from H_b and H_c, and $F_i^{(\Sigma^0\Lambda^0)}$ means the form function corresponding to the vertex $\Gamma_\mu^{(\Sigma^0\Lambda^0)}$ which describes the decay of Σ^0 into a Λ^0 and a photon. In deriving the relations (2), the mass difference between Λ and Σ has been neglected, and the even relative parity has been assumed between them. It should be noted that the relations (2) are rigorously true to all orders in the strong interactions if the above proposed symmetries are correct.

The e^2-order self-energy of Σ hyperons is calculated as usual in terms of the vertex $\Gamma_\mu^{(\Sigma)}$. It has been remarked[5] that the Σ^0 hyperon has also a considerable additional contribution from the vertex $\Gamma_\mu^{(\Sigma^0\Lambda^0)}$, which is given by the diagram of Fig. 1. Moreover,

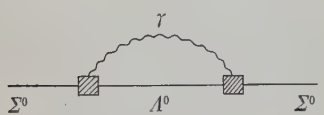


Fig. 1

judging from the dispersion theoretical argument on the nucleon form function,[6] in which the two-pion contribution has been considered to dominate the isovector part of the form function, it seems plausible to assume that the isovector part of the Σ hyperon form function may be approximated by the contribution C_i only that contains the direct pion current. Thus δM_Σ's are written as follows,

$$\delta M_{\Sigma\pm} \simeq e^2 \int d^4k [(A_1 \pm C_1)^2 L - (A_1 \pm C_1)(A_2 \pm C_2) Q - (A_2 \pm C_2)^2 Q'], \quad (3)$$

$$\delta M_{\Sigma^0} \simeq e^2 \int d^4k [(A_1^2 + C_1^2) L - (A_1 A_2 + C_1 C_2) Q - (A_2^2 + C_2^2) Q'], \quad (4)$$

where L gives a logarithmically divergent term, and Q and Q' give quadratically divergent terms, if the integrations are carried out with omission of form factors. Here we have neglected again the mass difference between Λ^0 and Σ^0 in the intermediate state of the diagram shown in Fig. 1, but an error for δM_{Σ^0} is estimated to be within 1%. It follows from (3) and (4), without performing the integrations, that $(1/2) \times (\delta M_{\Sigma^+} + \delta M_{\Sigma^-}) - \delta M_{\Sigma^0} \simeq 0$. If we regard the above δM_Σ's as shifts of a hypothetical degenerate Σ hyperon mass in the absence of the electromagnetic interaction, we have

$$(1/2)(M_{\Sigma^+} + M_{\Sigma^-}) - M_{\Sigma^0} \simeq 0, \quad (5)$$

which is to be compared with the experimental value (1). It should be remarked that the mass relation (5) is obtained without using any explicit form function, while on the other hand the mass difference $M_{\Sigma^-} - M_{\Sigma^+}$ requires an introduction of the explicit form function for calculating its finite value.[5]

A small but nonvanishing value of the observed mass difference (1) would be ascribed to several effects which are neglected in the above. One of these may be the effect of B_i, which contains the direct K current, the indirect pion contribution through a baryon pair etc. In fact, $F_1^{(\Sigma)}(0) = T_3$ and $F_1^{(\Sigma^0\Lambda^0)}(0) = 0$

do not hold simultaneously with neglect of B_1, but the e^2-self-energy is not very sensitive to the low momentum behavior of F_1. Besides this effect, there may be some other contributions from the strong interactions which cannot be represented by the vertex correction diagram. The lowest order calculation suggests,[5] however, that the strong interaction correction seems to be less than 0.5 Mev, in case of the even relative parity between Λ and Σ. Furthermore, the mass difference between Λ and Σ in the intermediate state would produce a small contribution, probably less than 1 Mev.

The electromagnetic mass shift of Λ^0 turns out to be equal to δM_{Σ^0}, without assuming the predominance of C_i, and then the observed mass difference of Λ^0 and Σ^0 should be understood as an effect of the strong interactions other than the electromagnetic one.

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Progress of Theoretical Physics, Vol. 25, No. 2, February 1961

A Theory of the Magneto-Optical Absorption

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A theory of the magneto-optical absorption is developed by means of the Luttinger and Kohn theory of the effect of magnetic field on the energy bands in the case of a simple band model. The main purpose is to study the absorption line shape. As the width of the absorption line we get a value which agrees with experiment. It does not exhibit appreciable dependence on the magnetic field and temperature. The absorption coefficient in the absence of a magnetic field is also discussed.

§1. Introduction

Recently, optical absorption experiments in semi-conductors have been performed in a strong magnetic field. These results show that absorption edges shift to shorter wave-length side in proportion to the magnetic field and the absorption spectrum shows an oscillation which is not observed in the absence of a magnetic field. These facts can be interpreted as the optical absorption which is accompanied with transitions of electrons between Landau levels of the valence band and of the conduction band. This is the oscillatory magneto-optical absorption effect.[1)]

In this effect we are interested in i) positions of absorption peaks, ii) selection rules and iii) absorption line shapes, which are determined by level scheme, symmetry of band structure and scattering mechanism of carriers. Elliott et al.[2)] discussed the problems i) and ii). In this paper the electron-phonon interaction is taken into account and the absorption line shape is discussed as the main problem. We aim to gain some information about the relation between absorption line shapes and scattering mechanisms in the case of a strong magnetic field. In order to avoid the complexity, treatment is done for a simple spherical band model. In § 2, we shall make general considerations. In § 3, we shall discuss the absorption coefficient by taking account of phonon effects. We will give numerical examples and discussions in § 4.

§2. General considerations

Here we summarize the facts concerning the wave functions and the energy levels of the carriers in a magnetic field, and we give the general formula of the absorption coefficient and its application to the no-phonon systems.

Throughout this paper, we assume both valence and conduction bands are spherically symmetric in a $\boldsymbol{k}$ space. We neglect the spin-orbit interaction and band degeneracy.

If the gauge of the vector potential is chosen as $\boldsymbol{A}=(-yH, 0, 0)$, the wave function of a hole, which is obtained from the effective mass equation of Luttinger and Kohn,[3] can be expressed in the form

$$\Psi_v=\exp[i(k_x x+k_z z)]\phi_n(y, k_x)\psi_{v0}, \tag{1}$$

$$\phi_n(y, k_x)=C_n H_n\left[\sqrt{\frac{eH}{\hbar c}}(y+\lambda^2 k_x)\right]\exp\left[-\frac{eH}{2\hbar c}(y+\lambda^2 k_x)^2\right], \tag{2}$$

where $C_n=(2^n n!\sqrt{\pi})^{-1/2}$, $\lambda^2=\hbar c/eH$, n is the Landau magnetic quantum number, H_n is the harmonic oscillator function and ψ_{v0} is the Bloch function of the valence band at $\boldsymbol{k}=0$. The quantum numbers which specify a state of a carrier are n, k_x and k_z. Similarly, the wave function of an electron is given by

$$\Psi_c=\exp[i(k_x' x+k_z' z)]\phi_{n'}(y, k_x')\psi_{c0}, \tag{3}$$

where ψ_{c0} is the Bloch function of the conduction band at $\boldsymbol{k}=0$.

The energies for holes and electrons are respectively given by

$$\epsilon_v=-\hbar\omega_v(n+1/2)-\hbar^2 k_z^2/2m_v, \tag{4}$$

and

$$\epsilon_c=E_0+\hbar\omega_c(n'+1/2)+\hbar^2 k_z'^2/2m_c. \tag{5}$$

The subscripts v and c refer to the valence and conduction bands, $\omega_0=eH/mc$, m is an effective mass, E_0 is the energy gap in the absence of a magnetic field.

We use the general formula of the absorption coefficient given by Toyozawa.[4] It is expressed as a function of the frequency of photon ω in the form

$$A(\omega)=\frac{2\pi\hbar e^2}{m^2 v_0 c n_0 \omega_K}\sum_M\sum_{M'}F_{MM'}(\boldsymbol{K})\int_{-\infty}^{\infty}dt\exp\left(i\omega t-i\varepsilon_M\frac{t}{\hbar}\right)\sum_n U_n(t; MM'; K), \tag{6}$$

where v_0 is the volume of a unit cell, $\omega_K=(c/n_0)K$ and n_0 is a refractive index of the crystal.

$$F_{MM'}(\boldsymbol{K})\equiv(1/2)\sum_\sigma(\boldsymbol{e}_{\sigma K}\cdot g_{MK}^*)(\boldsymbol{e}_{\sigma K}\cdot g_{M'K}), \tag{7}$$

$$g_{MK}\equiv\int\Psi_M^*\nabla\exp(i\boldsymbol{K}\boldsymbol{r})\Psi_t dv, \tag{8}$$

$$\varepsilon_M=E_0+\frac{\hbar^2}{2}\left(\frac{k_z'^2}{m_c}+\frac{k_z^2}{m_v}\right)+\hbar\omega_c(n'+1/2)+\hbar\omega_v(n+1/2), \tag{9}$$

and

$$U_n(t; MM'; K)=(i\hbar)^{-n}\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n(MK|\mathcal{H}'(t_1)\cdots\mathcal{H}'(t_n)|M'K)_{av}, \tag{10}$$

where $\sigma(=1, 2)$ refer to the polarization, the direction of which is denoted by a unit vector $\boldsymbol{e}_{\sigma K}$ and $\boldsymbol{K}$ to wave number of the photon. We denote a set of quantum numbers, (n, k_x, k_z) of a hole and (n', k'_x, k'_z) of an electron, by M, that is, the state in which one electron-hole pair exists is characterized by M. Averaging is done over the initial distribution of phonons. Making use of the fact that the wave length of the radiation field is larger than the lattice constant a, we replace $\exp(i\boldsymbol{K}\boldsymbol{r})$ in (8) by 1 hereafter. $\mathcal{H}'(t)$ represents the electron-phonon interaction Hamiltonian in the interaction representation:

$$\mathcal{H}'(t)=\exp[i(\mathcal{H}_e+\mathcal{H}_L)t/\hbar]\mathcal{H}_{eL}\exp[-i(\mathcal{H}_e+\mathcal{H}_L)t/\hbar], \tag{11}$$

where $\mathcal{H}_e$ and $\mathcal{H}_L$ are Hamiltonians of electrons and phonons.

For convenience, we consider the acoustic mode. Its interaction with an electron is written in the approximation of deformation potential as follows:

$$\mathcal{H}_{eL}=E_1\Delta, \tag{12}$$

where

$$\Delta=\nabla\boldsymbol{R}(\boldsymbol{r})=\sum_{q}(\hbar/2\mu V)^{1/2}(i\boldsymbol{q}/\omega_q^{1/2})\{\boldsymbol{p}_q\exp(i\boldsymbol{q}\boldsymbol{r})-\boldsymbol{p}_q^*\exp(-i\boldsymbol{q}\boldsymbol{r})\}. \tag{13}$$

$\boldsymbol{R}(\boldsymbol{r})$ denotes a displacement of an atom from the equilibrium point. μ is the mass density, $\boldsymbol{p}_q^*$ and $\boldsymbol{p}_q$ mean the creation and annihilation operators of phonons of the wave number $\boldsymbol{q}$, respectively. U_n vanishes for odd n when it is averaged over the phonon states.

Now we calculate the absorption coefficient neglecting the electron-phonon interaction. In the presence of a magnetic field, (6) becomes simply

$$A_H(\omega)=\frac{2\pi\hbar e^2}{m^2v_0cn_0\omega_K}\frac{1}{2}\sum_{M,\sigma}|\boldsymbol{e}_{\sigma K}\cdot g_{MK}|^2 2\pi\delta(\omega-\varepsilon_M/\hbar)$$

$$=\frac{\sqrt{2}e^2}{\pi\hbar^{3/2}cn_0}\frac{eH}{\hbar c}m^{*1/2}\sum_n\frac{\omega}{\sqrt{\omega-(E_0/\hbar)-(\omega_c+\omega_v)(n+1/2)}}\left|\int_{cell}\psi_{c0}\psi_{v0}\,r\,dv\right|^2, \tag{14}$$

where $1/m^*=1/m_c+1/m_v$ and the factor 2 was multiplied because of spin. (14) agrees with the results derived by Elliott et al. In derivation of (14) we obtained the following selection rules,

$$\Delta n=0, \tag{15}$$

$$k_x=k_x', \tag{16}$$

and

$$k_z=k_z'. \tag{17}$$

For comparison, the absorption coefficient in the absence of a magnetic field is obtained by the use of (6) as follows,

$$A_0(\omega)=\frac{2^{5/2}e^2m^{*3/2}}{3cn_0\hbar^3}\omega\sqrt{\hbar\omega-E_0}\left|\int_{cell}\psi_{c0}\psi_{v0}\,r\,dv\right|^2. \tag{18}$$

(18) coincides with the result given by Bardeen et al.[5] as expected. The ratio of the absorption coefficient in the presence of a magnetic field to that in the absence of a magnetic field is obtained from (14) and (18) as

$$\frac{A_H(\omega)}{A_0(\omega)}=\frac{3}{4\pi}\frac{eH}{m^*c}\frac{1}{\sqrt{\omega-E_0/\hbar}}\sum_n\frac{1}{\sqrt{\omega-E_0/\hbar-(\omega_c+\omega_v)(n+1/2)}}. \tag{19}$$

(19) does not include any unknown parameter such as $|\int_{cell}\psi_{c0}\psi_{v0}r\,dv|^2$. Actually, we observe the transmission intensity of the light given by $I_t=\exp[-A(\omega)d]$, where d is the thickness of a sample. Using (14) and (18), we obtain

$$\frac{I_t(H)}{I_t(0)}=\exp\{-(A_H-A_0)d\}=\exp\left[-|\int_{cell}\psi_{c0}\psi_{v0}r\,dv|^2\frac{\sqrt{2}e^2}{cn_0\hbar^{3/2}}\omega\right.$$
$$\left.\times\left\{m^{*1/2}\frac{eH}{\pi c}\sum_n\frac{1}{\sqrt{\omega-E_0/\hbar-(\omega_c+\omega_v)(n+1/2)}}-\frac{4m^{*3/2}}{3\hbar^{1/2}}\sqrt{\hbar\omega-E_0}\right\}d\right]. \tag{20}$$

Until now we did not take into account the scattering of carriers with phonons in the calculation of the absorption coefficient. As already shown by many authors, we can obtain information about the position of the absorption peaks and about the peak-to-peak distances $\hbar(\omega_c+\omega_v)$ by comparing the experimental data with (14). The effective mass of electrons in germanium at $\boldsymbol{k}=0$ is known from this $(\omega_c+\omega_v)$, using the known value of the effective mass of holes at $\boldsymbol{k}=0$. Also the selection rules for the absorption is obtained in deriving (14). However, concerning the absorption line shape, which is our main problem, the effect of the scattering plays an essential role as we shall see in the next section.

§ 3. The absorption coefficient with phonon

(i) $H\neq 0$

Now we discuss the line shape of the absorption. Taking the broadening of the levels by scattering of carriers with phonons into account, the height of an absorption line will become finite and the absorption line will be broadened of course.

In the approximation of taking by second order in $\mathcal{H}_{eL}$, we need only to calculate $U_2(t;MM';K)$. First we take into account the diagonal term $M=M'$ in (6), namely

$$U_2(t;MM;K)=-(1/\hbar^2)\int_0^t dt_1\int_0^{t_1}dt_2(M|\mathcal{H}'(t_1)\mathcal{H}'(t_2)|M)_{av}$$
$$=-(1/\hbar^2)\int_0^t dt_1\int_0^{t_1}dt_2\sum_{M''}(M|\exp[i(\mathcal{H}_e+\mathcal{H}_L)t_1/\hbar]\mathcal{H}_{eL}\exp[-i(\mathcal{H}_e+\mathcal{H}_L)t_1/\hbar]|M'')$$
$$\times(M''|\exp[i(\mathcal{H}_e+\mathcal{H}_L)t_2/\hbar]\mathcal{H}_{eL}\exp[-i(\mathcal{H}_e+\mathcal{H}_L)t_2/\hbar]|M)_{av}. \tag{21}$$

There are the following four processes in this case: (1) An electron absorbs a phonon and then emits it, (2) a hole absorbs a phonon and then emits it, (3) an electron emits a phonon and then absorbs it, and (4) a hole emits a phonon and then absorbs it. Rewriting (21) by taking the four processes into account, we obtain

$$U_2(t\,;\,MM\,;\,K)=-\frac{E_{1c}^2}{\hbar^2}\frac{\hbar}{2\mu V}\sum_q\frac{q^2}{\omega_q}\int_0^t dt_1\int_0^{t_1}dt_2\sum_{M''}(M|\exp[i(\varepsilon_M-\varepsilon_{M''})t_1/\hbar]$$
$$\times\exp(i\omega_q t_1)\boldsymbol{p}_q\exp(i\boldsymbol{qr})|M'')$$
$$\times(M''|\exp[-i(\varepsilon_M-\varepsilon_{M''})t_2/\hbar]\exp(-i\omega_q t_2)\boldsymbol{p}_q^*\exp(-i\boldsymbol{qr})|M)_{av}$$
$$+\text{terms corresponding to the processes (2), (3) and (4).}$$

E_{1c} and E_{1v} are constants of deformation potential for an electron and for a hole, respectively. In the above equation we may extend the region of summation over q_x and q_y to all space with negligible error. We will also assume the distribution of phonons to be classical, namely $N_q\simeq N_q+1\simeq k_0T/\hbar\omega_q$, where k_0 is the Boltzmann constant, and we will put $\omega_q=sq$ in which s is the sound velocity. Then we get from the previous equation

$$U_2(t\,;\,MM\,;\,K)=-\sum_{mm'}\int_0^t d\tau(t-\tau)\frac{k_0T}{2(2\pi)^2\mu\hbar^2s^2}\int dq_x dq_z\Big[\Big\{E_{1c}^2\int dy\,\phi_{m'}^2(y,\,k_x')$$
$$\times\phi_m^2(y,\,k_x'+q_x)+E_{1v}^2\int dy\,\phi_n^2(y,\,k_x)\phi_m^2(y,\,k_x-q_x)\Big\}\exp[i(\varepsilon_M-\varepsilon_{M''}+\hbar\omega_q)\tau/\hbar]\Big]$$
$$+\Big\{E_{1c}^2\int dy\,\phi_{n'}^2(y,\,k_x')\phi_{m'}^2(y,\,k_x'-q_x)+E_{1v}^2\int dy\,\phi_m^2(y,\,k_x+q_x)\phi_n^2(y,\,k_x)\Big\}$$
$$\times\exp[i(\varepsilon_M-\varepsilon_{M''}-\hbar\omega_q)\tau/\hbar]\Big].$$

Using the relation $\int dq_x\phi_m^2(y,\,k_x\pm q_x)=eH/\hbar c$, we obtain

$$U_2(t\,;\,MM\,;\,K)=-\sum_{mm'}\int_0^t d\tau(t-\tau)\frac{k_0T}{2(2\pi)^2\mu s^2\hbar^2}\frac{eH}{\hbar c}\int dq_z(E_{1c}^2+E_{1v}^2)$$
$$\times\Big\{\exp[i(\varepsilon_M-\varepsilon_{M''}+\hbar\omega_q)\tau/\hbar]+\exp[i(\varepsilon_M-\varepsilon_{M''}-\hbar\omega_q)\tau/\hbar]\Big\}\qquad(22)$$
$$=-\frac{1}{\hbar^2}\int_0^t d\tau(t-\tau)\int_{-\infty}^{\infty}dE\exp(-iE\tau/\hbar)f_M(E),\qquad(23)$$

where

$$f_M(E)=\sum_{mm'}\frac{k_0T}{2(2\pi)^2\mu s^2}\frac{eH}{\hbar c}(E_{1c}^2+E_{1v}^2)\int dq_z\Big\{\delta(\varepsilon_{M''}-\varepsilon_M+\hbar\omega_q-E)$$
$$+\delta(\varepsilon_{M''}-\varepsilon_M-\hbar\omega_q-E)\Big\}.\qquad(24)$$

In the case that the electron-phonon interaction is sufficiently weak, we can approximate, according to Toyozawa's theory, as

$$U_0(t\,;MM\,;K)+U_2(t\,;MM\,;K)+\cdots$$
$$=1+\left\{-\frac{\Gamma_{MK}}{2}|t|-it\frac{\Delta_{MK}}{\hbar}\right\}+\cdots\simeq\exp\left\{-\frac{\Gamma_{MK}}{2}|t|-it\frac{\Delta_{MK}}{\hbar}\right\}. \tag{25}$$

Using this approximation and neglecting $\hbar\omega_q$ in the arguments of the δ-functions in (24), the absorption coefficient is given by Lorentzian shape. It should be remarked that the absorption coefficient thus obtained includes the contribution only from the diagonal term $M=M'$. The result is

$$A_M(\omega)=\frac{4\pi\hbar e^2}{m^2v_0cn_0\omega_K}|\boldsymbol{e}_{\sigma K}\cdot\boldsymbol{g}_{MK}|^2\frac{\Gamma_{MK}}{(\omega-\varepsilon_M/\hbar-\Delta_{MK}/\hbar)^2+(\Gamma_{MK}/2)^2}, \tag{26}$$

where

$$\Gamma_{MK}\equiv\frac{2\pi}{\hbar}f_M(0), \tag{27}$$

$$\Delta_{MK}\equiv-\mathcal{P}\int\frac{f_M(E)}{E}dE. \tag{28}$$

$\mathcal{P}$ means to take the principal value of the integral. The width Γ_{MK} and the shift Δ_{MK} are given if the state of an electron-hole pair M is determined. The absorption coefficient has none of them if no electron-phonon interaction is considered. In order to compare with an observed value of the absorption coefficient, we must sum the absorption coefficient with such Γ_{MK} and Δ_{MK} over all the states.

Now we consider the photon with $\boldsymbol{K}=0$, then Γ_{M0} is given from (24) and (27), by

$$\Gamma_{M0}=\left(\frac{k_0T}{\sqrt{2}\pi\mu s^2\hbar^2}\right)\left(\frac{eH}{\hbar c}\right)(E_{1c}^2m_c+E_{1v}^2m_v)\sum_m\frac{1}{\sqrt{\hbar\omega_cm_c(n-m)+\hbar^2k_z^2/2}}. \tag{29}$$

Here the summation extends over all non-negative integers that leave the summand real. This limitation means that the carrier in a given initial state n, can transfer only to a state m in the restricted energy range because of the energy conservation law. (29) agrees with the inverse of the relaxation time τ^{-1} given by Argyres[6] in his quantum mechanical galvanomagnetic theory. Of course, when we compare (29) with his result we must replace $E_{1c}^2m_c+E_{1v}^2m_v$ by E_1^2m in his result, since his calculation is the case of one carrier conduction.

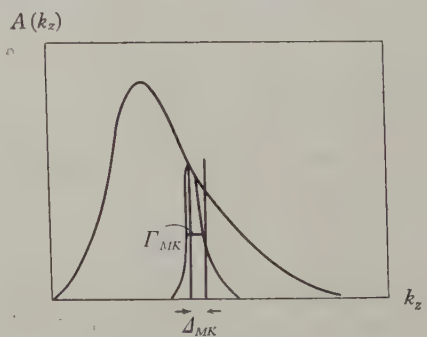


Fig. 1. Absorption coefficient at a fixed ω as a function of k_z.

Now we assume that when an electron

in a Landau level absorbs or emits phonons, the probability for the electron to transfer to another Landau level is small. Then we can take only the term $m=n$ in the summation in (29), so (29) becomes

$$\Gamma_{M0}=\left(\frac{k_0T}{\pi\mu s^2\hbar^2}\right)\left(\frac{eH}{\hbar c}\right)(E_{1c}^2m_c+E_{1v}^2m_v)\,(1/k_z). \tag{30}$$

This Γ_{M0} includes k_z as a parameter. In (30) Γ_{M0} is proportional to T and H which arise from the terms of the phonon distribution and the electron state density, respectively. However. the width of the absorption line does not necessarily show such dependence, because it is obtained by superposing the absorption coefficient thus obtained over the range of k_z.

Next we will calculate the shift of the absorption energy given by (28)

$$\Delta_{M0}=-\frac{k_0T}{2(2\pi)^2\mu s^2}\left(\frac{eH}{\hbar c}\right)(E_{1c}^2+E_{1v}^2)$$

$$\times\mathcal{P}\int\frac{dE}{E}\sum_{mm'}\int dq_z\{\delta(\varepsilon_{M'}-\varepsilon_M+\hbar\omega_q-E)+\delta(\varepsilon_{M'}-\varepsilon_M-\hbar\omega_q-E)\}.$$

Here we also neglect the transition of an electron between Landau levels by the phonon perturbation. Thus we obtain

$$\begin{aligned}\Delta_{M0}=&-\frac{k_0T}{2(2\pi)^2\mu s^2}\left(\frac{eH}{\hbar c}\right)\Big[E_{1c}^2\int dq_z\{(a_c\overline{k_z+q_z}^2-a_ck_z^2+\hbar sq_z)^{-1}\\&+(a_c\overline{k_z-q_z}^2-a_ck_z^2-\hbar sq_z)^{-1}\}+E_{1v}^2\int dq_z\{(a_v\overline{k_z+q_z}^2-a_vk_z^2+\hbar sq_z)^{-1}\\&+(a_v\overline{k_z-q_z}^2-a_vk_z^2-\hbar sq_z)^{-1}\}\Big]\\=&-\frac{k_0T}{(2\pi)^2\mu s^2}\left(\frac{eH}{\hbar c}\right)\left\{\frac{E_{1c}^2}{a_c}\left(2k_z+\frac{\gamma}{a_c}\right)^{-1}\log\left|\frac{(\pi/a)-(2k_z+\gamma/a_c)}{(\pi/a)+(2k_z+\gamma/a_c)}\right|\right.\\&\left.+\frac{E_{1v}^2}{a_v}\left(2k_z+\frac{\gamma}{a_v}\right)^{-1}\log\left|\frac{(\pi/a)-(2k_z+\gamma/a_v)}{(\pi/a)+(2k_z+\gamma/a_v)}\right|\right\},\end{aligned}$$

where $a_c=\hbar^2/2m_c$, $a_v=\hbar^2/2m_v$ and $\gamma=\hbar s$. We expand terms of logarithmus under the condition $\pi/a>(2k_z+\gamma/a_c)$. As we will show below, this condition is correct if we confine ourselves to small k_z. Namely, for small k_z the above-mentioned condition is written as $\hbar^2/2m_c\cdot q_{max}>\hbar sq_{max}$. This means that the energy of an electron with the wave vector q_{max} is larger than the energy of phonon with same magnitude of wave vector q_{max}. This is obviously satisfied.

Thus we obtain

$$\Delta_{M0}=\left(\frac{ak_0T}{2\pi^2\mu s^2}\right)\left(\frac{eH}{\hbar c}\right)\left(\frac{E_{1c}^2}{a_c}+\frac{E_{1v}^2}{a_v}\right). \tag{31}$$

For the usual experimental condition this term is too small to contribute to the absorption coefficient.

Now we will take into account the off-diagonal term $(M \neq M')$ of $U_2(t; MM'; K)$. But the off-diagonal term is expected to be much smaller than the diagonal term from physical consideration. In this case four processes exist: (5) An electron emits a phonon and then a hole absorbs it, (6) a hole emits a phonon and then an electron absorbs it, (7) an electron absorbs a phonon and then a hole emits it, (8) a hole absorbs a phonon and then an electron emits it. The term $U_2(t; MM'; K)$ is a correction to the diagonal term and gives an asymmetry to the absorption line. From (10) we get

$$U_2(t; MM'; K) = -(1/\hbar^2)\int_0^t dt_1 \int_0^{t_1} dt_2 (M|\mathcal{H}'(t_1)\mathcal{H}'(t_2)|M')_{av}. \tag{32}$$

Repeating the same procedure as used in the diagonal term, we get

$$U_2(t; MM'; K) = -\frac{i}{\hbar(\varepsilon_M - \varepsilon_{M'})}\int_{-\infty}^{\infty} dE \int_0^t d\tau \exp(-iE\tau/\hbar)$$
$$\times\Big[\Big\{\exp[i(\varepsilon_M - \varepsilon_{M'})t/\hbar] f_{M'M}(E) - f_{MM'}(E)\Big\}$$
$$+\text{terms corresponding to the processes (6), (7) and (8)}\Big], \tag{33}$$

where

$$f_{M'M}(E) = \sum_{mm'} \frac{E_{1c}E_{1v}k_0T}{2(2\pi)^2\mu s^2}\left(\frac{eH}{\hbar c}\right)\int dq_z \delta(\varepsilon_{M''} - \varepsilon_{M'} \pm \hbar\omega_q - E), \tag{34}$$

$$f_{MM'}(E) = \sum_{mm'} \frac{E_{1c}E_{1v}k_0T}{2(2\pi)^2\mu s^2}\left(\frac{eH}{\hbar c}\right)\int dq_z \delta(\varepsilon_{M''} - \varepsilon_{M} \pm \hbar\omega_q - E). \tag{35}$$

The contribution from the term $F_{MM'}\exp(-i\varepsilon_M t/\hbar)U_2(t; MM'; K) + F_{M'M} \times\exp(-i\varepsilon_{M'}t/\hbar)U_2(t; M'M; K)$ in (6) to the M peak causes an asymmetry which is given by

$$\mathcal{A}_M = \frac{1}{2}\sum_{M'} \frac{\mathrm{Re}(F_{MM'}\hbar\Gamma_{MM'})}{F_{MM}(\varepsilon_M - \varepsilon_{M'})}, \tag{36}$$

where $\Gamma_{MM'}$ is given by replacing the final state of Γ_M by M'. From (33), (34), (35) and (36)

$$\mathcal{A}_M = -\frac{E_{1c}E_{1v}k_0T}{\pi\mu s^2}\left(\frac{eH}{\hbar c}\right)\frac{m^*}{\hbar\gamma}\varepsilon_M^{-1}\Big[\frac{\{E_0 + \hbar(\omega_c+\omega_v)(n+1/2) + a^*(k_z - \gamma/a_c)^2\}}{(\gamma/a_c - 2k_z)|\gamma/a_c - 2k_z|}$$
$$+\frac{\{E_0 + \hbar(\omega_c+\omega_v)(n+1/2) + a^*(k_z + \gamma/a_v)^2}{(\gamma/a_v + 2k_z)|\gamma/a_v + 2k_z|}\Big], \tag{37}$$

where $$a^* = \hbar^2/2m^*.$$

Adding the off-diagonal terms of U_2 to the diagonal terms obtained before, the absorption coefficient is given by

$$A_M(\omega)=\frac{4\pi}{3}\frac{\hbar e^2|g_{M0}|^2}{m^2 v_0 c n_0 \omega}\left(\frac{eH}{\hbar c}\right)\frac{v_0}{(2\pi)^3}\int dk_z \frac{(\Gamma_M/2)+\mathcal{O}_M(\omega-\varepsilon_M/\hbar)}{(\omega-\varepsilon_M/\hbar)^2+(\Gamma_M/2)^2},$$

$$=\text{diagonal part of } A_M(\omega)$$

$$-\frac{e^2|g_{M0}|^2}{6\pi^3 m^2 c n_0 \omega}\left(\frac{eH}{\hbar c}\right)^2\frac{E_{1c}E_{1v}k_0 T m^*}{\gamma\mu s^2}\int dk_z \varepsilon_M^{-1}\frac{\omega-\varepsilon_M/\hbar}{(\omega-\varepsilon_M/\hbar)^2+(\Gamma_M/2)^2}$$

$$\times[C_c(k_z)+C_v(k_z)], \tag{38}$$

where

$$C_c(k_z)=\frac{\{E_0+\hbar(\omega_c+\omega_v)(n+1/2)+a^*(k_z-\gamma/a_c)^2\}}{|\gamma/a_c-2k_z|(\gamma/a_c-2k_z)},$$

and

$$C_v(k_z)=\frac{\{E_0+\hbar(\omega_c+\omega_v)(n+1/2)+a^*(k_z+\gamma/a_v)^2\}}{(\gamma/a_v+2k_z)|\gamma/a_v+2k_z|}.$$

When we perform the integration in the off-diagonal terms in (38), the most predominant contribution to the integral comes from the neighbourhood of $k_z=\pm\gamma/2a_c$, so that we may carry out the integration by putting $k_z=\gamma/2a_c$ (or $-\gamma/2a_c$) except the main factor $1/(\gamma/a_c-2k_z)|\gamma/a_c-2k_z|$. After calculations we can see that the contribution of diagonal terms to the $A_M(\omega)$ is 10^5 times larger than that of the off-diagonal terms, and therefore we neglect the off-diagonal terms.

Finally, neglecting the shift in (26), inserting (30) into (26) and carrying out integration over k_z, we have an absorption coefficient for a given Landau level n. We can rewrite it using dimensionless quantities as follows:

$$A_n(\epsilon)=\frac{2e^2}{\pi\hbar c n_0 a}\left(\frac{eH}{\hbar c}\right)\left|\int_{cell}\psi_{c0}\psi_{v0}\,r\,dv\right|^2\epsilon\int_0^1\frac{(\epsilon_0 C_T/\kappa)\,d\kappa}{\{\epsilon-1-\epsilon_\kappa-(\epsilon_c+\epsilon_v)(n+1/2)\}^2+\epsilon_0^2 C_T^2/\kappa^2}, \tag{39}$$

where

$$\epsilon=\frac{\hbar\omega}{E_0},\quad \epsilon_0=\frac{\hbar\omega_0}{E_0},\quad \epsilon_c=\frac{\hbar\omega_c}{E_0},\quad \epsilon_v=\frac{\hbar\omega_v}{E_0},\quad \epsilon_\kappa=\frac{\hbar^2\pi^2\kappa^2}{2m^* a^2}\Big/E_0,$$

$$\kappa=\frac{k_z}{\pi/a},\quad C_T=\frac{ak_0 T}{2\pi^2\mu s^2}\frac{m}{\hbar^4}(E_{1c}^2 m_c+E_{1v}^2 m_v).$$

The upper limit of κ is taken as 1. It may not be correct to extend the above formula to $\kappa=1$ using the mass at $\kappa=0$, m_c. However, since the contribution of the integrand in (39) is dominant in the region of $\kappa<10^{-3}$, we might use this carrier mass.

(ii) $H=0$

For comparison we consider the case in the absence of a magnetic field. By the same reason as in the presence of a magnetic field, we take into account only the diagonal terms:

$$U_2(t\,;\,MM\,;\,K)=-\frac{1}{\hbar^2}\int_0^t dt_1\int_0^{t_1} dt_2(M|\mathcal{H}'(t_1)\mathcal{H}'(t_2)|M)_{av}$$

$$=-\frac{1}{\hbar^2}\int_0^t d\tau\int_{-\infty}^{\infty} dE(t-\tau)\exp(-iE\tau/\hbar)f_M(E), \tag{40}$$

$$f_M(E)=\frac{k_0T}{2\mu Vs^2}E_{1c}^2\sum_q[\delta(a_c\overline{k+q}^2-a_ck^2+\hbar\omega_q-E)+\delta(a_c\overline{k-q}^2-a_ck^2-\hbar\omega_q-E)]$$

$$+\frac{k_0T}{2\mu Vs^2}E_{1v}^2\sum_q[\delta(a_v\overline{k+q}^2-a_vk^2+\hbar\omega_q-E)+\delta(a_v\overline{k-q}^2-a_vk^2-\hbar\omega_q-E)]. \tag{41}$$

Thus

$$\Gamma_{M0}\equiv\frac{2\pi}{\hbar}f_M(0)=\frac{k_0T}{\pi\hbar^3\mu s^2}(m_cE_{1c}^2+m_vE_{1v}^2)k. \tag{42}$$

This is a sum of the inverse of the relaxation times of electron and hole in the ordinary acoustic scattering theory.

We calculate the shift from (28). We can expand (40) using the relations $1+\gamma/a_iq>2/q\times k\cos\theta$ and $|1-\gamma/a_iq|>2/q\cdot k\cos\theta$, where θ is the angle between vectors $\boldsymbol{k}$ and $\boldsymbol{q}$. These conditions are fulfilled from the fact that phonon energy $\hbar sq$ is equal to the energy gain of an electron. To the second order of k, we have the following expression for the shift:

$$\Delta_{M0}=-\frac{k_0T}{(2\pi)^2\mu s^2}\frac{E_{1c}^2}{a_c}\left[2q_0+\log\frac{q_0-\gamma/a_c}{q_0+\gamma/a_c}-\frac{4}{3}k^2\left\{\frac{q_0+\gamma/a_c}{(q_0+\gamma/a_c)^2}+\frac{q_0-\gamma/a_c}{(q_0-\gamma/a_c)^2}\right\}\right]$$

$$-\frac{k_0T}{(2\pi)^2\mu s^2}\frac{E_{1v}^2}{a_v}\left[2q_0+\log\frac{q_0-\gamma/a_v}{q_0+\gamma/a_v}-\frac{4}{3}k^2\left\{\frac{q_0+\gamma/a_v}{(q_0+\gamma/a_v)^2}+\frac{q_0-\gamma/a_v}{(q_0-\gamma/a_v)^2}\right\}\right], \tag{43}$$

where $$q_0=\sqrt{6\pi^2}/a.$$

Particularly, if we put $k=0$ it gives the shift of the absorption edge:

$$\Delta_{M0}(\boldsymbol{k}=0)=-\frac{k_0T}{(2\pi)^2\mu s^2}\left[\frac{E_{1c}^2}{a_c}\left(2q_0+\frac{\gamma}{a_c}\log\frac{a_cq_0-\gamma}{a_cq_0+\gamma}\right)\right.$$

$$\left.+\frac{E_{1v}^2}{a_v}\left(2q_0+\frac{\gamma}{a_v}\log\frac{a_vq_0-\gamma}{a_vq_0+\gamma}\right)\right]. \tag{44}$$

It will be seen that (44) agrees with the result reported by Muto-Ohyama,[7] if we notice that $E_{1c}=2/3\cdot C_c$ and if we take into account the fact that we have used the approximation of $N_q\simeq N_q+1$.

$\Gamma_{M0}=\alpha\kappa$ and $\Delta_{M0}/\hbar=-\beta+\delta\kappa^2$ being set, the general absorption coefficient becomes from (26)

$$A_0(\omega)=\frac{2e^2\pi^2}{3a^3cn_0\hbar}\left|\int_{cell}\psi'_{c0}\psi'_{v0}\,r\,dv\right|^2\int\frac{\omega\alpha\kappa^3\,d\kappa}{(\omega-\varepsilon_M/\hbar+\beta-\delta\kappa^2)^2+\alpha^2\kappa^2/4}. \tag{45}$$

Further if we put $\delta+\hbar\pi^2/2m^*a^2=\delta'$ and $\omega-E_0/\hbar+\beta=w$, after integration (45) appears to be

$$A_0(\omega)=\frac{\pi^2e^2}{3a^3cn_0\hbar}\left|\int_{cell}\psi'_{c0}\psi'_{v0}\,r\,dv\right|^2\left\{\frac{1}{2\delta'}\log\left|\delta'^2t^2+\left(\frac{\alpha^2}{4}-2w\delta'\right)t+w^2\right|\right.$$
$$\left.-\left(\frac{\alpha^2}{4}-2w\delta'\right)\frac{I_B}{2\delta'^2}\right\}\Bigg|_0^1, \tag{46}$$

where

$$I_B=\begin{cases}\dfrac{2}{\sqrt{\alpha^2w\delta'-\alpha^4/16}}\tan^{-1}\left[\dfrac{2\delta't+(\alpha^2/4-2w\delta')}{\sqrt{\alpha^2w\delta'-\alpha^4/16}}\right] & \dfrac{\alpha^4}{16}-\alpha^2w\delta'<0\\[2ex] \dfrac{1}{\sqrt{\alpha^4/16-\alpha^2w\delta'}}\log\left|\dfrac{2\delta'^2t+(\alpha^2/4-2w\delta')-\sqrt{\alpha^4/16-\alpha^2w\delta'}}{2\delta'^2t+(\alpha^2/4-2w\delta')+\sqrt{\alpha^4/16-\alpha^2w\delta'}}\right| & \dfrac{\alpha^4}{16}-\alpha^2w\delta'>0.\end{cases} \tag{47}$$

As we shall know later, relations $\delta'>w>\alpha$ hold for real substances. Since the first term of (46) gives a correction smaller than 1% to the second term as we can easily show, we may neglect the first term in (46). As a result of this approximation, the absorption coefficient is given as

$$A_0(\omega)\simeq\frac{2e^2}{3cn_0\hbar}\left|\int_{cell}\psi'_{c0}\psi'_{v0}\,r\,dv\right|^2\left(\frac{\hbar}{2m^*}+\frac{a^2}{\pi^2}\delta\right)^{-3/2}\omega\sqrt{\omega-E_0/\hbar+\beta}. \tag{48}$$

By introducing phonons the absorption coefficient at each $\boldsymbol{k}$ has a width and a shift. Superposition of such absorption coefficient curve shifts its absorption edge to the lower energy side as much as $\hbar\beta$ and changes the curvature of the absorption spectrum by the factor $(\hbar/2m^*+a^2/\pi^2\delta)^{-3/2}$ (see Fig. 2). This change of the curvature corresponds to the fact that carriers have smaller effective masses than those in the case of no electron-phonon interactions. Usually, this change of the curvature is very small.

(iii) *The ratio of the absorption coefficient in the presence of a magnetic field to that in the absence of a magnetic field*

From (40) and (48) we obtain

$$\frac{A_H(\omega)}{A_0(\omega)}\simeq\frac{3eH\hbar^{1/3}}{\pi acE_0^{3/2}}\left(\frac{\hbar}{2m^*}+\frac{a^2}{\pi^2}\delta\right)^{3/2}$$
$$\times\sum_n\int_0^1\frac{\epsilon_0C_T/\kappa\,d\kappa}{\{\epsilon-1-\epsilon_\kappa-(\epsilon_c+\epsilon_v)(n+1/2)\}^2+\epsilon_0^2C_T^2/\kappa^2}\cdot\frac{1}{\sqrt{\epsilon-1+\epsilon_\Delta}}, \tag{49}$$

where $$\epsilon_\Delta=\hbar\beta/E_0.$$

As we can see that the integral depends on H slightly, the ratio is nearly propor-

tionl to H^2. The ω-dependence of (49) shows also oscillations because of a factor $\{\epsilon-1-\epsilon_\kappa-(\epsilon_c+\epsilon_v)(n+1/2)\}$ but it has not infinite peaks. In the same way as in the case of no-phonon, the ratio of the transmission intensity is given by

$$\frac{I_t(H)}{I_t(0)} \simeq \exp\Big[-\frac{2e^2}{cn_0\hbar}\Big|\int_{cell}\psi_{c0}\psi_{v0}\,r\,dv\Big|^2\epsilon \times\Big\{\frac{eH}{\hbar\pi ac}\sum_n\int_0^1\frac{(\epsilon_0 C_T/\kappa)\,d\kappa}{\{\epsilon-1-\epsilon_\kappa-(\epsilon_c+\epsilon_v)(n+1/2)\}^2+\epsilon_0^2\overline{C_T^2}/\kappa^2} -\frac{E_0^{3/2}}{3\hbar^{3/2}}\Big(\frac{\hbar}{2m^*}+\frac{a^2}{\pi^2}\delta\Big)^{-3/2}\sqrt{\epsilon-1+\epsilon_\kappa}\Big\}d\Big]. \tag{50}$$

§ 4. Numerical examples and discussions

When we apply the above-mentioned theory to real crystals, there will be

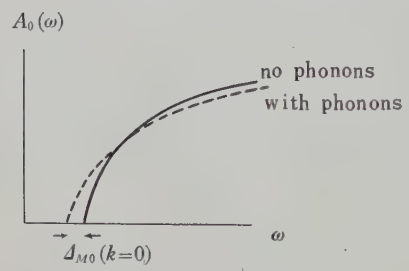


Fig. 2. Absorption coefficient in the absence of a magnetic field.

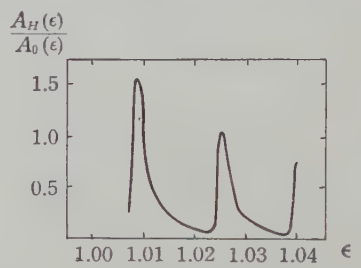


Fig. 3. Ratio of the absorption coefficient in the presence and the absence of a magnetic field considering phonons.

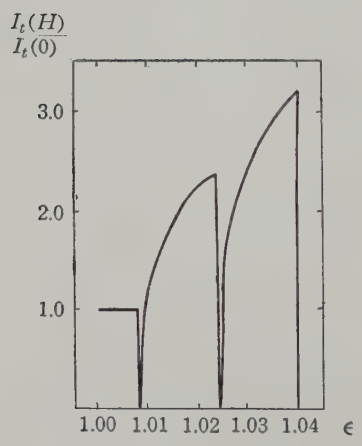


Fig. 4. Ratio of the transmission intensity in the presence and the absence of a magnetic field with no phonons.

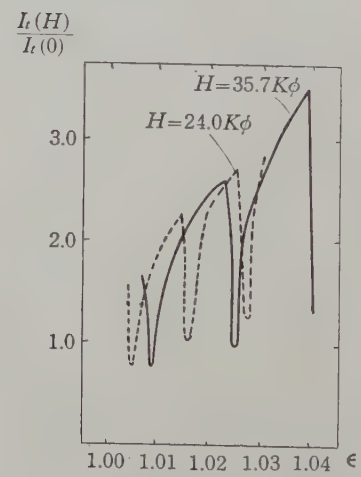


Fig. 5. Ratio of the transmission intensity in the presence and the absence of a magnetic field with phonons.

many problems to be justified. Especially the assumption of the simple spherical band model is not realistic for germanium and silicon. As is well known, the valence band of germanium has fourfold degeneracy at $\boldsymbol{k}=0$ if the spin-orbit coupling is taken into account. In the presence of a magnetic field this degeneracy is lifted, so that there are four sets of levels of the hole. And the energy level designated by n in each set is given by a linear combination of states of harmonic oscillator with n and $n-2$, as being treated by Luttinger and Kohn[3] and by Luttinger.[8] For this reason the selection rule concerning n becomes $\Delta n=0$ and -2. Therefore the results of this paper cannot be compared with the experimental data. In order to get a measure of criterion, however, we calculate numerically for a simplified model of germanium. We will use the following numerical values.

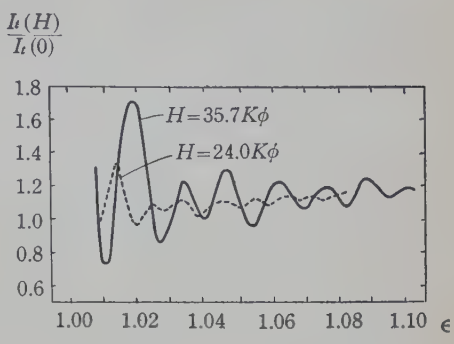


Fig. 6. Experimental data of ratio of the transmission intensity in the presence and the absence of a magnetic field (after Lax et al.).

$T=298°\mathrm{K}$, $\mu_s^2=2.41\times10^{11}\,g/\mathrm{sec}^2\,\mathrm{cm}^2$, $m_v=0.33\,m_0$, $m_c=0.036\,m_0$, $m^*=0.033\,m_0$, $E_0=0.803\,\mathrm{ev}$, $a=5.62\,\text{Å}$, $d=7\,\mu$, $H=35.7K\phi$, $24.0K\phi$ and $n_0=4$.

For parameters of the deformation potential we made use of the values determined by Bardeen and Shockley[9] from their mobility data. So we get

$$E_{1c}=1.7\,\mathrm{ev} \quad \text{and} \quad E_{1v}=2.4\,\mathrm{ev}.$$

$|\int_{cell}\psi'_{c0}\psi'_{v0}r\,dv|^2$ is obtained by fitting the first minimum of $I_t(H)/I_t(0)$ to the value observed by Lax et al. In Fig. 3 $A_H(\omega)/A_0(\omega)$ is plotted by taking into account the interactions of electrons and holes with phonons. In Fig. 4 and Fig. 5 $I_t(H)/I_t(0)$ is plotted as a function of ω with neglect of phonon effects and in consideration of phonon effects, respectively. The experimental data of $I_t(H)/I_t(0)$ is shown in Fig. 6. In Figs. 5 and 6 the graphs for $H=35.7K\phi$ and $24.0K\phi$ are given.

Let us now consider how the width depends upon H and T. In the peak which corresponds to the transition of $n=0$, the width of the spectrum at which $I_t(H)/I_t(0)=1$ is 0.002 ev in Fig. 5 for $H=35.7K\phi$. The experimental value given by Lax et al. is also 0.002 ev. As we have seen from Fig. 5, the absorption line does not depend on H appreciably. The observations agree with this fact. From the form of (26) and (30), we can see that the absorption constant does not depend on temperature T for a constant magnetic field. The width does not, therefore, show the same temperature dependence as the color center absorption. Although the experiments at 4.2°K is reported, it cannot be compared with

the theory at once because the impurity scattering must be important at that temperature. The above-mentioned slight dependence of the absorption constant upon H and T is understood as follows (see Figs. 7 and 8). Let us consider a fixed frequency ω. The absorption coefficient at this ω is given by an area which is enveloped by $A(k_z)$ calculated at each k_z. Then we get the actual absorption line width by plotting the area against ω. When H (or T) increases, however, the area does not appreciably increase so that the width also does not. We can thus conclude that the change of H (or T) under practical experimental conditions gives only a small effect on a total absorption coefficient.

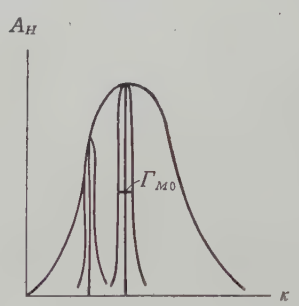


Fig. 7. H-dependence of the absorption coefficient (at smaller H).

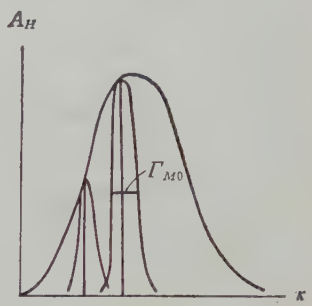


Fig. 8. H-dependence of the absorption coefficient (at larger H).

The above-mentioned results are different from the experimental facts particularly in the following two points: The experimental results show that $I_t(H)/I_t(0)$ does not increase with frequency ω, and that the low energy side of peak has a larger width than that obtained by theory. Here we cannot give the explanation of these discrepancies.

Burstein et al. assumed that the width is independent of the wave vector of electrons, κ, and they obtained $2\times10^{-4}E_0$ as the value of the width by fitting their theory to the experiment. In our theory the width is given by superposing Γ_{M0} at various κ. In doing so, we obtained the value which agrees with that obtained by Burstein et al.

Recently, Ohta et al.[11] investigated the same problem. They used nearly the same approximation as ours, but treated the δ-funtions in $f_{ss'}^{s'}(E)$ in (4·1) of reference 11) exactly. This made further calculations very complicated. Therefore, they calculated only the line width of the first absorption peak and obtained the dependence of Γ_{s0} on H by calculating $J_\alpha(k_0, H)$ ((4·9) of reference 11)) numeri cally. On the other hand, we neglected $\hbar\omega_q$ in the arguments of the δ-functions appearing in the $f_M(E)$ in (24), because the energy $\hbar\omega_q$ of the phonons emitted or absorbed with appreciable probability is much smaller than that of the electron. Therefore, we could obtain in analytical forms how Γ_{M0} depends on H and T and conclude its nearly linear dependence on H and the linear dependence on T in (29).

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Approximate Solution of the Bethe-Salpeter Equation for Two Fermions

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The eigenfunction and the binding energy of the ground state are obtained for the bound state of two fermions, which are interacting through photons or scalar photons in the ladder approximation. The Bethe-Salpeter equation is separated into four independent sets of equations by applying the Tani-Foldy-Wouthuysen transformations. These equations are further simplified, if a part of recoil is neglected. Here retardation is completely taken into account. The wave function is expanded in a form convenient to obtain spin singlet and triplet solutions. As an example, the ground state of positronium is solved using Wick's method.

§ 1. Introduction

The Bethe-Salpeter equation for the relativistic two-body problem is the basis of the covariant formalism of bound states. This equation has, however, been solved only for special cases so far, for instance, the bound state of two spinless particles interacting through scalar photons in the ladder approximation.[1] More realistic applications have been done for the mass corrections to the hydrogen spectrum[2] and the energy levels of positronium[3] for Coulomb interactions.

In the present paper, we would like to solve the Bethe-Salpeter equation for two fermions interacting through photons, or, more simply, scalar photons in the ladder approximation. Our approach is different from the works on similar problems[2),3] in the following point; we take into account relative time, in other words, retardation, as much as possible. On the other hand, we are obliged to neglect higher power terms in p/m, where p and m are the relative momentum and the mass of the fermions. This approximation means that a considerable part of recoil is neglected. In this sense, the present method is still less elegant than Wick's one because of the complicated natures of spinors.

§ 2 is devoted to the expansion of wave functions. This new representation is convenient to separate out the part corresponding to spin singlet and triplet states in the nonrelativistic limit.

In § 3 the Tani-Foldy-Wouthuysen transformation[4),5] is applied in an extended form to the Bethe-Salpeter equation. First of all, the original equations, which are simultaneous equations for the sixteen components of the wave function, are separated into two sets, each having eight components. Second, they are further re-

duced to equations for four components, by using unitary transformations characteristic of two-body problem. This reduction can be achieved for any interaction, though infinitely many times transformations must be carried out.

The expansion of the wave function given in § 2 is applied to the reduced equations. Solutions are immediately found for the spin singlet and triplet states in § 4 for both positive and negative energy states.

In § 5 the lowest energy eigenvalue is found for positronium by the straightforward application of Wick's method.[1] Since the spin dependent terms are neglected here as higher order in p/m, the separation of the ^1S and ^3S states is not obtained. The deviation of energy spectrum from the nonrelativistic Balmer formula is of order 10^{-5} ev.

§ 2. Expansion of the wave function

In this section mathematical preliminaries are given in order to make discussions of two-fermion problems[6] easy. Although some of them are not always necessary to our simple examples in the later sections, they are shown for the sake of convenience in further applications.

1. *Expansion of two-body wave function*

A two-body wave function $g_{\alpha\beta}(12)$ is defined by

$$g_{\alpha\beta}(12) \equiv (\Psi_0,\ T[\psi_\alpha(1)\psi_\beta(2)]\Psi), \tag{2·1}$$

where $\psi(1)$ is the field operator of the fermion labelled (1). $g_{\alpha\beta}(12)$ is expanded as

$$\begin{aligned} g_{\alpha\beta}(12) &= \sum_s a^s_{\alpha\beta} B^s(12), \\ B^s(12) &\equiv (\Psi_0,\ T[\psi(1)C^{-1}O^s\psi(2)]\Psi), \end{aligned} \tag{2·2}$$

where O^s is one of the 16 independent γ-matrices whose explicit forms are given in Table I. Note that γ's are all hermitian matrices, C is the charge conjugation matrix which satisfies

$$C^+C=1,\quad C^T=-C,$$

and

$$C^{-1}O^sC=\pm(O^s)^T \qquad \begin{matrix} +\text{for } S,\ PS,\ PV, \\ -\text{for } V,\ T. \end{matrix} \tag{2·3}$$

It is readily seen from the definitions (2·1) and (2·2) that

Table I. Explicit forms of O^s in γ-matrices and ρ-σ formalism

	S	PS	PV	V	T
O^s	1	γ_5	$i\gamma_5\gamma_\mu$	γ_μ	$(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i$
ρ-σ	1	$-\rho_1$	$\rho_3\boldsymbol{\sigma}, -\rho_2$	$\rho_2\boldsymbol{\sigma},\ \rho_3$	$\boldsymbol{\sigma},\ \rho_1\boldsymbol{\sigma}$

$$\sum_s a^s_{\alpha\beta}(C^{-1}O^s)_{\mu\nu}=\delta_{\mu\alpha}\delta_{\nu\beta}. \tag{2·4}$$

In order to find the coefficients $a^s_{\alpha\beta}$, the matrix $(O^{s'}C)_{\nu\mu}$ is multiplied and we take trace. One finds

$$a^{s'}_{\alpha\beta}=(O^{s'}C)_{\beta\alpha}/4,$$

or combining this with (2·3),

$$a^s_{\alpha\beta}=\mp(O^sC)_{\alpha\beta}/4, \tag{2·5}$$

where $-$ is valid for S,PS,PV, and $+$ for V,T.

2. *Spin exchange operator*

In the nonrelativistic theory the spin exchange operator $P_{\boldsymbol{\sigma}}$ is related with the space coordinate exchange operator $P_{\boldsymbol{x}}$, namely

$$P_{\boldsymbol{\sigma}}=P_{12}P_{\boldsymbol{x}}, \tag{2·6}$$

where P_{12} is the exchange operation of the particles 1 and 2. In the relativistic theory the spin exchange operator is defined, related with the parity operator for two identical particles in the center of mass system. The parity operator is

$$\Pi=\rho_3^{(1)}\rho_3^{(2)}P_{\boldsymbol{x}}. \tag{2·7}$$

$P_{\boldsymbol{x}}$ denotes the exchange of the space coordinates, and (1) and (2) refer to the particles 1 and 2. Accordingly, the spin exchange operator is

$$P_{\boldsymbol{\sigma}}=P_{12}\Pi=P_\gamma\rho_3^{(1)}\rho_3^{(2)}P_t, \tag{2·8}$$

where P_γ is the exchange operator of spin and energy state, and is of the form[6]

$$P_\gamma=\frac{1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{2}\cdot\frac{1+\boldsymbol{\rho}^{(1)}\boldsymbol{\rho}^{(2)}}{2}. \tag{2·9}$$

P_t is the relative time exchange operator.

3. *Transformation properties of* B^s

In this subsection we only consider the special case where P_t does not play any role ($P_t=1$). For example, two free particles in the center of mass system is the case. Then the parity operation is written as

$$\Pi g(12)=\rho_3^{(1)}\rho_3^{(2)}P_x g(12),$$

where P_x is the exchange operator of four-dimensional coordinates. From the definition (2·2) one sees

$$\begin{aligned}P_x g(12)&=\sum_s a^s B^s(21)\\&=\sum_s a^s(\Psi_0,\ T[\psi(2)C^{-1}O^s\psi(1)]\Psi)\\&=\sum_s a^s(\Psi_0,\ T[-\psi(1)(C^{-1}O^s)^T\psi(2)]\Psi).\end{aligned}$$

Using the relation (2·3), we obtain

$$P_x g(12) = \sum_s \pm a^s B^s(12)$$
$$+\text{for } S,\ PS,\ PV,$$
$$-\text{for } V,\ T. \qquad (2\cdot10)$$

In order to complete the parity operation, (2·4) is helpful, namely

$$(\rho_3^{(1)})_{\alpha\alpha'}(\rho_3^{(2)})_{\beta\beta'}\sum_s a^s_{\alpha'\beta'}(C^{-1}O^s)_{\mu\nu} = (\rho_3^{(1)})_{\alpha\mu}(\rho_3^{(2)})_{\beta\nu}$$

and

$$\sum_s a^s_{\alpha\beta}(C^{-1}O^s)_{\mu'\nu'}(\rho_3^{(1)})_{\mu'\mu}(\rho_3^{(2)})_{\nu'\nu} = (\rho_3^{(1)})_{\alpha\mu}(\rho_3^{(2)})_{\beta\nu}.$$

Combining these two relations, we finally get

$$\Pi g(12) = \rho_3^{(1)}\rho_3^{(2)}\sum_s \pm a^s B^s(12)$$
$$= \sum_s \pm a^s(\Psi_0,\ T[\psi(1)\rho_3{}^T C^{-1}O^s\rho_3\psi(2)]\Psi)$$
$$= \sum_s \mp a^s(\Psi_0,\ T[\psi(1)(C^{-1}\rho_3 O^s\rho_3)\psi(2)]\Psi)$$
$$= \sum_s (-\varepsilon^s)a^s B^s(12). \qquad (2\cdot11)$$

Here ε^s is displayed in Table II. For the spin exchange operator the result is

$$P_\Sigma g(12) = \sum_s \varepsilon^s a^s B^s(12). \qquad (2\cdot12)$$

Table II.

	S	PS	PV		V		T	
O^s	1	γ_5	$i\gamma_5\gamma_i$	$i\gamma_5\gamma_4$	γ_i	γ_4	$(1/i)\gamma_i\gamma_j$	$(1/i)\gamma_4\gamma_i$
ε^s	+	−	+	−	+	−	−	+

It is found that ten components, S, $V(1, 2, 3)$., $PV(1, 2, 3)$, and $T(14, 24, 34)$, belong to spin triplet states, and six components, PS, $V(4)$, $PV(4)$, and $T(12, 23, 31)$, describe spin singlet states, when the wave function is independent of the relative time.

4. *Properties of* $g(12)$

If $g(12)$ is independent of the relative time t we can find some relations among components of $g(12)$. For triplet states, one sees from (2·8)

$$g_{\alpha\beta}(12) = P_\Sigma g_{\alpha\beta}(12) = P_\gamma \rho_3^{(1)}\rho_3^{(2)} g_{\alpha\beta}(12). \qquad (2\cdot13)$$

Then $g_{\alpha\beta}$ satisfies

$$g_{12} = g_{21},\ g_{13} = -g_{31},\ g_{14} = -g_{41},\ g_{23} = -g_{32},\ g_{24} = -g_{42},\ g_{34} = g_{43}. \qquad (2\cdot14)$$

It is again seen that 10 independent components belong to triplet states.

Since we have $a_{\alpha\beta}=\mp a_{\beta\alpha}$, $-$ for S, PS, PV, $+$ for V, T, from (2·5), $g_{\alpha\alpha}$, g_{12} and g_{34}, are expressed by combinations of $V(1, 2, 3)$ and $T(14, 24, 34)$. And g_{13}, g_{14}, g_{23}, and g_{24} are given by S and $PV(1, 2, 3)$.

For singlet states we have

$$g_{11}=g_{22}=g_{33}=g_{44}=0,\ g_{12}=-g_{21},$$
$$g_{34}=-g_{43},\ g_{13}=g_{31},\ g_{14}=g_{41},\ g_{23}=g_{32},\ g_{24}=g_{42}, \tag{2·15}$$

where g_{12} and g_{34} are expressed by PS and $PV(4)$, and g_{13}, g_{14}, g_{23}, and g_{24} are given by $V(4)$ and $T(12, 23, 31)$.*

§ 3. Tani-Foldy-Wouthuysen transformation

The Bethe-Salpeter equation for two identical fermions, which are interacting through $S(s)$ photons in the ladder approximation, is

$$(\gamma\partial+m)^{(1)}(\gamma\partial+m)^{(2)}g(12)=f^2D_F(12)g(12), \tag{3·1}$$

f being the coupling constant. In momentum space the total momentum and energy are separated, and we have the equation for the relative four-momentum $p(\boldsymbol{p}, \varepsilon)$ in the center of mass system,

$$\left\{i\boldsymbol{\gamma}^{(1)}\boldsymbol{p}-\gamma_4^{(1)}\left(\frac{1}{2}E+\varepsilon\right)+m\right\}\left\{-i\boldsymbol{\gamma}^{(2)}\boldsymbol{p}-\gamma_4\left(\frac{1}{2}E-\varepsilon\right)+m\right\}g(p)$$
$$=\frac{-if^2}{(2\pi)^4}\int\frac{d^4k}{(p-k)^2}g(k), \tag{3·2}$$

E being the total energy of the system. $g(p)$ is the Fourier component of the B-S wave function of the relative coordinates.

Now we apply the Tani-Foldy-Wouthuysen transformation[4),5)] to the respective particles, i.e. for the fermion (1)

$$U^{(1)}=\exp\left\{\frac{i}{2}\frac{(\boldsymbol{\sigma}^{(1)}\boldsymbol{p})}{|\boldsymbol{p}|}\rho_2^{(1)}\tan^{-1}\left(\frac{|\boldsymbol{p}|}{m}\right)\right\}. \tag{3·3}$$

The (3·2) is transformed as follows:

$$\left\{\rho_3^{(1)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E+\varepsilon\right)\right\}\left\{\rho_3^{(2)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E-\varepsilon\right)\right\}h(p)$$
$$=-\frac{if^2}{(2\pi)^4}\rho_3^{(1)}\rho_3^{(2)}\int\frac{d^4kh(k)}{(p-k)^2}\{1+O((p/m)^2)\}, \tag{3·4}$$

where we put

$$h(p)=U^{(1)}U^{(2)}g(p). \tag{3·5}$$

* When $g(12)$ depends on the relative time, the component g_{12}, for instance, for singlet state is no longer given by a combination of PS, $PV(4)$ only. In general, it may be given by a combination of PS, $PV(4)$, $V(1, 2, 3)$, and $T(14, 24, 34)$. Note that $V(4)$ never appears in g_{12}.

Hereafter we neglect higher order terms in p/m.

For electromagnetic interaction, instead of Eq. (3·4) one obtains

$$\left\{\rho_3^{(1)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E+\varepsilon\right)\right\}\left\{\rho_3^{(2)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E-\varepsilon\right)\right\}h(p)$$

$$=-\frac{ie^2}{(2\pi)^4}(1-\rho_1^{(1)}\rho_1^{(2)}\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})\int\frac{d^4k\,h(k)}{(p-k)^2}\{1+O((p/m)^2)\}. \qquad (3\cdot6)$$

(3·6) is the sum of the product of even operators for both of the particles, $\varepsilon^{(1)}\varepsilon^{(2)}$, and of odd operators, $\mathcal{O}^{(1)}\mathcal{O}^{(2)}$, but $\varepsilon^{(1)}\mathcal{O}^{(2)}$ and $\mathcal{O}^{(1)}\varepsilon^{(2)}$ are not involved. Consequently, one sees that (3·6) has already been separated into two independent sets of equations: one is coupled between "*large*"×"*large*" and "*small*"×"*small*" components, and the other is coupled between "*small*"×"*large*" and "*large*"×"*small*".

Next task is to find how to separate "*large*"×"*large*" from "*small*"×"*small*". In the present case, the $\mathcal{O}^{(1)}\mathcal{O}^{(2)}$ term in (3·6) is apparently smaller than 1, so that the final form of the equation is given by

$$\left\{\rho_3^{(1)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E+\varepsilon\right)\right\}\left\{\rho_3^{(2)}\sqrt{\boldsymbol{p}^2+m^2}-\left(\frac{1}{2}E-\varepsilon\right)\right\}h(p)$$

$$=-\frac{ie^2}{(2\pi)^4}\int d^4k\frac{h(k)}{(p-k)^2}\{1+O((p/m)^2)\}. \qquad (3\cdot7)$$

In general, this procedure is done by a unitary transformation.* In (3·4) and (3·7) each component of the wave function is solved independently.

The recoil terms neglected in (3·4) or (3·7) is unfortunately of the same order of magnitude with the retardation, which we take into account. Both of them are of order $(f^2/4\pi)^2$. In order to obtain complete information on order $(f^2/4\pi)^2$ in binding energy, and so on, $(p/m)^2$ terms in interaction and $(f^2/4\pi)^2$ diagrams must be taken in addition to retardation.

In spite of these simplification of the equation, we are still going to expand $h(p)$ as (2·2), in order to make the interpretation of solutions easy. Of course, the argument in § 2 is approximately valid.

(3·4) is expanded as

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* Actually, the unitary transformation in this case is given by

$$V=\exp\left[-\frac{\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}}{4E\sqrt{\boldsymbol{p}^2+m^2}}(\rho_2^{(1)}\rho_1^{(2)}+\rho_1^{(1)}\rho_2^{(2)})\frac{e^2}{(2\pi)^4}\int\frac{d^4k}{(p-k)^2}\right],$$

where

$$\int\frac{d^4k\,h(k)}{(p-k)^2}\approx h(p)\int\frac{d^4k}{(p-k)^2}+O\left(\frac{p}{m}\right)$$

is used. Note that the relations

$$\rho_3^{(1)}=\rho_3^{(2)}, \quad \text{and} \quad \rho_3^{(1)}\rho_3^{(2)}=1$$

hold for "*large*"×"*large*" and "*small*"×"*samll*" components.

$$\sum_s \sum_{\alpha'\beta'} \Big[\Big\{ \rho_3^{(1)} \sqrt{\boldsymbol{p}^2+m^2} - \Big(\frac{1}{2}E+\varepsilon\Big)\Big\}_{\alpha\alpha'} \Big\{ \rho_3^{(2)} \sqrt{\boldsymbol{p}^2+m^2} - \Big(\frac{1}{2}E-\varepsilon\Big)\Big\}_{\beta\beta'} B^s(p)$$

$$+ \frac{if^2}{(2\pi)^4} (\rho_3^{(1)})_{\alpha\alpha'} (\rho_3^{(2)})_{\beta\beta'} \int \frac{d^4k B^s(k)}{(p-k)^2} \Big] a^s_{\alpha'\beta'} = 0,$$

or

$$\sum_s \Big[\Big\{ \rho_3 \sqrt{\boldsymbol{p}^2+m^2} - \Big(\frac{1}{2}E+\varepsilon\Big)\Big\} a^s \Big\{ \rho_3 \sqrt{\boldsymbol{p}^2+m^2} - \Big(\frac{1}{2}E-\varepsilon\Big)\Big\}^T B^s(p)$$

$$+ \frac{if^2}{(2\pi)^4} \rho_3 a^s \rho_3^T \int \frac{d^4k B^s(k)}{(p-k)^2} \Big]_{\alpha\beta} = 0.$$

Multiply $C^{-1}O^{s'}$, and take trace taking (2·5) into account,

$$\mathrm{Tr} \sum_s O^{s'} \Big[\Big\{ \rho_3 \sqrt{\boldsymbol{p}^2+m^2} - \Big(\frac{1}{2}E+\varepsilon\Big)\Big\} \Big(\mp\frac{1}{4}O^s\Big) \Big\{ -\rho_3 \sqrt{\boldsymbol{p}^2+m^2}$$

$$- \Big(\frac{1}{2}E-\varepsilon\Big)\Big\} B^s(p) - \frac{if^2}{(2\pi)^4} \rho_3 \Big(\mp\frac{1}{4}O^s\Big) \rho_3 \int d^4k \frac{B^s(k)}{(p-k)^2} \Big] = 0. \quad (3\cdot8)$$

Finally, we obtain

$$\Big(\boldsymbol{p}^2+m^2+\frac{E^2}{4}-\varepsilon^2\Big) B_4^{PV}(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B_4^{PV}(k)}{(p-k)^2} - iE \sqrt{\boldsymbol{p}^2+m^2} B^{PS}(p) = 0, \quad (3\cdot9a)$$

$$\Big(\boldsymbol{p}^2+m^2+\frac{E^2}{4}-\varepsilon^2\Big) B^{PS}(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B^{PS}(k)}{(p-k)^2} + iE \sqrt{\boldsymbol{p}^2+m^2} B_4^{PV}(p) = 0, \quad (3\cdot9b)$$

$$\Big(\boldsymbol{p}^2+m^2+\frac{E^2}{4}-\varepsilon^2\Big) B_{4i}^T(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B_{4i}^T(k)}{(p-k)^2} + iE \sqrt{\boldsymbol{p}^2+m^2} B_i^V(p) = 0, \quad (3\cdot9c)$$

$$\Big(\boldsymbol{p}^2+m^2+\frac{E^2}{4}-\varepsilon^2\Big) B_i^V(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B_i^V(k)}{(p-k)^2} - iE \sqrt{\boldsymbol{p}^2+m^2} B_{4i}^T(p) = 0, \quad (3\cdot9d)$$

$$\Big(\boldsymbol{p}^2+m^2-\frac{E^2}{4}+\varepsilon^2\Big) B_4^V(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B_4^V(k)}{(p-k)^2} + 2\varepsilon \sqrt{\boldsymbol{p}^2+m^2} B^S(p) = 0, \quad (3\cdot9e)$$

$$\Big(\boldsymbol{p}^2+m^2-\frac{E^2}{4}+\varepsilon^2\Big) B^S(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B^S(k)}{(p-k)^2} + 2\varepsilon \sqrt{\boldsymbol{p}^2+m^2} B_4^V(p) = 0, \quad (3\cdot9f)$$

$$\Big(\boldsymbol{p}^2+m^2-\frac{E^2}{4}+\varepsilon^2\Big) B_{ij}^T(p) + \frac{if^2}{(2\pi)^4} \int \frac{d^4k B_{ij}^T(k)}{(p-k)^2}$$

$$- 2\varepsilon \sqrt{\boldsymbol{p}^2+m^2}\, \varepsilon(ijk4) B_k^{PV}(p) = 0, \quad (3\cdot9g)$$

$$\left(\boldsymbol{p}^2+m^2-\frac{E^2}{4}+\varepsilon^2\right)B_i^{PV}(p)+\frac{if^2}{(2\pi)^4}\int\frac{d^4kB_i^{PV}(k)}{(p-k)^2}$$
$$-2\varepsilon\sqrt{\boldsymbol{p}^2+m^2}\,\varepsilon(ijk4)\,B_{jk}^{T}(p)=0. \qquad (3\cdot9h)$$

For electromagnetic interaction f^2 is replaced by $-e^2$ in (3, 9a, b, c, d). Other eight equations are not given here.*

§ 4. Physical meaning of solutions

The compatibility condition of (3·9a) and (3·9b) is found to be

$$\left(\boldsymbol{p}^2+m^2+\frac{E^2}{4}-\varepsilon^2\right)B^{PS}(p)+\frac{if^2}{(2\pi)^4}\int\frac{d^4kB^{PS}(k)}{(p-k)^2}\pm E\sqrt{\boldsymbol{p}^2+m^2}B^{PS}(p)=0, \qquad (4\cdot1)$$

and exactly the same equation for $B_4^{PV}(p)$.

The sign + or − must be the same for B_4^{PV} and B^{PS}. Incidentally, this equation is directly obtained from (3·4).

If (4·1) holds, a non-vanishing solution of (3·9a) and (3·9b) is found. The ratio of B_4^{PV} and B^{PS} reads

$$B_4^{PV}(p)=\mp iB^{PS}(p), \qquad (4\cdot2)$$

according to ± in (4·1). The situation is the same for (3·9c) and (3·9d).

Let us consider the case

$$B_4^{PV}(p)=iB^{PS}(p),$$

and other B's are chosen to be zero. The wave function $h(12)$ in (3·5) is written as

$$h(12)=\sum_s a^sB^s(12)=(a^{PS}+ia_4^{PV})B^{PS}(12)$$
$$=\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)B^{PS}(12). \qquad (4\cdot3)$$

Here explicit representations are employed for the Dirac matrices and $C=i\gamma_4\gamma_2$. (4·3) is just the ordinary singlet wave function $(\alpha^{(1)}\beta^{(2)}-\beta^{(1)}\alpha^{(2)})/\sqrt{2}$.

On the other hand, the + sign in (4·1) corresponds to the singlet state for two fermions both in negative energy state,

$$h(12)=(a^{PS}-ia_4^{PV})B^{PS}(12)$$
$$=\left(\begin{array}{c|cc} 0 & 0 & \\ \hline & 0 & 1 \\ 0 & -1 & 0 \end{array}\right)B^{PS}(12). \qquad (4\cdot4)$$

---

* Green and Biswas have simplified the *B-S* equation for two fermions in a different way.[7]

It is needless to repeat discussions for triplet states.*

The meaning of (4·1) is clarified in the following way: First the retardation in the interaction term is disregarded, and integrate over the relative energy. Defining

$$b(\boldsymbol{p})=\int d\varepsilon B(p), \tag{4·5}$$

one sees

$$b(\boldsymbol{p})+\frac{if^2}{(2\pi)^4}\int d\varepsilon\int\frac{d^3kb(\boldsymbol{k})}{(\boldsymbol{p}-\boldsymbol{k})^2}\times\frac{1}{(\sqrt{\boldsymbol{p}^2+m^2}\pm E/2)^2-\varepsilon^2}=0.$$

This yields, using the ordinary contour integral for ε,

$$(2\sqrt{\boldsymbol{p}^2+m^2}\pm E)\,b(\boldsymbol{p})-\frac{f^2}{(2\pi)^3}\int\frac{d^3kb(\boldsymbol{k})}{(\boldsymbol{p}-\boldsymbol{k})^2}=0, \tag{4·6}$$

which is nothing but the Schrödinger equation in the nonrelativistic limit.

(3·9e) and (3·9f) are compatible provided that the equation

$$\left(\boldsymbol{p}^2+m^2-\frac{E^2}{4}+\varepsilon^2\right)B(p)+\frac{if^2}{(2\pi)^4}\int\frac{d^4kB(k)}{(p-k)^2}\pm 2\varepsilon\sqrt{\boldsymbol{p}^2+m^2}B(p)=0 \tag{4·7}$$

is satisfied, and we have

$$B_4{}^V(p)=\pm B^S(p). \tag{4·8}$$

Accordingly, the explicit form of the solutions are

$$\left(\begin{array}{cc|c} 0 & & 0 \\ & & \\ \hline 0 & 1 & \\ -1 & 0 & 0 \end{array}\right)B^s(12),\quad\text{and}\quad\left(\begin{array}{c|cc} 0 & 0 & 1 \\ \hline & -1 & 0 \\ 0 & & \\ & 0 & \end{array}\right)B^s(12), \tag{4·9}$$

which indicate the spin singlet states of a fermion-antifermion system. In this case the argument concerned with singlet and triplet states given in the end of § 2-3 fails, because the relative time plays an important role.

§ 5. Energy level of the bound state

The remaining work is to find $B(p)$ and energy eigenvalues which satisfy (4·1), at least, to find the lowest energy level of the bound state.

First we take the square of the operator of (4·1) and neglect the f^4 term, then we have

$$\left(\boldsymbol{p}^2+m^2+\frac{1}{4}E^2-\varepsilon^2\right)^2B(p)+\frac{2if^2}{(2\pi)^4}\left(\boldsymbol{p}^2+m^2+\frac{1}{4}E^2-\varepsilon^2\right)\int\frac{d^4kB(k)}{(p-k)^2}-$$
$$-E^2(\boldsymbol{p}^2+m^2)B(p)=0. \tag{5·1}$$

* It seems to be fortuitous that PS and $PV(4)$ represent purely the singlet state, nevertheless the relative time plays a role in general.

It is noted that there might be redundant solutions of (5·1) which are not solutions of (4·1).

(5·1) is rewritten in a covariant form introducing the total momentum P,

$$\left(p^2-pP+\frac{P^2}{4}+m^2\right)\left(p^2+pP+\frac{P^2}{4}+m^2\right)B(p)$$
$$+\frac{2if^2}{(2\pi)^4}\left(p^2-\frac{P^2}{4}+m^2\right)\int\frac{d^4kB(k)}{(p-k)^2}=0. \tag{5·2}$$

In (5·2) p^2 in $(p^2-P^2/4+m^2)$ can be neglected compared with $-P^2/4+m^2$, while other p cannot be disregarded.

Now we can apply Wick's method.[1),8)] Let the solution be representable as

$$B(p)=\int_{-1}^{1}\frac{g(z)\,dz}{[p^2+zpP+P^2/4+m^2]^3}. \tag{5·3}$$

From (5·2) we have the ordinary differential equation for $g(z)$,

$$(1-z^2)g''(z)+\frac{2f^2}{(4\pi)^2}\frac{m^2+E^2/4}{m^2-(1-z^2)E^2/4}g(z)=0 \tag{5·4}$$

with the boundary condition

$$g(\pm1)=0. \tag{5·5}$$

Put $E=2m\eta$, then η is very approximately equal to 1, $0\ll\eta<1$, since the binding energy E_B is small. Hence we can approximate

$$\frac{m^2+E^2/4}{m^2-(1-z^2)E^2/4}=\frac{1+\eta^2}{1-\eta^2+\eta^2z^2}\approx\pi\left(\frac{1+\eta^2}{\eta^2}\right)\left(\frac{1-\eta^2}{\eta^2}\right)^{-1/2}\delta(z). \tag{5·6}$$

The lowest eigenfunction is given by

$$g(z)\approx\pi\left(\frac{1+\eta^2}{\eta^2}\right)\left(\frac{1-\eta^2}{\eta^2}\right)^{-1/2}g(0)\frac{f^2}{(4\pi)^2}(1-|z|) \tag{5·7}$$

with the eigenvalue

$$\frac{f^2}{4\pi}=\frac{4(1-\eta^2)^{1/2}\eta}{1+\eta^2}. \tag{5·8}$$

Most of the argument given in this paper is valid for two different particles. We apply, therefore, the same method to the lowest energy level of positronium, where the electron and positron are treated as different particles. Any way, pair annihilation connot be treated in our approximation. In this case f^2 is replaced by e^2, and the binding energy of the ground state is immediately obtained from (5·8),

$$E_B=\frac{\alpha^2}{4}m\left(1+\frac{1}{16}\alpha^2+\cdots\right)\approx6.8\text{ ev}. \tag{5·9}$$

where $\alpha = e^2/4\pi$.

The second term is considered to be the deviation from the nonrelativistic Balmer formula, which is of order 10^{-5} ev. In our approximation ^3S and ^1S separation is not obtained, which is slightly larger than the above-mentioned deviation. It is, therefore, desirable to calculate the energy spectrum of positronium more precisely, for example, the ^3S and ^1S separation as a small perturbation using the above-solved wave function as the unperturbed one.

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On the Electromagnetic Mass Differences among the Sigma Hyperons

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The problem of the mass difference of the triplet sigma is discussed in terms of an electromagnetic self-energy by taking account of the virtual process $\Sigma^0 \to \Lambda^0 + \gamma \to \Sigma^0$ and the process $\Sigma^0 \to \Sigma^0 + \gamma \to \Sigma^0$ for the self-mass of the neutral sigma on an equal footing. The electromagnetic form factor is introduced to serve as a cutoff factor. With the e^2-approximation, it can be seen that the process $\Sigma^0 \to \Lambda^0 + \gamma \to \Sigma^0$ plays an important role to study the mass splitting of Σ's along this line.

§ 1. Introduction

In the strong interactions among the strange particles and nucleons, there is no definite experimental evidence against the Nakano-Nishijima-Gell-Mann scheme. When the charge independence of such a scheme for the strong interactions holds, it will be natural to assume that the observed mass differences between the baryons belonging to the same charge multiplet will originate, to a large extent, from the electromagnetic self-energy. In this point of view the theoretical explanation of the large mass splittings[1] among Σ^+, Σ^- and Σ^0,

$$M_{\Sigma^-} - M_{\Sigma^+} = (6.84 \pm 0.28) \text{ Mev},$$

$$M_{\Sigma^0} - M_{\Sigma^+} = (2.39 \pm 0.37) \text{ Mev},$$

will be especially interesting.

When we assume the mass splitting is due to the electromagnetic self-energy, it may be reasonable to reproduce the above mass differences along the line of approach to the problem by Feynman and Speisman[2] who succeeded in explaining the neutron-proton mass difference with a cutoff perturbation theory by using the experimental values of nucleon anomalous magnetic moments. Here the anomalous magnetic moment of Σ is not yet measured and the mass differences depend on the assumed values of the magnetic moments. About the general properties of Σ anomalous magnetic moments, study in the non-relativistic cutoff theory may be instructive. For instance, we know the fact that the experimental values of the nucleon anomalous magnetic moment can be reproduced with the non-relativistic cutoff theory in static approximation. In the case of Σ's too, this situation would

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not be so largely converted. By the calculation[3),4)] along this line, it may be expected that

i) the absolute values of the charged sigma's anomalous magnetic moments will be of the order of nucleon anomalous magnetic moments, and

ii) the sign of μ_+ and μ_- will be positive and negative respectively.* On this standpoint we shall again view the papers on the Σ mass splittings published up to now.

Many attempts[3),5)-8)] have been made to explain the mass differences of Σ by taking account of the contributions from Figs. 1(a) and 1(b). To reproduce the observed mass difference of Σ, with a straight cutoff $\lambda=K/M\simeq 0.5$ (K is the cutoff momentum and M is the unperturbed Σ hyperon mass), Katsumori[5)] required rather large absolute values for μ_+ and μ_-, i. e. $\mu_+<-8.02$ or $\mu_+>5.96$. With $\lambda=1.0$ and by assuming $\mu_-=1.27$, he obtained $\mu_+=1.66$ or -3.83 and $\mu_0=0.915$ or -0.915. In second order perturbation approximation Sudarshan and Marshak[6)] demanded that $\mu_++\mu_-\simeq 3\sim 4$ with Feynman cutoff parameter $1.4M\sim 2M$, where μ_- is always assumed to be positive. Kato and Takeda[3)] evaluated the mass differences by using the exponential form function with r.m.s. (root mean square) radius $0.8y$ ($1y$ (yukawa) $=10^{-13}$ cm). With $\mu_+=1.93$ and $\mu_-=-1.77$ which are their estimates of μ's from the static theory (they assumed the global symmetry of the *pi*-mesonic interaction and the contribution from the K-mesonic interaction was assumed to be small), they obtained the mass difference between Σ^+ and Σ^-,

$$|M_{\Sigma^-}-M_{\Sigma^+}|\simeq 0.08 \text{ Mev}.$$

Recently Zimerman[8)] made consideration about some possibilities to explain the mass splittings of the Σ multiplet for reasonable signs and magnitudes of the anomalous magnetic moments. He introduced a form factor which is the superposition of two Yukawa-type form functions and assumed that the charge distribution of Σ^0 is zero. If all the sigmas have the same form factor except the charge distribution of Σ^0, there should necessarily result that $\mu_->0$, $\mu_0>0$ or $\mu_-<0$, $\mu_0<0$ to realize the experimental values of the mass differences. On the other hand $\mu_-<0$, $\mu_0>0$ can explain the mass differences when Σ^0 has a different structure from charged sigmas; e. g. the neutral sigma's distribution is an exponential type with

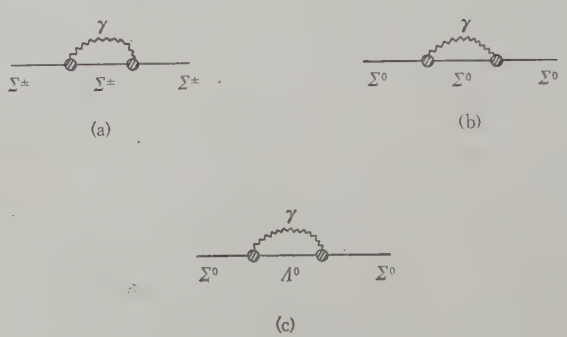


Fig. 1. The self-energy of Σ in e^2-approximation.

* Hereafter we shall these terms i) and ii) as our "conditions" which μ_+ and μ_- may satisfy. $\mu_\pm$ and μ_0 refer to the anomalous magnetic moments of $\Sigma^\pm$ and Σ^0 in unit of the hyperon magneton, respectively.

a r.m.s. radius $r_0(\approx r_c/8$, where r_c is the r.m.s. radius of charged sigma which he assumed to be $0.8y$). For $\mu_-<0$, $\mu_0<0$, it is necessary to assume different structures for the three particles. All the above but Zimerman could not obtain the observed mass splittings with such $\mu_\pm$ that fulfil our conditions. They[3),5)-8)] did not pay attention to the contribution from the virtual process $\Sigma^0\to\Lambda^0+\gamma\to\Sigma^0$ (Fig. 1(c)), but this will have to be taken into account.* As far as the strangeness conservation law is correct, Figs. 1(a), 1(b) and 1(c) are to be considered on an equal footing. We shall note in parentheses that: If the global symmetry exists, $\mu_0\cong(1/2)(\mu_p+\mu_n)$, $\mu_0'\cong(1/2)(\mu_p-\mu_n)$ and $|\mu_0'|\gg|\mu_0|$.** So the contribution of μ_0' cannot be neglected. Here μ_0' is a contribution from the vertices Fig. 2. The mass difference would come from the internal structure of the particle and the type of the form factor is responsible for the results,[8),9),10)] but at present we are too ignorant about the Σ hyperon structure of inner region, so here we shall not enter into the discussion of the form factor. To gain a clear understanding of our position, in that we wish to suggest the importance of Fig. 1(c), we shall introduce a simple type form function to play only the role of cutoff of divergence.

§ 2. Calculation

We shall discuss the problem in the second order perturbation approximation. The interactions of the sigma with the photon are described by the Hamiltonians, $(e>0)$

$$
\begin{aligned}
H_{\Sigma^+} &= -ie\int dx\,dy\,\bar{\Sigma}^+(x)\gamma_\mu\Sigma^+(x)F_+(x-y)A_\mu(y)\\
&\quad -\frac{1}{2}\frac{e\mu_+}{2M}\int dx\,dy\,\bar{\Sigma}^+(x)\sigma_{\mu\nu}\Sigma^+(x)G_+(x-y)F_{\mu\nu}(y),\\
H_{\Sigma^-} &= +ie\int dx\,dy\,\bar{\Sigma}^-(x)\gamma_\mu\Sigma^-(x)F_-(x-y)A_\mu(y)\\
&\quad -\frac{1}{2}\frac{e\mu_-}{2M}\int dx\,dy\,\bar{\Sigma}^-(x)\sigma_{\mu\nu}\Sigma^-(x)G_-(x-y)F_{\mu\nu}(y), \qquad (1)^{***}\\
H_{\Sigma^0} &= -\frac{1}{2}\frac{e\mu_0}{2M}\int dx\,dy\,\bar{\Sigma}^0(x)\sigma_{\mu\nu}\Sigma^0(x)G_0(x-y)F_{\mu\nu}(y)
\end{aligned}
$$

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* When this calculation was finished it came into our notice that Katsumori,[9)] who tried to explain the observed mass differences by examination of the type of form factor, suggested that there was the possibility that the situation might be changed by taking account of this diagram. The author wishes to express her thanks to Dr. H. Katsumori who sent her a preprint of his paper before its publication.

** Unfortunately, a description in footnote 4) of the reference paper 7) that $\mu_0'\simeq\mu_0$ is a mistake.

*** In this article we adopt the natural units: $\hbar=c=1$.

$$-\frac{1}{2}\frac{e\mu_0'}{2M}\int dx\,dy\,\bar{\Lambda}^0(x)\gamma_5'\sigma_{\mu\nu}\Sigma^0(x)G_0'(x-y)F_{\mu\nu}(y)+\text{H. c.},^*$$

where it has been assumed that i) the spin of Σ and Λ are 1/2, ii) the charge distribution of the neutral hyperon is identically zero and iii) the propagation functions of the hyperons, in the strong interaction with the other fields, can satisfactorily be approximated by the conventional propagation function for the

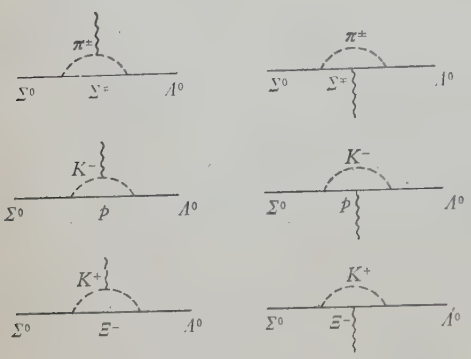


Fig. 2. The eg^2-order vertices producing the Pauli moment-like interaction between Σ^0 and Λ^0.

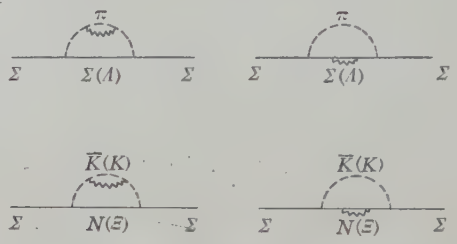


Fig. 3. The contribution from the mass difference of baryon multiplet and of meson multiplet to the Σ self-energy.

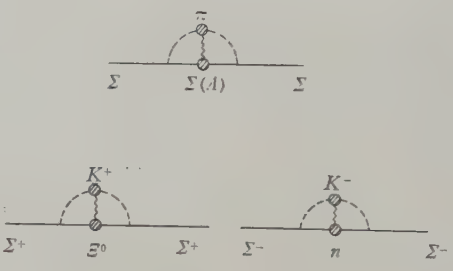


Fig. 4. The magnetic self-energy of Σ.

point hyperon. μ_0' represents the transient magnetic moment between Σ^0 and Λ^0 which comes from the diagram of Fig. 2 and causes the transition $\Sigma^0 \rightleftarrows \Lambda^0+\gamma$. M is the unperturbed mass of the Σ hyperon. The notation γ_5' is to be read 1 or $i\gamma_5$ according to the even or odd relative parity between Σ and Λ: $P(\Sigma\Lambda)$. We shall take as that the effects of the higher order corrections of the strong interaction (e. g. Figs. 3, 4), i. e. the influence of the pion interaction and of the kaon interaction, etc., are small.[3),8),9)] So we shall carry on this calculation in the following with neglect of these effects.

The self-masses of the sigma hyperons are given in e^2-approximation,

$$\Delta M^+ = -\frac{ie^2}{(2\pi)^4}\int\frac{d^4k}{k^2}\left[\pm\gamma_\mu F_\pm(k)+\frac{i\mu_\pm}{4M}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_\pm(k)\right]\frac{1}{i\gamma p-i\gamma k+M}$$

* The vertex part which describes the process $\Sigma^0\to\Lambda^0+\gamma$ (real or virtual) is generally written as the sum of three terms which are proportional to γ_μ, k_μ and $\sigma_{\mu\nu}$.[11)] For the case of the emission of a real photon the first two terms do not contribute since $k^2=0$ and $\varepsilon_\mu k_\mu=0$. It was shown that even in the process of a virtual photon emission the term of $\sigma_{\mu\nu}$ gives the main contribution.[11),12)]

$$\times\left[\pm\gamma_\mu F_\pm(k)-\frac{i\mu_\pm}{4M}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_\pm(k)\right]\Big|_{i\gamma p=iM}, \tag{2}$$

$$\Delta M^0=-\frac{ie^2}{(2\pi)^4}\left(\frac{\mu_0}{4M}\right)^2\int\frac{d^4k}{k^2}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)\frac{1}{i\gamma p-i\gamma k+M}(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_0^2(k)\Big|_{\gamma p=iM}$$

$$-\frac{ie^2}{(2\pi)^4}\left(\frac{\mu_0'}{4M}\right)^2\int\frac{d^4k}{k^2}\gamma_5'(\gamma_\mu\gamma k-\gamma k\gamma_\mu)\frac{1}{i\gamma p-i\gamma k+M_\Lambda}\gamma_5'(\gamma_\mu\gamma k-\gamma k\gamma_\mu)G_0'^2(k)\Big|_{\gamma p=iM}.$$

For the sake of simplicity we shall restrict ourselves to the consideration of 3-dimensional form factors, namely $F(k)=F(\boldsymbol{k})$. This will not cause qualitative difference for the present problem. After carrying out the integration over k_0 and solid angle, the mass differences become

$$\Delta M^- - \Delta M^+ = \frac{e^2}{(2\pi)^2}M\int dQ\Big[\{F_-^2(MQ)-F_+^2(MQ)\}\left(-\frac{Q^2}{\sqrt{Q^2+1}}+\frac{1}{\sqrt{Q^2+1}}+Q\right)$$
$$+3\{\mu_-F_-(MQ)G_-(MQ)+\mu_+F_+(MQ)G_+(MQ)\}\frac{Q^2}{\sqrt{Q^2+1}}$$
$$+\frac{5}{4}\{\mu_+^2G_+^2(MQ)-\mu_-^2G_-^2(MQ)\}\frac{Q^2}{\sqrt{Q^2+1}}\Big], \tag{3}$$

$$\Delta M^0 - \Delta M^+ = \frac{e^2}{(2\pi)^2}M\int dQ\Big[F_+^2(MQ)\left(\frac{Q^2}{\sqrt{Q^2+1}}-\frac{1}{\sqrt{Q^2+1}}-Q\right)$$
$$+3\mu_+F_+(MQ)G_+(MQ)\frac{Q^2}{\sqrt{Q^2+1}}+\frac{5}{4}\{\mu_+^2G_+^2(MQ)-\mu_0^2G_0^2(MQ)\}\frac{Q^2}{\sqrt{Q^2+1}}$$
$$-\frac{2M\pm3M_\Lambda}{4M}\mu_0'^2G_0'^2(MQ)\frac{Q^2}{\sqrt{Q^2+\kappa}}$$
$$-\frac{M^2-M_\Lambda^2}{4M^2}\mu_0'^2G_0'^2(MQ)\,\mathcal{P}\left(\frac{1}{Q+\sqrt{Q^2+\kappa}-1}-\frac{1}{Q+\sqrt{Q^2+\kappa}+1}\right)\frac{Q^2}{\sqrt{Q^2+\kappa}}\Big]. \tag{4}$$

Here $\kappa=(M_\Lambda/M)^2$, $Q=|\boldsymbol{k}|/M$, and $\mathcal{P}$ indicates that the principal value of the four dimensional Fourier integral is to be taken. In Eq. (4) we must read the upper sign when $P(\Sigma\Lambda)$ is even, the lower sign for $P(\Sigma\Lambda)$ odd.

For simplicity's sake we assume that the spatial extents of the charges of Σ^+ and Σ^- are the same, and also the same distribution to the charge distribution occurs for the anomalous magnetic moments of Σ^+, Σ^- and Σ^0:

$$F_+=G_+=F_-=G_-=G_0=G_0'=F. \tag{5}$$

This assumption will be seen not to injure the conclusion of this article. In the following we assume the distribution has the exponential shape, and this density distribution is expressed by means of Fourier transformation as

$$F(MQ)=1/\{1+(M^2a^2/12)Q^2\}^2, \tag{6}$$

where a gives a root mean square radius. After Eqs. (3) and (4) are arranged, we obtain the relations (7) and (8) of the a.m.m. to the mass differences corresponding to the evenness and oddness of the relative parity of Λ to Σ, respectively.

[I] $P(\Sigma\Lambda)$: even

$$\left(\mu_+ + \frac{6}{5}\right)^2 - \left(\mu_- - \frac{6}{5}\right)^2 = \frac{[\Delta M^- - \Delta M^+]_{obs.}}{(e^2/(2\pi)^2)MR},$$

$$\left(\mu_+ + \frac{6}{5}\right)^2 - \mu_0^2 - \frac{T}{R}\mu_0'^2 = \frac{[\Delta M^0 - \Delta M^+]_{obs.}}{(e^2/(2\pi)^2)MR} + \frac{16}{25} + \frac{S}{R} + \frac{1}{6AR}. \qquad (7)$$

[II] $P(\Sigma\Lambda)$: odd

The same relations as (7) hold except that T is replaced by $-T'$. (8)

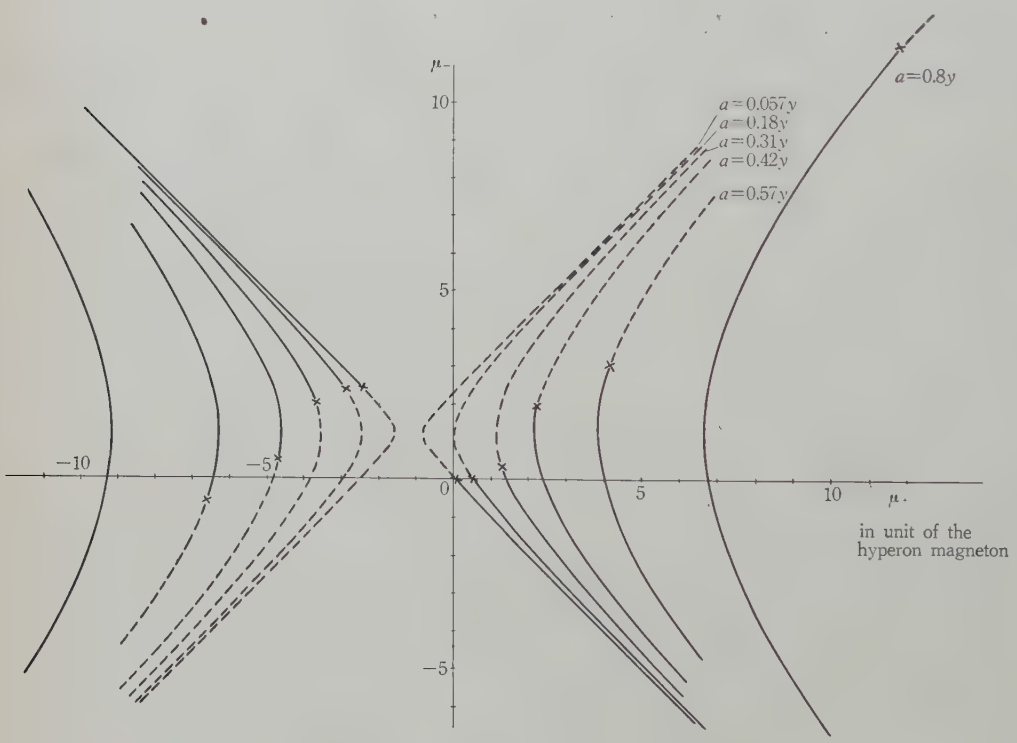


Fig. 5. The points which can explain the observed mass difference $(M_{\Sigma^-} - M_{\Sigma^+})$.
×: With $\mu_0' = 0$, these points give the observed mass differences for $(M_{\Sigma^-} - M_{\Sigma^+})$ and $(M_{\Sigma^0} - M_{\Sigma^+})$ at the same time.
——: Points on this curve (μ_+, μ_-) with corresponding μ_{0I}' $(|\mu_{0I}'| \ll 0)$ in Fig. 6 give the observed mass differences $(M_{\Sigma^-} - M_{\Sigma^+})$ and $(M_{\Sigma^0} - M_{\Sigma^+})$ at the same time.
- - -: Points on this curve (μ_+, μ_-) with corresponding μ_{0II}' $(|\mu_{0II}'| \ll 0)$ in Fig. 7 give the observed mass differences $(M_{\Sigma^-} - M_{\Sigma^+})$ and $(M_{\Sigma^0} - M_{\Sigma^+})$ at the same time.

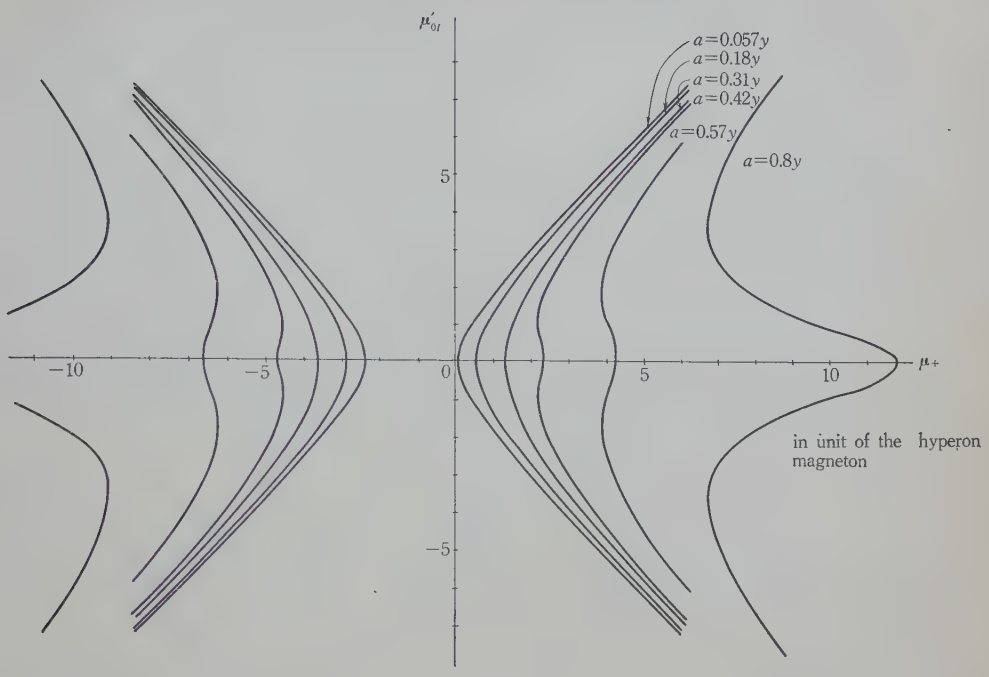


Fig. 6. The values of μ'_{0I} which can explain the observed mass difference $(M_{\Sigma^0}-M_{\Sigma^+})$ with corresponding solid curve in Fig. 5. The intersecting point with μ_+-axis corresponds to the point × in Fig. 5.

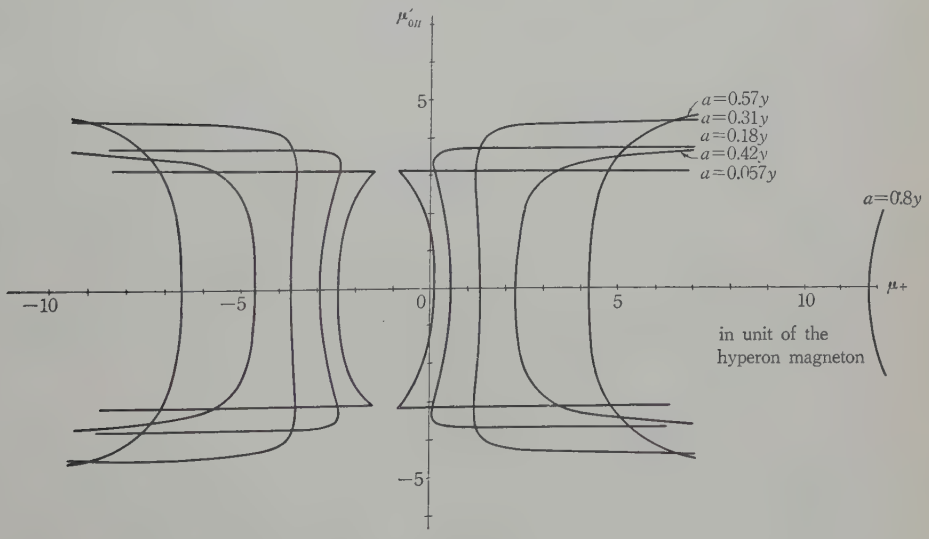


Fig. 7. The values of μ'_{0II} which can explain the observed mass difference $(M_{\Sigma^0}-M_{\Sigma^+})$ with corresponding dashed curve in Fig. 5. The intersecting point with μ_+-axis corresponds to the point × in Fig. 5.

Here

$$R=\frac{5}{4}\int dQ\frac{Q^2}{\sqrt{Q^2+1}}\frac{1}{(1+AQ^2)^4},$$

$$S=\int dQ\frac{1}{\sqrt{Q^2+1}}\frac{1}{(1+AQ^2)^4}, \tag{9}$$

$$T=\int dQ\left\{\frac{2M+3M_\Lambda}{4M}+\frac{M^2-M_\Lambda^2}{4M^2}\mathcal{P}\left(\frac{1}{Q+\sqrt{Q^2+\kappa}-1}-\frac{1}{Q+\sqrt{Q^2+\kappa}+1}\right)\right\}$$
$$\times\frac{Q^2}{\sqrt{Q^2+\kappa}}\frac{1}{(1+AQ^2)^4},$$

$$T'=\int dQ\left\{\frac{3M_\Lambda-2M}{4M}-\frac{M^2-M_\Lambda^2}{4M^2}\mathcal{P}\left(\frac{1}{Q+\sqrt{Q^2+\kappa}-1}-\frac{1}{Q+\sqrt{Q^2+\kappa}+1}\right)\right\}$$
$$\times\frac{Q^2}{\sqrt{Q^2+\kappa}}\frac{1}{(1+AQ^2)^4},$$

and

$$A=M^2a^2/12.$$

The corresponding values of the integrations in Eq. (9) to the various r.m.s. radii are listed in Table I.

Table I. Calculated values of the integrations in Eq. (9) for various r.m.s. radii.

A	a ($y=10^{-13}$ cm)	R	S	T	T'
0.01	0.057 y	19.8	2.12	19.1	3.16
0.1	0.18 y	1.64	1.13	1.59	0.251
0.3	0.31 y	0.450	0.763	0.438	0.0621
0.54	0.42 y	0.215	0.602	0.210	0.0282
1.0	0.57 y	0.0956	0.457	0.0956	0.0116
1.95	0.8 y	0.0393	0.334	0.0404	0.00341

In the Appendix we present the relations of μ to the mass differences corresponding to the various r.m.s. radii. Figs. 5, 6 and 7 give the curves (μ_+, μ_-), (μ_+, μ'_{0I}) and (μ_+, μ'_{0II}) which can explain the observed mass differences $(M_{\Sigma^-}-M_{\Sigma^+})$ and $(M_{\Sigma^0}-M_{\Sigma^+})$ for several values of a, respectively. Here suffix I represents the case when $P(\Sigma\Lambda)$ is even, II when $P(\Sigma\Lambda)$ is odd.

§ 3. Discussion

The problem of the nucleon mass difference has long been investigated[13] from the point of view similar to the present one. If the same exponential type of distributions is assumed for the protonic charge and the magnetic moments of the proton and the neutron, while the charge distribution of the neutron is taken to be zero everywhere, it is required that their r.m.s. radius should be $0.42y$ in order

to obtain the observed n-p mass difference with the given parameters: μ_p, μ_n.

To get a clue we first analyze the sigma mass difference by using a cutoff which corresponds to a r.m.s. radius of $0.42y$. In this case, if we neglect μ_0', the effect of the diagrams Fig. 2, only the values of ($\mu_+>0$, $\mu_->0$) or ($\mu_+<0$, $\mu_-<0$) can explain the observed mass difference ($M_{\Sigma^-}-M_{\Sigma^+}$) and ($M_{\Sigma^0}-M_{\Sigma^+}$) (see Table II). Neither of these two pairs of μ's satisfies our conditions i) and ii). For obtaining the values which satisfy our conditions it is necessary to introduce suitable μ_0' at this cutoff. The points (μ_+, μ_-) on the solid curve in Fig. 5 with corresponding μ_{0I}' in Fig. 6 or the points (μ_+, μ_-) on the dashed curve in Fig. 5 with corresponding μ_{0II}' in Fig. 7 can explain the observed mass differences. With suitable μ_{0I}' we can get $\mu_+>0$, $\mu_-<0$ (see Table III and Fig. 5). If we assume the conditions i) and ii) in § 1, there is no real μ_{0II}' which can explain the observed mass differences. However, if we give up these conditions, both μ_{0I}' and μ_{0II}' can play roles to explain the mass differences. As far as we demand the conditions stated in § 1, the relative parity between Σ and Λ has to be even in order to explain the observed mass differences ($M_{\Sigma^-}-M_{\Sigma^+}$) and ($M_{\Sigma^0}-M_{\Sigma^+}$) at the same time. We can put the foregoing in another way that: Even though we neglect and have no regard for the contribution of μ_0', the experimental values of Σ mass splitting can be reproduced by taking some μ_+, μ_- and a (even if these values may not be adequate). The sets of such parameters are given in Table II. But when we take the parameter a that can explain the nucleon mass difference, it is the conclusive factor to consider the problem with μ_{0I}' for reproducing the observed Σ mass splittings by μ_+ and μ_- which fulfil the conditions.

We do not know whether the structure of Σ is the same as that of nucleon

Table II. Values of pair (μ_+, μ_-) which, with $\mu_0'=0$, can explain the observed mass differences ($M_{\Sigma^-}-M_{\Sigma^+}$) and ($M_{\Sigma^0}-M_{\Sigma^+}$).

a	μ_+	μ_-
0.057 y	0.078	−0.028
	or −2.48	2.43
0.18 y	0.515	−0.003
	or −2.92	2.40
0.31 y	1.29	0.354
	or −3.69	2.05
0.42 y	2.26	1.90
	or −4.66	−0.497
0.57 y	4.20	3.03
	or −6.60	−0.627
0.8 y	11.7	11.3
	or −14.1	−8.93

or not, so it may be necessary to examine the situation with the other parameters. As is seen, with any parameter a which appears in Table II, we cannot get the values that satisfy our conditions. These conditions can be fulfilled only by introducing μ'_{0I}. As examples we shall quote some values of the a.m.m. which can explain the observed mass differences by adding μ'_{0I} in Table III (see also Figs. 5 and 6). In any case of these parameters a as well as in the case of $0.42y$, we have no real μ'_{0I} corresponding to (μ_+, μ_-) which satisfy the conditions for explaining the observed mass differences. Here $2\mu_0=\mu_+ + \mu_-$: this is a consequence

Table III. Values of $\mu_\pm$, μ_0 and $\mu_0{}'_I$ which can explain the observed mass differences $(M_{\Sigma^-}-M_{\Sigma^+})$ and $(M_{\Sigma^0}-M_{\Sigma^+})$. I is the case when $P(\Sigma\Lambda)$ is even.

a	μ_+	μ_-	μ_0	$\mu_0{}'_I$
0.18 y	1.8	−1.52	0.140	2.48
	2	−1.76	0.122	2.75
	2.5	−2.28	0.110	3.33
	3	−2.82	0.092	3.90
0.31 y	1.8	−0.652	0.574	1.81
	2	−0.981	0.510	2.15
	2.5	−1.68	0.420	2.85
	3	−2.29	0.357	3.52
0.42 y	2.7	−0.70	1.0	2.63
	3	−1.28	0.859	3.09
	4	−2.75	0.628	4.42

of the charge independence of interactions and is perturbation independent.[5),7)] This relation can be extended for the complete magnetic moments including the form factors.[7)] As we have assumed the relations (5) among the form factors, this is exact. If the distribution of a Σ^0, which has the vertex as shown in Fig. 1(b) but not that of Fig. 1(c), is smaller than that of the charged sigma (in other words, a parameter a included in the form factor G_0 is smaller than that of the other form factors), the values of μ_0 which can reproduce the observed mass difference become smaller than that given in Table III with the values of $\mu_\pm$, μ'_{0I} in each corresponding row kept unchanged. This is not impossible, because there is no direct current contribution of pion interaction to μ_0. Though there are many sets of μ's to explain the mass differences (as is seen in Figs. 5, 6 and 7), in Table III we listed a part of the values which satisfy the condition that $\mu_+>0$ and $\mu_-<0$. When we demand $\mu_+>0$ and $\mu_-<0$, μ_0 is to be positive as far as we use our simple model which is defined by (5) and (6). For $\mu_0<0$, it is required that $\mu_+<0$. At any rate direct measurements of the anomalous magnetic moments of the Σ hyperons, especially of the sign of the anomalous magnetic moment μ_0, would be interesting.

The decay of Σ^0, $\Sigma^0 \to \Lambda^0 + \gamma$, is considered as being caused through μ_0'-interaction and the lifetime τ of Σ^0 is expressed as

$$1/\tau = (e^2/4\pi)\,(\mu_0'/4M)^2\,2(M^2 - M_\Lambda^2)^3/M^3 \cong 0.317\times 10^{19}\,\mu_0'^2\ (\mathrm{sec}^{-1}). \qquad (10)$$

The experimental upper limit of τ, $\tau < 10^{-11}$ sec,[1),14)] tells that

$$|\mu_0'| > 1.8\times 10^{-4}.$$

On the other hand we shall have a relation between the experimental uncertainty δM of Σ^0-mass determination and τ:

$$\delta M > 1/\tau. \qquad (11)$$

Eqs. (10) and (11) will combine and can give an upper limit of μ_0'. From the present experimental status of $\delta M = 0.53$ Mev and $\delta M/M \approx 1/2300$, we obtain

$$|\mu_0'| < 16.$$

In future, the measurements of μ_+, μ_-, μ_0 and μ_0' will be established and the structure of the hyperon Σ will be made clear. Then the importance of the contribution from the process $\Sigma^0 \to \Lambda^0 + \gamma \to \Sigma^0$ will be realized more clearly.

Acknowledgements

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Appendix

$$(\mu_+ + 6/5)^2 - (\mu_- - 6/5)^2 = X,$$

$$(\mu_+ + 6/5)^2 - \mu_0^2 - Y\mu_0'^2 = Z, \quad \text{for case I,}$$

or

$$(\mu_+ + 6/5)^2 - \mu_0^2 + Y'\mu_0'^2 = Z, \quad \text{for case II.}$$

Table IV. Values of X, Y, Y' and Z.

A	X	Y	Y'	Z
0.01	0.125	0.962	0.160	1.63
0.1	1.51	0.969	0.153	2.87
0.3	5.48	0.973	0.138	5.49
0.54	11.5	0.979	0.131	7.59
1.0	25.8	1.00	0.121	16.2
1.95	62.8	1.03	0.0868	33.3

In Table IV, we took the following values as the mass differences of Σ's and the bare mass of sigma hyperon:

$$[\Delta M^- - \Delta M^+]_{obs.} = 6.84 \text{ Mev},$$

$$[\Delta M^0 - \Delta M^+]_{obs.} = 2.39 \text{ Mev},$$

and

$$M = 1193 \text{ Mev}.$$

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On the $KK\pi\pi$ Interactions

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Low energy K meson-nucleon interaction is re-investigated under the following assumptions.

1) K meson is scattered via $KK\pi\pi$ interactions of the isospin independent and dependent type in addition to the direct K meson-hyperon-nucleon interactions.

2) In the low energy range (up to 250 Mev), S-wave K-nucleon scattering is dominant.

3) The source of S-wave K-mesons is extended to a comparable size as the potential range via the exchange of two pions.

The characteristic features of low energy K-nucleon scattering are shown to be reproduced under the above assumptions. It turns out that the $KK\pi\pi$ interaction of isospin dependent type plays an important role in K-nucleon scattering.

§ 1. Introduction

There is some evidence which might be considered to show the existence of $KK\pi\pi$ interactions. For instance, the pion production in K^+ collisions on protons above the threshold at 220 Mev was reported,[1] and some calculations have been successfully done through the mechanism via $KK\pi\pi$ interaction.[2] On the other hand, the existence of the effective strong $KK\pi\pi$ interactions is theoretically expected from the strong pion-nucleon and K meson-hyperon-nucleon interactions.

Such an interaction, if it exists, will play an important role in various strange particle reactions such as K meson-nucleon scattering, K meson-hyperon pair production, and so on. However, we have no direct method to attack this interaction, hence little has been known about it until now.

In this paper the K meson is considered to be scattered via the exchange of two pions between the nucleon and the K meson as well as via the direct K meson-hyperon-nucleon interactions (Fig. 1, Fig. 2). We shall discuss the results of the application of such a theoretical scheme to the K meson-nucleon scattering problem and get some information concerning the $KK\pi\pi$ interactions such as coupling types and the magnitude of the effective coupling

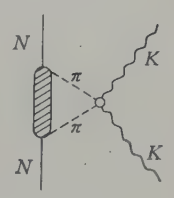


Fig. 1. Contribution to K-N scattering via the mechanism of the boson-boson interaction between two pions and two K mesons.

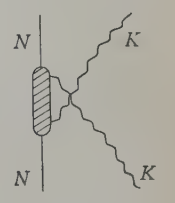


Fig. 2. Contribution to K-N scattering via the mechanism of the direct K meson-hyperon-nucleon interactions.

constant through the analysis of the K-nucleon scattering. We derive the conclusion that $KK\pi\pi$ interactions of both the isospin-independent type and the isospin-dependent type amalgamate into the direct K meson-hyperon-nucleon interactions to play an important role in the three K-nucleon scattering.

The several features of the recent experimental analysis[3] which we consider significant for our present purposes are:

a) The S wave is dominant at the energy region from zero to 250 Mev at least.

b) The effective K^+-p potential is repulsive.

c) The elastic K^+-n cross section increases up to 100 Mev and then decreases, while the charge exchange scattering cross section is very small at low energies and then increases rapidly and becomes equal to the K^+-n elastic cross section at about 240 Mev (Fig. 3). In § 2, we shall calculate the K-nucleon scattering with a proposed scheme, in § 3 $KK\pi\pi$ interactions are investigated by making use of the results in § 2 and § 4 is devoted to the discussion for the results thus obtained.

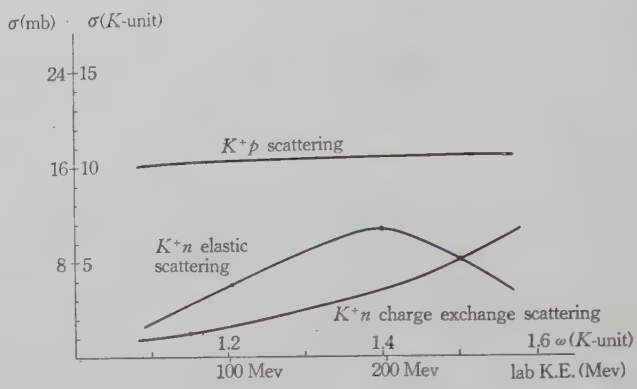


Fig. 3. Energy dependence of the cross section for the K^+-nucleon scattering.

§ 2. *K* meson-nucleon scattering

In this section, K-nucleon scattering is re-investigated under the following assumptions.

1) K meson is scattered via the exchange of two pions between the nucleon and K meson in addition to direct K meson-hyperon-nucleon interactions.

2) In the low energy range (up to 250 Mev), S-wave K-nucleon scattering is dominant.

3) The source of S-wave K mesons is extended to a comparable size as the potential range via the exchange of two mesons (Fig. 1).

In assumption 1), a possible form of $KK\pi\pi$ interaction will be derived by requiring the Lorentz invariance and the charge independence in the framework that no derivative of order higher than second will be admitted. $KK\pi\pi$ interactions under the above requirements are of the following types,

$$H_1=\lambda_1 K_i^* K_i \pi_j \pi_j \text{[4]}, \tag{2·1}$$

$$H_2=\lambda_2 i(K^* \tau_\alpha \partial_\mu K-\partial_\mu K^* \tau_\alpha K)(\pi\times\partial_\mu\pi)_\alpha, \tag{2·2}$$

$$H_3=\lambda_3(K_i^* \partial_\mu K_i)(\pi_j \partial_\mu \pi_j). \tag{2·3}$$

Here τ_α is the isospin operator for the K meson treated as a doublet $K=\begin{pmatrix}K^+\\K^0\end{pmatrix}$, $K^*=\begin{pmatrix}K^{+*}\\K^{0*}\end{pmatrix}$. We choose only H_1 as the isospin independent and H_2 as the isospin dependent $KK\pi\pi$ interactions to see the characteristic features. Another isospin independent interaction H_3 is not essentially different from H_1.

Assumption 3) means that the source of the S-wave K mesons which is probably made up of nucleon-antihyperon pairs extends to a potential range via exchanging two pions analogous to a treatment in S-wave pion-nucleon scattering.[5] We consider two cases of range $\approx(4\mu)^{-1}$ and $(6\mu)^{-1}$.[6] Now the effective Hamiltonian can be expressed as

$$H=A\left(\int\rho(r)\boldsymbol{K}(r)\,d^3r\right)^2+\int\boldsymbol{K}(r)\,V(r)\,\boldsymbol{K}(r)\,d^3r+\int\boldsymbol{K}(r)\,U(r)\,\boldsymbol{K}(r)\,d^3r, \tag{2·4}$$

where $\boldsymbol{K}(r)$ means the wave function of K meson

and $$A=-\frac{g_\Lambda^2}{2M}+\frac{3g_\Sigma^2}{2M} \quad \text{in the case of total isospin } I=0 \text{ state,} \tag{2·5}$$

$$=\frac{g_\Lambda^2}{2M}+\frac{g_\Sigma^2}{2M} \quad \text{in the case of total isospin } I=1 \text{ state.} \tag{2·6}$$

The first term corresponds to the S-wave scattering via the intermediate pair nucleon-antihyperon state through the direct K-hyperon-nucleon interaction. g_Λ^2 and g_Σ^2 are rationalized, renormalized pseudoscalar K-Λ-N and K-Σ-N coupling constant respectively. $V(r)$ means potential induced by the isospin independent four boson interaction of the type (2·1) and $U(r)$ means potential induced by the isospin dependent interaction of the type (2·2).

For simplicity of calculation we replace the second and the third term of (2·4) by a separable source version without losing the essential features of the problem. Then

$$H=A\bar{\boldsymbol{K}}\cdot\bar{\boldsymbol{K}}+\hat{V}(0)\bar{\boldsymbol{K}}\cdot\bar{\boldsymbol{K}}+\hat{U}(0)\boldsymbol{K}\cdot\boldsymbol{K} \tag{2·7}$$

with $\bar{\boldsymbol{K}}=\int\rho(r)\boldsymbol{K}(r)\,d^3r$, $V(r)=\hat{V}(0)\rho(r)$ and $U(r)=\hat{U}(0)\rho(r)$. Here the same function $\rho(r)$ is used for the second and the third term since the range of $V(r)$ and $U(r)$ is assumed by 3) to the same as that of $\rho(r)$. $\hat{V}(0)$ can be calculated to the lowest order in λ_1, by calculating the transition matrix for zero-momentum transfer in the static theory[7] in terms of the interaction (2·1) (Fig. 1).

$$\hat{V}(0)\cong\left(\frac{\lambda_1}{4\pi}\right)\left\{\left(\frac{f^2}{4\pi}\right)\frac{24}{\mu^2}\int_0^M\frac{k^4}{\omega^4}\,dk\right.$$

$$\left.+\frac{8}{3\pi^2}\int_0^M k^4\,dk\int\frac{dp}{\omega_p}\,\sigma_{33}(p)\,\frac{1}{\omega^2(\omega+\omega_p)^2}\left(1+\frac{\omega_p}{2\omega}\right)\right\}. \tag{2·8}$$

Here we assumed that the (3, 3) state dominates the integral. The first term in curly

brackets comes from the Born approximation expressed in terms of the rationalized, renormalized pion-nucleon coupling constant, $f^2/4\pi=0.08$; the second term involving an integral over the pion-nucleon large cross section σ_{33} contains corrections from the multiple scattering of the exchanged pions on the physical nucleon.

Quite similarly, $\hat{U}(0)$ follows from the isospin dependent $KK\pi\pi$ interaction through the mechanism of Fig. 1.

$$\hat{U}(0)=-\left(\frac{\lambda_2}{4\pi}\right)\left\{\left(\frac{f^2}{4\pi}\right)\frac{64}{\mu^2}\int_0^M\frac{k^4}{\omega^3}\,dk \right.$$
$$\left.-\frac{32}{9\pi^2}\int k^4\,dk\int\frac{dp\,\sigma_{33}(p)}{\omega_p}\,\frac{1}{\omega(\omega_p+\omega)^2}\right\}\omega\,\tau_N\cdot\tau_K \qquad (2\cdot9)$$

with $\tau_N\cdot\tau_K=1$ in the case of $I=1$

$=-3$,, $I=0$.

Using the above effective Hamiltonian (2·7), (2·8) and (2·9), the effective potential $\langle k|V|k'\rangle$ can be obtained in the Born approximation. In the case of zero total isospin of K-nucleon system,

$$\langle k|V|k'\rangle_0=\left[-\frac{g_A^2}{2M}+\frac{3g_\Sigma^2}{2M}+\left(\frac{\lambda_1}{4\pi}\right)\left\{\left(\frac{f^2}{4\pi}\right)\frac{24}{\mu^2}\int_0^M\frac{k^4}{\omega^4}\,dk+\text{rescattering term}\right\}\right.$$
$$\left.+\left(\frac{\lambda_2}{4\pi}\right)3\left\{\left(\frac{f^2}{4\pi}\right)\frac{32}{\mu^2}\int_0^M\frac{k^4}{\omega^3}\,dk-\text{rescattering term}\right\}(\omega+\omega')\right]\frac{1}{(\omega\omega')^{1/2}}. \qquad (2\cdot10)$$

In the case of the total isospin one,

$$\langle k|V|k'\rangle_1=\left[\frac{g_A^2}{2M}+\frac{g_\Sigma^2}{2M}+\left(\frac{\lambda_1}{4\pi}\right)\left\{\left(\frac{f^2}{4\pi}\right)\frac{24}{\mu^2}\int_0^M\frac{k^4}{\omega^4}\,dk+\text{rescattering term}\right\}\right.$$
$$\left.-\left(\frac{\lambda_2}{4\pi}\right)\left\{\left(\frac{f^2}{4\pi}\right)\frac{32}{\mu^2}\int_0^M\frac{k^4}{\omega^3}\,dk-\text{rescattering term}\right\}(\omega+\omega')\right]\frac{1}{(\omega\omega')^{1/2}}. \qquad (2\cdot11)$$

Here the rescattering term contributes only about 30%, so that we shall retain the Born approximation term only in the following analysis.

In order to take higher order effects into account we propose to solve an integral equation that the scattering amplitudes $\langle k|K|k_0\rangle$ satisfy.

The scattering amplitudes $\langle k|K|k_0\rangle$ satisfy an integral equation of the form[8]

$$\langle k|K|k_0\rangle=\langle k|V|k_0\rangle+\frac{1}{2\pi^2}P\int dk'\,k'^2\frac{\langle k|V|k'\rangle\langle k'|K|k_0\rangle}{\omega_0-\omega'}, \qquad (2\cdot12)$$

where k_0 is the initial momentum of K meson in the center-of-mass system and $\omega_0=(k_0^2+m_K)^{1/2}$ its initial energy. The phase shifts δ_I for K-nucleon scattering are related to $\langle k|K|k_0\rangle$ as follows,

$$\tan\delta_I = -\frac{k_0\omega_0}{2\pi}\langle k_0|K|k_0\rangle_I \quad (I=0,\ 1). \tag{2·13}$$

Then the total cross section can be obtained from δ_I.

Now let us solve the integral equation (2·12). For the effective potential $\langle k|V|k'\rangle$ of the form (2·10) and (2·11), namely

$$\langle k|V|k'\rangle = \frac{C_1 + C_2(\omega+\omega')}{(\omega\omega')^{1/2}}, \tag{2·14}$$

the exact solution for $\langle k_0|K|k_0\rangle$ is given by

$$\begin{aligned}
&\langle k_0|K|k_0\rangle \\
&= \left(\frac{C_1}{\omega_0} + C_2\right) \frac{\begin{vmatrix} 1+\frac{C_2\omega_0}{2\pi^2}M(\omega_0), & -\frac{C_2 M(\omega_0)}{2\pi^2} \\ \frac{C_2\omega_0}{2\pi^2}N(\omega_0), & 1-\frac{C_2 N(\omega_0)}{2\pi^2} \end{vmatrix}}{\begin{vmatrix} 1-\frac{C_1 M(\omega_0)}{2\pi^2}-\frac{C_2 N(\omega_0)}{2\pi^2}, & -\frac{C_2 M(\omega_0)}{2\pi^2} \\ -\frac{C_1 N(\omega_0)}{2\pi^2}-\frac{C_2 L(\omega_0)}{2\pi^2}, & 1-\frac{C_2 N(\omega_0)}{2\pi^2} \end{vmatrix}} \\
&+ C_2 \frac{\begin{vmatrix} 1-\frac{C_1 M(\omega_0)}{2\pi^2}-\frac{C_2 N(\omega_0)}{2\pi^2}, & \frac{C_1 M(\omega_0)+C_2 N(\omega_0)}{2\pi^2\omega_0} \\ -\frac{C_1 N(\omega_0)}{2\pi^2}-\frac{C_2 L(\omega_0)}{2\pi^2}, & 1+\frac{C_1 N(\omega_0)+C_2 L(\omega_0)}{2\pi^2\omega_0} \end{vmatrix}}{\begin{vmatrix} 1-\frac{C_1 M(\omega_c)}{2\pi^2}-\frac{C_2 N(\omega_0)}{2\pi^2}, & -\frac{C_2 M(\omega_0)}{2\pi^2} \\ -\frac{C_1 N(\omega_0)}{2\pi^2}-\frac{C_2 L(\omega_0)}{2\pi^2}, & 1-\frac{C_2}{2\pi^2}N(\omega_0) \end{vmatrix}},
\end{aligned} \tag{2·15}$$

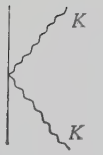
where

$$L(\omega_0) = -\int_0^K dk'\,k'^2 \frac{\omega'}{\omega'-\omega_0}, \tag{2·16}$$

$$M(\omega_0) = -\int_0^K dk'\,k'^2 \frac{1}{\omega'(\omega'-\omega_0)}, \tag{2·17}$$

$$N(\omega_0) = -\int_0^K dk'\,k'^2 \frac{1}{\omega'-\omega_0}, \tag{2·18}$$

and K is the cutoff momentum which corresponds to the extension of the potential range. Details of the calculation are shown in the Appendix. The solution (2·15) involves all Feynman graphs which are combined with the three typical graphs (Fig. 4. a, b, c) in every order.



(a) Foldy type.



(b) Graph via the mechanism of the boson-boson interaction between two pions and two K mesons of the isospin independent type (2·1).



(c) Same as (b) but with the interaction of the isospin dependent type (2·2).

Fig. 4

§ 3. Calculation of the coupling constants λ_1 and λ_2

In this section we shall examine the four-field boson-boson interaction by applying the proposed scheme stated in § 1, to the K-nucleon scattering. We note that $KK\pi\pi$ interactions of both the isospin-independent type and the isospin-dependent type are important for reproducing the three K-nucleon scattering. For determining the coupling constants λ_1 and λ_2 of the $KK\pi\pi$ interactions, the following experimental data for K-nucleon scattering, § 1, a), b) and c) will be used as the most reliable.

1) $\sigma(pK^+|pK^+)=16\text{mb}=10$ (K meson mass unit) at zero K. E.

2) $\sigma(nK^+|pK^0)=2\text{mb}=1.25$ at 80 Mev K.E. (3·1)

3) $\sigma(nK^+|nK^+)=\sigma(nK^+|pK^0)=8\text{mb}=5$ at 240 Mev K. E.

4) The effective K^+-p potential is repulsive.

Evaluation of the coupling constants using (2·13) and (2·15) together with the above experimental data 1), 2), 3) and 4) will be given in the two cases of the cutoff momentum $K\approx 4\mu$ corresponding to the potential range $\approx 1/4\mu$ and $K\approx 6\mu$ corresponding to that $\approx 1/6\mu$. Here the second order terms with respect to $M(\omega_0)$, $N(\omega_0)$ and $L(\omega_0)$ in the solution of the scattering amplitude (2·15) were neglected because they are rather small than the first order terms.

I) The case of the cutoff momentum $K=4\mu$,

$$\frac{g_\Sigma^2}{4\pi}-\frac{g_\Lambda^2}{4\pi}\approx 1.7,$$

$$\frac{\lambda_2}{4\pi}\approx -0.13,$$

$$\frac{g_\Sigma^2}{4\pi}+5.0\frac{\lambda_1}{4\pi}\approx 1.67.$$

Here $g_A^2/4\pi$, $g_\Sigma^2/4\pi$ and $\lambda_1/4\pi$ cannot be separately determined only by the experimental data (3·1) ; however, if we use a plausible value $g_\Sigma^2/4\pi+g_A^2/4\pi\approx 7$ obtained by the dispersion relation approach,[8] the following solutions will be derived,

$$\begin{aligned}
&\frac{g_A^2}{4\pi}\approx 2.6,\\
&\frac{g_\Sigma^2}{4\pi}\approx 4.4,\\
&\frac{\lambda_1}{4\pi}\approx -0.55,\\
&\frac{\lambda_2}{4\pi}\approx -0.13.
\end{aligned}\tag{3·2}$$

II) The case of the cutoff momentum $K=6\mu$,

$$\begin{aligned}
&\frac{g_\Sigma^2}{4\pi}-\frac{g_A^2}{4\pi}\approx 2.64,\\
&\frac{\lambda_2}{4\pi}\approx -0.20,\\
&\frac{g_\Sigma^2}{4\pi}+5.0\frac{\lambda_1}{4\pi}\approx 2.56.
\end{aligned}$$

Taking into account $g_\Sigma^2/4\pi+g_A^2/4\pi\approx 7$ as in the case I), the solution will be obtained as follows,

$$\begin{aligned}
&\frac{g_A^2}{4\pi}\approx 2.2,\\
&\frac{g_\Sigma^2}{4\pi}\approx 4.8,\\
&\frac{\lambda_1}{4\pi}\approx -0.45,\\
&\frac{\lambda_2}{4\pi}\approx -0.20.
\end{aligned}\tag{3·3}$$

Of course the above values of the coupling constants would not be taken seriously. With these choices of the coupling constants (3·2), (3·3), the general behavior of the K^+-p scattering, K^+-n elastic scattering and K^+-n charge exchange scattering can be reproduced (see Fig. 5). The phase shifts $\delta_I(I=0, 1)$ given by the expression (2·13) shows the following tendency with these solutions (see Fig. 6). In low energy regions, the K-nucleon potential is repulsive in both the isospin states zero and one. However, in the isospin zero, it becomes attractive at about 240 Mev kinetic energy in the laboratory system. If $\lambda_2=0$, the tendency that

the charge exchange scattering cross section is very small at low energies and then increases rapidly while K-n elastic scattering stays constant cannot be reproduced, but the above two scatterings show the same tendency, hence it seems necessary to take into account the $KK\pi\pi$ interaction of the type (2·2).

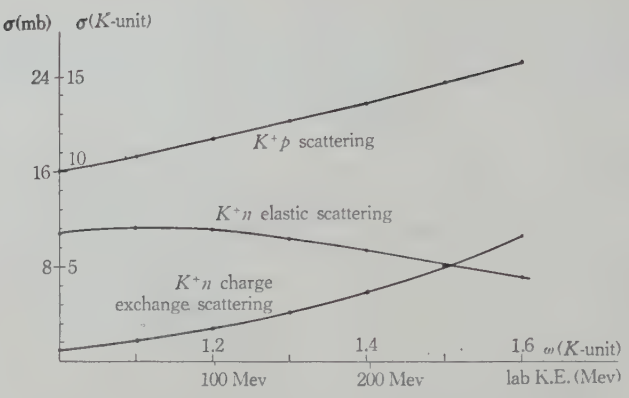


Fig. 5. The calculation of the energy dependences of three K^+-nucleon cross sections with the choice of the coupling constants (3·2) and (3·3).

§ 4. Discussions and concluding remarks

As was discussed in § 3, $KK\pi\pi$ interactions of the isospin-independent type and the isospin-dependent type, within the framework of the proposed theoretical scheme, seems capable of reproducing the three K^+-nucleon scattering. Barshay[9] also reproduced the experimentally observed features of K-$^+$nucleon scattering by the scheme that K-$^+$nucleon scattering mainly takes place through the exchange of two pions between K mesons and nucleons of the only isospin-independent type $K_i^* K_i \pi_j \pi_j$ and the direct scattering may be damped at low energies for some reason. However, we get the effective $KK\pi\pi$ interaction of the isospin-dependent type as far as there exist the strong pion-nucleon and K meson-hyperon-nucleon interactions, hence it would be necessary to take into account the interaction (2·2) in the K^+-nucleon scattering.

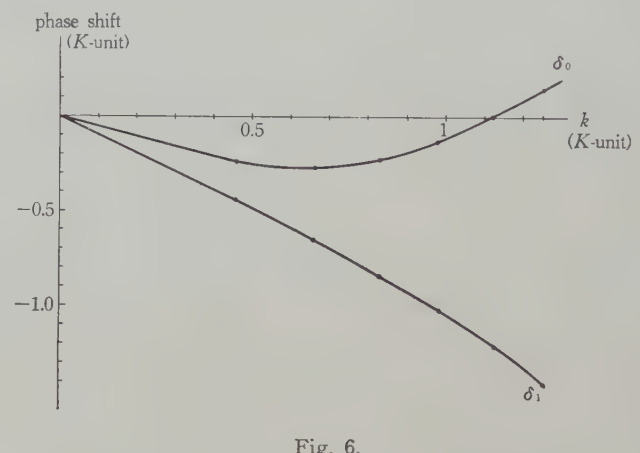


Fig. 6.

Such an interaction might play an important role also in various strange particle phenomenon. For example, the anomaly in $\pi^- + p \to \Lambda^0 + K^0$, $\pi^- + p \to \Sigma^- + K^+$ and $\pi^+ + p \to \Sigma^+ + K^+$ production angular distribution might be explained by the interaction. The difference of the electromagnetic structure between K^+ and K^0 will also be deduced. More information about $KK\pi\pi$ interaction will be obtained by applying the general method of "Unstable Particles as Targets in Scattering Experiments" given by G. F. Chew and F. E. Low.[10] The role of the $KK\pi\pi$ interac-

tion seems less important in the K^--nucleon scattering because the K^- absorption process reflects rather significant effect on scattering, so this case has not been discussed.

In this paper we investigated $KK\pi\pi$ interaction from the point of view of the Hamiltonian formalism, but the analysis based on the Mandelstam representation for K-nucleon scattering is now in progress.

The author expresses his sincere thanks to Professor H. Miyazawa for his valuable discussions and continual encouragement.

Appendix

We consider to solve the integral equation

$$\langle k|K|k_0\rangle=\langle k|V|k_0\rangle+\frac{1}{2\pi^2}P\int dk'\,k'^2\frac{\langle k|V|k'\rangle\langle k'|K|k_0\rangle}{\omega_0-\omega'} \tag{2·12}$$

where the potential is

$$\langle k|V|k'\rangle=\frac{C_1+C_2(\omega+\omega')}{(\omega\omega')^{1/2}}. \tag{2·14}$$

With the expression for $\langle k|V|k'\rangle$, Eq. (2·12) may be solved exactly as follows: Let

$$X(k_0)\equiv\int dk'\,k'^2\frac{\langle k'|K|k_0\rangle}{\sqrt{\omega'}(\omega'-\omega_0)}, \tag{A·1}$$

$$Y(k_0)\equiv\int dk'\,k'^2\frac{\sqrt{\omega'}\langle k'|K|k_0\rangle}{\omega'-\omega_0}. \tag{A·2}$$

These are as yet undetermined functions of k_0. Then (2·12) becomes

$$\langle k'|K|k_0\rangle=\frac{C_1}{(\omega'\omega_0)^{1/2}}+\frac{C_2\omega'^{1/2}}{\omega_0^{1/2}}+\frac{C_2\omega_0^{1/2}}{\omega'^{1/2}}+\frac{C_1X(k_0)}{2\pi^2\omega'^{1/2}}+\frac{C_2\omega'^{1/2}}{2\pi^2}X(k_0)+\frac{C_2}{2\pi^2\omega'^{1/2}}Y(k_0). \tag{A·3}$$

Substituting this in the previous equations (A·1), (A·2),

$$\left[1-\frac{C_1M(\omega_0)}{2\pi^2}-\frac{C_2N(\omega_0)}{2\pi^2}\right]X(k_0)-\frac{C_2M(\omega_0)}{2\pi^2}Y(k_0)=\frac{C_1+C_2\omega_0}{\sqrt{\omega_0}}M(\omega_0)+\frac{C_2}{\sqrt{\omega_0}}N(\omega_0), \tag{A·4}$$

$$-\frac{1}{2\pi^2}[C_1N(\omega_0)+C_2L(\omega_0)]X(k_0)+\left[1-\frac{C_2}{2\pi^2}N(\omega_0)\right]Y(k_0)=\frac{C_1+C_2\omega_0}{\sqrt{\omega_0}}N(\omega_0)+\frac{C_2}{\omega_0^{1/2}}L(\omega_0), \tag{A·5}$$

where $M(\omega_0)$, $N(\omega_0)$ and $L(\omega_0)$ are given by (2·16), (2·17) and (2·18) respectively.

Eqs. (A·4), (A·5) are easily solved for $X(\omega_0)$, $Y(\omega_0)$. From (A·3)

$$\langle k_0|K|k_0\rangle=\left(\frac{C_1}{\omega_0}+C_2\right)\left[1+\frac{\omega_0^{1/2}}{2\pi^2}X(k_0)\right]+C_2\left[1+\frac{1}{2\pi^2\omega_0^{1/2}}Y(k_0)\right]. \quad \text{(A·6)}$$

Substituting the above solution for $X(\omega_0)$, $Y(\omega_0)$ into (A·6) the exact solution of $\langle k_0|K|k_0\rangle$ will be derived as (2·15).

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Note on the Integral Representation of Commutators

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The difference between Dyson's integral representation and Deser et al.'s one of commutators is discussed. It is shown that we cannot determine the vanishing region of the weight function for the latter representation from the spectral condition. Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in Deser et al.'s sense from the integral representation alone.

§ 1

Recently, Deser, Gilbert and Sudarshan[1)-3)] (DGS) have found an integral representation as a function of invariants for the following quantity,*

$$f(q)\equiv\int dx \exp iqx \langle\alpha|[A(x/2),\ B(-x/2)]|\beta\rangle. \tag{1}$$

Here both $|\alpha\rangle$ and $|\beta\rangle$ are statevectors which describe the states of one particle with the same energy-momentum p, or one of them is the state of one particle with the energy-momentum p and the other is the vacuum state. The same representation has been found by Ida[4)] for the Bethe-Salpeter amplitude under the stability condition.

This representation makes use of the information derivable from the Lorentz invariance, the local commutativity of the operators and the spectral condition. From the expression (1) and the spectral condition, we see that $f(q)=0$ in some region in q-space. We denote such a region by R.

The representation they have found is

$$f(q)=\int d\mu\, d\beta\ H(\mu,\beta)\,\epsilon(pq+\beta p^2)\,\delta(q^2+2pq\beta-\mu), \tag{2}$$

where $\epsilon(x)$ is the sign function and $H(\mu,\beta)$ is a weight function. Let us consider the straight line

$$q^2+2pq\beta-\mu=0 \tag{3}$$

in the (q^2, pq) plane for fixed μ and β. We call such a straight line "admissible" if this line does not pass the region in the (q^2, pq) plane which corresponds to R. This determines an admissible maximum region S in (μ,β) plane such that

* In the first paper of DGS,[1)] the arguments of operators A and B are x and zero.

every point in S gives rise to an admissible line. The main result due to DGS is that $H(\mu, \beta)=0$ at the outside of S.

Using this representation, DGS[2] have proved that there exists an analytic function which coincides with the forward nucleon-nucleon scattering amplitude on real axis of the energy larger than the rest mass of the nucleon and which is regular in the upper half plane of the energy of the nucleon in the laboratory system. However, so far as we use Dyson's representation[5] instead of DGS's one, we never come to the same conclusion in spite of the use of the same information, that is, the Lorentz invariance, the local commutativity, the spectral condition and the number of the constant momenta (in this case only one p).

In order to make clear the above stated contrariety, we shall discuss the difference between the two kinds of representations. The conclusion of this paper is that the result of DGS ($H(\mu, \beta)=0$ at the outside of S) is incorrect. At the present stage of our knowledge, it is impossible to incorporate the spectral conditions into the integral representation (2). Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in DGS's sense from the integral representation (2) alone.

§ 2

Dyson's representation[5] is

$$f(q)=\int_0^\infty d\kappa^2 \int d\boldsymbol{u} \int du_0 \epsilon(q_0-u_0)\,\delta[(q_0-u_0)^2-(\boldsymbol{q}-\boldsymbol{u})^2-\kappa^2]\,\varphi(\boldsymbol{u}, u_0, \kappa^2) \tag{4}$$

instead of (2) for the matrix element of the commutator between arbitrary states. The information on the statevector is included in the invariant weight function $\varphi(\boldsymbol{u}, u_0, \kappa^2)$. Let us consider the hyperboloid

$$(q_0-u_0)^2-(\boldsymbol{q}-\boldsymbol{u})^2-\kappa^2=0 \tag{5}$$

in q-space for fixed $\boldsymbol{u}$, u_0 and κ^2. We call such a hyperboloid admissible if both sheets do not invade the region R. This determines an admissible maximum region S in $(\kappa^2, \boldsymbol{u}, u_0)$-space such that every point in S gives rise to an admissible hyperboloid. The main result due to Dyson is that $\varphi(\boldsymbol{u}, u_0, \kappa^2)=0$ at the outside of S.

In what follows, we shall investigate the matrix element between special states as discussed in DGS's representation and without loss of generality we choose the special Lorentz frame where the space components of the four-vector p are zero. If we put

$$\varphi(\boldsymbol{u}, u_0, \kappa^2)=\delta(\boldsymbol{u})\psi(u_0, \kappa^2), \tag{6}$$

we can easily obtain DGS's representation for the same special frame. The weight function $H(\mu, \beta)$ is expressible in terms of $\psi(u_0, \kappa^2)$. DGS's representation is a special case of Dyson's one. Because the representation (2) with the weight func-

tion $H(\mu, \beta)$ and Dyson's representation with the specialization (6) of the weight function (namely the representation with the weight function $\psi(u_0, \kappa^2)$) are equivalent, we shall use the more convenient one between the two, as the case may be, for the discussion of DGS's representation.

In configuration space, DGS's one is

$$\tilde{f}(x) = \int_0^\infty d\kappa^2 \Delta(x, \kappa) \tilde{\psi}(x_0, \kappa^2). \tag{7}$$

Dyson's one is

$$\tilde{f}(x) = \int_0^\infty d\kappa^2 \Delta(x, \kappa) \tilde{\varphi}(\boldsymbol{x}, x_0, \kappa^2). \tag{8}$$

Here $\Delta(x, \kappa)$ is usual invariant commutator function of the free field operator with mass κ. $\tilde{f}(x)$ depends only on x^2 and $x_0 = px/(p^2)^{1/2}$, and the integral representation (7) is always possible, because this is essentially the Fourier Bessel transform. Therefore, if we can conclude that for every $f(q)$ which vanishes in R it is possible to put $\psi(u_0, \kappa^2) = 0$ at the outside of S, DGS's representation is correct. Thus we shall discuss the possibility of putting $\psi(u_0, \kappa^2) = 0$ at the outside of S.

§ 3

The first proof by DGS on the possibility of putting $H(\mu, \beta) = 0$ is not precise. It should be noted that $\tilde{\psi}(x_0, \kappa^2)$ is not uniquely determined because of arbitrariness of $\tilde{f}(x)$ in the unphysical region $x^2 = x_0^2 - \boldsymbol{x}^2 > x_0^2$. Then, even if it is possible to put $\psi(u_0, \kappa^2) = 0$ at the outside of S, $\psi(u_0, \kappa^2) = 0$ is not a necessary condition for $f(q)$ to vanish in R. The problem we must solve is whether a set of the functions, whose weight function $\psi(u_0, \kappa^2)$ or $H(\mu, \beta)$ vanishes at the outside of S, covers all the functions which vanish in R and whose Fourier transforms are zero in $x^2 < 0$.

An answer to this problem has been given in the appendix[1] by DGS. Their proof is made in two stages: First they assert that if $f(q) = 0$ in R' (which is contained in R), then $H(\mu, \beta) = 0$ at the outside of the intersection of S' and S'', where the region R' is defined by $(q-p)^2 < 0$ and $q^2 < 0$, and S' and S'' are defined by $0 \geq \beta \geq -1$ and $\mu \geq -\beta^2 p^2$ respectively. No assumption concerning the mass spectrum is involved in the requirement of the vanishing of $f(q)$ in R'. In the next stage they have incorporated the mass spectral condition by using a lemma of Jost and Lehmann.[6] If the first stage of their proof is correct, we can come to the conclusion that $H(\mu, \beta) = 0$ at the outside of S. Now, in the first stage, the assertion that one can set $H(\mu, \beta) = 0$ at the outside of S'' is obviously unqestionable in view of the relation $\kappa^2 = \mu + \beta^2 p^2$ and Eq. (7). Hence the important result in the first stage is that, even though $H(\mu, \beta) = 0$ at the outside of S', a

set of $f(q)$ can cover all the functions which vanish in R' and their Fourier transform is zero for $x^2<0$.

However, this conclusion is not correct. In the proof of the first stage, they have used the following identity given by Dyson,[7]

$$\int d\boldsymbol{u}\, \Delta(q-\tilde{\boldsymbol{u}}, m)\delta(\boldsymbol{u}^2-b^2)=\pi\int_{-b}^{b} dv\; I_0(m[b^2-v^2]^{1/2})\, \Delta(q-\tilde{v}, m), \tag{9}$$

where $\tilde{\boldsymbol{u}}$ is a four-vector with the vanishing time component and the space components $\boldsymbol{u}$, ann $\tilde{v}$ is a four-vector with the vanishing space components and the time component v. Now we consider the following integral,

$$\int_0^\infty dm\int_{-b}^{b} dv\chi(m)\, I_0(m[b^2-v^2]^{1/2})\, \Delta(q-\tilde{v}, m) \tag{10}$$

where $\chi(m)$ is an arbitrary function. We cannot invert the order of integration in (10), since the asymptotic behaviour of $I_0(x)$ is exponential. In the proof of the first stage, there appears the integral like (10). The range $0\geq\beta\geq-1$ (S') of β corresponds to the range $b\geq v\geq-b$ of v in (10). We cannot come to the conclusion of the first stage without requiring the inversion of the order of the integration like (10). Therefore the discussion in the appendix[1] is incorrect.

By using the integral representation of the double commutator,[7] Ida[4] has arrived at the same conclusion as that of DGS. However, as was revised by Dyson himself,[8] the integral representation of the double commutator is incorrect by the same reason as that of the defect of the proof of the first stage in DGS.

In conclusion, the integral representation (2) is correct, however, we cannot put $H(\mu, \beta)=0$ at the outside of S. At the present stage of our knowledge, it is impossible to incorporate the spectral conditions into the integral representation (2). Hence, we cannot prove the forward dispersion relation for nucleon-nucleon scattering in DGS's sense from the integral representation (2) alone.

After completion of this work, we find the similar work.[9] However, the assertion and contents seem to be somewhat different from ours. Recently, Nakanishi[10] has pointed out that for the vertex function $H(\mu, \beta)=0$ at the outside of S is valid in every order of perturbation theory.

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The Polarity of the Diatomic Molecule

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The relation between the ionicity of the bond and the electronegativity difference is discussed by the use of the semi-localized orbital method. It is found that the ionicity of the bond depends not only on the electronegativity difference but also on the overlap integral between the atomic valence orbitals. According to the magnitude of the overlap integral the diatomic molecules are divided into the two groups which show the different features with respect to the dependence of the ionicity of the bond on the electronegativity difference. The *S*-shaped curve given by Townes and Dailey from the analysis of the results of the measurements of eQq seems to be the composite of the above two characteristic groups.

§ 1. Introduction

Starting from Pauling's original application to the ionic resonance energy, the concept of the electronegativity scale of the elements has many and undoubtful successes in the discussion on the polarity of the bond.[1] There have been many attempts to give a relation of the ionic character of the bond to the electronegativity difference. We shall briefly describe the following three relations. It must be noted here that the ionic character or the ionicity of the bond means the degree of the polarity of the bond. Pauling's relation[2] of the ionic character of the bond to the electronegativity difference is deduced from the values of the dipole moments of the hydrogen halides. There is, however, no reason that the ionic character of the bond can be exclusively determined from the value of the dipole moment of the molecule, since it was suggested by Coulson[3] that the value of the dipole moment depends essentially on the atomic dipole which is due to the polarization of the atom. Gordy[4] proposed the following linear relation from the results of the measurements of eQq:

$$\text{Ionic character of the bond A}-\text{B}=\frac{1}{2}(x_A-x_B), \qquad (1\cdot1)$$

where x_A and x_B are the values of the electronegativity of the atoms A and B, respectively. On the other hand, Townes and Dailey[5] gave the *S*-shaped curve for the ionic character *versus* the electronegativity difference from the more detailed analysis of the observed values of eQq. The above three relations are shown in Fig. 1 for the comparison. It must be noted that all of the above relations are given empirically, but the justification of those relations in general term from

the first principle has not been given. This article aims to show a theoretical relation between the ionicity of the bond and the electronegativity difference by the use of the semi-localized orbital method. In addition we shall compare the theoretical result with the empirical relation given by Townes and Dailey.

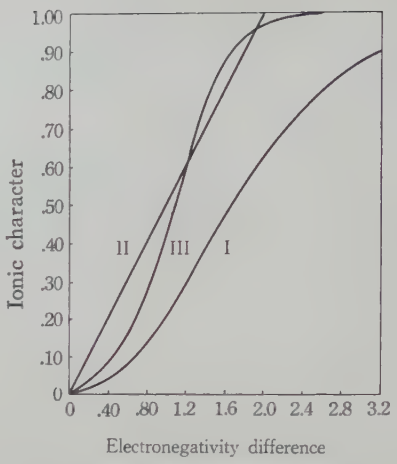


Fig. 1. The relations between the ionic character of the bond and the electronegativity difference given by (I) Pauling, by (II) Gordy and by (III) Townes and Dailey.

§ 2. Diatomic molecule

We shall consider a diatomic molecule AB which is bound with a single bond and describe the wave function ψ^{AB} of the bond A-B as follows:

$$\psi^{AB}=\frac{1}{\sqrt{M_{ij}}}\{\psi_i(1)\psi_j(2)+\psi_j(1)\psi_i(2)\}\times\frac{1}{\sqrt{2}}\{\alpha(1)\beta(2)-\beta(1)\alpha(2)\},\qquad(2\cdot1)$$

where M_{ij} is the normalization constant and ψ_i and ψ_j are the semi-localized orbitals which are responsible for the bond A-B. These orbitals are expressed as

$$\left.\begin{aligned}\psi_i&=\frac{1}{\sqrt{N_i}}(\phi_i^A+\lambda\phi_j^B),\\ \psi_j&=\frac{1}{\sqrt{N_j}}(\phi_j^B+\mu\phi_i^A),\end{aligned}\right\}\qquad(2\cdot2)$$

where ϕ_i^A and ϕ_j^B are the atomic valence orbitals of the atoms A and B, respectively, and N_i and N_j are the normalization constants of the orbitals which are expressed as

$$N_i=1+\lambda^2+2\lambda S_{ij},\quad N_j=1+\mu^2+2\mu S_{ij},\qquad(2\cdot3)$$

where S_{ij} is the overlap integral between the atomic orbitals ϕ_i^A and ϕ_j^B, and λ and μ are the parameters in the semi-localized orbitals (2·2), whose values should be determined by the condition of minimizing the total energy of the molecule. Thus, the factor M_{ij} in Eq. (2·1) is represented as

$$M_{ij}=2N_{ij}/N_iN_j,$$

where

$$N_{ij}=N_iN_j+R_{ij}^2 \text{ and } R_{ij}=\lambda+\mu+(1+\lambda\mu)S_{ij}.\qquad(2\cdot4)$$

The Hamiltonian of the diatomic molecule is

$$\mathcal{H}=h(1)+h(2)+\frac{1}{r_{12}}+V_{AB},$$

where $h(1)=-(1/2)\Delta(1)+V^A(1)+V^B(1)$, $V^A(1)$ and $V^B(1)$ are the core potentials of the atom A and B, respectively, for the first electron, and V_{AB} is the repulsive potential between the atomic cores A^+ and B^+. Thus, the total energy of the system is written as

$$E=\frac{1}{N_{ij}}[A_i I_{ii}+A_j I_{jj}+B_{ij}(I_{ij}+I_{ji})+(1+\lambda\mu)^2 C_{ij} + \{(1+\lambda\mu)^2+4\lambda\mu\}K_{ij}+2\lambda^2 C_{jj}+2\mu^2 C_{ii}+4(1+\lambda\mu)(\lambda D_{j,ij}+\mu D_{i,ij})]+V_{AB}, \tag{2·5}$$

where

$$\left.\begin{aligned} I_{ij}&=\int \phi_i^{A*}(1)\,h(1)\,\phi_j^B(1)\,dv_1,\\ C_{ij}&=\iint \phi_i^{A*}(1)\,\phi_j^{B*}(2)\,\frac{1}{r_{12}}\,\phi_i^A(1)\,\phi_j^B(2)\,dv_1 dv_2,\\ K_{ij}&=\iint \phi_j^{B*}(1)\,\phi_i^{A*}(2)\,\frac{1}{r_{12}}\,\phi_i^A(1)\,\phi_j^B(2)\,dv_1 dv_2,\\ D_{j,ij}&=\iint \phi_j^{B*}(1)\,\phi_j^{B*}(2)\,\frac{1}{r_{12}}\,\phi_i^A(1)\,\phi_j^B(2)\,dv_1 dv_2 \end{aligned}\right\} \tag{2·6}$$

and

are the molecular energy integrals involving the atomic valence orbitals, while

$$\left.\begin{aligned} &A_i=\mu^2 N_i+2\mu R_{ij}+N_j,\quad A_j=\lambda^2 N_j+2\lambda R_{ij}+N_i,\\ \text{and}\quad &B_{ij}=\lambda N_j+\mu N_i+(1+\lambda\mu)R_{ij} \end{aligned}\right\} \tag{2·7}$$

are the coefficients depending only on λ, μ and S_{ij}. It is easily seen from the definition (2·4) of N_{ij} and R_{ij} that the coefficients A_i, A_j and B_{ij} have the relation

$$2N_{ij}=A_i+A_j+2S_{ij}B_{ij}. \tag{2·8}$$

As seen from Eqs. (2·5) and (2·8), the quantities A_i/N_{ij} and A_j/N_{ij} are interpreted to be the charge densities of the atom A and B, respectively, and the quantity $2S_{ij}B_{ij}/N_{ij}$ is the charge density in the overlap region of the bond A-B.

We shall now assume that ϕ_i^A and ϕ_j^B satisfy the following equations:

$$\left.\begin{aligned} \left\{-\frac{1}{2}\Delta(1)+V^A(1)\right\}\phi_i^A(1)&=\varepsilon_i^A\phi_i^A(1),\\ \left\{-\frac{1}{2}\Delta(1)+V^B(1)\right\}\phi_j^B(1)&=\varepsilon_j^B\phi_j^B(1), \end{aligned}\right\} \tag{2·9}$$

where ε_i^A and ε_j^B are the eigenvalues of the equations, respectively, and correspond to the ionization potentials of the electron in the orbital ϕ_i^A and ϕ_j^B, respectively, though they have the reversed signs. We shall also assume that the molecular

energy integrals can be approximated by the following formulae:

$$\left.\begin{aligned} &K_{ij}=\frac{1}{4}S_{ij}^2(C_{ii}+C_{jj}+2C_{ij}),\\ &D_{j,ij}=\frac{1}{2}S_{ij}(C_{ij}+C_{jj}),\quad D_{i,ij}=\frac{1}{2}S_{ij}(C_{ij}+C_{ii}),\\ &V_{ij}^{A}=\frac{1}{2}S_{ij}(V_{ii}^{A}+V_{jj}^{A}) \end{aligned}\right\} \qquad (2\cdot10)$$

and

$$V_{ij}^{B}=\frac{1}{2}S_{ij}(V_{ii}^{B}+V_{jj}^{B}),$$

where

$$V_{ij}^{A}=\int\phi_i^{A*}(1)\,V^{A}(1)\,\phi_j^{B}(1)\,dv_1,\quad V_{ij}^{B}=\int\phi_i^{A*}(1)\,V^{B}(1)\,\phi_j^{B}(1)\,dv_1,$$

and so on. Under the above assumptions Eq. (2·5) is reduced to the from

$$\begin{aligned} E=&\varepsilon_i^{A}+\varepsilon_j^{B}+\frac{1}{2}(V_{ii}^{A}+V_{jj}^{B}+V_{ii}^{B}+V_{jj}^{A})+\frac{1}{4}(C_{ii}+C_{jj}+2C_{ij})\\ &+V_{AB}-\frac{1}{2}q_{ij}\{(V_{ii}^{A}+V_{jj}^{B})-(V_{ii}^{B}+V_{jj}^{A})\}\\ &+\frac{1}{4}\left(-\frac{k_{ij}}{1-S_{ij}^2}+\frac{p_{ij}^2}{1-k_{ij}}\right)(C_{ii}+C_{jj}-2C_{ij})+p_{ij}(\chi_i-\chi_j), \end{aligned} \qquad (2\cdot11)$$

where

$$p_{ij}=(A_i-A_j)/2N_{ij},\quad q_{ij}=(A_i+A_j)/2N_{ij}, \qquad (2\cdot12)$$

$$k_{ij}=\sqrt{(1-S_{ij}^2)\{q_{ij}^2-(1-q_{ij})^2/S_{ij}^2-p_{ij}^2\}}, \qquad (2\cdot13)$$

and

$$\chi_i=\varepsilon_i^{A}+\frac{1}{2}C_{ii}+C_{ij}+V_{ii}^{B}. \qquad (2\cdot14)$$

The reduction to Eq. (2·11) will be discussed in the Appendix. From the physical meaning of the quantities A_i/N_{ij} and A_j/N_{ij}, which is already mentioned, p_{ij} can be interpreted to represent the amount of charges transferred from the atom B to the atom A through the bond, in other word, it denotes the ionic character of the bond A-B. χ_i is approximately equal to $\eta_i^{A}=\varepsilon_i^{A}+(1/2)C_{ii}$, as $C_{ij}+V^{B}{}_{ii}$ represents the effective potential of the neutral atom B for the electron in the orbital ϕ_i^{A} and may be approximately neglected. $-\eta_i^{A}$ is approximately equal to the electronegativity in the Mulliken scale[6] which is defined as the arithmetic mean value of the ionization potential and the electroaffinity of the atom A. We shall particularly discuss the dependence of the value of the quantity p_{ij} on the value of the difference $\chi_i-\chi_j$. For this purpose we shall use p_{ij} and q_{ij}

instead of λ and μ as the independent parameters and determine their values to minimize the total energy of the molecule. Thus, we derive the following two equations which give the optimum values of p_{ij} and q_{ij}:

$$\frac{\partial E}{\partial q}=\frac{1}{2}Q+\frac{1}{4}\frac{p^2(1-S^2)-(1-k)^2}{S^2k(1-k)^2}\{1-(1-S^2)q\}D=0, \quad (2\cdot15)$$

$$\frac{\partial E}{\partial p}=\frac{1}{4}\frac{1-k^2-(1-k^2)p^2}{k(1-k)^2}pD-P=0, \quad (2\cdot16)$$

where

$$\left.\begin{aligned} Q_{ij}&=-\{(V_{ii}^A+V_{jj}^B)-(V_{ii}^B+V_{jj}^A)\},\\ D_{ij}&=C_{ii}+C_{jj}-2C_{ij},\\ P_{ij}&=-(\chi_i-\chi_j). \end{aligned}\right\} \quad (2\cdot17)$$

and

In Eqs. (2·15) and (2·16) the subscripts i and j are omitted for simplicity. Putting as

$$\beta=Q/D \text{ and } \gamma=P/D, \quad (2\cdot18)$$

we can reduce Eqs. (2·15) and (2·16) to the equations

$$(1-S^2)p^3-(1-k^2)p+4\gamma k(1-k)^2=0, \quad (2\cdot19)$$

and

$$S^2\beta^2p^3-4\gamma kp^2+16\gamma^2k(1-k)p-16\gamma^3k(1-k)^2=0, \quad (2\cdot20)$$

where we use the relation (2·13). It is easily seen from these equations that p is an odd function of γ.

§ 3. The ionic character of the bond

It is the aim of this section to solve Eqs. (2·19) and (2·20) with respect to the parameters p and q as the functions of β, γ and S, but we shall at first discuss some special cases.

1) For the homopolar bond we start from Eqs. (2·15) and (2·16) and put $p=0$ in those equations. We then get the equations

$$\frac{2S^2\beta}{1-(1-S^2)q}=\frac{1}{k}=\frac{1}{\sqrt{(1-S^2)\{q^2-(1-q)^2/S^2\}}}. \quad (3\cdot1)$$

The solution of these equations is

$$q=\frac{1}{1-S^2}(1-2S^2\beta k), \quad (3\cdot2)$$

where

$$k^2=\frac{1}{1+4S^2\beta^2}. \quad (3\cdot3)$$

Thus, we can get the value of q, if we know the values of S and β. If the value of $S\beta$ is very large ($4S^2\beta^2 \gg 1$), q is almost equal to $1/(1+S)$. Then, it is found that the bond orbital approximation discussed later is a good approximation in this case.

2) If the polarity of the bond is very small, we may neglect the term of p^3 in Eq. (2·19). Then, we get the relation

$$p=\frac{4k(1-k)}{1+k}\gamma. \tag{3·4}$$

Substituting this relation to Eq. (2·20), we get

$$k^2=\frac{1}{1+4S^2\beta^2}. \tag{3·5}$$

This form of k is the same as that in the homopolar bond (Eq. (3·3)). It is, then, found that p is almost proportional to γ, if the value of $S\beta$ does not radically vary from molecule to molecule and the value of γ is small.

3) We shall discuss the results in the bond orbital approximation. In this case we should use a bond orbital instead of a pair of the semi-localized orbitals (2·2) and then put a condition $\lambda\mu=1$ in the orbitals (2·2). This corresponds to put $k=0$ in Eq. (2·11), as seen from the Appendix, and then we cannot use Eqs. (2·15) and (2·16). Considering the relation (2·13), which gives a relation between p and q, we get the equation

$$\left\{\frac{S\beta}{1-(1-S^2)p^2}+1\right\}p=2\gamma, \tag{3·6}$$

instead of Eqs. (2·15) and (2·16). Using this equation, we can obtain the p-γ diagram for the given values of β and S. Fig. 2 shows the curves for ($\beta=4$, $S=0.1$) and ($\beta=4$, $S=0.5$). From this figure we can easily find the saturational effect of the ionic character of the bond.

4) Lastly we shall discuss the solutions of Eqs. (2·19) and (2·20). If we put $x=\gamma/p$, we get the equations

$$p\sqrt{(1-S^2)} = (1-k)\sqrt{\frac{1+k}{1-k}-4kx} \tag{3·7}$$

and

$$16k(1-k)^2x^3-16k(1-k)x^2+4kx-S^2\beta^2=0 \tag{3·8}$$

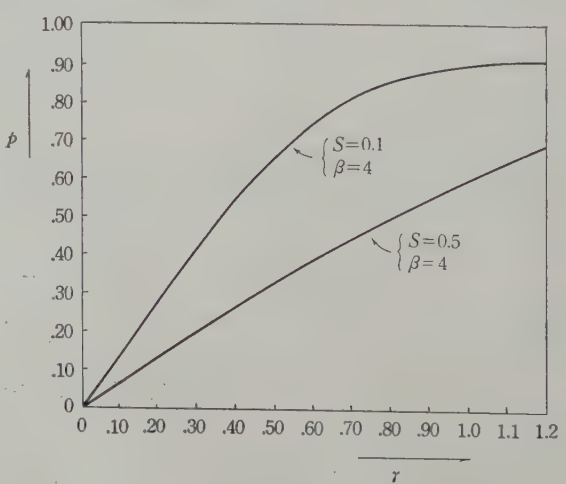


Fig. 2. The p-γ diagram in the bond orbital approximation.

from Eqs. (2·19) and (2·20), respectively. From Eq. (3·8) x is determined as a function of k. Then, we get a relation between $p\sqrt{1-S^2}$ and $\gamma\sqrt{1-S^2}$ for a given value of $S^2\beta^2$, taking k as a variable parameter. The result is shown in Fig. 3 for the several values of $S^2\beta^2$. From this figure we can determine the value of p, if we know the values of S, β and γ. The value of q is determined by Eq. (2·13), that is,

$$q=\frac{1}{1-S^2}\{1-S\sqrt{1-k^2-(1-S^2)p^2}\}, \tag{3·9}$$

where we have used a condition $q<1$.

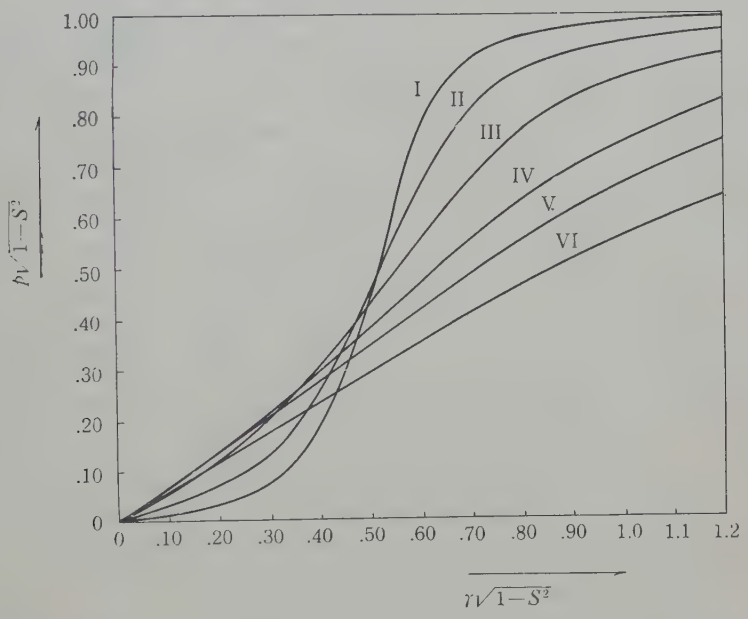


Fig. 3. The relations between $p\sqrt{1-S^2}$ and $\gamma\sqrt{1-S^2}$ in the semilocalized orbital approximation for the several values of $S^2\beta^2$. The values of $S^2\beta^2$ are (I) 1/27, (II) 1/9, (III) 1/3, (IV) 1, (V) 2, and (VI) 4.

We here describe the general features of the results in Figs. 2 and 3.

i) For the large value of $S^2\beta^2$, $(S\beta>1)$, p is almost linearly dependent on γ and the result in the semi-localized orbital approximation is almost equivalent to that in the bond orbital approximation.

ii) For the small value of $S^2\beta^2$, $(S\beta<1)$, the p-γ diagram shows evidently the S-shaped curve and the result in the semi-localized orbital method is quite different from that in the bond orbital method in the region of the small value of γ, although the curve in the former method tends to coincide with that in the latter method for the large value of γ.

§ 4. Nuclear quadrupole coupling constants

In the same way as we derived the expression (2·5) for the total energy of the molecule, we can easily derive the expression for the nuclear quadrupole coupling constant of the atom A, which is assumed to be a negatively charged atom, in the diatomic molecule AB as follows:

$$eQ\boldsymbol{q}^{A}_{mol.}=eQ\Big[\frac{A_i}{N_{ij}}\boldsymbol{q}_{ii}+\frac{A_j}{N_{ij}}\boldsymbol{q}_{jj}+\frac{2B_{ij}}{N_{ij}}\boldsymbol{q}_{ij}+2\sum_{l}\boldsymbol{q}_{ll}+2\sum_{l'}\boldsymbol{q}_{l'l'}\Big], \quad (4\cdot1)$$

where

$$\boldsymbol{q}_{ij}=\Big(\phi_i^{A}\Big|\frac{3\cos^2\theta-1}{r^3}\Big|\phi_j^{B}\Big), \quad (4\cdot2)$$

and so on. The bold-faced notation is used to be distinguished from the parameter q. The molecule AB is taken to lie along the z axis with the atom A at the origin. $\boldsymbol{q}_{ll}$ and $\boldsymbol{q}_{l'l'}$ denote the average values with respect to the non-bonding electrons of the atoms A and B, respectively, and the summations $\sum_{l}$ and $\sum_{l'}$ extend over all the non-bonding orbitals of the atoms A and B, respectively. Using Eqs. (2·8) and (2·12), we can rewrite Eq. (4·1) as

$$eQ\boldsymbol{q}^{A}_{mol.}=eQ\Big[(p+q)\boldsymbol{q}_{ii}+(q-p)\boldsymbol{q}_{jj}+\frac{2}{S}(1-q)\boldsymbol{q}_{ij}+2\sum_{l}\boldsymbol{q}_{ll}+2\sum_{l'}\boldsymbol{q}_{l'l'}\Big]. \quad (4\cdot3)$$

Now we consider the atom A to be the halogen atom and take account of the s-p hybridized orbitals

$$\left.\begin{aligned}\phi_i^{A}&=\phi_s^{A}\cos\alpha+\phi_{pz}^{A}\sin\alpha,\\ \phi_1^{A}&=\phi_s^{A}\sin\alpha-\phi_{pz}^{A}\cos\alpha,\end{aligned}\right\} \quad (4\cdot4)$$

where ϕ_s^{A} and ϕ_{pz}^{A} are the ns and npz atomic orbitals of the halogen atom and $\cos^2\alpha$ denotes the s character of the bonding orbital. The other non-bonding orbitals are npx and npy. Then, we have the relations

$$\boldsymbol{q}_{ii}=\boldsymbol{q}_{zz}\sin^2\alpha,\ \boldsymbol{q}_{ll}=\boldsymbol{q}_{zz}\cos^2\alpha,\ \boldsymbol{q}_{xx}\text{ and }\boldsymbol{q}_{yy}, \quad (4\cdot5)$$

where

$$\boldsymbol{q}_{zz}=\Big(\phi_{pz}^{A}\Big|\frac{3\cos^2\theta-1}{r^3}\Big|\phi_{pz}^{A}\Big),$$

and so on. Considering the relation $\boldsymbol{q}_{xx}+\boldsymbol{q}_{yy}+\boldsymbol{q}_{zz}=0$, we can reduce Eq. (4·3) to the relation

$$eQ\boldsymbol{q}^{A}_{mol.}=(-2+p+q)(\sin^2\alpha)eQ\boldsymbol{q}_{zz}=-(1+r-p)(1-s)eQ\boldsymbol{q}_{zz}, \quad (4\cdot6)$$

where we put $r=1-q$ and $s=\cos^2\alpha$, and assumed that the contribution from the terms $\boldsymbol{q}_{ij}$, $\boldsymbol{q}_{jj}$ and $\boldsymbol{q}_{l'l'}$ may be small enough to be neglected. We can also obtain the similar relation to Eq. (4·6) for the nuclear coupling constant of the positively charged atom B in the molecule, that is,

$$eQq^{B}_{mol.} = -(1+r+p)(1-s')eQq_{z'z'}, \qquad (4\cdot7)$$

where s' is the s character of the bonding orbital ϕ_j^B of the atom B and $q_{z'z'}$ denotes the average value with respect to the electron of the $n'pz'$ atomic valence orbital of the atom B.

Comparing the relations (4·6) and (4·7) with those used by Gordy or by Townes and Dailey in the analysis of the results of the measurements of eQq, we find that the ionicity of the bond obtained from the data of the microwave measurements should correspond to the quantities $(p-r)$ and $(p+r)$ for the negatively and positively charged atoms, respectively, though Townes and Dailey considered that $q_{z'z'}$ is different from that of the neutral atom. However, the value of r is very small in the cases in which the state of the bond is nearly ionic ($p \simeq 1$) or the overlap integral between the atomic orbitals ϕ_i^A and ϕ_j^B is very small. In this case we may use the relations

$$\left.\begin{aligned} &eQq^{A}_{mol.} = -(1-p)(1-s)eQq_{zz}, \text{ for the negatively charged atom,} \\ &\text{and} \\ &eQq^{B}_{mol.} = -(1+p)(1-s')eQq_{z'z'}, \text{ for the positively charged atom,} \end{aligned}\right\} \qquad (4\cdot8)$$

as good approximations. If the overlap integral between the atomic orbitals is not small and the state of the bond is fairly covalent, we must consider the contribution from the neglected terms q_{ij} and q_{jj} for the atom A. Particularly, the contribution of q_{jj} is due to the non-orthogonality between the orbital ϕ_j^B and the core orbital of the atom A, as noted by Itoh.[7] Thus, q_{jj} is approximately estimated by $T_{j\mu}{}^2\, q_{\mu\mu}$, where $T_{j\mu}$ is the overlap integral between the orbitals ϕ_j^B and ϕ_μ^A, ϕ_μ^A is the $(n-1)pz$ orbital of the halogen atom A, and $q_{\mu\mu}$ is the average value of $(3\cos^2\theta-1)/r^3$ with respect to the orbital ϕ_μ^A. $T_{j\mu}$ is usually very small but $q_{\mu\mu}/q_{zz}$ is very large. Then, the above contribution is by no means negligible. Now the atom A is assumed to be the Cl atom. Employing the Slater orbitals, we have verified that the contribution of q_{jj} amounts to about 5 and 15% of eQq_{zz} in the molecule Cl_2 and HCl, respectively. The same state of affairs holds for the positively charged atom B. Then, the contribution from r in Eqs. (4·6) and (4·7) is diminished largely and the relations (4·8) may be in good approximation, even if the overlap integral between the atomic valence orbitals is not small. However, the above effect may not completely cancel out with the r term in either case. Townes and Dailey found that the ionicity of the positively charged atom is a little larger than that of the negatively charged atom in one molecule. This difference may be considered to be arisen from the residual effect of the r term in Eqs. (4·6) and (4·7), as the ionicity of the bond should be determined uniquely in one bond. At any rate the quantity p is considered to correspond to the ionicity of the bond obtained from the analysis of the microwave measurements.

§ 5. Discussion

Using the semi-localized orbital method, we found that the ionicity of the bond is determined by the three quantities β, γ and S of the bond in the diatomic molecule. There are two characteristic groups in the p-γ diagram, that is, if the value of $S\beta$ is large ($S\beta>1$), the p-γ diagram shows an almost straight line, but if the value of $S\beta$ is small ($S\beta<1$), the diagram shows the S-shaped curve. It will be necessary to compute the three quantities β, γ and S for the individual molecules, but we shall discuss only the general feature of the polarity of the diatomic molecules by the use of the roughly estimated values of the above three quantities in this article.

The value of β, which is defined by Eqs. (2·17) and (2·18), is approximately estimated in the following way. The relation $V_{ii}^{B}+C_{ij}\simeq V_{jj}^{A}+C_{ij}\simeq 0$ may hold approximately, as mentioned in § 2.

Assuming that the virial theorem is approximately applicable to the valence electron of the orbital ϕ_i^A, we take the value of V_{ii}^{A} to be about $2\varepsilon_i^A$. Due to the same reason we use the relation $V_{jj}^{B}\simeq 2\varepsilon_j^B$. If we express the electroaffinities of the atoms A and B as $-\alpha_i^A$ and $-\alpha_j^B$, respectively, the values of C_{ii} and C_{jj} can be estimated by the relations

$$C_{ii}\simeq \alpha_i^A-\varepsilon_i^A,\quad C_{jj}\simeq \alpha_j^B-\varepsilon_j^B, \tag{5·1}$$

as proposed by Parr and Pariser.[8] According to the above consideration the value of β may be estimated by the equation

$$\beta=\frac{-2(\varepsilon_i^A+\varepsilon_j^B)-2C_{ij}}{(\alpha_i^A-\varepsilon_i^A)+(\alpha_j^B-\varepsilon_j^B)-2C_{ij}}. \tag{5·2}$$

From this equation it is found that the value of β is about 2.0 or more. The quantity γ defined by Eq. (2·18) may not be considered to be used as a measure of the electronegativity difference, since the quantity D_{ij} defined by Eq. (2·17) may not be constant from molecule to molecule. However, we have found that the quantity D_{ij} is roughly constant in the σ bond of the diatomic molecules constituted from the atoms belonging to the first and second rows in the periodic table, where the molecular integrals were given by the use of the Slater or Hartree-Fock atomic orbitals, and its value is of the order of 10 eV. While the quantity P_{ij} defined also by Eq. (2·17) is approximately equal to the electronegativity difference in Mulliken's scale, as mentioned already. Then, it seems to be probable that we make the assumption $\gamma<1$ for large number of molecules and take the quantity γ as a rough measure of the electronegativity difference. According to the above consideration and Fig. 3, it seems that the nearly ionic bond occurs in the small value of $S\beta$ and then of the small value of the overlap integral between the atomic orbitals.

The values of the overlap integrals are $S_{ij}\simeq 0.1$ for the alkali halides and $S_{ij}\simeq 0.5$ for the interhalogen compounds (BrCl, ICl, etc.) and the hydrogen halides. (These values can be verified by the use of the Slater orbitals.) Now we assume the

value of $\beta \simeq 2.0$ and the relation $\gamma = 0.4(x_A - x_B)$, where x_A and x_B are the electronegativities in Pauling's scale of the atom A and B, respectively. Under these assumptions we get the relations between the ionic character of the bond and the electronegativity difference for the above molecules. The result is shown in Fig. 4 with the relation given by Townes and Dailey and the results of the microwave measurements analyzed by the same authors. From this figure it is found that our two curves fit into the empirical data in their respective molecules and the curve given by Townes and Dailey is the composite of our two curves. The assumed value of β might, however, be rather small. If we use the values of the ionization potentials and the electroaffinities of the atoms given by Mulliken[6] and estimate the value of C_{ij} by the use of the relation[9]

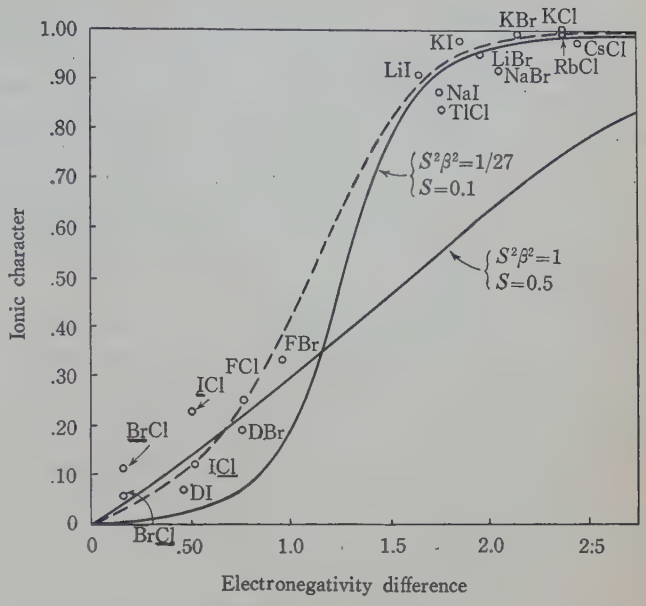


Fig. 4. The relations between the ionic character of the bond and the electronegativity difference. The broken line is the relation given by Townes and Dailey. The small circles show the results of the microwave measurements analyzed by the same authors.

$$C_{ij} = \frac{1}{a + r_0}, \qquad (5\cdot3)$$

where $1/a = (C_{ii} + C_{jj})/2$ and r_0 is the equilibrium nuclear distance, we obtain the value of β to be about 3.5 to 4.0 for the molecules in Fig. 4 from Eq. (5·2). The assumed relation $\gamma = 0.4(x_A - x_B)$ may also be rather crude. This corresponds to taking D_{ij} to be about 7 eV, if we take account of the empirical relation

$$x_A - x_B = 0.36(X_A - X_B),$$

where X_A and X_B are the electronegativities in Mulliken's scale of atom A and B, respectively. This value of D_{ij} seems to be probable for the alkali halides but rather small for the interhalogen compounds and the hydrogen halides, and then the ratio of γ to $(x_A - x_B)$ might be smaller than 0.4 for the latter molecules. Thus, the ionicity of the bond theoretically predicted will be a little smaller than that shown in Fig. 4. However, the general feature in Fig. 4 does not alter and

we may divide the above molecules into the two characteristic groups with respect to the dependence of the ionic character of the bond on the electronegativity difference. According to the relation given by Townes and Dailey the ionicity of the HF molecule is about 90%. This value is surprisingly large. In fact Karo's calculation[10] shows that the ionicity of this molecule is 48% according to the population analysis proposed by Mulliken.[11] This result seems to support our opinion. As the result it must be noted that the ionic character of the bond is not determined by the electronegativity difference alone, but it depends on the values of S and β, particularly on the value of $S\beta$.

It is easy to extend the discussion in the preceding sections to the case of the saturated polyatomic molecule. We already noted that the interaction term between the polar bonds appeared in the polyatomic molecule.[12] Considering this point, Maeda[13] discussed the inductive effect of the haloalkane molecules in the bond orbital approximation.

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Appendix

We shall explain the relation between two pairs of parameters (p, q) and (λ, μ). In this article it has always been assumed that the variable region of the parameters

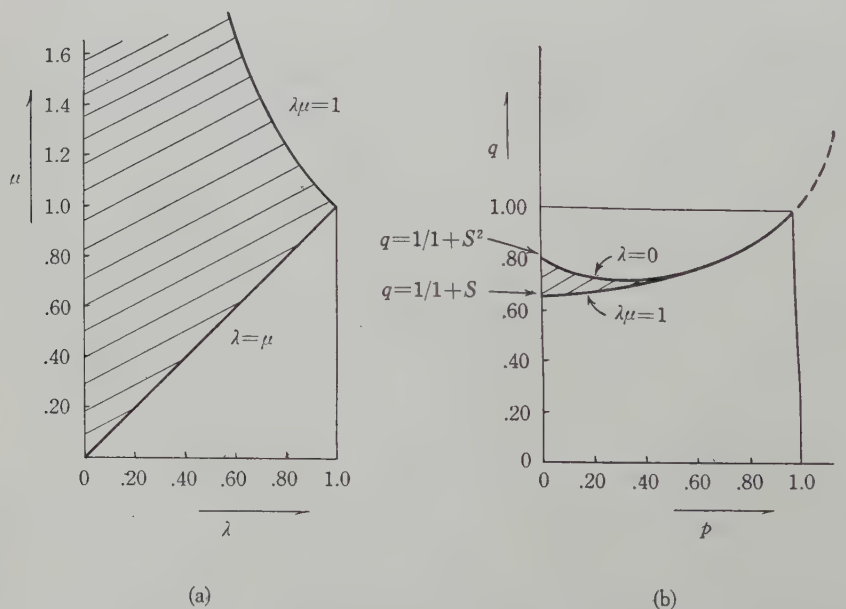


Fig. 5. The allowed regions of the parameters (λ, μ) and (p, q). The figure (b) is drawn under the condition $S_{ij}=0.5$.

λ and μ is taken to be $\mu \geqq \lambda \geqq 0$ and $\lambda\mu \leqq 1$, since we have considered that the wave function (2·1) gives the bonding state of the molecule and the atom A is more electronegative than the atom B. Then, the allowed region of the parameters λ and μ is the shaded part in Fig. 5(a) and the corresponding region of the parameters p and q is shown in Fig. 5(b) by the shaded part, as easily seen from the following discussion. We can, then, put the conditions

$$1 \geqq p \geqq 0, \quad 1 \geqq q \geqq \frac{1}{1+S} \quad \text{and then} \quad 0 \leqq r \leqq \frac{S}{1+S}. \tag{A·1}$$

By the use of Eqs. (2·3), (2·7), (2·8) and (2·12) the quantities p and q are also written as

$$p = -\frac{2(\lambda-\mu)R}{N}, \quad q = \frac{2(\lambda+\mu)R + (1-\lambda\mu)^2}{N}, \tag{A·2}$$

and the quantity k defined by Eq. (2·13) is expressed as

$$k = (1-S^2)\frac{(1-\lambda\mu)^2}{N}. \tag{A·3}$$

From this equation it is found that the bond orbital approximation ($\lambda\mu=1$) is equivalent to putting $k=0$.

From Eqs. (A·2) and (A·3) it is easily found that the relations

$$\frac{(\lambda+\mu)R}{N} = \frac{1}{2}\left[q - \frac{k}{1-S^2}\right] \tag{A·4}$$

and

$$\frac{R^2}{N} = \frac{1}{2}(1-k) \tag{A·5}$$

hold. Using Eq. (A·5), we can find the relation

$$\frac{(\lambda-\mu)^2}{N} = \frac{p^2}{2(1-k)}. \tag{A·6}$$

Using Eqs. (A·4) and (A·6), we derived Eq. (2·11) as the approximate expression for the total energy of the diatomic molecule.

Considering the above equations, we can easily express the total energy of the molecule as the function of the parameters p and q without the assumptions made in § 2, but the discussion will be more complicated than that given in this article.

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Electromagnetic Structure of the Nucleon and the Composite Model for Pion

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The electromagnetic structure of the nucleon is investigated based on the composite model for the pion. The mean square radius of charge distribution of the nucleon is calculated by the second order perturbation and the same coupling constants are used with those in the calculation of the pion-nucleon interaction and the anomalous magnetic moment of the nucleon on this model. The results are qualitatively in agreement with experiment and it is possible to explain all these phenomena consistently from the standpoint of the composite model.

§ 1. Introduction

Many authors have so far investigated the electromagnetic structure of the nucleon and in particular have calculated the mean square radii of charge distribution and anomalous magnetic moment of the nucleon in detail by meson theory because the relatively outside region of the nucleon contributes to them contrary to the fact that the inside region plays an important role for the anomalous magnetic moment, but a satisfactory result is not yet obtained in spite of their great efforts. It seems to us that one of the main reasons of their failure is the improper treatment of the virtual nucleon pair. Therefore in the present paper we study the electromagnetic structure from the viewpoint of the composite model for the pion in which the virtual nucleon pair plays the main role.

In our previous papers,[1] we have investigated the pion-nucleon interaction and the anomalous magnetic moment of the nucleon on the basis of the composite model where the fundamental interaction has been assumed to be a certain linear combination of scalar (s), tensor (t) and pseudoscalar (p) (Case I), or vector (v) and pseudovector (a) (Case II) types in the Fermi couplings* and therefore the same model is used also in this paper. We calculate here only the mean square radius of charge distribution of the nucleon considering the degree of accuracy of our approximation. In section 2 we calculate the mean square radius of charge distribution of the nucleon in second order of perturbation theory and in section 3 we decide the coupling constants to be able to explain the pion-nucleon interaction, the anomalous magnetic moment and the mean square radius of charge distribution of the nucleon consistently.

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* We here consider only the special combination[2] of s, v, t, a and p for convenience of calculation.

The notation hereafter is the same as that in I and II, unless specially mentioned.

§ 2. Mean square radius of charge distribution of the nucleon

We calculate the mean square radius of charge distribution of the nucleon by means of the second order perturbation,* adopting a similar method as has been used in the analysis of neutron-electron interaction by K. Ishida.[3]

The fundamental interaction Hamiltonians are given by

$$H'=\sum_k\sum_{i=1}^{3}\frac{g_{1k}}{M^2}\int(\bar{\psi}(\boldsymbol{r})O_k\tau_i\psi(\boldsymbol{r}))(\bar{\psi}(\boldsymbol{r})O_k\tau_i\psi(\boldsymbol{r}))\,d\boldsymbol{r}$$

$$+\sum_k\frac{g_{0k}}{M^2}\int(\bar{\psi}(\boldsymbol{r})O_k\psi(\boldsymbol{r}))(\bar{\psi}(\boldsymbol{r})O_k\psi(\boldsymbol{r}))\,d\boldsymbol{r}, \tag{II·1}$$

$$H_{N\gamma}=-ie\int\bar{\psi}(\boldsymbol{r})\gamma_\mu\frac{1+\tau_3}{2}\psi(\boldsymbol{r})A_\mu(\boldsymbol{r})\,d\boldsymbol{r}. \tag{II·21}$$

If we take up only the static electric field for convenience of our purpose, we have

$$A_\mu(\boldsymbol{r})=(0,\ 0,\ 0,\ i\phi_0(\boldsymbol{r})), \tag{1}$$

$$H_{N\gamma}=e\int\psi^*(\boldsymbol{r})\frac{1+\tau_3}{2}\psi(\boldsymbol{r})\phi_0(\boldsymbol{r})\,d\boldsymbol{r}. \tag{2}$$

The covariant nucleon vertex operator $\Gamma_\mu(p_1,\ p_2)$ must take the form

$$\Gamma_\mu(p_1,\ p_2)=\gamma_\mu F_1(q^2)+\sigma_{\mu\nu}q_\nu F_2(q^2) \tag{3}$$

according to the requirement of the relativistic and gauge invariance of the theory, if p_1 and p_2 are free momenta and $p_1-p_2=q$, $\sigma_{\mu\nu}=\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu$. The first term in (3) is expanded in powers of q^2 and has relation to the mean square radius of charge distribution of the nucleon as follows:

$$F_1(q^2)=F_1(0)-\frac{1}{6}q^2\langle r^2\rangle+\cdots, \tag{4}$$

where $F_1(0)$ is zero for neutron.

Now we may take out only the terms proportional to $\boldsymbol{q}^2$ in $\Gamma_\mu(p_1,\ p_2)$ in order to obtain $\langle r^2\rangle$ where $\boldsymbol{q}$ is 3-dimensional momentum, as the electric field is assumed to be static.

The diagrams contributing to the mean square radius of charge distribution are the same as those in the calculation of the anomalous magnetic moment (cf. Fig. 1 in II) and the contribution from the diagram (A1) in Fig. 1 is shown, as an example, in the case of pseudoscalar coupling as follows.

* We assume here that the higher order processes do not so much contribute to the results. However the conclusion from the following calculation will be meaningless if they play the essential role.

$$(A1)_p=\frac{e}{(2\pi)^{3/2}}\frac{2}{(2\pi)^3}\sum_i\frac{g_{1p}}{M^2}\int d\boldsymbol{p}_1d\boldsymbol{p}_2d\boldsymbol{p}_3d\boldsymbol{p}_4d\boldsymbol{q}\delta(\boldsymbol{p}_1-\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)$$
$$[a^*_{p_1}\{u^*(\boldsymbol{p}_1)(\beta\gamma_5\tau_i)u(\boldsymbol{p}_2)\}a_{p_2}\mathrm{Sp}\{u^*(\boldsymbol{p}_3)(\beta\gamma_5\tau_i)(-cv^*(\boldsymbol{p}_4))v(\boldsymbol{p}_4)c$$
$$\times\frac{1+\tau_3}{2}u(\boldsymbol{p}_3)\}\delta(\boldsymbol{p}_3+\boldsymbol{p}_4+\boldsymbol{q})-a^*_{p_1}\{u^*(\boldsymbol{p}_1)(\beta\gamma_5\tau_i)(-cv^*(\boldsymbol{p}_4))v(\boldsymbol{p}_4)c$$
$$\times\frac{1+\tau_3}{2}u(\boldsymbol{p}_3)u^*(\boldsymbol{p}_3)(\beta\gamma_5\tau_i)u(\boldsymbol{p}_2)\}a_{p_2}\delta(\boldsymbol{p}_3+\boldsymbol{p}_4+\boldsymbol{q})]\phi_0(\boldsymbol{q})$$
$$\times(E_{p_2}-E_{p_1}-E_{p_3}-E_{p_4})^{-1}$$
$$=\frac{e}{(2\pi)^{9/2}}\sum_i\frac{2g_{1p}}{M^2}\int d\boldsymbol{p}_1d\boldsymbol{p}_2d\boldsymbol{p}_3d\boldsymbol{p}_4d\boldsymbol{q}\delta(\boldsymbol{p}_1-\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)\delta(\boldsymbol{p}_3+\boldsymbol{p}_4+\boldsymbol{q})$$
$$[a^*_{p_1}\{u^*(\boldsymbol{p}_1)(\beta\gamma_5\tau_i)u(\boldsymbol{p}_2)\}a_{p_1}\mathrm{Sp}\{(\beta\gamma_5\tau_i)\frac{E_{p_4}+(\boldsymbol{\alpha}\boldsymbol{p}_4)-M\beta}{2E_{p_4}}$$
$$\times\frac{1+\tau_3}{2}\frac{E_{p_3}+(\boldsymbol{\alpha}\boldsymbol{p}_3)+M\beta}{2E_{p_3}}\}-a^*_{p_1}\{u^*(\boldsymbol{p}_1)(\beta\gamma_5\tau_i)\frac{E_{p4}+(\boldsymbol{\alpha}\boldsymbol{p}_4)-M\beta}{2E_{p_4}}$$
$$\times\frac{1+\tau_3}{2}\frac{E_{p_3}+(\boldsymbol{\alpha}\boldsymbol{p}_3)+M\beta}{2E_{p_3}}(\beta\gamma_5\tau_i)u(\boldsymbol{p}_2)\}a_{p_2}]\phi_0(\boldsymbol{q})$$
$$\times(E_{p_2}-E_{p_1}-E_{p_3}-E_{p_4})^{-1}, \tag{5}$$

where

$$u(\boldsymbol{p})u^*(\boldsymbol{p})=(E_p+(\boldsymbol{\alpha}\boldsymbol{p})+M\beta)/2E_p, \tag{II·23}$$
$$-cv^*(\boldsymbol{p})v(\boldsymbol{p})c=(E_p+(\boldsymbol{\alpha}\boldsymbol{p})-M\beta)/2E_p. \tag{II·24}$$

The first term in (5) vanishes because the spur in it is equal to zero. Using the relations

$$(\boldsymbol{\alpha}\boldsymbol{p}+M\beta)u(\boldsymbol{p})=E_pu(\boldsymbol{p}), \tag{II·25}$$
$$u^*(\boldsymbol{p})(\boldsymbol{\alpha}\boldsymbol{p}+M\beta)=E_pu^*(\boldsymbol{p}), \tag{II·26}$$

we obtain

$$(A1)_p\sim\frac{e}{(2\pi)^{9/2}}\frac{2g_{1p}}{M^2}\int d\boldsymbol{p}_1d\boldsymbol{p}_2d\boldsymbol{p}_3d\boldsymbol{p}_4d\boldsymbol{q}d\boldsymbol{p}\delta(\boldsymbol{p}_1-\boldsymbol{p}_2+\boldsymbol{p}_3+\boldsymbol{p}_4)\delta(\boldsymbol{p}_3+\boldsymbol{p}+\frac{\boldsymbol{q}}{2})$$
$$\times\delta(\boldsymbol{p}_4-\boldsymbol{p}+\frac{\boldsymbol{q}}{2})a^*_{p_1}u^*(\boldsymbol{p}_1)\frac{(E_{p_1}+E_{p_4})-(\boldsymbol{\alpha}\cdot\boldsymbol{p}_1+\boldsymbol{p}_4)}{2E_{p_4}}$$
$$\times\frac{(E_{p_3}-E_{p_2})-(\boldsymbol{\alpha}\cdot\boldsymbol{p}_3-\boldsymbol{p}_2)}{2E_{p_3}}\frac{3-\tau_3}{2}u(\boldsymbol{p}_2)a_{p_2}\phi_0(\boldsymbol{q})(E_{p_2}-E_{p_1}-E_{p_3}-E_{p_4})^{-1}. \tag{6}$$

Next making the approximation $E_{p_1}\sim E_{p_2}\sim M$ and expanding $E_{p+(q/2)}$ and $E_{p-(q/2)}$ in (6) in powers of $\boldsymbol{q}$, we pick up the terms proportional to $\boldsymbol{q}^2$ in $(A1)_p$.

$$(A1)_p \sim \frac{e}{(2\pi)^{3/2}} \int d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, d\boldsymbol{q}\, \delta(\boldsymbol{p}_1-\boldsymbol{p}_2-\boldsymbol{q})\, a^*_{p_1} u^*(\boldsymbol{p}_1)$$

$$\times \left\{ -\frac{1}{6}\boldsymbol{q}^2 \frac{3}{2(2\pi)^2} \frac{g_{1p}}{M^2} \int \left(1-\frac{p^2}{6E_p^2}\right) \frac{p^2 dp}{E_p^3} \frac{3-\tau_3}{2} \right\} u(\boldsymbol{p}_2)\, a_{p_2} \phi_0(\boldsymbol{q}). \tag{7}$$

Then we get

$$\langle r^2 \rangle = \frac{3}{2(2\pi)^2} \frac{g_{1p}}{M^2} \int \left(1-\frac{p^2}{6E_p^2}\right) \frac{p^2 dp}{E_p^3} \frac{3-\tau_3}{2}. \tag{8}$$

Carrying out the similar procedure for the other diagrams and making the approximation $E_p \sim M+\boldsymbol{p}^2/2M$ and cutting off integrals on $\boldsymbol{p}$ at the nucleon mass, we have the mean square radius of charge distribution of the nucleon $\langle r^2 \rangle_N$ as follows:

$$\langle r^2 \rangle_N = (1/2)\{\langle r^2 \rangle_s + \langle r^2 \rangle_v \tau_3\} \tag{9}$$

where

Case I (s, t, p)

$$\langle r^2 \rangle_s = \frac{1}{(2\pi)^2} \frac{1}{5} \{-(3g_{1s}+g_{0s})+(3g_{1t}+g_{0t})\}$$

$$+\frac{1}{(2\pi)^4} \frac{1}{120} (111 g_{1p}^2 - 54\, g_{1p}\, g_{0p} + 19 g_{0p}^2 + 6 g_{1s}^2 - 12 g_{1s} g_{0s} - 2 g_{0s}^2$$

$$+357\, g_{1t}^2 - 2178\, g_{1t} g_{0t} - 607\, g_{0t}^2 - 12\, g_{1p} g_{1s} + 12\, g_{1p} g_{0s}$$

$$+12\, g_{0p}\, g_{1s} + 4\, g_{0p}\, g_{0s} + 456\, g_{1s}\, g_{1t} + 2304\, g_{1s}\, g_{0t}$$

$$+920\, g_{0s} g_{0t} + 582\, g_{1p}\, g_{1t} - 126\, g_{1p}\, g_{0t} - 126\, g_{0p} g_{1t} + 110 g_{0p} g_{0t}), \tag{10a}$$

$$\langle r^2 \rangle_v = \frac{1}{(2\pi)^2} \frac{1}{5} \{(g_{1s}-g_{0s})-(g_{1t}-g_{0t})\}$$

$$+\frac{1}{(2\pi)^4} \frac{1}{120} (-329\, g_{1p}^2 - 54 g_{1p} g_{0p} + 19 g_{0p}^2 - 1362 g_{1s}^2 + 4 g_{1s} g_{0s} - 2 g_{0s}^2$$

$$-4617 g_{1t}^2 - 670 g_{1t} g_{0t} + 161 g_{0t}^2 + 4 g_{1p} g_{1s} - 4 g_{1p} g_{0s}$$

$$+12 g_{0p} g_{1s} + 4 g_{0p} g_{0s} - 816 g_{1s} g_{1t} + 200 g_{1s} g_{0t} + 920 g_{0s} g_{1t}$$

$$+286 g_{1p} g_{1t} - 134 g_{1p} g_{0t} + 26 g_{0p} g_{1t} - 42 g_{0p} g_{0t}). \tag{11a}$$

Case II (v, a)

$$\langle r^2 \rangle_s = \frac{1}{(2\pi)^2} \frac{1}{5} \{(3g_{1v}+g_{0v})+(3g_{1a}+g_{0a})\}$$

$$+\frac{1}{(2\pi)^4} \frac{1}{120} (339\, g_{1v}^2 - 390 g_{1v} g_{0v} - 785 g_{0v}^2 - 1170 g_{1v} g_{1a} + 258 g_{1v} g_{0a}$$

$$-126g_{0v}g_{1a}-346g_{0v}g_{0a}+255g_{1a}^2-174g_{1a}g_{0a}+27g_{0a}^2), \tag{10b}$$

$$\langle r^2\rangle_v=\frac{1}{(2\pi)^2}\frac{1}{5}\{-(g_{1v}-g_{0v})-(g_{1a}-g_{0a})\}$$

$$+\frac{1}{(2\pi)^4}\frac{1}{120}(-3621g_{1v}^2+98g_{1v}g_{0v}+111g_{0v}^2-33g_{1v}g_{1a}-118g_{1v}g_{0a}$$

$$-166g_{0v}g_{1a}+86g_{0v}g_{0a}-4713g_{1a}^2-118g_{1a}g_{0a}+27g_{0a}^2). \tag{11b}$$

§ 3. Results

The coupling constants g_{1k} and g_{0k} must satisfy the assumptions (II·8a), (II·8b), (II·9a) and (II·9b) to obtain one of the solutions of the eigenvalue equation for the one nucleon pair excited state $\omega_K=\sqrt{\boldsymbol{k}^2+\mu^2}$ (μ is the pion mass) (and further (II·14b) to get the free Hamiltonian of the pion H_π in Case II) and the assumptions (II·36a), (II·36b) and (II·37a) must also be fulfilled by them so that the effective s-wave interaction Hamiltonian H_s has the desirable form and the anomalous magnetic moment of the nucleon μ_N gives the experimental result.

Case I (s, t, p) We find out the coupling constants to satisfy the relations (II·8a), (II·9a), (II·36a),

$$\mu_s=-\frac{1}{31}\mu_v \tag{12a}$$

instead of (II·37a) and

$$\langle r^2\rangle_s=\langle r^2\rangle_v. \tag{13a}$$

At first H_s becomes (II·38a) by the use of (II·36a). Next we assume the conditions $g_{1p}=g_{1s}$ and $g_{0p}=g_{0s}$.* Then the numerical values for the anomalous magnetic moment and the mean square radius of charge distribution are shown in Table I and they are qualitatively in agreement with experimental results.

Table I

g_{1p}	g_{0p}	g_{1s}	g_{0s}	g_{1t}	g_{0t}	μ_p $(e/2M)$	μ_n $(e/2M)$	$\langle r^2\rangle_p$ (cm^2)	$\langle r^2\rangle_n$ (cm^2)
2.18	−38.15	2.18	−38.15	−9.17	−2.82	1.78	−1.90	$(0.14\times10^{-13})^2$	0

Case II (v, a) Assuming the relations (II·8b), (II·9b), (II·14b) and (II·36b), we get

$$g_{1v}=7.5,\quad g_{0v}=-15,\quad g_{1a}=7.5,\quad g_{0a}=0.$$

* If these conditions are fulfilled, the ambiguity in the correction of electric charge on this model is eliminated (cf. reference 4)).

Then the results obtained by use of these coupling constants are shown in Table II. It is seen that they are not reasonable.*

Table II

g_{1v}	g_{0v}	g_{1a}	g_{0a}	μ_p $(e/2M)$	μ_n $(e/2M)$	$\langle r^2\rangle_p$ (cm^2)	$\langle r^2\rangle_n$ (cm^2)
7.5	−15	7.5	0	1.06	−1.92	$-(0.08\times10^{-13})^2$	$(0.1\times10^{-13})^2$

Thus it seems to us that it is possible to account for all these phenomena consistently from the viewpoint of the composite model for the pion, though we cannot definitely conclude so because of the approximation used here, and especially it is interesting that the results obtained above support the restriction of the interaction type proposed by Iso.[2] Our next task is to consider the reason why the composite model can explain the experimental results which can hardly be accounted for by the meson theory, but we cannot clarify this problem on the stage of the above calculation in spite of its importance. Though we do not consider the effects of Λ-particle in this paper, this will also be one of our future tasks.

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* If the effects of the higher order processes and Λ-particle play the important role,[5] these results are not always unreasonable.

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Compound Nucleus Processes in the Reactions between Complex Nuclei

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A modification is made for the statistical theory of nuclear reactions so as to take account of the angular momenta of the nuclear states relevant in the reactions. Since the average values of the angular momenta of the compound nuclei formed by different projectiles at the same excitation energy will be different, various aspects of the decay process will also depend on the incident particles. For our purpose it is essential to take the spin dependence of the nuclear level density not as $(2I+1)$, but $(2I+1)\exp[-I(I+1)/2cT]$. In this paper, the energy spectra of emitted particles are treated. The average kinetic energy of evaporating neutrons from a high spin nucleus becomes higher than twice the nuclear temperature T. It is shown that the anomalies found in the branching ratios or excitation functions in the heavy ion reactions can be explained by the angular momentum effect. The degree of these anomalies depends sensitively on the nuclear moment of inertia $c\hbar^2$.

§ 1. Introduction

In the region of overlapping compound nuclear levels, it is usually assumed that the decay of the compound nucleus is independent of the mode of its formation. This independence hypothesis can be stated plausibly in the following semi-classical argument.[1] "The incident particle and the constituents of the target nucleus interact so strongly with each other that, in a time short compared with the free passage time of the particle through the nucleus, the energy is distributed among a large number of constituents. Second, since the particles are constantly and rapidly exchanging energy, a state of statistical equilibrium is reached before the compound state is broken up; and finally, because the decay takes place from a statistical distribution, the probabilities of the different break-ups do not depend upon the nature of the initial delivery of energy".

Experiments were made by Ghoshal[2] and John[3] to test the independence hypothesis directly. Their experiments were designed to determine whether the various products of the compound nucleus were produced in the same relative proportions, when the same compound nucleus was formed at the same excitation energy by proton or α-particle bombardment. It was found that the cross sections of α-particle reactions were shifted by $1\sim2$ Mev to higher excitation energy than the proton reactions. And it remained unsettled whether this discrepancy is significant or not.

Since then, the experimental investigation of the compound nucleus processes

by accelerated ions heavier than helium has developed and the possibility to check up the independence hypothesis has increased. For a given excitation energy, the heavy ions bring in much more angular momenta than the nucleons and α-particles. Associated with this angular momentum is some rotational kinetic energy, and in computing the probability for particle emission in terms of the density of states in the residual nucleus, one should subtract from the excitation energy this unavailable rotational energy. Then, it seems reasonable to apply the above-mentioned classical argument concerning the independence hypothesis with respect to the intrinsic energy of the nucleus with definite angular momentum. Because the amount of the angular momentum deposited in the compound nucleus varies with the nature of the projectiles, the various aspects of its decay will do depend on the incident particles.

Indeed, following this line, Ericson and Strutinski,[4] and Douglas and MacDonald[5] have studied the angular distribution of evaporating particles. The classical statement that a part of the total excitation energy, $I^2/2c$, is lost in the rotational energy, is expressed by taking the spin dependence of the nuclear level density as $(2I+1)\exp[-I(I+1)/2cT]$, where I is the nuclear spin, $c\hbar^2$ the moment of inertia of the nucleus, and T the nuclear temperature. According to the above authors, a mode of particle emission is favored in which its orbital angular momentum is parallel to the spin of the compound nucleus, making the residual nuclear spin as low as possible. For this reason, from the compound nucleus with high angular momentum, particles come off 90°-symmetrically and with greatest probability forward and backward along the incident beam. Recent experiments by Knox et al.[6] indicate that in the bombardments of Ni with 162 Mev O^{16} ions the angular distributions of α-particles emitted are all predominant forward and backward. This data can be considered to support the above calculations.

In the heavy ion induced reaction which may be regarded as proceeding via compound nucleus, curious effects are shown up in other aspects. For example, in the reaction $Cu^{63}+N^{14}$ with mean excitation energy between 40 and 60 Mev, Beydon et al.[7] have found that, while the usual statistical theory agrees well with the experiment for the nuclides involving the evaporation of four or more nucleons, there is a large experimental excess of the nuclides involving a loss of only two or three nucleons. Also, examining various secondaries from the nucleus with about 40 Mev excitation energy formed by $K^{39}+N^{14}$, Pinajian and Halbert[8] point out that one-particle emissions account for a surprisingly large fraction (14%) of the estimated total cross section.

Pursuing the same kind of argument as Ericson and Strutinski, we shall inquire whether a clue to these phenomena could be found. In § 2 and § 3, we shall calculate the change caused by the angular momentum effect in the energy spectrum of the evaporating neutrons for the case of heavy ion bombardment. In the conventional statistical theory, the energy spectrum of neutrons follows the classical Maxwell distribution determined by the temperature T of the residual

nucleus, and its average kinetic energy is given by $2T$. Since the probability to find high spin states in the residual nucleus is reduced by taking account of the spin dependence of density of states correctly, it becomes favorable for particles evaporating from a compound state with high spin to carry off larger angular momentum (i.e. higher energy) than usual. Then the average number of nucleons evaporated may be lowered and branching ratio will be modified. In § 4, under certain assumptions, we treat multiple neutron evaporation reaction by heavy ions. There, the energy shift as found in Ghoshal's experiment will also be explained as due to the difference between the average angular momenta which are introduced by different projectiles into the same compound nucleus. In § 5, discussion of our results will be made.

§ 2 Theory

We follow the classical approximation of Ericson and Strutinski.[4] To begin with, the incident, the outgoing particles and the target nucleus are assumed to have zero spin. Quantities connected with the incident and outgoing particles are distinguished by suffices 1 and 2, respectively. Assuming the independence hypothesis for the compound state with spin I, the cross section for the emission of particle 2 with energy E_2 in the $\boldsymbol{n}$ direction can be written as

$$\sigma(\boldsymbol{n}, E_2) = \int dI \sigma_I^{(1)} G_I(\boldsymbol{n}, E_2). \tag{1}$$

Here, $\sigma_I^{(1)}$ is the cross section for the formation of the compound nucleus with spin I,

$$\sigma_I^{(1)} = \pi \lambda\!\!\!^{-}_1{}^2 2I T_1(I) \tag{2}$$

where $\lambda\!\!\!^{-}_1$ is the wavelength in the incident channel, $T_1(I)$ is the Ith partial wave transmission coefficient. The branching ratio $G_I(\boldsymbol{n}, E_2)$ is given by

$$G_I(\boldsymbol{n}, E_2) = \int_0^{2\pi} d\varphi/2\pi \int d\boldsymbol{l}\, \Gamma_I(\boldsymbol{l}, \boldsymbol{n}, E_2)/\Gamma_I, \tag{3}$$

where φ is the azimuthal angle of $\boldsymbol{I}$ with respect to the incident beam, $\boldsymbol{l}$ is the angular momentum of the outgoing particle, Γ_I is the total width of the compound state with spin I. $\Gamma_I(\boldsymbol{l}, \boldsymbol{n}, E_2)$, the partial width for emission in the $\boldsymbol{n}$ direction of particle 2 with E_2, $\boldsymbol{l}$ from that state, can be expressed in terms of the capture cross section of particle 2 with E_2, $-\boldsymbol{l}$ incident from the $-\boldsymbol{n}$ direction, $d\sigma_l(\boldsymbol{n}, E_2)$, as follows:

$$\Gamma_I(\boldsymbol{l}, \boldsymbol{n}, E_2)\, d\boldsymbol{l} = \frac{\mu_2}{4\pi^3} \cdot \frac{E_2}{\hbar^2} \cdot \frac{\rho_f}{\rho_i} d\sigma_l(\boldsymbol{n}, E_2), \tag{4}$$

where

$$d\sigma_l(\boldsymbol{n}, E_2) = \lambda\!\!\!^{-}_2{}^2 T_2(l)\, \delta(\boldsymbol{n}\cdot\boldsymbol{l})\, d\boldsymbol{l}, \tag{5}$$

and μ_2 is the reduced mass of the emitted particle, $\delta(x)$ is the one-dimensional δ-function, ρ_i and ρ_f are the level density of the compound and residual nucleus, respectively. Denoting the spin of the residual nucleus by $\boldsymbol{j}$, ρ_f is given by

$$\rho_f = \int \rho(\boldsymbol{j})\, \delta^3(\boldsymbol{l}+\boldsymbol{j}-\boldsymbol{I})\, d\boldsymbol{j} \tag{6}$$

with

$$\rho(\boldsymbol{j}) = \rho_0(E_f)\, e^{-\alpha_f j^2}. \tag{7}$$

Here E_f is the energy of the residual nucleus. As to the spin independent factor $\rho_0(E_f)$, we may assume the form[9]

$$\rho_0(E_f) = \rho_0 E_f^{-2}\, e^{2(aE_f)^{1/2}} \tag{8}$$

in which ρ_0 is a slowly varying function of E_f. The constant a connects the nuclear temperature T and the excitation energy E by the relation $E=aT^2$. α_f and α_i are defined by

$$\alpha = (2cT)^{-1}, \tag{9}$$

where $c\hbar^2$ is the nuclear moment of inertia in the excited state. According to Lane and Thomas,[10] the replacement of $\rho(\boldsymbol{j})$ by spin independent $\rho_0(E_f)$ in the above discussion is essential to the independence of the decay of the compound nucleus from its formation.

Integration of (1), (2) over φ and direction of $\boldsymbol{l}$ gives

$$\sigma(\boldsymbol{n}, E_2) = \text{const} \times \rho_0(E_f) \int_0^\infty I T_1(I)/\Gamma_I dI \times \int_0^\infty \exp[\alpha_i I^2 - \alpha_f (I^2+l^2)]\, W_n(\theta, E_2)\, l T_2(l)\, dl, \tag{10}$$

where

$$W_n(\theta, E_2) = \sum_{k=0}^{\infty} (-1)^k (4k+1) \left[\frac{(2k)!}{(2^k k!)^2}\right]^2 j_{2k}(2i\alpha_f Il)\, P_{2k}(\cos\theta). \tag{11}$$

If we integrate (10) over the angle θ between the direction of emission and the incident beam the energy spectrum of emitted particles becomes

$$\sigma(E_2) = \text{const} \times \rho_0(E_f) \int_0^\infty I T_1(I)/\Gamma_I dI \times \int_0^\infty \exp[\alpha_i I^2 - \alpha_f (I^2+l^2)]\, j_0(2i\alpha_f Il)\, l T_2(l)\, dl. \tag{12}$$

Now we shall assume the sharp cutoff approximation for the transmission coefficient, that is,

$$T_1(I)=\begin{cases}1 & \text{for } I\leq I_m\equiv[2\mu_1(E_1-B_1)R_1^2/\hbar^2]^{1/2}\\ 0 & \text{for } I>I_m,\end{cases} \tag{13}$$

where $\hbar I_m$ is that angular momentum which the particle classically can just overcome the potential barrier and penetrate to the interior of the nucleus. A similar equation holds for $T_2(l)$, by replacing I_m by $l_m=[2\mu_2(E_2-B_2)R_2^2/\hbar^2]^{1/2}$. Here B_i is the Coulomb barrier, and R_i the interaction radius. Inserting the above approximation into (12), one can rewrite this in the following form,

$$\sigma(E_2)=\text{const}\times\rho_0(E_f)\,\alpha_f^{-3/2}\int_0^{I_m}\Gamma_I^{-1}e^{\alpha_i I^2}f(I,\,E_2)\,dI \tag{14}$$

with

$$f(I,\,E_2)=2\Phi(\sqrt{\alpha_f}I)-\Phi(\sqrt{\alpha_f}(I+l_m))-\Phi(\sqrt{\alpha_f}(I-l_m)), \tag{15}$$

$$\Phi(x)=2/\sqrt{\pi}\cdot\int_0^x e^{-y^2}dy. \tag{16}$$

If α_i and α_f were both zero, $\sigma(E_2)$ would reduce to constant$\times\rho_0(E_f)E_2$, the energy spectrum given by the ordinary statistical theory. We define the ratio of (14) to this original distribution as $R(E_2)$.

In the following, we shall make the assumption that neutron emission is the only effective process for de-excitation. Then, the total width Γ_I becomes

$$e^{\alpha_i I^2}/I\times\int_0^\infty\rho_0(E_f)\,\alpha_f^{-3/2}f(I,\,E_2)\,dE_2 \tag{17}$$

except for a multiplying constant.

§ 3. The energy spectrum of evaporating neutrons

Now, let us calculate the energy spectra of evaporating neutrons from the compound nucleus Zn^{65} formed in $N^{14}+V^{51}$ reaction. The excitation energy of Zn^{65} is fixed at 60 Mev, then, the maximum angular momentum introduced by N^{14} amounts to as high as $28\hbar$. This example is utilized in the next section.

3.1. *The parameters*

(a) The moment of inertia of the residual nucleus Zn^{64}. Various values are suggested for $c\hbar^2$. For instance, using the Fermi gas model, LeCouteur and Lang[11] derived the value $c=6a\langle m^2\rangle/\pi^2$. The mean square value of the magnetic quantum number of individual nucleons is given by $\langle m^2\rangle\approx 0.146A^{2/3}$ from the sequence of states in the shell model. If $a=A/8\,\text{Mev}^{-1}$ is adopted, we find $c=A^{5/3}/90\text{Mev}^{-1}$, which corresponds to a moment of inertia equal to 0.8 times the rigid body value for a nucleus with radius $1.22A^{1/3}\times10^{-13}$cm (denoted by $c_R\hbar^2$). Then, taking into account of the pairing interaction we change the moment of inertia

multiplied by the nuclear temperature, cT, to $c'T=cTe^{-b/T}$, where $b=0.437\Delta$, Δ is the doubled pairing energy given by $(3.36-0.0084A)$ Mev.[12] On the other hand, the value derived by Newton[9] including shell effect corresponds to $a=A/13$ Mev^{-1}, $c=0.45c_R$ for Zn^{64}. Although a has been determined from low energy neutron reaction data in these cases, other authors propose considerably different values. In view of these uncertainties of a and c, they were left as adjustable parameters. The value of $c\hbar^2$ based on LeCouteur and Lang, and Newton's formula are listed in Table I for two choices of a.

Table I. Moment of inertia of Zn^{64} (in units of $c_R\hbar^2$)

	$a=A/8$ Mev^{-1}	$a=A/20$ Mev^{-1}
LeCouteur and Lang[11]	0.8	0.32
Lang and LeCouteur[12]	0.34	0.23
Newton[9]	0.72	0.29

(b) The value of $\alpha_f=(2cT_f)^{-1}$. Assuming that c of the final nucleus does not depend much upon the energy E_f, we extract a factor β from the expression for α_f by

$$\beta=(\alpha_f c_R E_f^{1/2})^{-1}=2c/(c_R a^{1/2}). \tag{18}$$

As is seen from (14), if Γ_I is independent of I, the ratio of the energy distributions $R(E_2)$ depends only on β, not on c and a separately. If Γ_I depends on I, we still continue to consider β as a suitable parameter, and compute the energy spectra for two values of β, that is, for $\beta=0.57$, 0.28. The values of c/c_R corresponding to these β are shown in Table II for $a=A/8$, $A/20$(Mev^{-1}).

Table II.

	$a=A/8$ Mev^{-1}	$a=A/20$ Mev^{-1}
$\beta=0.57$	0.8	0.5
$\beta=0.28$	0.4	0.25

(c) The interaction radii. R_1 is taken as $1.5\,(A_1^{1/3}+A_2^{1/3})\times10^{-13}$ cm, the determination of which may be possible by the measurement of the compound nucleus formation cross sections by heavy ions. In a somewhat arbitrary manner, R_2 is supposed to be equal to R_1. If R_2 turns out to be smaller than R_1, the angular momentum of outgoing particle will become smaller, and the angular momentum effect will decrease.

3.2. *Calculation*

In the case of $\beta=0.28$, $a=A/20$ Mev^{-1}, the calculated energy spectra are

shown in Fig. 1 for the compound states of Zn^{65} with definite spin I. This exhibits the expected displacement of the spectrum towards higher energies for higher I. In Fig. 2, the neutron spectrum from Zn^{65} having spin ranging from zero to $28\hbar$, calculated by (14) is compared with that based on the conventional statistical theory. $R(E_2)$ and the ratio of energy spectra to a Maxwell distribution with $T=4$ Mev are plotted in Fig. 3.

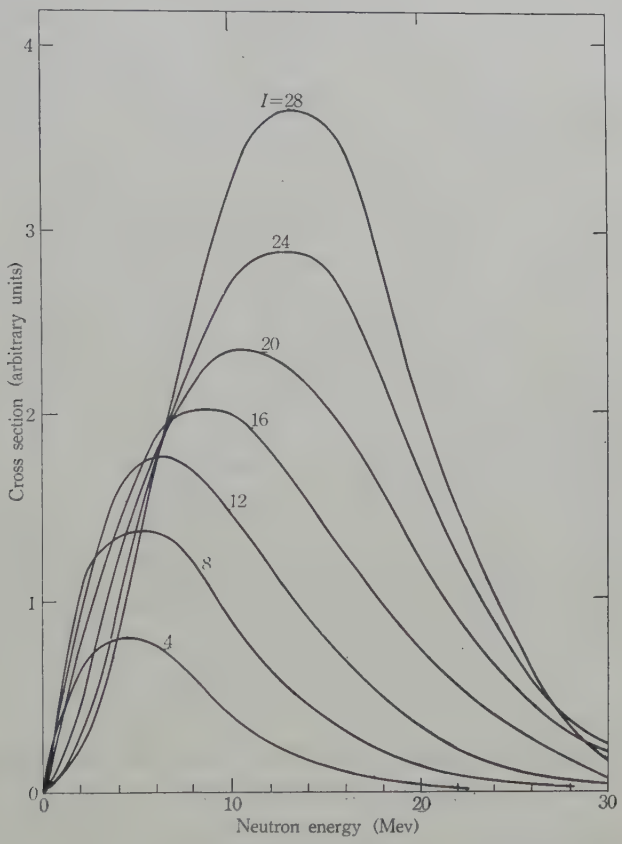


Fig. 1. Energy spectra of first neutrons to be emitted from Zn^{65} with definite spin I calculated for 60 Mev excitation, $\beta=0.28$, $a=A/20$ and $R_1=R_2=9.2\times10^{-13}$ cm.

Next, fixing β at 0.28, we have changed a to $A/8$ Mev^{-1}. From Fig. 3, it is seen that $R(E_2)$ are not sensitive on a as stated in § 3. 1. (b).

Increasing β from 0.28 to 0.57 makes the energy dependence of $R(E_2)$ insignificant (Fig. 3). For $a=A/8$ Mev^{-1}, the average kinetic energy of emitted neutrons is 5.1 Mev in the ordinary theory. The present effect gives an increase of only about 1 Mev for this when $\beta=0.57$.

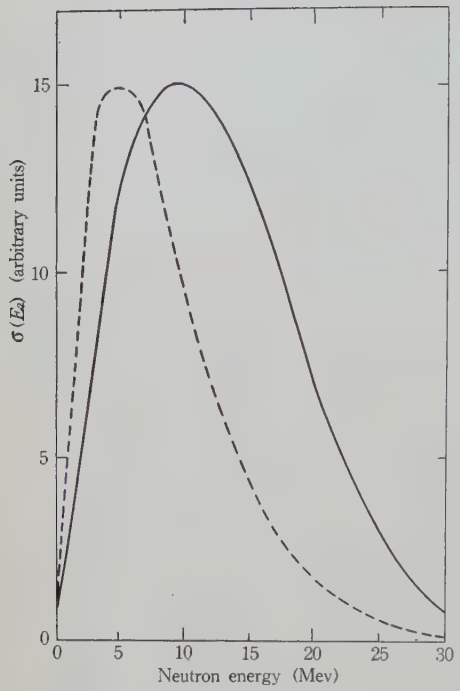


Fig. 2. Energy spectrum of first neutrons to be emitted from V^{51} (N^{14}, n) Zn^{64} reaction calculated for 60 Mev excitation, $\beta=0.28$, and $a=A/20$. The spectrum based on the ordinary statistical theory is also shown for 60 Mev excitation, $a=A/20$ (dashed curve).

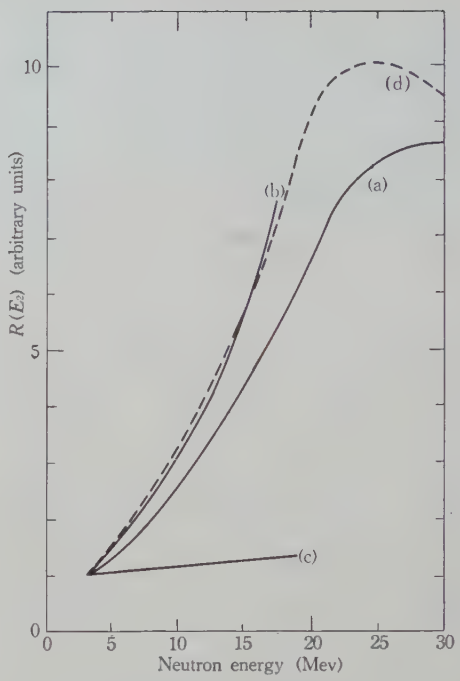


Fig. 3. The ratios $R(E_2)$ of the new spectra to that obtained by the ordinary statistical theory as a function of neutron energy E_2 calculated for various parameters. (a) $\beta=0.28$, $a=A/20$; (b) $\beta=0.28$, $a=A/8$; (c) $\beta=0.57$, $a=A/8$. In (d), the ratio of the new spectrum to $E_2 \exp(-E_2/4.0)$ (E_2 in Mev) is plotted for $\beta=0.28$, $a=A/20$.

§ 4. Excitation function of $V^{51}(N^{14}, 3n)Zn^{62}$ as compared with $Cu^{65}(p, 3n)Zn^{63}$

In a study of the excitation function of $V^{51}(N^{14}, 3n)$ Zn^{62} (Zn^{65} compound nucleus), Karamyan et al.[13] have found that, while this curve is characterized by a maximum indicating compound nucleus formation followed by neutron evaporation, this is shifted as a whole towards higher excitation energy by approximately 8 Mev relative to the $Cu^{65}(p, 3n)Zn^{63}$ reaction (Zn^{66} compound nucleus). This is reproduced in Fig. 4. There are two factors which contribute to this discrepancy: First, the strong Coulomb barrier to the incident heavy ion distorts the cross section curve, cutting down its lower side; second, as both reactions do not involve the same compound nucleus, the differences in neutron binding energies are important. However, even after applying these corrections, there remains an energy shift of several Mev unexplained. Under these circumstances, (N^{14}, $3n$) and

$(p, 3n)$ reactions cannot be fitted by the Jackson formula with the same value of nuclear temperature. The evaporation model devised by Jackson[14] involves the following assumptions: (1) The neutron energy spectrum is given by $E_2 \exp \times(-E_2/T)$, (2) neutron emission occurs whenever it is energetically possible, and the competition from other modes of de-excitation can be neglected, (3) the nuclear temperature T is independent of excitation energy, regarded as a parameter.

Now, let us modify the Jackson model along the lines presented in the preceding sections and consider whether $(N^{14}, 3n)$ and $(p, 3n)$ reactions could be fitted with the same T. For this purpose, we shall make the following two assumptions in place of (1): (1)′ The energy spectrum of the first neutron emitted has the form of

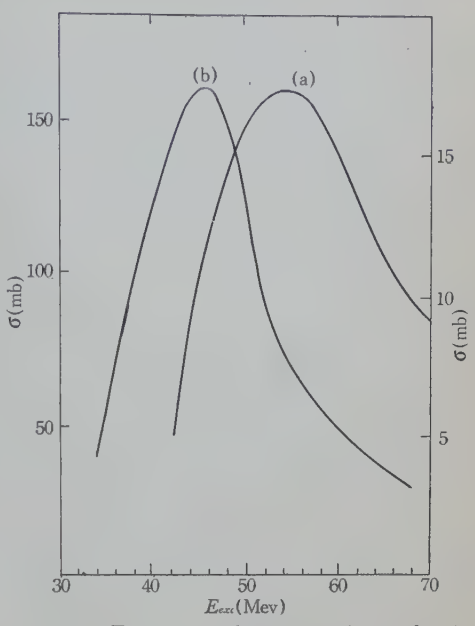


Fig. 4. Experimental cross sections of (a) $V^{51}(N^{14}, 3n)Zn^{62}$ (right hand scale) and (b) $Cu^{65}(p, 3n)Zn^{63}$ (left hand scale) as a function of the excitation energy of the respective compound nucleus (ref. 13)).

$$(pE_2^2+qE_2+r)E_2\exp(-E_2/T),$$

approximating the exact spectrum calculated in § 2 and § 3. (1)″ the spin of the residual nucleus decreases so rapidly that the spectra of the second and third neutrons reduce to $E_2 \exp(-E_2/T)$. Then one can express $(N^{14}, 3n)$ cross section as

$$\sigma(N^{14}, 3n)=\pi(R_1+\lambda\!\!\!^{-}_1)^2(1-B_1(R_1+\lambda\!\!\!^{-}_1)/E_1)P(E^*, 3). \qquad (19)$$

Here $B_1(R_1+\lambda\!\!\!^{-}_1)$ is the Coulomb barrier at the interaction distance $(R_1+\lambda\!\!\!^{-}_1)$, and the probability $P(E^*, 3)$ of exactly three neutrons evaporating from the compound state with energy E^* is given by

$$\begin{aligned} P(E^*, 3)=\{&6pT^2[I(\delta_3, 5)-I(\delta_4, 7)]+2qT[I(\delta_3, 4)-I(\delta_4, 6)] \\ &+r[I(\delta_3, 3)-I(\delta_4, 5)]\}/N, \end{aligned} \qquad (20)$$

where

$$\begin{aligned} &N=6pT^2+2qT+r, \\ &\delta_3=[E^*-\textstyle\sum_{i=1}^{3}B_i]/T, \quad \delta_4=[E^*-\sum_{i=1}^{4}B_i]/T, \\ &I(z, n)=(1/n!)\int_0^z y^n e^{-y}dy. \end{aligned} \qquad (21)$$

B_i is the binding energy of ith neutron that is emitted.

We determine T so that the position of the maximum of the calculated $Cu^{65}(p, 3n)$ reaction agrees with the experimental data. The value chosen is 2.8 Mev, corresponding roughly to $a=A/20\,\mathrm{Mev}^{-1}$. The correction factor $(pE_2^2+qE_2+r)$ is given by approximating the curve of the ratio $R(E_2)$ in Fig. 3 for $a=A/20$, $\beta=0.28$ $(c=0.25\,c_R)$, up to $E_2=20$ Mev. The results are shown in Fig. 5. The peak position of $P(E^*, 3)$ of $(N^{14}, 3n)$ reaction which lies lower than $E^*=44$ Mev in the ordinary Jackson model, moves to $E^*=47$ Mev after taking into account the angular momentum effect. Finally, because of the energy dependence of the compound nucleus formation cross section, the maximum of $V^{51}(N^{14}, 3n)$ excitation function is placed at $E^*=52$ Mev.

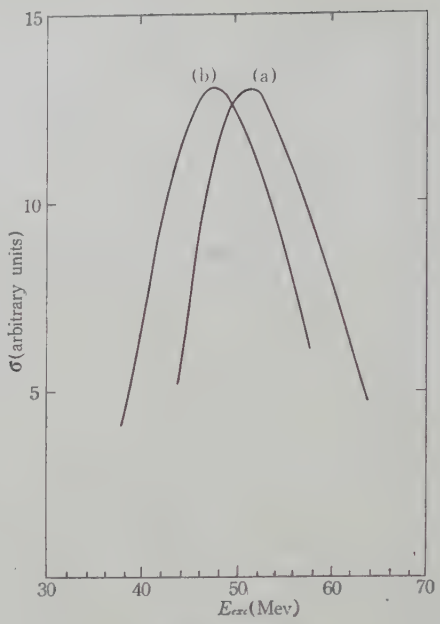


Fig. 5. Calculated cross sections of (a) $V^{51}(N^{14}, 3n)Zn^{62}$ and (b) $Cu^{65}(p, 3n)Zn^{63}$ as a function of the excitation energy of the compound nucleus for $\beta=0.28$, $a=A/20$.

§ 5. Discussion and concluding remarks

From Fig. 5 it is seen that in spite of the strong Coulomb distortion, the upward energy shift of $(N^{14}, 3n)$ cross section can be reproduced by the present model with suitable parameters. The displacement of the proton and α-particle curves on the energy axis in Ghoshal and John's experiments will also be explained likewise. The energy shift becomes smaller than the heavy ion case (several Mev), corresponding to the lower angular momentum introduced by α-particle. It should be noted that the angular-momentum effect depends on the moment of inertia of the nucleus in the excited state, and seems to bring the significant results only when $c\hbar^2$ is much smaller than the rigid body value. Whether the interparticle correlation causing this reduction remains the same in these high energy regions, should be confirmed from other sides of the evaporation process such as angular distribution.

Now, let us suppose that the evaporated neutrons can carry off only a small part of the angular momentum, and the nucleus after evaporation is left in the excited state with rather large angular momentum. Then there are two ways open. First, since (the spin of the intermediate nucleus) divided by (cT of the final nucleus) may become large for the last neutron emitted, the angular-momentum effect may be largest for it. In this case, it may be possible to fit the experimental energy shift with c larger than that used in the calculation of § 4. Second, even

if the nucleus is left with excitation energy higher than the neutron binding energy, it cannot decay to low-lying states by neutron emission because of the angular momentum conservation, while the successive emission of γ-ray will be enhanced. For the investigation of the manner in which the initial angular momentum dissipates itself, and the degree in which γ-ray de-excitation is enhanced, we must await further experimental works.

In this paper, we have considered that, since the extra centrifugal barrier caused by high spin suppresses the emission of low energy particles, average kinetic energy of evaporated neutrons exceeds $2T$. This effect is expected to decrease average number of nucleons evaporated than the usual statistical theory. On the other hand, the inclusion of γ-ray competition to the statistical theory was called forth as an alternative attempt for this lowering. That is, Fisher et al.[15] measured the ratio of p emission to $2p$ emission in the case of 27 Mev nitrogen ion bombardment on S^{32}. In their comparison with the statistical theory, two values of the level density parameter, a, were employed. The value $a=A/10.5\,\mathrm{Mev}^{-1}$ fits the ratio when gamma emission is considered, whereas without γ-ray de-excitation, $a=2(\approx A/23)$ Mev^{-1} gives equally good agreement. Although the experiments concerning the branching ratios include both the angular-momentum effect and the γ-ray competition, only the former appears in the energy shift experiments, otherwise the γ-ray competition must depend on the projectiles like that induced by the angular-momentum effect. It may be necessary to perform the experiments which single out the effect of the γ-ray competition.

As to the case of charged particle emission, we cannot study in a quantitatively correct manner as far as our sharp cutoff approximation for the transmission coefficient. Moreover, the situation is more complex owing to the fact that the enhancement of the low energy particle emission introduced by the diffuseness of the nuclear potential barrier[16] must be combined with our viewpoints.

So far our discussions have rather been concentrated on medium weight nuclei. Finally, we have to mention the experiment performed by Tamers and Wolfgang[17] in the region of light nuclei. For example, they compared the excitation function of $C^{12}(C^{12}, np)$ with $Na^{23}(p, np)$ leading to the same compound nucleus Mg^{24}, and pointed out that the former has a maximum in the excitation energy of Mg^{24} lower than the latter. This is just opposite to our viewpoints. However, in the elastic scattering of C^{12} by carbon, the existence of unexpected resonances in the excited states of Mg^{24} higher than 24 Mev has been found, but not in the scattering of O^{16} by oxygen.[18] These complex situations may indicate that the above energy shift cannot be explained by a simple modification to the conventional statistical theory such as ours, but must invoke the structural study of the colliding nuclei.

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A Semi-phenomenological Neutron-Proton Potential

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A $T=0$ two-nucleon potential is found which, when combined with the $T=1$ potential reported previously, can reproduce all experimental neutron-proton data below 300 Mev. The triplet even parity potential is not strictly energy independent. The required energy dependence is, however, very small and confined in the core region. In the triplet even parity state, it is found that both linear and quadratic LS potentials are required. The linear LS potential is weak and repulsive. The quadratic LS potential is here stronger than that required in the $T=1$ state and it is attractive. The singlet odd parity potential is slightly more repulsive than that of the one-pion-exchange potential. We conclude that all two-nucleon data up to 300 Mev can be understood in terms of a potential picture which is consistent with the current implications of the pion theory of nuclear forces.

§ 1. Introduction

In a previous paper[1] we have reported on a $T=1$ ($T=$isospin of the two-nucleon system) potential which reproduces all available p-p data up to 310 Mev. We have been led to the conclusion that the p-p scattering in the non-relativistic region can be understood in terms of an energy independent potential which is not inconsistent with the pion theory of nuclear forces. In the present paper we report on the result of the investigation into the corresponding situation in the n-p system below 300 Mev.

On the basis of the charge independence of nuclear forces, the $T=1$ (singlet even and triplet odd parity) potentials are taken over from I. We are then concerned in the present paper with the $T=0$ (triplet even and singlet odd parity) potentials of the form

$$ {}^3V^+(x) = {}^3V_C^+(x) + {}^3V_T^+(x) S_{12} + {}^3V_{LS}^+(x) (\boldsymbol{L}\cdot\boldsymbol{S}) + {}^3V_Q^+(x) Q_{12}, \tag{1} $$

and

$$ {}^1V^-(x) = {}^1V_C^-(x) + {}^1V_Q^-(x) Q_{12}, \tag{2} $$

in the same notations as in I.

The potentials of the form (1) and (2) with $V_Q \equiv 0$ have been applied to the n-p scattering by Gammel-Thaler (GT)[2] and Signell-Marshak (SM).[3] GT have

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not attempted an energy independent fit. The $T=1$ potential of SM is now considered inadequate above about 150 Mev.[1),4)]

Watari and Tamagaki[5),6)] analysed the n-p scattering below 150 Mev to find no evidence of V_{LS} or V_Q. It has now become clear, however, that the $T=1$ static potential alone does not account for the p-p data above 150 Mev. We also note that there exists evidence in the 300 Mev n-p data for non-static forces in (1).[7)]

To be consistent with I we assume the following forms for V_i in (1) and (2) :

triplet even:

$$
\begin{aligned}
&{}^3V_C^+(x) = -(g^2/4\pi)\mu(e^{-x}/x)[1+{}^3a_C^+(e^{-x}/x)+{}^3b_C^+(e^{-x}/x)^2],\\
&{}^3V_T^+(x) = -(g^2/4\pi)\mu(e^{-x}/x)(1+3/x+3/x^2)[1+{}^3a_T^+(e^{-x}/x)+{}^3b_T^+(e^{-x}/x)^2],\\
&{}^3V_{LS}^+(x) = {}^3G_{LS}^+\mu(e^{-x}/x)^2[1+{}^3b_{LS}^+(e^{-x}/x)],\\
&{}^3V_Q^+(x) = {}^3G_Q^+\mu(e^{-x}/x)[1+{}^3a_Q^+(e^{-x}/x)+{}^3b_Q^+(e^{-x}/x)^2].
\end{aligned}
\tag{3}
$$

singlet odd:

$$
\begin{aligned}
&{}^1V_C^-(x) = 3(g^2/4\pi)\mu(e^{-x}/x)[1+{}^1a_C^-(e^{-x}/x)+{}^1b_C^-(e^{-x}/x)^2],\\
&{}^1V_Q^-(x) = {}^1G_Q^-\mu(e^{-x}/x)[1+{}^1a_Q^-(e^{-x}/x)+{}^1b_Q^-(e^{-x}/x)^2].
\end{aligned}
\tag{4}
$$

with $g^2/4\pi=0.08$, and a hard core of radius ${}^3x_0^+$ and ${}^1x_0^-$ for (3) and (4), respectively. Some relevant discussions on the assumed functional forms of V_i are found in I.

As to the consistency with the pion theory of nuclear forces, we expect ${}^3a_T^+\simeq 0$, i. e., the ${}^3V_T^+(x)$ is very nearly that of OPEP at $x\gtrsim 1$. It is known that this condition is indeed required from the deuteron data.[8)] As we shall see in § 2 the deuteron data and the behaviour of the triplet D-wave phase shifts combined give some insight into other parameters in (3). Parameters in (4) will be determined purely phenomenologically.

The available experimental data on the n-p scattering are very much less in number and poorer in accuracy than those on the p-p scattering. Consequently, there exists no n-p counter-part of the extensive phase shift analysis of MacGregor et al.,[9)] which provided a very convenient starting point of the whole work in I. Because of this situation the determination of the $T=0$ potential is considerably more ambiguous and we can hardly claim the quantitative uniqueness of the potential described below. It is nevertheless significant that we have been able to find an approximately (see below) energy independent potential which reproduces the experimental data reasonably well over the whole energy range, and that the potential agrees with the pion theoretical implications.

Another more practical difficulty encountered in the actual attempt to fit the data is the following. Once ${}^3G_i^+$, ${}^3a_i^+$ and ${}^3b_i^+$ in (3) are specified, the core

radius ${}^3x_0^+$ is uniquely determined by fitting the deuteron binding energy. Thus the potential (3) is less flexible than the triplet odd parity potential considered in I as far as the scattering is concerned. Such an inflexibility of the potential in the core region does affect the result since here we have to deal with the triplet S-wave which feels the inner-most interaction. Because of this situation we had to introduce an energy dependent hard core radius in (3) in order to attain satisfactory fit over the whole energy range. We do not consider this a serious difficulty since it concerns the region where the potential picture itself is likely to lose its significance. Rather, we feel that it is remarkable that the required energy dependence is so small.

§ 2. Non-static forces in the triplet even parity state

In I the non-static potentials, particularly ${}^3V_{LS}^-$, have been determined mainly by studying the triplet P-wave phase shifts at 310 Mev. Although there exists no available phase shift solution for the $T=0$ state, the study of the triplet D-wave phase shifts at 300 Mev, combined with the deuteron problem, can give useful information on the non-static forces in (1).

Before going into any detail, we point out the following situation. The first partial wave affected by the LS potential in the triplet even parity state is the D-wave, the impact parameter of which being about 0.9 at 300 Mev. Hence, it is at once clear that the V_{LS} in the $T=0$ state is not so important as in the $T=1$ state where the P-wave is the first partial wave affected by the V_{LS}. There may be a strong ${}^3V_{LS}^+$ at small x but its effect is small as far as the n-p system below 300 Mev is concerned.* As we shall see later, this statement holds for the deuteron problem as well. Then we expect that the long ranged V_Q may play an important role in the $T=0$ state.**

GT's failure to find an energy independent potential to fit the n-p data up to 300 Mev is most likely due to this situation. The Bryan type potential,[4] which does not include the V_Q, is also likely to meet the same difficulty if applied to the n-p scattering.

We now consider the triplet D-wave phase shifts in conjunction with the experimental n-p total cross section at 300 Mev, 35 mb.[14] The ${}^3D_1$ phase shift is less important in this respect compared with the ${}^3D_2$ and ${}^3D_3$ phase shifts because of the small statistical weight. Also the ${}^3D_1$ phase shift is not sensitive to the non-static potentials. Therefore, we shall be concerned mainly with the ${}^3D_2$ and ${}^3D_3$ phase shifts in the following.

---

* GT[2] use the 90 Mev n-p data to determine the ${}^3V_{LS}^+$. At 90 Mev the D-wave impact parameter is about 1.7 where ${}^3V_{LS}^+$ is already negligible. Thus we can not take GT's conclusion on the ${}^3V_{LS}^+$ seriously.

** The introduction of ${}^3V_Q^+$ has been suggested by Feshbach[10] in connection with the apparent difficulty concerning the LS interaction moment in the deuteron ground state.[11] Notice also that the argument developed here supersedes the one given in reference 7).

In the same notations as in I, define

$$\Delta_i = -\kappa \int x^2 j_2^2(\kappa x)\, U_i(x)\, dx, \tag{5}$$

where j_L is the spherical Bessel function of order L. Then in the Born approximation* (see Eqs. (7) and (8) of I)

$$\delta(^3D_2) = \Delta_C + 2\Delta_T - \Delta_{LS} - 5\Delta_Q, \tag{6}$$

$$\delta(^3D_3) = \Delta_C - (4/7)\,\Delta_T + 2\Delta_{LS} + 4\Delta_Q. \tag{7}$$

Previous calculations[8] with ${}^3V_{LS}^+ = {}^3V_Q^+ \equiv 0$ show that the ${}^3V_T^+(x)$ must be close to that of the OPEP for $x \gtrsim 1$ in order to reproduce the correct deuteron data. Such a potential gives the $\delta(^3D_2)$ of at least about 0.8 at 300 Mev making the total cross section too large. It is clear from (6) that the large $\delta(^3D_2)$ is due to the strong negative OPEP type ${}^3V_T^+$. It appears, on the other hand, that the same phase shift should be at most about 0.5 to be consistent with the observed total cross section.[7] Since the ${}^3V_{LS}^+$ cannot be very effective as pointed out above, the negative ${}^3V_Q^+$ of considerable strength must be present in order to effect the large reduction in $\delta(^3D_2)$. Because of the large eigenvalue 5 of Q_{12} [see (6)], we can indeed expect that the ${}^3V_Q^+$ will be quite effective in this respect.

Let us now turn to the $\delta(^3D_3)$. The static potentials adjusted to fit the deuteron data predict $\delta(^3D_3)$ of at most 0.2 at 300 Mev. From (7) it is seen that the same negative ${}^3V_Q^+$ introduced in order to reduce the $\delta(^3D_2)$ will inevitably increase the $\delta(^3D_3)$ making the total cross section higher. The increase should, however, be slightly less in magnitude than the decrease in $\delta(^3D_2)$. Further, from (6) and (7) we expect that a positive ${}^3V_{LS}^+$ will counteract the ${}^3V_Q^+$ in $\delta(^3D_3)$ without appreciably affecting the $\delta(^3D_2)$.

Summarising our findings on the non-static potentials in (1), it appears that a negative long ranged ${}^3V_Q^+$ of substantial strength is indispensable. The ${}^3V_{LS}^+$ is not important but a positive ${}^3V_{LS}^+$ is likely to be useful.

Very recently, Hoshizaki and Machida[12] have derived the non-static OPEP which may be regarded exact except in the core region. According to them the non-static OPEP contains the term

$$\pm (g^2/4\pi)\,\mu(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)[(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\boldsymbol{L}^2 - Q_{12}][(1/2)(\mu/M)^2(1+3/x+3/x^2)(e^{-x}/x^3)], \tag{8}$$

where upper (lower) sign corresponds to the case of the $ps(pv)$ coupling. The ${}^1V_Q^+$ found in I has been shown to be consistent with the pv coupling.[12),13] It is interesting to note that the required large reduction in $\delta(^3D_2)$ at high energies can also be attributed to (8) if the coupling is pseudovector.

The difference between Q_{12} and $L_{12} = (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\boldsymbol{L}^2 - Q_{12}$ is trivial in the singlet state. In the triplet state, however, the choice of L_{12} in place of Q_{12} can make

* The Born approximation suffices for our qualitative argument developed here.

significant difference. If one chooses $L_{12}{}^3V_L{}^+$ in place of $Q_{12}{}^3V_Q{}^+$, (6) and (7) are replaced by

$$\delta({}^3D_2)=\Delta_C+2\Delta_T-\Delta_{LS}+11\Delta_L, \tag{9}$$

$$\delta({}^3D_3)=\Delta_C-(4/7)\Delta_T+2\Delta_{LS}+2\Delta_L. \tag{10}$$

Then the positive ${}^3V_L{}^+$ can reduce $\delta({}^3D_2)$ without increasing the $\delta({}^3D_3)$ and hence there may be no need for a positive ${}^3V_{LS}^+$. Thus our conclusion on the ${}^3V_{LS}^+$ should be regarded preliminary. On the other hand, the conclusion that there should be a long ranged non-static force which is repulsive in the ${}^3D_2$ state seems to be on the firm ground. The fact that GT had to reduce the ${}^3V_T{}^+$ appreciably above 150 Mev points to the same effect.[2]

§ 3. The deuteron problem

The deuteron properties resulting from the potentials of the form (3) but with ${}^3V_{LS}^+={}^3V_Q{}^+=0$ have been reported previously.[8] The code has since been extended to include both V_{LS} and V_Q, and at present we have tabulated results for some 100 combinations of parameters appearing in (3). In Table I we show several of them.

Table I. The deuteron properties. For the specification of the potential see (3) of text. ${}^3x_0{}^+$=hard core radius, adjusted to give correct binding energy, in the unit of pion Compton wave length 1.42×10^{-13} cm; Q=electric quadrupole moment in 10^{-27} cm^2; ${}^3r_e$=triplet effective range in 10^{-13} cm; P_D=D-state probability in %; A_D/A_S=asymptotic D to S wave ratio. The contribution to the deuteron magnetic moment from non-static potentials considered in this Table is $|\Delta\mu|<0.006$ nuclear magneton. Relevant experimental values are: $Q=2.738\pm0.014$, ${}^3r_e=1.704\pm0.030$, $\Delta\mu=\mu_d-\mu_p-\mu_n=-0.02231\pm0.00012$ n. m.

Identification No.	${}^3a_C^+$	${}^3b_C^+$	${}^3a_T^+$	${}^3b_T^+$	${}^3G_{LS}^+$	${}^3b_{LS}^+$	${}^3G_Q^+$	${}^3a_Q^+$	${}^3b_Q^+$	${}^3x_0^+$	Q	${}^3r_e$	P_D	A_D/A_S
0	0	0	0	0	0	0	0	0	0	0.340	2.88	1.72	7.42	0.02716
1	−8	6	0	0	0	0	0	0	0	0.315	2.78	1.59	7.94	0.02729
2	−8	6	−0.3	0.1	0	0	0	0	0	0.276	2.66	1.53	7.55	0.02666
3	−8	6	−0.3	0.1	0.05956	4	0	0	0	0.292	2.72	1.54	8.34	0.02717
4	−8	6	−0.3	0.1	0.05956	4	−0.005956	15	0	0.309	2.88	1.55	9.49	0.02855
5	−8	4	−0.4	0.1	0.05956	4	−0.007445	12	2	0.275	2.83	1.50	9.89	0.02856
6	−0.5	0.5	−0.4	0.2	0.08934	−4	−0.007445	10	2	0.302	2.85	1.67	7.47	0.02749

In Table I, the potential 0 is the pure OPEP with $g^2/4\pi=0.08$. The entries 1, 2, 3 and 4 show effects of individual forces on the deuteron properties. As expected the non-static forces are not very important. The potential 5 with a slightly energy dependent hard core radius will be shown in § 4 to be capable of reproducing the n-p data below 300 Mev. We have plotted this potential in Fig. 1. The deuteron wave function is shown in Table II. The potential 6 is an example with a negative ${}^3V_{LS}^+$.

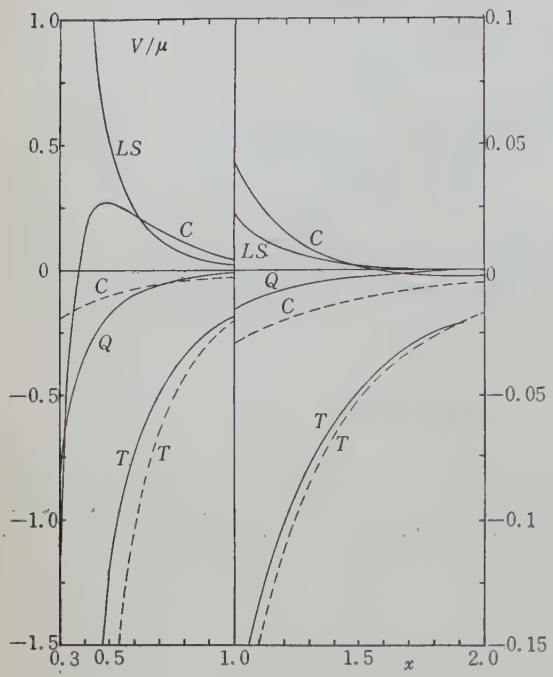


Fig. 1. Triplet even parity potential defined by (3) and (11) in the unit of pion mass. The dashed curves represent OPEP with $g^2/4\pi=0.08$.

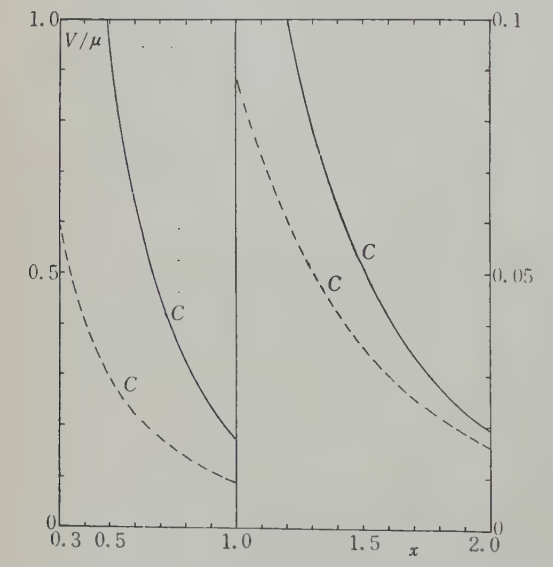


Fig. 2. Singlet odd parity potential defined by (4) and (12) in the unit of pion mass. The dashed curve represents the OPEP with $g^2/4\pi=0.08$.

Table II. The deuteron wave function. Entries are the radial S and D wave functions (multiplied by x) calculated using the potential defined by (3) and (11) (core radius ${}^3x_0^+=0.275$). They are normalized to $\int_0^\infty (u^2+w^2)\,dx=0.99689$. See Table I for the deuteron properties derived from this wave function.

x	$u(x)$	$w(x)$
0.3	0.0648	0.0474
0.4	0.2777	0.1894
0.5	0.4161	0.2580
0.6	0.5020	0.2851
0.7	0.5543	0.2910
0.8	0.5849	0.2860
0.9	0.6013	0.2754
1.0	0.6080	0.2619
1.1	0.6081	0.2473
1.2	0.6036	0.2323
1.3	0.5957	0.2175
1.4	0.5856	0.2033
1.5	0.5739	0.1897
1.6	0.5611	0.1769
1.7	0.5476	0.1649
1.8	0.5336	0.1537
1.9	0.5193	0.1432
2.0	0.5049	0.1335
2.1	0.4906	0.1245
2.2	0.4736	0.1162
2.3	0.4623	0.1084
2.4	0.4484	0.1013
2.5	0.4348	0.0946
2.6	0.4215	0.0885
2.7	0.4085	0.0828
2.8	0.3959	0.0776
2.9	0.3835	0.0727
3.0	0.3715	0.0681
3.2	0.3484	0.0600
3.4	0.3267	0.0530
3.6	0.3062	0.0470
3.8	0.2869	0.0417
4.0	0.2688	0.0372
4.2	0.2519	0.0332
4.4	0.2359	0.0297
4.6	0.2210	0.0266
4.8	0.2070	0.0239
5.0	0.1939	0.0216
5.2	0.1816	0.0194
5.4	0.1701	0.0176
5.6	0.1593	0.0159
5.8	0.1492	0.0144
6.0	0.1397	0.0131
6.5	0.1186	0.0104
7.0	0.1007	0.0083
7.5	0.0854	0.0066
8.0	0.0725	0.0053
8.5	0.0615	0.0043
9.0	0.0522	0.0035
9.5	0.0443	0.0029
10.0	0.0376	0.0024

The D-state probability of 9.9% resulting from the potential 5 of Table I is slightly higher than the usually quoted values.[15] It is still not inconsistent in view of large uncertainties in the estimation of this quantity.[16]

Wong[17] has recently applied the Chew-Mandelstam method to estimate the asymptotic ratio of D to S-wave amplitudes. His result, $A_D/A_S=0.029$ for $g^2/4\pi=0.08$, is in good agreement with the one predicted by the potential 5. His crude estimation of the electric quadrupole moment based on this A_D/A_S, $Q\sim 4\times 10^{-27}\text{cm}^2$, seems to be an overestimation. Our results show that Q can be consistent with the experimental value even for this rather large A_D/A_S.

§ 4. The *n-p* scattering

We first studied the quantity IP ($I=$ unpolarized differential cross section, $P=$polarization) at 300 Mev.[14),18] This quantity is independent of the singlet scattering so that we are not disturbed by ambiguities in the singlet odd parity potential to start with. The triplet odd parity potential is of course taken over from I.

We have tried some 50 triplet even parity potentials consistent with the deuteron data. We have insisted that the ${}^3V_T^+$ be close to that of OPEP at $x\gtrsim 1$. Following the observations made in § 2, the ${}^3V_Q^+$ was chosen to be negative in all of them. About one third of the potentials had a negative ${}^3V_{LS}^+$.

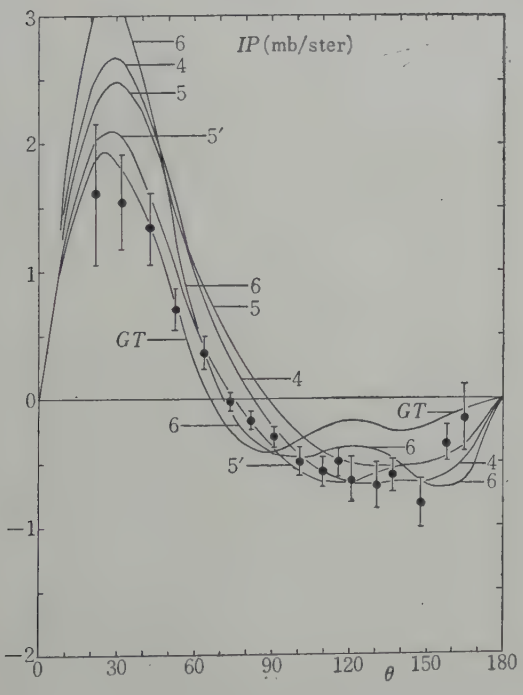


Fig. 3. $I(\theta)P(\theta)$ at 300 Mev calculated from potentials given in Table I ($T=1$ phase shifts are taken from I).

First of all we found that the qualitative features of IP at 300 Mev, i. e., a maximum at about 30°, passing 0 near 80° and reaching a minimum at about 120°, can be reproduced without any difficulty. In fact all tried triplet even parity potentials, combined with the triplet odd parity potential found in I, did reproduce these qualitative features. In particular, the fit at $\theta>90°$ was found quite satisfactory for most of the potentials tested.

In Fig. 3 the curves numbered as 4, 5 and 6 are calculated from the correspondingly numbered poten-

tials in Table I. The most serious discrepancy with the data is seen to be in the maximum value of IP near 30°. For the pure OPEP, the potential 0 of Table I, the value becomes as high as 3.8 mb, which is more than twice the experimental value. Both $\delta(^3D_2)$ and $\delta(^3D_3)$ had to be made as small as possible to reduce $IP(30°)$. The potential 5 of Table I is the best in this respect but $IP(30°)$ is still too high. Also for this potential IP becomes negative at too large an angle. Experience has shown that the latter situation is due to too large $\delta(^3S_1)$.

At this point we have given up the idea of a strictly energy independent fit as mentioned in § 1. In order to reduce the $\delta(^3S_1)$ at 300 Mev we considered a slightly energy dependent hard core radius in the potential 5 of Table I. The curve 5′ of Fig. 3 is calculated from the potential 5 but with ${}^3x_0{}^+=0.300$ instead of 0.275 as required for the correct deuteron binding. This small increase in the hard core radius has very little effect on the phase shifts other than $\delta(^3S_1)$, $\delta(^3D_1)$ and ϵ_1.* Yet, we notice a marked improvement in the fit at $\theta\simeq 90°$ and at the same time at $\theta\simeq 30°$. Further increase in ${}^3x_0{}^+$ and consequently further reduction in $\delta(^3S_1)$ make IP negative at too small an angle. This feature may be seen in GT who give $\delta(^3S_1)=-0.258$ (the curve marked as GT in Fig. 3).

Having thus decided on the potential 5 of Table I with an energy dependent hard core radius, we proceeded to fit I and P separately at 300 Mev, and also other n-p data at lower energies. We simply quote the results.

Triplet even:

$$\begin{aligned} &g^2/4\pi=0.08,\ {}^3G_{LS}^{+}=0.05956,\ {}^3G_{Q}^{+}=-0.007445,\\ &{}^3x_0^{+}=0.275+0.00144\sqrt{E(\text{Mev})},\\ &{}^3a_C^{+}=-8,\ {}^3b_C^{+}=4,\ {}^3a_T^{+}=-0.4,\ {}^3b_T^{+}=0.1,\\ &{}^3b_{LS}^{+}=4,\ {}^3a_Q^{+}=12,\ {}^3b_Q^{+}=2. \end{aligned} \tag{11}$$

Singlet odd:

$$\begin{aligned} &g^2/4\pi=0.08,\ {}^1G_Q^{-}=0,\ {}^1x_0^{-}=0.30,\\ &{}^1a_C^{-}=3,\ {}^1b_C^{-}=-1. \end{aligned} \tag{12}$$

The potentials are plotted in Figs. 1 and 2. The phase shifts are calculated as functions of energy and shown in Tables III and IV. I, P and D calculated at several energies from these phase shifts are shown in Figs. 4 to 7. Wherever the comparison with the experiment is possible, the agreement is reasonable.

We conclude this section with a few remarks. Firstly, ${}^1G_Q{}^-=0$ in (12) should be interpreted to mean that the present experimental data on the n-p scattering does not warrant the detailed determination of presumably weak quadratic LS potential in the singlet odd parity states. Secondly, we are convinced that

* At 300 Mev the potential 5 of Table I gives $\delta(^3S_1)=0.410$, $\delta(^3D_1)=-0.480$ and $\epsilon_1=0.045$. When ${}^3x_0{}^+$ is increased to 0.300 we have $\delta(^3S_1)=0.177$, $\delta(^3D_1)=-0.528$ and $\epsilon_1=0.192$.

Table III. Triplet even parity phase shifts calculated from the potential defined by (3) and (11). Entries are nuclear Blatt-Biedenharn phase shifts in radians.

E (Mev)	${}^3x_0^+$	${}^3S_1$	${}^3D_1$	ϵ_1	${}^3D_2$	${}^3D_3$	${}^3G_3$	ϵ_3	${}^3G_4$	${}^3G_5$	${}^3I_5$	ϵ_5	${}^3I_6$
10	0.280	1.732	−0.011	0.031	0.014	0.002	−0.001	0.664	...	...	...	...	...
20	0.281	1.437	−0.035	0.043	0.044	0.008	−0.006	0.739	0.001	0.001	−0.000	0.490	...
30	0.283	1.297	−0.063	0.053	0.072	0.016	−0.012	0.763	0.003	0.002	−0.000	0.583	...
42	0.284	1.164	−0.095	0.058	0.107	0.026	−0.022	0.773	0.006	0.004	−0.001	0.643	...
60	0.286	1.008	−0.141	0.066	0.155	0.041	−0.036	0.777	0.011	0.009	−0.003	0.694	...
90	0.289	0.817	−0.211	0.080	0.220	0.064	−0.058	0.770	0.020	0.017	−0.006	0.737	0.001
130	0.291	0.645	−0.288	0.093	0.285	0.091	−0.082	0.755	0.032	0.030	−0.011	0.763	0.002
160	0.293	0.535	−0.339	0.108	0.323	0.109	−0.098	0.742	0.039	0.039	−0.014	0.773	0.003
200	0.295	0.415	−0.399	0.128	0.364	0.131	−0.115	0.726	0.047	0.050	−0.019	0.781	0.004
250	0.298	0.283	−0.467	0.160	0.402	0.156	−0.133	0.707	0.055	0.064	−0.024	0.786	0.005
300	0.300	0.177	−0.528	0.192	0.432	0.178	−0.148	0.690	0.060	0.078	−0.029	0.789	0.005

Table IV. Singlet odd parity phase shifts calculated from the potential defined by (4) and (12). Entries are nuclear Blatt-Biedenharn phase shifts in radians.

E (Mev)	${}^1P_1$	${}^1F_3$	${}^1H_5$
10	−0.063	−0.001	...
20	−0.117	−0.005	...
30	−0.157	−0.011	−0.001
42	−0.196	−0.018	−0.003
60	−0.241	−0.028	−0.004
90	−0.298	−0.043	−0.009
130	−0.354	−0.061	−0.015
160	−0.388	−0.072	−0.019
200	−0.427	−0.085	−0.025
250	−0.468	−0.099	−0.031
300	−0.503	−0.112	−0.037

some energy dependence must be introduced in the triplet even parity potential. The energy dependence must be such that reduces $\delta({}^3S_1)$ at high energies. Admittedly our calculations are rather limited. We believe, nevertheless, that this conclusion is independent of the detailed potential forms. It is an interesting question whether this effect can be accounted for by the non-static forces other than (8).[12] The final remark concerns the parameter D. The measurement of this parameter in the p-p scattering at 150 Mev has been of decisive importance particularly in the determination of V_{LS} in the triplet odd parity state.[1] In the n-p scattering, however, D is insensitive to the triplet even parity potential. Even at 300 Mev most potentials we have tried in the present work predict very similar angular dependence of D. Even the GT potential predicts D practically identical to those shown

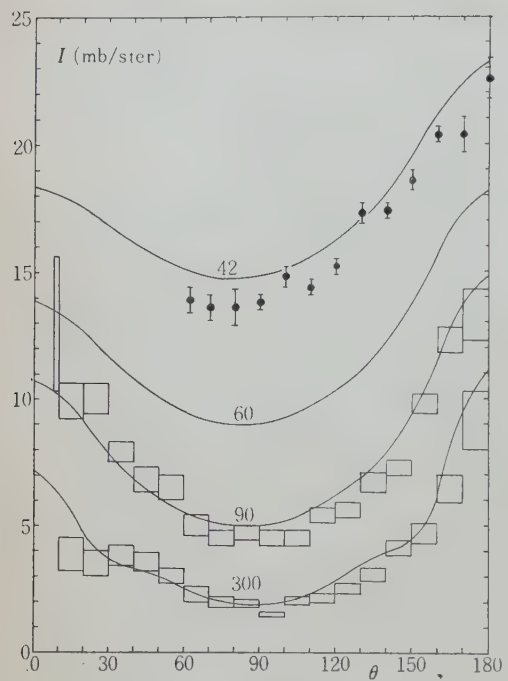


Fig. 4. $I(\theta)$ at 42, 60, 90 and 300 Mev calculated from the phase shifts given in Tables III and IV. $T=1$ phase shifts are taken from I. Experimental points are taken from reference 14).

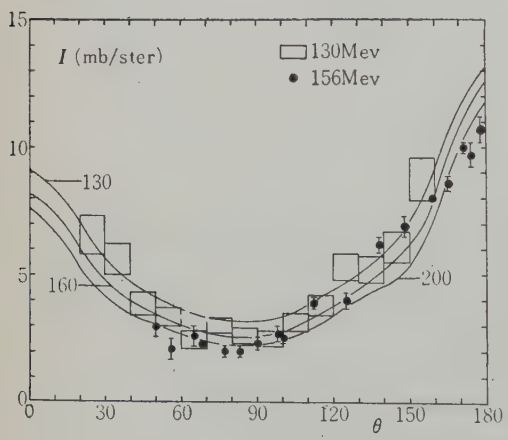


Fig. 5. $I(\theta)$ calculated at 130, 160 and 200 Mev from the phase shifts of Tables III and IV. $T=1$ phase shifts are taken from I. Experimental points are taken from reference 14).

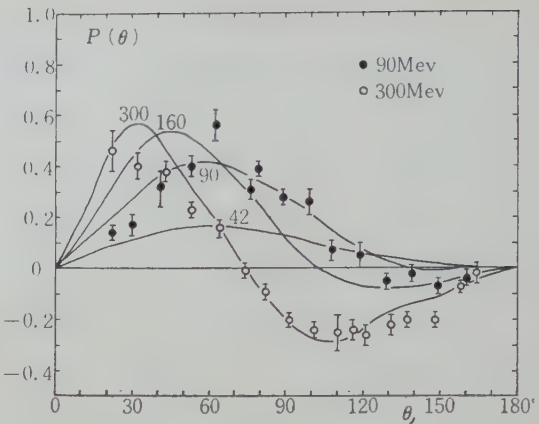


Fig. 6. $P(\theta)$ at 42, 90, 160 and 300 Mev calculated from the phase shifts of Tables III and IV. $T=1$ phase shifts are taken from I. Experimental points are taken from references 14) and 18).

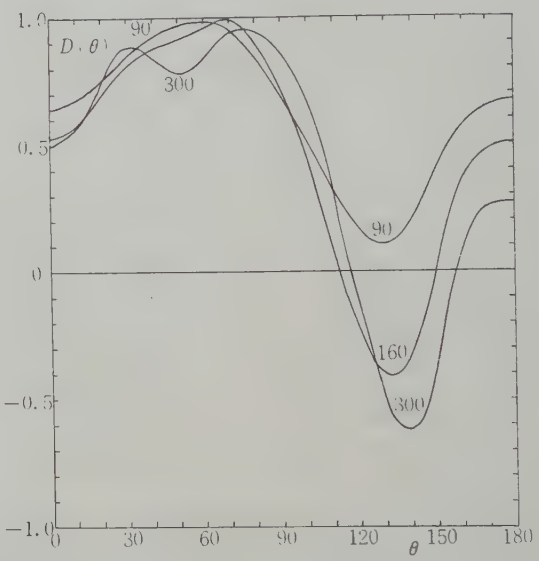


Fig. 7. $D(\theta)$ at 90, 160 and 300 Mev calculated from the phase shifts given in Tables III and IV. $T=1$ phase shifts are taken from I.

in Fig. 7. It is also slowly varying function of energy (see Fig. 7). Thus the measurement of D in the n-p scattering is not likely to provide very useful information regarding nuclear forces in the triplet even parity state.

§ 5. Conclusions

We have shown that the $T=0$ two-nucleon potential defined by (3), (4), (11) and (12) can reproduce the experimental n-p data below 300 Mev reasonably well when combined with the $T=1$ potential found in I. The triplet even parity potential is not strictly energy independent. The required energy dependence is, however, very small and confined in the core region. This may be an indication for the non-static potentials of other types than those considered in the present work.

One of the important findings in recent development of the pion theory of nuclear forces is that the non-static part of the potential is likely to depend on the type of pion-nucleon coupling,[12),19)] pseudoscalar or pseudovector. Our conclusion on the non-static effect in the singlet even parity state drawn in I has already proved to be useful in the attempt to distinguish between the two couplings.[12)] The suppression of the ${}^3D_2$ phase shift at high energies, which we found necessary to account for the small total cross section at 300 Mev, is also suggestive in this respect as discussed in § 2. Obviously, however, such an important question as the type of pion-nucleon coupling requires further works. In particular, the non-static potentials of the type other than those considered in the present work have to be studied. We also note that our choice of Q_{12} in place of the pion theoretical L_{12} [see § 2] makes our conclusion on the ${}^3V_{LS}^+$ inconclusive when it comes to the comparison with the pion theory.

Combining our findings in the present paper with those of I, we conclude that there exists a potential picture which is compatible with all experimental two-nucleon data below 300 Mev, and that the potential is not inconsistent with the current pion theoretical implications. Our results on the non-static potentials in I and the present paper are incomplete in many ways. They are nevertheless highly suggestive in several respects and certainly justify further careful investigations into this subject.

Acknowledgements

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On the Charge Exchange Collision in Nucleon-Deuteron*

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The charge exchange collision in nucleon-deuteron is investigated by using the two-nucleon scattering phase shifts, in order to supplement some information regarding the two-nucleon interaction with the singlet isotopic spin state, and also to study a source of the partially polarized neutron beam which is very useful to investigate the interaction between the neutron and a certain nucleus. The differential cross section, polarization and triple scattering parameters of the nucleon ejected in the charge exchange nucleon-deuteron collision are calculated by using the Gammel-Thaler and Signell-Marshak phase shifts. The energy distribution of the ejected nucleon is calculated by means of the cross section of magnetic dipole photo-disintegration of the deuteron, instead of the explicit use of the wave functions of the di-neutron (or di-proton) and the deuteron to evaluate the nuclear matrix element. The polarization and triple scattering parameter of the nucleon quasi-elastically ejected from the nucleus, say, C^{12}, in the charge exchange collision and of the nucleon caused by the free $n(p, n)p$- and $p(n, p)n$-processes are calculated for comparison.

§ 1. Introduction

Recently the information regarding the nucleon-nucleon interactions has been greatly increased in the analyses of experimental data for the two-nucleon scattering at high energies. In the framework of the impulse approximation, the interactions between the high energy nucleon and the nucleus, and between two nucleons in the nucleus can be understood on the basis of the information on the two-nucleon system.[2] This makes possible to calculate the differential cross sections and polarizations of nucleons scattered from nuclei by using sets of the two-nucleon scattering phase shifts. By comparing the calculated results with the experimental data for every set of the phase shifts, one can reject some of the sets of the phase shifts.[3),4)] On the same lines, in the present paper we consider the spin dependent interaction between two nucleons from somewhat different point of view from the free two-nucleon scattering. The information regarding the two-nucleon system is supplemented by the investigation on the spin dependent interaction. The investigation is that of the high energy nucleon ejected in the high energy charge exchange nucleon-deuteron collision.

The investigation on the collision gives a certain way to study the spin dependence of the nucleon forces between two nucleons. This comes from the fact that

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* The present paper is partly reported in ref. 1).

the differential cross section of the nucleon ejected in the charge exchange nucleon-deuteron collision gives some information on the spin dependent nuclear forces which is not obtained by the cross section of the free two-nucleon scattering. Chew[5] applied the impulse approximation to the neutron-deuteron collision and parametrically introduced the ratio of singlet and triplet amplitudes of proton-neutron scattering, instead of the ratio of the amplitude of the ejected proton when the final di-neutron is left in the singlet spin state to the one when the final di-neutron is in the triplet spin state. Then he reproduced the experimental cross section for the ejected proton and showed that the differential cross section of the ejected proton is explained by the parameter chosen suitably. A similar consideration was given by Lapidus.[6] However, when the phase shifts of the two-nucleon scattering are obtained, the ratio of the amplitude of the ejected proton when the final di-neutron is left in the singlet to the one when the di-neutron is in the triplet spin state is automatically fixed by the set of the nuclear phase shifts.

We present the expressions for the polarization and triple scattering parameters for the nucleon ejected in the charge exchange nucleon-deuteron collision. And we show how these expressions are connected with the nucleon quasi-elastically scattered from the nucleus, say, C^{12}, in the charge exchange collision within the framework of the impulse approximation, or with the free nucleon-nucleon scattering, and how the measurements of the nucleon-deuteron collision supplement the information on the nucleon-nucleon interaction with the isotopic spin singlet state ($T=0$).

In the succeeding sections we restrict ourselves to the neutron for the incident nucleon to avoid the complexity due to the Coulomb interaction. However, the results given in the present paper can equally be applied to the neutron ejected in the proton-deuteron collision, except for the Coulomb interaction between two protons.

§ 2. Differential cross section

In high energy neutron-deuteron collision, there are two main groups of the protons: A group at low energy resulting from the neutron-neutron and ordinary neutron-proton collisions (favoring small momentum transfers) which break up the deuteron, and a group of the high energy protons which results from the neutron-proton exchange collisions. The low energy group will roughly be isotropic, since the momentum distribution of the nucleons in the deuteron is isotropic, but the high energy proton group will be confined to the forward direction as well as in the scattering of the neutron by the proton. Thus, the two groups of the protons are distinguished experimentally.

Since the high energy proton results from the neutron-proton exchange collision, we may use the t-matrix to describe the interaction between the incident neutron and the proton in the deuteron. In order to make clear the spin states of the interacting two nucleons, we adopt the two-nucleon scattering spin matrix, which contains the singlet and triplet projection operators, S and T, defined by[7]

$$t=BS+C(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_t)\bar{\boldsymbol{n}}+\Big[N(\boldsymbol{\sigma}_1\bar{\boldsymbol{n}})\,(\boldsymbol{\sigma}_t\bar{\boldsymbol{n}})+\frac{1}{2}\,G\{(\boldsymbol{\sigma}_1\bar{\boldsymbol{p}})\,(\boldsymbol{\sigma}_t\bar{\boldsymbol{p}})+(\boldsymbol{\sigma}_1\bar{\boldsymbol{k}})\,(\boldsymbol{\sigma}_t\bar{\boldsymbol{k}})\}$$
$$+\frac{1}{2}H\{(\boldsymbol{\sigma}_1\bar{\boldsymbol{k}})\,(\boldsymbol{\sigma}_t\bar{\boldsymbol{k}})-(\boldsymbol{\sigma}_1\bar{\boldsymbol{p}})\,(\boldsymbol{\sigma}_t\bar{\boldsymbol{p}})\}\Big]T, \qquad (2\cdot1)$$

where

$$\bar{\boldsymbol{n}}=\frac{\boldsymbol{k}_i\times\boldsymbol{k}_f}{|\boldsymbol{k}_i\times\boldsymbol{k}_f|},\quad \bar{\boldsymbol{k}}=\frac{\boldsymbol{k}_f-\boldsymbol{k}_i}{|\boldsymbol{k}_f-\boldsymbol{k}_i|},\quad \bar{\boldsymbol{p}}=\bar{\boldsymbol{k}}\times\bar{\boldsymbol{n}}$$

where $\boldsymbol{k}_i$ and $\boldsymbol{k}_f$ are momenta of the incident and scattered nucleon in center of mass system, respectively. The suffix 1 denotes the incident neutron and t the proton in the deuteron, in the present case.

The incident neutron, say, 1, interacts with the proton in the deuteron, say, 3. After the collision, the incident neutron and the neutron in the deuteron, say, 2, are left in the singlet or triplet spin states and the high energy proton is ejected. Then, the transition matrix is given by

$$-\frac{m}{4\pi\hbar^2}\langle(1-P_{12})\Psi(12)\psi(3)\,|\,V_{13}|\psi(1)\Phi(23)\rangle, \qquad (2\cdot2)$$

where $\Phi(23)$ and $\Psi(12)$ are the wave functions of the deuteron and di-neutron, respectively, and P_{12} is the exchange operator for two like nucleons.

The scattering matrix coefficients, B, C, N, G and H, actually depend on three variables, q^2, $p^2+q^2=(2m/\hbar^2)(E+E')$, $\boldsymbol{q}\cdot\boldsymbol{p}=(m/\hbar^2)(E'-E)$ where $\boldsymbol{q}=\boldsymbol{k}_f-\boldsymbol{k}_i$, $\boldsymbol{p}=\boldsymbol{k}_f+\boldsymbol{k}_i$ and E and E' are the energy of the incident and scattered nucleons. $\bar{\boldsymbol{p}}$ is equal to $\boldsymbol{p}/|\boldsymbol{p}|$ only on the energy shell where $\boldsymbol{q}\cdot\boldsymbol{p}=0$ and p^2+q^2 is proportional to the energy. As the dependence of these coefficients on the energy is weak compared with their variation with q^2, only the dependence on q^2 will be considered.

Then, if one approximates V_{13} by $t_{13}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_3)$, the transition matrix is given by

$$T=\langle(1-P_{12})\Psi(12)\psi(3)\,|\,t_{13}|\psi(1)\Phi(23)\rangle, \qquad (2\cdot3)$$

or

$$T=\langle\Psi(12)\psi(3)\,|\,(1-P_{12})t_{13}|\psi(1)\Phi(23)\rangle. \qquad (2\cdot4)^*$$

After some algebraic calculations for three-body spin wave functions, the differential cross section is given, in the laboratory system,** by

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* The formulas reported in the previous note[1] are derived from this transition matrix. Setting B_{23}, C_{23}, etc., by B_{12}, C_{12}, etc., in the previous note, one obtains the same equations derived in the present paper.

** In the present paper we assume the pure S-state deuteron wave function. This approximation introduces very little error, roughly estimated at about 1%. Because for small $\boldsymbol{k}_1+\boldsymbol{k}_2$, the D-wave part of $\Phi(23)$ neglected in the present paper contributes mainly through the D-state part of $\Psi(12)$, and for larger $\boldsymbol{k}_1+\boldsymbol{k}_2$, it contributes more through the S-wave of $\Psi(12)$. In both cases the overlap integral is small.

$$d\sigma(\theta)=4\frac{k_f}{k_i}\left\{\left[|M_1|^2+|M_2|^2+\frac{2}{3}(|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2)\right]|I_{12}^t|^2\right.$$
$$+\left[\frac{1}{3}(|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2)\right]|I_{12}^s|^2$$
$$-\left(\left[|M_1|^2+|M_2|^2+\frac{2}{3}(|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2)\right]I_{12}^{t*}I_{21}^t\right.$$
$$\left.\left.-\left[\frac{1}{3}(|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2)\right]I_{12}^{s*}I_{21}^s\right)\right\}\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}d\Omega, \qquad (2\cdot5)$$

where

$$I_{12}^{t,s}=\int\Psi^{t,s}(12)^*e^{i[(k_1+k_2)/2]r}\Phi(23)\,d\boldsymbol{r}, \qquad (2\cdot6)$$

while t and s indicate the triplet and singlet spin states of the di-neutron, and $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are momenta of the two neutrons in the di-neutron, respectively, and $\boldsymbol{k}_{12}$ is the momentum conjugate to the relative coordinate of neutrons in the di-neutron, and Ω is the direction associated with the momentum of the ejected proton $\boldsymbol{k}_p$. M_1, M_2, M_3, M_4 and M_5 are given as follows,

$$M_1=\frac{1}{4}(-B+N+G),\quad M_2=C, \qquad (2\cdot7)$$
$$M_3=\frac{1}{4}(G-N+B+2H),\quad M_4=\frac{1}{4}(G-N+B-2H),$$
$$M_5=\frac{1}{4}(3N+B-G).$$

Here, the coefficients of the scattering spin matrix for the nucleon ejected in the charge exchange collision is given by

$$B=\frac{1}{2}(B_1-B_0),\quad C=\frac{1}{2}(C_1-C_0),\quad N=\frac{1}{2}(N_1-N_0),$$
$$G=\frac{1}{2}(G_1-G_0),\quad H=\frac{1}{2}(H_1-H_0). \qquad (2\cdot8)$$

The matrix $t_1=t_{T=1}$ is meant to be the matrix for the neutron-neutron scattering, or the proton-proton scattering without electro-magnetic effects. The coefficients of the matrix t_1 are B_1, C_1, N_1, G_1 and H_1. The quantities B_0, C_0, N_0, G_0 and H_0 contain the contributions of the neutron-proton interaction with $T=0$.

If one approximates $\Psi(12)$ by the plane wave and integrates by $\boldsymbol{k}_{12}$ approximately, one obtains

$$\frac{d\sigma}{d\Omega}(\theta)_{nd}=4\cos\theta[(\mathcal{Y}_1+\mathcal{Y}_2)-(\mathcal{Y}_1-\mathcal{Y}_2)F(q)], \qquad (2\cdot9)$$

where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are coefficients of $|I_{12}^t|^2$ and $|I_{12}^s|^2$ in Eq. (2·5), respectively. $F(q)$ is the interference factor, which is given by the Fourier transform of square

of the deuteron wave function.[4] q is the momentum transfer of the incident nucleon to the ejected nucleon in the laboratory system.

By using the equation

$$4\cos(\mathcal{Y}_1+\mathcal{Y}_2)=\frac{d\sigma}{d\Omega}(\theta)_{np}, \qquad (2\cdot10)$$

Eq. (2·9) is rewritten as

$$\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_C F(q)]\times\frac{d\sigma}{d\Omega}(\theta)_{np}, \qquad (2\cdot11)$$

where

$$K_C=\frac{\mathcal{Y}_1-\mathcal{Y}_2}{\mathcal{Y}_1+\mathcal{Y}_2}. \qquad (2\cdot12)$$

While $d\sigma(\theta)_{np}/d\Omega$ is the cross section for the free two-nucleon scattering having the charge exchange; $n(p, n)p$, or $p(n, p)n$ processes, which is equivalent to the cross section for the neutron-proton scattering at backward direction. If the two-nucleon scattering phase shifts are determined by analyses of the complete experimental data of the two-nucleon scattering, the value of K_C is uniquely fixed by the set of the nuclear phase shifts. K_C may be used to reject some sets of the nuclear phase shifts if one compares the results with the experimental data.[8]

For nucleon ejected at forward direction, one obtains

$$\frac{d\sigma}{d\Omega}(\theta)_{nd}\Big|_{\theta\to0}\simeq 8\mathcal{Y}_2, \qquad (2\cdot13)$$

since $F(q)=1$. Here, it is important to note that the triplet state interaction between two neutrons in the di-neutron is certainly sufficiently weak

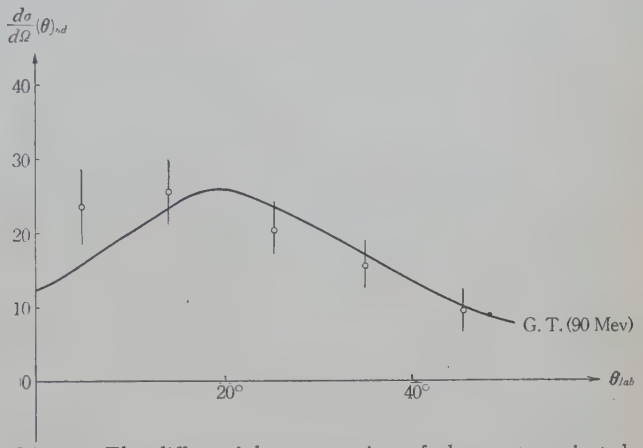


Fig. 1. The differential cross section of the proton ejected in the neutron-deuteron collision at 90 Mev. The curve is calculated by the use of the Gammel-Thaler phase shifts at 90 Mev.

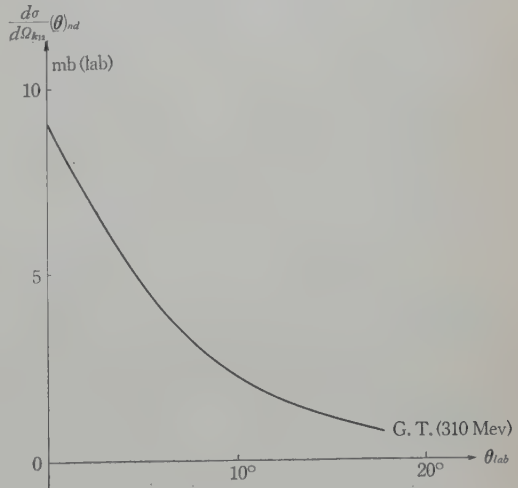


Fig. 2. The differential cross section of the nucleon ejected in the charge exchange nucleon-deuteron collision at 310 Mev. The final two like nucleons are left in the singlet spin state. The curve is calculated by using the Gammel-Thaler phase shifts.

but the singlet interaction in the S-state is by no means weak. However, at large angles where the proton moves so slowly that one of the neutrons has a good chance to capture it or at least perturb it strongly, one does not specify the spin state of the di-neutron. On the other hand, for very small angle scattering, one may specify the spin state of the di-neutron with the relative S-state of low momentum $\boldsymbol{k}_{12}$. In this case, the cross section for very small scattering angles is given, in the laboratory system, by

$$\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\cdot\mathscr{Y}_2(|I_{12}^s|^2+I_{12}^{s*}I_{21}^s). \tag{2·14}$$

Now, the differential cross section of the nucleon, say, neutron, scattered elastically from the deuteron is given, in the laboratory system,[4] by

$$\frac{d\sigma}{d\Omega}(\theta)_{nd,el}=\frac{4(5+4\cos\theta)^{3/2}}{9(2+\cos\theta)}\left[|\alpha|^2+|\gamma|^2+\frac{2}{3}(|\gamma|^2+|\delta|^2+|\varepsilon|^2+|\beta|^2)\right]S(q), \tag{2·15}$$

where

$$\alpha=\frac{1}{4}(\bar{B}+\bar{N}+\bar{G}),\quad \gamma=\bar{C}, \tag{2·16}$$

$$\delta=\frac{1}{4}(\bar{G}-\bar{N}-\bar{B}-2\bar{H}),\quad \varepsilon=\frac{1}{4}(\bar{G}-\bar{N}-\bar{B}+2\bar{H}),\quad \beta=\frac{1}{4}(3\bar{N}-\bar{B}-\bar{G}),$$

while

$$\bar{B}=\frac{1}{2}(3B_1+B_0),\quad \bar{C}=\frac{1}{2}(3C_1+C_0),\quad \bar{N}=\frac{1}{2}(3N_1+N_0),$$

$$\bar{H}=\frac{1}{2}(3H_1+H_0),\quad \bar{G}=\frac{1}{2}(3G_1+G_0). \tag{2·17}$$

$S(q)$ is the sticking factor defined by[9]

$$F^2\left(\frac{q}{2}\right)=S(q). \tag{2·18}$$

Here it should be noted that the coefficients of $|I_{12}^t|^2$ in Eq. (2·5) are very similar to the ones of $S(q)$ in Eq. (2·15). However, the signs of $\bar{B}$ contained in Eq. (4·16) are opposite to the ones of B in Eq. (2·7). If one replaces $\bar{B}$ (or $-\bar{B}$) in Eq. (2·15) by $-\bar{B}$ (or $\bar{B}$), puts terms associated with the neutron scattered by the neutron in the deuteron equal to zero, and exchanges the signs of coefficients of spin matrix associated with $T=0$, one obtains the coefficients of $|I_{12}^t|^2$ in Eq. (2·5).[1]

§ 3. Polarization

Some information about the two-nucleon scattering with $T=0$ is obtained by the investigation on the polarization and triple scattering parameters of the

nucleon ejected in the charge exchange collision.

The polarized cross section of the nucleon ejected in the charge exchange nucleon-deuteron collision is given, after some algebraic calculations, in the laboratory system, by

$$P(\theta)\,d\sigma(\theta)=4\,\frac{k_f}{k_i}\,\mathrm{Re}\Big[2M_2^*\Big(M_1+\frac{2}{3}M_5\Big)|I_{12}^t|^2+2M_2^*\,\frac{1}{3}M_5|I_{12}^s|^2$$
$$-\Big\{2M_2^*\Big(M_1+\frac{2}{3}M_5\Big)I_{12}^{t*}I_{21}^t-2M_2^*\frac{1}{3}M_5\,I_{12}^{s*}I_{21}^s\Big\}\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}\,d\Omega. \quad (3\cdot1)$$

When one approximates the di-neutron by the plane wave and integrates by $\boldsymbol{k}_{12}$, one obtains

$$P(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=4\cos\theta[(\mathcal{A}_1+\mathcal{A}_2)-(\mathcal{A}_1-\mathcal{A}_2)F(q)], \quad (3\cdot2)$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are coefficients of $|I_{12}^t|^2$ and $|I_{12}^s|^2$ in Eq. (3·1), respectively.

Now, the polarized cross section for the free two-nucleon scattering having the charge exchange, $n(p,n)p$- or $p(n,p)n$-processes, is given by

$$P(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}=4\cos\theta(\mathcal{A}_1+\mathcal{A}_2), \quad (3\cdot3)$$

where $P(\theta)_{np}$ is the same as the polarization of the nucleon quasi-elastically ejected from the nucleus, say, C^{12}, in the charge exchange collision within the framework of the impulse approximation.

Substituting Eq. (3·3) into Eq. (3·2), one obtains

$$P(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_PF(q)]P(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \quad (3\cdot4)$$

where

$$K_P=\frac{\mathcal{A}_1-\mathcal{A}_2}{\mathcal{A}_1+\mathcal{A}_2}. \quad (3\cdot5)$$

For the nucleon ejected at very small angular region in the charge exchange collision, where the deuterons are bombarded with sharp beams and the final di-neutron is left in the relative S-state of low momentum $\boldsymbol{k}_{12}$, one obtains

$$P(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\cdot\mathcal{A}_2(|I_{12}^s|^2+I_{12}^{s*}I_{21}^s). \quad (3\cdot6)$$

In this case the polarization is given by

$$P(\theta)=\mathcal{A}_2/\mathcal{I}_2. \quad (3\cdot7)$$

On the other hand, the polarized cross section of the nucleon, say, neutron, elastically scattered from the deuteron is expressed as follows,[4]

$$P(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd,el}$$
$$=\frac{4(5+4\cos\theta)^{3/2}}{9(2+\cos\theta)}$$
$$\times\mathrm{Re}\left[2\gamma^*\left(\alpha+\frac{2}{3}\beta\right)\right]S(q). \tag{3·8}$$

It is also noted that the coefficients of $S(q)$ in Eq. (3·8) are very similar to the one of $|I_{12}^t|^2$ in Eq. (3·1), and the signs of $\bar{B}$ in α, β and γ are opposite to the ones of B in M_1, M_5 and M_2.

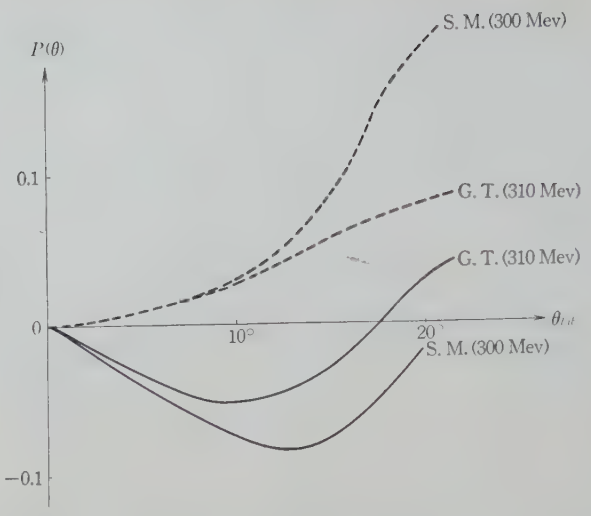


Fig. 3. Polarizations of the nucleon ejected in the charge exchange nucleon-deuteron collision, when the final two like nucleons are left in the singlet spin state. Solid and dashed curves indicate the polarization of the nucleon ejected in the charge exchange nucleon-deuteron and the one of the nucleon ejected in the quasi-elastic charge exchange nucleon-nucleus collision within the framework of the impulse approximation, respectively. (G. T. at 310 Mev, and S. M. at 300 Mev.)

§ 4. Triple scattering parameters

The triple scattering parameters defined by Wolfenstein[7] are the depolarization $D(\theta)$, and $A(\theta)$, $R(\theta)$ $A'(\theta)$ and $R'(\theta)$ which describe the rotation of the direction of the polarization. If the incident beam of the nucleon is polarized, the triple scattering parameters of the nucleon ejected in the charge exchange nucleon-deuteron collision are obtained. The quantities $D(\theta)$, $A(\theta)$, $R(\theta)$, $A'(\theta)$ and $R'(\theta)$ for the ejected nucleon furnish supplementary information on the two-nucleon scattering matrix with $T=0$ in excess of that supplied by the measurement of $P(\theta)$.

The triple scattering parameters for the nucleon ejected in the charge exchange nucleon-deuteron collision are given, in the laboratory system, as follows:

$$D(\theta)\,d\sigma(\theta)=4\frac{k_f}{k_i}\Big[(\mathsf{Y}_1+\mathsf{Y}_5)\,|I_{12}^t|^2+(\mathsf{S}_1+\mathsf{S}_5)\,|I_{12}^s|^2$$
$$-\{(\mathsf{Y}_1+\mathsf{Y}_5)\,I_{12}^{t*}\,I_{21}^t-(\mathsf{S}_1+\mathsf{S}_5)\,I_{12}^{s*}\,I_{21}^s\}\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}\,d\Omega, \tag{4·1}$$

$$R(\theta)\,d\sigma(\theta)=4\frac{k_f}{k_i}\Big[\{-\mathsf{Y}_2\sin\theta+(\mathsf{Y}_1+\mathsf{Y}_3)\cos\theta\}\,|I_{12}^t|^2$$
$$+\{-\mathsf{S}_2\sin\theta+(\mathsf{S}_1+\mathsf{S}_3)\cos\theta\}\,|I_{12}^s|^2-[\{-\mathsf{Y}_2\sin\theta+(\mathsf{Y}_1+\mathsf{Y}_3)\cos\theta\}\,I_{12}^{t*}\,I_{21}^t$$
$$-\{-\mathsf{S}_2\sin\theta+(\mathsf{S}_1+\mathsf{S}_3)\cos\theta\}\,I_{12}^{s*}\,I_{21}^s]\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}\,d\Omega, \tag{4·2}$$

$$A(\theta)\,d\sigma(\theta)=4\frac{k_f}{k_i}\Big[\{-\mathsf{Y}_2\cos\theta-(\mathsf{Y}_1+\mathsf{Y}_3)\sin\theta\}\,|I_{12}^t|^2$$
$$+\{-\mathsf{S}_2\cos\theta-(\mathsf{S}_1+\mathsf{S}_3)\sin\theta\}\,|I_{12}^s|^2-[\{-\mathsf{Y}_2\cos\theta-(\mathsf{Y}_1+\mathsf{Y}_3)\sin\theta\}\,I_{12}^{t*}I_{21}^t$$
$$-\{-\mathsf{S}_2\cos\theta-(\mathsf{S}_1+\mathsf{S}_3)\sin\theta\}\,I_{12}^{s*}I_{21}^s]\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}d\Omega, \tag{4·3}$$

$$R'(\theta)\,d\sigma(\theta)=4\frac{k_f}{k_i}\Big[\{\mathsf{Y}_2\cos\theta+(\mathsf{Y}_1+\mathsf{Y}_4)\sin\theta\}\,|I_{12}^t|^2$$
$$+\{\mathsf{S}_2\cos\theta+(\mathsf{S}_1+\mathsf{S}_4)\sin\theta\}\,|I_{12}^s|^2-[\{\mathsf{Y}_2\cos\theta+(\mathsf{Y}_1+\mathsf{Y}_4)\sin\theta\}\,I_{12}^{t*}I_{21}^t$$
$$-\{\mathsf{S}_2\cos\theta+(\mathsf{S}_1+\mathsf{S}_4)\sin\theta\}\,I_{12}^{s*}I_{21}^s]\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}d\Omega, \tag{4·4}$$

$$A'(\theta)\,d\sigma(\theta)=4\frac{k_f}{k_i}\Big[\{-\mathsf{Y}_2\sin\theta+(\mathsf{Y}_1+\mathsf{Y}_4)\cos\theta\}\,|I_{12}^t|^2$$
$$+\{-\mathsf{S}_2\sin\theta+(\mathsf{S}_1+\mathsf{S}_4)\cos\theta\}\,|I_{12}^s|^2-[\{-\mathsf{Y}_2\sin\theta+(\mathsf{Y}_1+\mathsf{Y}_4)\cos\theta\}\,I_{12}^{t*}I_{21}^t$$
$$-\{-\mathsf{S}_2\sin\theta+(\mathsf{S}_1+\mathsf{S}_4)\cos\theta\}\,I_{12}^{s*}I_{21}^s]\Big]\frac{d\boldsymbol{k}_{12}}{(2\pi)^3}d\Omega, \tag{4·5}$$

where

$$\mathsf{Y}_1=|M_1|^2+\frac{1}{3}|M_2|^2-\frac{2}{3}[|M_2|^2+|M_3|^2+M_4|^2+|M_5|^2],$$
$$\mathsf{Y}_2=2\,\mathrm{Im}\,M_2^*\frac{2}{3}(M_5-M_1),\quad \mathsf{Y}_3=2\frac{2}{3}|M_3|^2,$$
$$\mathsf{Y}_4=2\frac{2}{3}|M_4|^2,\quad \mathsf{Y}_5=2[|M_2|^2+\frac{2}{3}|M_5|^2], \tag{4·6}$$

and

$$\mathsf{S}_1=\frac{2}{3}|M_2|^2-\frac{1}{3}[|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2],$$
$$\mathsf{S}_2=2\,\mathrm{Im}\,M_2^*\frac{1}{3}(M_5-M_1),\quad \mathsf{S}_3=2\frac{1}{3}|M_3|^2,$$
$$\mathsf{S}_4=2\frac{1}{3}|M_4|^2,\quad \mathsf{S}_5=2\frac{1}{3}|M_5|^2. \tag{4·7}$$

If one approximates the di-neutron by the plane wave and integrates by $\boldsymbol{k}_{12}$, one obtains, in the laboratory system,

$$D(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_D F(q)]D(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \tag{4·8}$$

$$R(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_R F(q)]R(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \tag{4·9}$$

$$A(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_A F(q)]A(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \tag{4·10}$$

$$R'(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_{R'}F(q)]R'(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \tag{4·11}$$

$$A'(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=[1-K_{A'}F(q)]A'(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}, \tag{4·12}$$

where

$$K_D=\frac{\{Y_1+Y_5-(S_1+S_5)\}}{\{Y_1+Y_5+S_1+S_5\}}, \tag{4·13}$$

$$K_R=\frac{\{(-Y_2+S_2)\sin\theta+(Y_1+Y_3-S_1-S_3)\cos\theta\}}{\{(-Y_2-S_2)\sin\theta+(Y_1+Y_3+S_1+S_3)\cos\theta\}}, \tag{4·14}$$

$$K_A=\frac{\{(-Y_2+S_2)\cos\theta+(-Y_1-Y_3+S_1+S_3)\sin\theta\}}{\{(-Y_2-S_2)\cos\theta+(-Y_1-Y_3-S_1-S_3)\sin\theta\}}, \tag{4·15}$$

$$K_{R'}=\frac{\{(Y_2-S_2)\cos\theta+(Y_1+Y_4-S_1-S_4)\sin\theta\}}{\{(Y_2+S_2)\cos\theta+(Y_1+Y_4+S_1+S_4)\sin\theta\}}, \tag{4·16}$$

$$K_{A'}=\frac{\{(-Y_2+S_2)\sin\theta+(Y_1+Y_4-S_1-S_4)\cos\theta\}}{\{(-Y_2-S_2)\sin\theta+(Y_1+Y_4+S_1+S_4)\cos\theta\}}, \tag{4·17}$$

while $D(\theta)_{np}$, $R(\theta)_{np}$, $A(\theta)_{np}$, $R'(\theta)_{np}$ and $A'(\theta)_{np}$ are the depolarization and triple scattering parameters of the nucleon ejected quasi-elastically from the nucleus, say, C^{12}, in the charge exchange collision within the framework of the impulse approximation.

Specifying the relative S-state of low momentum $\boldsymbol{k}_{12}$ for the di-neutron, one obtains

$$D(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta(S_1+S_5)[|I_{12}^s|^2+I_{12}^{s*}I_{21}^s], \tag{4·18}$$

$$R(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\{-S_2\sin\theta+(S_1+S_3)\cos\theta\}[|I_{12}^s|^2+I_{12}^{s*}I_{21}^s], \tag{4·19}$$

$$A(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\{-S_2\cos\theta-(S_1+S_3)\sin\theta\}[|I_{12}^s|^2+I_{12}^{s*}I_{21}^s], \tag{4·20}$$

$$R'(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\{S_2\cos\theta+(S_1+S_4)\sin\theta\}[|I_{12}^s|^2+I_{12}^{s*}I_{21}^s], \tag{4·21}$$

$$A'(\theta)\frac{d\sigma}{d\Omega_{k_{12}}}(\theta)_{nd}=4\cos\theta\{-S_2\sin\theta+(S_1+S_4)\cos\theta\}[|I_{12}^s|^2+I_{12}^{s*}I_{21}^s]. \tag{4·22}*$$

§ 5. Energy distribution of the ejected nucleon

In this section we investigate the energy distribution of the nucleon ejected in

* In the course of the calculation for the triple scattering parameters, the author was informed that Phillips[10] derived $D(0)$ and $A'(0)$ in the approximate forms more than Eqs. (4·18) and (4·22) at $\theta=0°$.

the charge exchange nucleon-deuteron collision.

The energy distribution of the ejected nucleon is given by

$$d\sigma=\frac{2}{3}\frac{k_{12}}{k_t}\left(\frac{1}{2\pi}\right)^3 d\boldsymbol{k}_p\int d\Omega_{12}|T|^2. \tag{5·1}$$

The transition matrix T is approximated as

$$T\simeq t_{13}(1-P_{12})\int d\boldsymbol{r}\,\Psi_{k_{12}}(12)\,\Phi(23)\,e^{i[(k_1+k_2)/2)]r}, \tag{5·2}$$

while Ω_{12} is the direction associated with $\boldsymbol{k}_{12}$. Since the total momentum of the di-neutron is very small at the present case, one may put

$$\boldsymbol{k}_1+\boldsymbol{k}_2\simeq 0. \tag{5·3}$$

When $\Psi_{k_{12}}(12)$ in Eq. (5·2) is separated to $\Psi^t_{k_{12}}$ and $\Psi^s_{k_{12}}$, $\Psi^t_{k_{12}}$ is orthogonal to the ground state Φ. The integral

$$\int\Psi^s_{k_{12}}(r)\,\Phi(r)\,d\boldsymbol{r} \tag{5·4}$$

is the same as the one that occurs in the photo-magnetic disintegration of the deuteron, and the one in the radiative capture of the proton by the neutron at the S-state. Then, Eq. (5·1) is approximated as

$$d\sigma=\frac{2}{3\pi^2}\frac{k_{12}}{k_t}d\boldsymbol{k}_p\,\mathscr{J}_2\left|\int\Psi^s_{k_{12}}(r)\,\Phi(r)\,d\boldsymbol{r}\right|^2. \tag{5·5}$$

Meanwhile, the cross section for the magnetic dipole photodisintegration of the deuteron is given by[11]

$$\sigma_m=\frac{e^2}{\hbar c}(\mu_p-\mu_n)^2\,\frac{k_{12}\hbar\nu}{mc^2}\,\frac{\pi}{3}\left|\int\Psi^s(r)\,\Phi(r)\,d\boldsymbol{r}\right|^2, \tag{5·6}$$

and the cross section for the radiative capture of the proton by the neutron, which is the inverse process of the dipole magnetic photodisintegration of the deuteron, is given by[11]

$$\sigma_c=\frac{3}{2}\left(\frac{\hbar\nu}{mc^2}\right)\left(\frac{\hbar\nu}{E}\right)\sigma_m, \tag{5·7}$$

where

$$E=\hbar\nu-E_g,$$

while E_g is the binding energy of the deuteron and $\hbar\nu$ is the energy of photon.

Gluckstern-Bethe[12] have used the effective range method to calculate the nuclear matrix element in Eq. (5·4). In the present paper, we do not attempt to estimate the nuclear matrix element explicitly. We replace the matrix element by the experimental data of the cross section for the magnetic dipole photodisintegration of the deuteron, or of the cross section for the proton radiatively captured by the

neutron in the S-state. For instance, substituting Eq. (5·6) in Eq. (5·1), one obtains

$$d\sigma=\frac{2}{\pi^3}\sigma_m\frac{dk_p}{k_t}\mathcal{Y}_2(\mu_p-\mu_n)^{-2}\left(\frac{\hbar c}{e^2}\right)\frac{mc^2}{\hbar\nu}. \tag{5·8}$$

Then, the energy distribution of the ejected nucleon can be calculated without any special information referring to the wave function of the di-neutron, when $\mathcal{Y}_2$ is fixed by the two-nucleon scattering phase shifts and σ_m or σ_c is measured experimentally.

§ 6. Relations between the ejected nucleon and the two-nucleon scattering

In this section, we show some examples of how the information on the two-nucleon interactions is obtained by the nucleon ejected in the charge exchange nucleon-deuteron collision.

Using the equation

$$\frac{d\sigma}{d\Omega}(\pi-\theta)=\frac{1}{4}|B|^2+2|C|^2+\frac{1}{2}|N|^2+\frac{1}{2}|H|^2+\frac{1}{4}|G-N|^2, \tag{6·1}$$

one obtains, from Eq. (2·9),

$$\frac{d\sigma}{d\Omega}(\theta)_{nd}=\left[1-\frac{2}{3}F(q)\right]\frac{d\sigma}{d\Omega}(\pi-\theta)$$
$$-\frac{2}{3}\cos\theta\left[\mathrm{Re}\,N^*(G-N-B)-|H|^2-\frac{1}{4}|B+G-N|^2\right]F(q), \tag{6·2}$$

which becomes for $\theta=0°$

$$\frac{d\sigma}{d\Omega}(0)_{nd}=\frac{1}{3}\frac{d\sigma}{d\Omega}(\pi)-\frac{2}{3}\left[\mathrm{Re}\,N^*(G-N-B)-|H|^2-\frac{1}{4}|B+G-N|^2\right], \tag{6·3}$$

where $d\sigma(\pi-\theta)/d\Omega$ is the cross section of the free neutron-proton scattering at backward direction, that is, $d\sigma(\pi-\theta)/d\Omega=d\sigma(\theta)_{np}/d\Omega$. Since the experimental value of the ratio of $d\sigma/d\Omega_{nd}$ to $d\sigma/d\Omega_{np}$ is $1/5$ at 380 Mev,[13] one obtains from Eq. (6·3)[6]

$$\frac{1}{5}\frac{d\sigma}{d\Omega_{np}}=\mathrm{Re}\,N^*(G-N-B)-|H|^2-\frac{1}{4}|B+G-N|^2, \tag{6·4}$$

for the free proton-neutron scattering at 380 Mev.

Using the equation

$$P(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}=2\,\mathrm{Re}\,N^*C, \tag{6·5}$$

one obtains

$$P(\theta)\frac{d\sigma}{d\Omega}(\theta)_{nd}=\left[1-\frac{1}{2}F(q)\right]P(\theta)_{np}\frac{d\sigma}{d\Omega}(\theta)_{np}-\frac{4}{3}\cos\theta[\mathrm{Re}(G-B)^{*}C]. \tag{6·6}$$

The coefficients of $|I_{12}^{s}|^2$ in Eq. (2·5) is rewritten as

$$\frac{1}{3}[|M_2|^2+|M_3|^2+|M_4|^2+|M_5|^2]$$
$$=\frac{1}{6}\left\{\frac{3}{2}\frac{d\sigma}{d\Omega}(\pi-\theta)-|\bar{\gamma}|^2-\mathrm{Re}(\bar{\alpha}^{*}\bar{\delta}+\bar{\alpha}^{*}\bar{\varepsilon}+\bar{\alpha}^{*}\bar{\beta}+\bar{\beta}^{*}\bar{\delta}+\bar{\beta}^{*}\bar{\varepsilon}+\bar{\delta}^{*}\bar{\varepsilon})\right\}, \tag{6·7}$$

where

$$\bar{\alpha}=\frac{1}{4}(B+N+G),\quad \bar{\gamma}=C,\quad \bar{\delta}=\frac{1}{4}(G-N-B-2H),$$
$$\bar{\varepsilon}=\frac{1}{4}(G-N-B+2H),\qquad \bar{\beta}=\frac{1}{4}(3N-B-G). \tag{6·8}$$

The right-hand side of Eq. (6·7) is easily connected with the correlation parameters[7] for the free neutron-proton scattering at backward direction:

$$\frac{1}{2}\frac{d\sigma}{d\Omega}C_{nn}=\mathrm{Re}(\bar{\alpha}\bar{\beta}^{*}-\bar{\delta}\bar{\varepsilon}^{*})+|\bar{\gamma}|^2, \tag{6·9}$$

$$\frac{1}{2}\frac{d\sigma}{d\Omega}C_{kk}=\mathrm{Re}(\bar{\alpha}\bar{\varepsilon}^{*}-\bar{\beta}\bar{\delta}^{*}), \tag{6·10}$$

$$\frac{1}{2}\frac{d\sigma}{d\Omega}C_{pp}=\mathrm{Re}(\bar{\alpha}\bar{\delta}^{*}-\bar{\beta}\bar{\varepsilon}^{*}). \tag{6·11}$$

The polarization and triple scattering parameters of the nucleon ejected in the charge exchange nucleon-deuteron collision give a certain information on the interaction of the two nucleon system with $T=0$, in similar manner as the cross section of the ejected nucleon above mentioned.

§ 7. Results and discussion

In this section, some values for the nucleon ejected in the charge exchange collision are calculated by the use of the Gammel-Thaler[14] and Signell-Marshak phase shifts.[15]

If one uses the sharp energy beam of the nucleons to measure the charge exchange nucleon-deuteron collision and specifies the final spin state of two like nucleons with the relative S-state of low momentum, one can obtain from these experiments more information on the two-nucleon scattering with $T=0$. When the final two like nucleons are left in the singlet spin state, the behavior of the ejected nucleon is very different from the one of the nucleon ejected in the charge

exchange quasi-elastic scattering or the one of the free two-nucleon scattering. Then, we show the results for the nucleon ejected in the charge exchange nucleon-deuteron collision when the final two like nucleons are left in the singlet spin state. The polarization and triple scattering parameters of the nucleon ejected in the charge exchange nucleon-deuteron collision do not depend on the wave functions of the deuteron and two like nucleons, when one specifies the final two like nucleons left in the singlet spin state. They depend on the wave function of the deuteron through the interference factor, $F(q)$, when the deuterons are bombarded with the broad energy beam of the nucleons. The interference factor depends on the momentum transfer of the ejected nucleon.

Fig. 1 shows the differential cross section of the proton ejected in the neutron-deuteron collision at 90 Mev. The neutron beam in the Berkeley experiment has an energy width of about 30 Mev, so that we approximate the final di-neutron by the plane wave, that is, we use Eq. (2·9) to reproduce the experimental data. To calculate the interference factor, we use the Hulthén wave function for the deuteron. If the experiment is done by the use of the sharp energy beam of the neutron, it is expected that the final di-neutron is specified in the singlet spin state for the proton ejected in very small angles. Fig. 2 shows the differential cross section of the ejected proton for the di-neutron left in the singlet spin state. The curve in Fig. 2 is very similar to the neutron scattered by the free proton in backward direction.

Dzhelepov et al.[13] measured the ratio of the cross section of the nucleon ejected forward in the charge exchange nucleon-deuteron collision to the one of the free neutron-proton scattering in backward direction. The measured ratio is 0.20 ± 0.035 at 380 Mev. The ratios calculated by the use of the Gammel-Thaler phase shifts at 310 Mev and Signell-Marshak phase shifts at 300 Mev are 0.43 and 0.45, respectively. We do not conclude which set of the nuclear phase shifts reproduces the experimental ratio better than the other, since the concerning energy

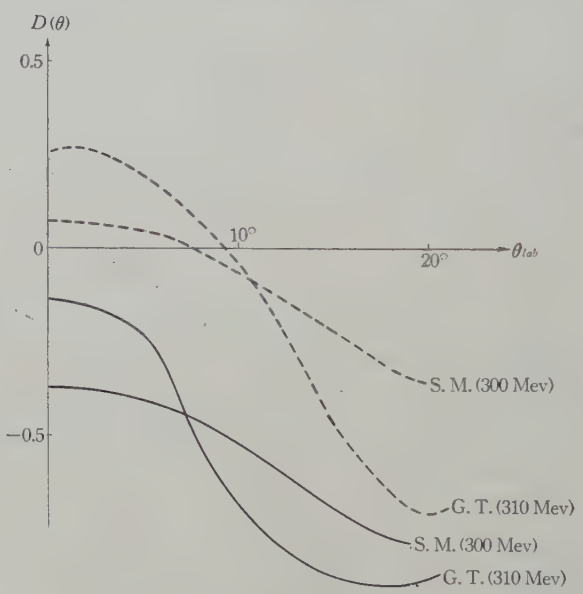


Fig. 4. The depolarization of the nucleon ejected in the charge exchange nucleon-deuteron collision, when the final two like nucleons are left in the singlet spin state. Solid and dashed curves indicate the depolarization of the nucleon ejected in the charge exchange nucleon-deuteron and the one of the nucleon ejected in the quasi-elastic charge exchange nucleon-nucleus collision within the framework of the impulse approximation, respectively.

is different from each other. The ratio estimated by using the Klein phase shifts at 330 Mev[16] is close to unity. This is in strong contradiction with the experimental ratio. This discrepancy is probably due to the fact that Klein's analysis which underestimates the role of states with $l \geqq 2$ agrees badly with the experimental data on neutron-proton scattering in the angular region close to 180°.

Fig. 3 contains the polarization of the ejected nucleon for the final di-neutron left in the singlet spin state. The polarization of the nucleon quasi-elastically ejected from the nucleus in the charge exchange collision, $p(C^{12}, C^{11}+p)n$, which is calculated within the framework of the impulse approximation, is shown in Fig. 3 for comparison.[17] The polarization of the nucleon ejected in the quasi-elastic scattering, $p(C^{12}, C^{11}+p)n$, is the same as that of the neutron in the free two-nucleon scattering, $p(n, p)n$. The curves G. T. and S. M. stand for the results calculated by the use of the Gammel-Thaler phase shifts at 310 Mev and the Signell-Marshak phase shifts at 300 Mev, respectively. The sign of the polarization of the nucleon ejected in the charge exchange collision, when two like nucleons are left in the singlet spin state, is opposite to that of the nucleon ejected in the charge exchange quasi-elastic collision.

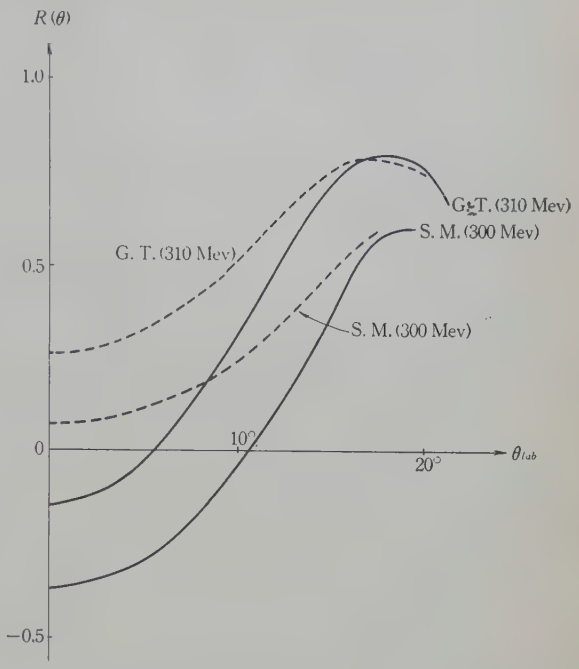


Fig. 5. $R(\theta)$ of the nucleon ejected in the charge exchange nucleon-deuteron collision. Solid and dashed curves correspond to the ones of Fig. 4.

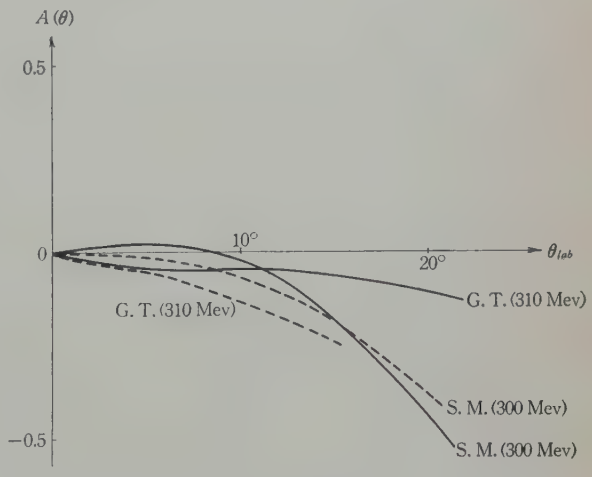


Fig. 6. $A(\theta)$ of the nucleon ejected in the charge exchange nucleon-deuteron collision. Solid and dashed curves correspond to the ones of Fig. 4.

Figs. 4, 5, 6, 7 and 8 show the angular variations of the depolarization and triple scattering parameters of the nucleon ejected in the charge exchange nucleon-deuteron collision, when the final two like nucleons are left in the singlet spin state, and those of the nucleon ejected in the charge exchange quasi-elastic scattering within the framework of the impulse approximation, respectively.

Fig. 9 shows the energy distribution of the nucleon ejected forward in the charge exchange nucleon-deuteron collision. Instead of an explicit estimation for the nuclear matrix element given in Eq. (5·4), we use the experimental data for the magnetic dipole photodisintegration of the deuteron[18] to calculate the energy distribution of the ejected nucleon.

According to Chew-Wick,[19] the impulse approximation introduces errors of order $a/(kR)^2$, where a is the two-body scattering amplitude (here meaning some average), k is the relative wave number of the incident nucleon and R is the average nucleon separation distance in the deuteron ($\sim 3.2\times 10^{-13}$ cm). This quantity is less than 3% at 300 Mev. These authors also state that the multiple scattering correction may be of order 1% at 300 Mev. The accuracy of substituting a mean value for t_{1t} in Eq. (2·3) depends on how smoothly t_{1t} varies with the energy of the incident nucleon. A rough estimation gives an error from

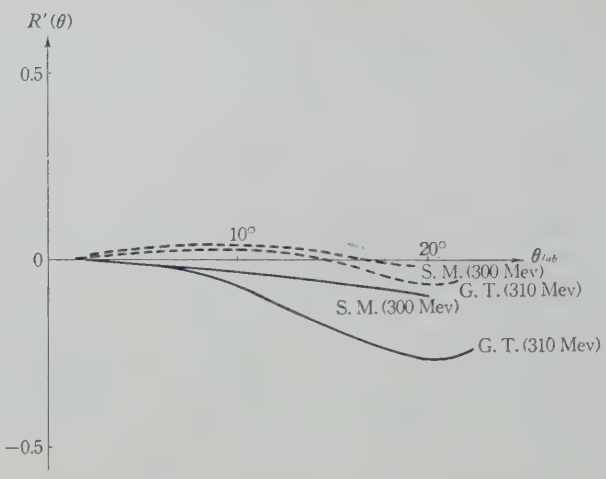


Fig. 7. $R'(\theta)$ of the nucleon ejected in the charge exchange nucleon-deuteron collision. Solid and dashed curves correspond to the ones of Fig. 4.

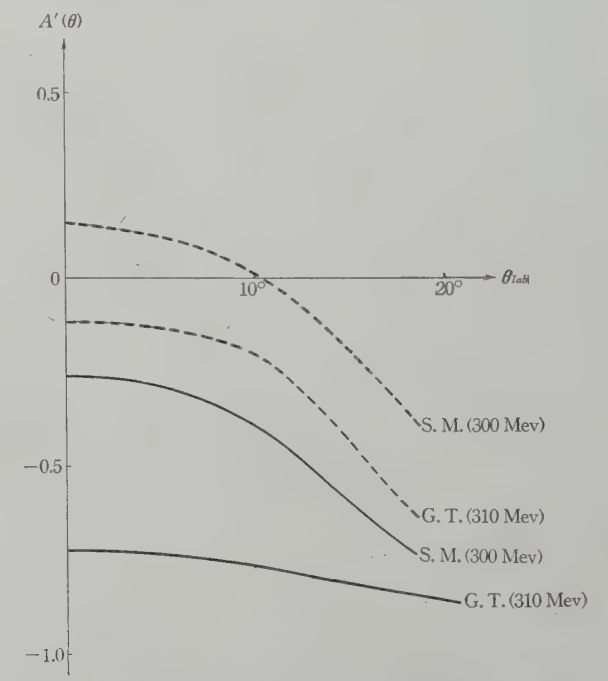


Fig. 8. $A'(\theta)$ of the nucleon ejected in the charge exchange nucleon-deuteron collision. Solid and dashed curves correspond to the ones of Fig. 4.

this cause of about 1% at 300 Mev in forward scattering angle.

§ 8. Concluding remarks

The high energy nucleon ejected in the charge exchange nucleon-deuteron collision as well as the nucleon quasi-elastically ejected from nucleus, say, C^{12}, in the charge exchange collision supplement the information on the two-nucleon scattering with $T=0$. That is, even if one restricts oneself to the cross section of the nucleon ejected in the charge exchange nucleon-deuteron collision, one obtains the information on the two-nucleon scattering which is not obtained by the free two-nucleon scattering. When the polarization and triple scattering parameters of the ejected nucleon are measured, the information would be greatly increased.

The sharp peak in the energy distribution of the neutron ejected in the proton-deuteron collision could be used in practice to obtain very nearly monochromatic beams of neutrons, which is very useful to investigate the interaction between two neutrons, and between neutrons and a certain nucleus. For this purpose, the proton of given energy E is permitted to fall on a deuteron. Then, in the forward direction, neutron will be emitted whose energy is slightly below $E-E_g$. Fig. 9 shows that the width of the energy distribution is only 1 Mev at forward direction. It is expected that the width would not be appreciably increased if the ejected neutron up to 10° angle with the incident proton beam is included.[12] The neutron ejected in θ at about 300 Mev is partially polarized about $P(\theta)=-0.4\theta$ (rad).

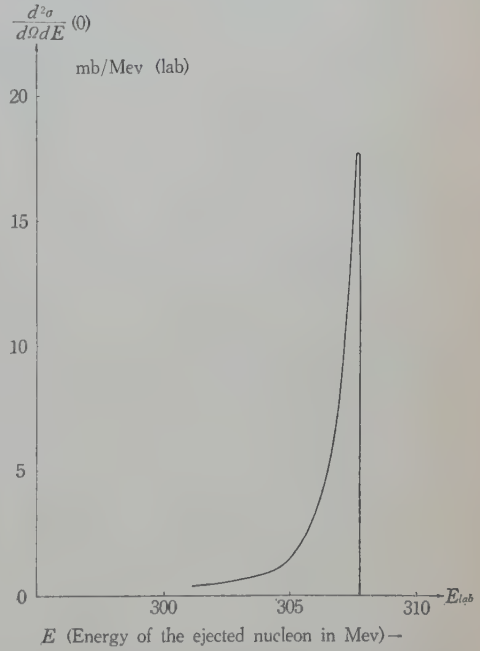


Fig. 9. Energy distribution of the nucleon ejected in the charge exchange nucleon-deuteron collision in the forward direction. The curve is calculated by the use of the Gammel-Thaler phase shifts at 310 Mev.

If the coefficients of the scattering spin matrix are reasonably parametrized on the basis of the complete two-nucleon scattering experiment, the investigation of the energy distribution of the nucleon ejected in the charge exchange nucleon-deuteron collision furnishes a certain way to study the magnetic dipole photo-disintegration of the deuteron and the proton radiatively captured by the neutron in the S-state. Furthermore, the nuclear matrix element, Eq. (5·4), gives the information on the effective range of the di-neutron in the virtual resonance state.

When the nucleon-nucleon scattering phase shifts are completely determined by the analysis of the experimental data, the formulas given in the present paper give a test for the impulse approximation applied to the rearrangement collision or the inelastic scattering.

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New Analyses of Anomalous Imaginary Part

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Using a new method of analytic continuation and the knowledge of parameter integration, anomalous imaginary part is written explicitly with the absorptive part (concerning t) of matrix element which connects initial or final state and intermediate one. The meaning of intermediate state is more clarified.

In the case of π-meson deuteron scattering, all the singularities are determined. It is confirmed that these singularities agree with those obtained by performing the direct parameter integration of fourth order diagram.

§ 1. Introduction

The analytic properties of scattering amplitude or vertex function have been discussed from various points of view.[1)-6)]

Their behaviour is restricted by the fundamental notions of current field theory, for instance, Lorentz invariance, microscopic causality and spectral condition.

To obtain the physical quantity more restriction for singularity should be added.

In this point of view, we must in the first place investigate the restriction for the most peculiar one, anomalous threshold,[7)-11)] which invalidates the usual expression conceived from the unitarity condition.

In this article we first carry out, in §3, the analytic continuation of imaginary part expressed by the unitarity condition in the physical region deforming the path of angle integration in the intermediate state. Then we compare the expression thus obtained with the parameter representation analysed in § 2, and determine the anomalous imaginary part.

With some assumptions, general expression for physical scattering amplitude is obtained in § 4.

The formula (28) in § 4 means that anomalous threshold occurs when the matrix element which connects initial or final state and intermediate state become singular for too small positive value of t (or u) (see § 3 for the notation). To determine its existence, the knowledge of external and intermediate masses and the minimum t (or u) singularity is sufficient. Moreover, its explicit form is also obtainable if we know the above-mentioned matrix elements.

It modifies the unitarity condition when s is less than some value that is larger than normal threshold. The modified part is the anomalous imaginary part and is

simultaneously the absorptive part of external mass variable. Its level is that of bound particle, but it may not necessarily correspond to real bound particle.

§ 5 concerns with the problem of π-D scattering. All the singularities are obtained. They agree with those obtained by the fourth order perturbation calculation which is done in § 6. Especially the curve singularity is of interest, because it may give rather large contribution to the total integral of absorptive part.

We neglect spins and isospins throughout in this note.

§ 2. Method of finding the singularities in perturbation theory

For a particular diagram of perturbation theory, we can perform momentum integration according to the Feynmann method. We get a scalar function, whose variables are scalar products of external momenta. It includes parameter integration.

Singularities concerning with some scalar variable appear at the point where the function that decides the end point or curve, etc., of the range of parameter integration is not analytic with this scalar variable.

Practically we can get every singularity (hereafter we only consider the singularity of the absorptive part) by using the method of Landau[10] and Nakanishi,[9] not restricting to the minimum singularity.

We show some examples of simple diagram.

1. *Vertex functioh with two external ends on the mass shell*

The notation for momenta, masses and parameters is shown in Fig. 1. Other definitions are

$$P_1^2=M_1^2,\quad P_2^2=M_2^2,\quad (P_1+P_2)^2=s,$$

$$y_{32}=\frac{m_3^2+m_2^2-M_2^2}{2m_3m_2},\quad y_{21}=\frac{m_1^2+m_2^2-s}{2m_1m_2},\quad y_{13}=\frac{m_1^2+m_3^2-M_1^2}{2m_1m_3},$$

$$p_{1,2}=\sqrt{\boldsymbol{p}_{1,2}^2},\quad q=\sqrt{\boldsymbol{q}^2},\quad \boldsymbol{p}_1+\boldsymbol{p}_2=0,$$

$$A_1=Q^2-m_1^2,\quad A_2=(P_1+P_2-Q)^2-m_2^2,\quad A_3=(P_1-Q)^2-m_3^2, \tag{1}$$

$$z_2'=\frac{s^2-s(m_1^2+m_2^2+M_1^2+M_2^2-2m_3^2)+(m_1^2-m_2^2)(M_1^2-M_2^2)}{\sqrt{\{s^2-2s(m_1^2+m_2^2)+(m_1^2-m_2^2)^2\}\{s^2-2s(M_1^2+M_2^2)+(M_1^2-M_2^2)^2\}}}$$

$$=\frac{-(P_{10}-Q_0)^2+\boldsymbol{p}_1^2+\boldsymbol{q}^2+m_3^2}{2p_1q}, \tag{2}$$

(with the condition $A_1=A_2=0$).

1. Case 1) $y_{13}+y_{32}>0$

There is only one singular point at $s=(m_1+m_2)^2$ where A_1, A_2, and α_3 are zero.

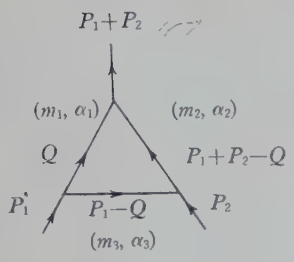


Fig. 1

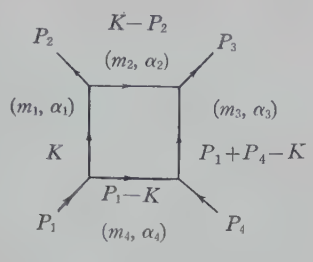


Fig. 2

1. Case 2) $y_{13}+y_{32}<0$

For the s-value smaller than $(m_1+m_2)^2$, the point $\alpha_3=0$ is excluded from the range of α_3's integration. This means a portion of the absorptive part whose extreme value with α_i is given by $A_1=A_2=0$, $\alpha_3=0$ has vanished.

When s becomes s_a given by (3),

$$\frac{m_1^2+m_2^2-s_a}{2m_1m_2}=y_{13}y_{32}-\sqrt{(1-y_{13}^2)(1-y_{32}^2)}\,, \tag{3}$$

the range of α_3 vanishes. This corresponds to $A_1=A_2=A_3=0$ and s minimum.

2. *Scattering diagram with four external ends on the mass shell*

The notation is shown in Fig. 2. Other definitions are

$P_i^2=M_i^2 \quad (i=1, 2, 3, 4)$,

$s=(P_1+P_4)^2$, $t=(P_1-P_2)^2$, $u=(P_1-P_3)^2$,

$\boldsymbol{p}_1+\boldsymbol{p}_4=0$, $p_1=\sqrt{\boldsymbol{p}_1^2}$, $k=\sqrt{\boldsymbol{k}^2}$, $p_4=\sqrt{\boldsymbol{p}_4^2}$, etc.,

$$y_{i\,i+1}=\frac{m_i^2+m_{i+1}^2-M_{i+1}^2}{2m_im_{i+1}} \quad (i=1, 2, 3), \qquad y_{41}=\frac{m_4^2+m_1^2-M_1^2}{2m_4m_1},$$

$A_1=K^2-m_1^2$, $A_2=(K-P_2)^2-m_2^2$, $A_3=(P_1+P_4-K)^2-m_3^2$,

$A_4=(P_1-K)^2-m_4^2$,

$$z_2'=\frac{-(P_{10}-K_0)^2+\boldsymbol{p}_1^2+\boldsymbol{k}^2+m_4^2}{2p_1k}$$

(with the condition $A_1=A_3=0$)

$$=\frac{s^2-s(m_1^2+m_3^2+M_1^2+M_4^2-2m_4^2)+(m_1^2-m_3^2)(M_1^2-M_4^2)}{\sqrt{\{s^2-2s(m_1^2+m_3^2)+(m_1^2-m_3^2)^2\}\{s^2-2s(M_1^2+M_4^2)+(M_1^2-M_4^2)^2\}}},$$

$$z_3'=\frac{-(P_{20}-K_0)^2+\boldsymbol{p}_2^2+\boldsymbol{k}^2+m_2^2}{2p_2k}$$

(with the condition $A_1=A_3=0$)

$$=\frac{s^2-s(m_1^2+m_3^2+M_2^2+M_3^2-2m_2^2)+(m_1^2-m_2^2)(M_2^2-M_3^2)}{\sqrt{\{s^2-2s(m_1^2+m_3^2)+(m_1^2-m_3^2)^2\}\{s^2-2s(M_2^2+M_3^2)+(M_2^2-M_3^2)^2\}}},$$

$$z_1 = \frac{-(P_{10}-P_{20})^2+\boldsymbol{p}_1^2+\boldsymbol{p}_2^2+t}{2p_1p_2}$$

$$= \frac{s^2-s(M_1^2+M_2^2+M_3^2+M_4^2-2t)+(M_1^2-M_4^2)(M_2^2-M_3^2)}{\sqrt{\{s^2-2s(M_1^2+M_4^2)+(M_1^2-M_4^2)^2\}\{s^2-2s(M_2^2+M_3^2)+(M_2^2-M_3^2)^2\}}}. \tag{4}$$

We assume t to be less than t_a, which is the minimum line singularity[12] with respect to t. Only when $y_{12}+y_{23}+y_{34}+y_{41}$ is negative, we assume $t<t_{a3}$ in cases 1) to 3) and $t_a \leq t \leq t_{a3}$ in case 4).

t_a is defined by (8).

2. Case 1) $y_{41}+y_{34}>0$, $y_{12}+y_{23}>0$

There is only one normal threshold, i.e. $(m_1+m_3)^2$.

2. Case 2) $y_{41}+y_{34}>0$, $y_{12}+y_{23}<0$

When s becomes smaller than $(m_1+m_3)^2$, the range of α_4 is larger than zero. The vanishing extreme value with α_i is $A_1=A_3=0$ and $\alpha_2=\alpha_4=0$.

When s is equal to s_{a2} (which is the anomalous threshold obtained when one puts α_2 zero; similarly we define s_{a1} for case 3) putting α_4 zero), the range of α_4 vanishes.

2. Case 3) $y_{41}+y_{34}<0$, $y_{12}+y_{23}<0$

We further assume for simplicity $s_{a1}>s_{a2}$ and define s_{a3} by (5),

$$z_1-z_2'z_3'=\sqrt{(1-(z_2')^2)(1-(z_3')^2)} \tag{5}$$
$$(A_1=A_3=0),$$

then the following inequality holds:

$$s_{a1} \leq s_{a3} < (m_1+m_3)^2. \tag{6}$$

When s becomes smaller than $(m_1+m_3)^2$, the point $(\alpha_2=\alpha_4=0)$ vanishes. At the point $s=s_{a3}$ the absorptive part whose extreme value with α_i is given by $A_1=A_2=A_3=A_4=0$ vanishes, this gives curve singularity in the sense of Eden,[12] but in such energy region as in 2. Case 3), no investigation has ever been done.

When s is smaller than s_{a3}, the range of α_2, α_4 is divided into two disconnected parts. They vanish each at s_{a1} and s_{a2}. See also Fig. 3 to Fig. 7.

2. Case 4) $y_{12}+y_{23}+y_{34}+y_{41}<0$

$$t_a>t>t_{a1},\ s \leq s_{a1}.$$

The minimum singularity is

$$z_1=z_2'z_3'-\sqrt{(1-(z_2')^2)(1-(z_3')^2)}, \tag{7}$$

and given by $A_1=A_2=A_3=A_4=0$, s minimum.[12]

Definition of t_a is

$$z_1=z_2',\ z_3'=1. \tag{8}$$

The range of α_2, α_4 is shown in Fig. 8.

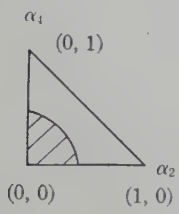


Fig. 3
$s>(m_1+m_3)^2$



Fig. 4
$(m_1+m_3)^2>s>s_a$
2. Case 2)



Fig. 5
$(m_1+m_3)^2>s>s_{a_3}$
2. Case 3)



Fig. 6
$s_{a_3}>s>s_{a_1}$
2. Case 3)



Fig. 7
$s_{a_1}>s>s_{a_2}$
2. Case 3)

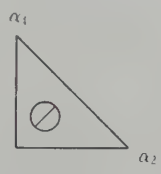


Fig. 8
2. Case 4)

§ 3. Extension of the unitarity condition to the unphysical region of s

1. Case 2) $s\geq(M_1+M_2)^2$

$$\mathrm{Im}\,\Gamma=-\frac{g^3}{64\pi^2\sqrt{s}\,p_1}\int_{-1}^{1}\frac{dz_2}{z_2-z_2'}=\frac{g^3}{64\pi^2\sqrt{s}\,p_1}\log\frac{z_2'+1}{z_2'-1}. \tag{9}$$

From the analyses of the parameter integral we know that singular points are only $s=(m_1+m_2)^2$ and $s=s_a$, so $\mathrm{Im}\,\Gamma$ must be analytically continued with s to the point $(m_1+m_2)^2$.

But when s varies from $(M_1+M_2)^2$ to $(m_1+m_2)^2$, z_2' varies from $-i\infty$ to $i\infty$ and becomes zero only once.

Therefore we must deform the path of integration (9) lest the integrand should be singular. Figs. 9 and 10 show the deformation. In case 1) $z_2'\neq0$ and the

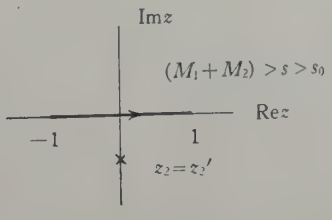


Fig. 9

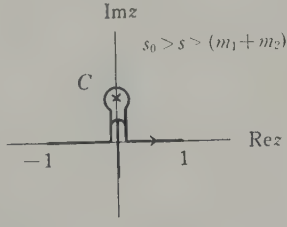


Fig. 10

point z_2' does not cross the path of integration so we can use the form (9) for all $s \geq (m_1+m_2)^2$.

$$s_0 > s \geq (m_1+m_2)^2, \quad z_2'(s_0)=0,$$

$$\mathrm{Im}\, \Gamma = -\frac{g^3}{64\pi^2 \sqrt{s}\, p_1} \left\{ \int_{-1}^{1} \frac{dz_2}{z_2-z_2'} + \int_{C} \frac{dz_2}{z_2-z_2'} \right\}. \tag{9'}$$

Further we can fix the correspondence of two terms to the two singularities. $s=s_a$ is the minimum point where $A_1=A_2=A_3=0$. Therefore s_a corresponds to the vanishing of the second term of (9'). Similarly $s=(m_1+m_2)^2$ is the point where the first term of (9') vanishes. We call hereafter the second term anomalous imaginary part, and the first term, normal one.

We can interpret the normal term from its form as the extension of that which was defined in the physical region from unitarity, and was considered as arising from the two particle real intermediate state.

We can regard Γ as the boundary value of P_1^2 (or P_2^2) ; the anomalous term is its absorptive part.

It has become clear that this intermediate state is composed of two particles on the mass shell and its level is that of bound particle.

But there is no base to assert any real bound particle exists, because it must suffice more difficult conditions, for instance, the Bethe-Salpeter equation.

We can only interpret the correspondence of particle level as arising from bound structure.[8)-11)]

These interpretations can be applied to the case of general scattering.

2. Case 1)

$$\mathrm{Im}\, A = \frac{g^4}{128\pi^2 \sqrt{s}\, p_{1e}\, p_{1o}\, p_1} \int_0^{2\pi} \int_{-1}^{1} \frac{d\phi\, dz_2}{(z_2-z_2')\,(z_3-z_3')}, \tag{10}$$

$$z_3 = z_1 z_2 - \sqrt{(1-z_1^2)\,(1-z_2^2)} \cdot \cos\phi, \tag{11}$$

$$s \geq (m_1+m_3)^2.$$

2. Case 2) $s > s_0$

$$\mathrm{Im}\, A = \frac{g^4}{128\pi^2 \sqrt{s}\, p_{1e}\, p_{1o}\, p_1} \int_0^{2\pi} \int_{-1}^{1} \frac{d\phi\, dz_2}{(z_2-z_2')\,(z_3-z_3')}. \tag{12}$$

s_0 is given by

$$z_2'=0, \tag{13}$$

$$s_0 > s \geq (m_1+m_3)^2,$$

$$\mathrm{Im}\,A=\frac{g^4}{128\pi^2\sqrt{s}\,p_{1e}p_{1o}p_1}\left[\int_0^{2\pi}\int_{-1}^{1}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}\right.$$
$$\left.+\int_0^{2\pi}\int_{C}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}\right], \qquad (14)$$
$$(m_1+m_3)^2>s\geq s_a,$$

$$\mathrm{Im}\,A=\frac{g^4}{128\pi^2\sqrt{s}\,p_{1e}p_{1o}p_1}\int_0^{2\pi}\int_{C}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}$$
$$=2\pi g^4/(128\pi^2\sqrt{s}\,p_{1e}p_{1o}p_1\sqrt{K}), \qquad (15)$$
$$K=(z_1-z_2'z_3')^2-(1-(z_2')^2)(1-(z_3')^2). \qquad (16)$$

2. Case 3) $s>\max\{(M_1+M_4)^2,\ (M_2+M_3)^2\}$

$$\mathrm{Im}\,A=\frac{g^4}{128\pi^2\sqrt{s}\,p_{1e}p_{1o}p_1}\int_0^{2\pi}\int_{-1}^{1}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')} \qquad (17)$$

$$=\frac{g^4}{64\pi\sqrt{s}\,p_{1e}p_{1o}p_1\sqrt{K}}\log\left\{\frac{z_1-z_2'z_3'+\sqrt{K}}{z_1-z_2'z_3'-\sqrt{K}}\right\}. \qquad (18)$$

Equation (18) is given by Mandelstam.[13]

We must deform the path of integration if necessary. (There is some arbitrariness, but hereafter we fix the path of $\cos\phi$ and deform z_2 only.)

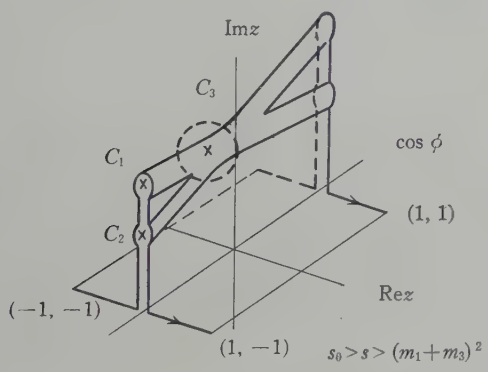


Fig. 11

The deformation of z_2's path is shown by Fig. 11.

As the curves $A_2=0$ and $A_4=0$ coincides at one value of $\cos\phi=\cos\phi_0$, we must write

$$s>(m_1+m_3)^2, \qquad t<t_a,$$

$$\cos\phi_0=\frac{z_1-z_2'z_3'}{\sqrt{(1-(z_2')^2)(1-(z_3')^2)}}, \qquad (19)$$

$$\mathrm{Im}\,A=\frac{g^4}{128\pi^2\sqrt{s}\,p_{1e}p_{1o}p_1}\left[\int_0^{2\pi}\int_{-1}^{1}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}\right.$$
$$\left.+\int_0^{2\pi}\int_{C_1}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}+\int_0^{2\pi}\int_{C_2}\frac{d\phi\,dz_2}{(z_2-z_2')(z_3-z_3')}\right.$$

$$-2\int\int_{C_3}\frac{d\phi\,dz_2}{(z_2-z_2')\,(z_3-z_3')}\Big]. \tag{20}$$

C_1, C_2 and C_3 mean the circle (its radius tending to zero) which encloses each the point $A_2=0$, $A_4=0$ and $A_2=A_4=0$.

The fourth term must vanish at $s=s_{a3}$, the second and the third terms vanish at $s=s_{a2}$ and s_{a1}. The first term vanishes at $s=(m_1+m_3)^2$. The results of explicit evaluation are

$$\int_0^{2\pi}\int_{C_1}\frac{d\phi\,dz_2}{(z_2-z_2')\,(z_3-z_3')}=\frac{2\pi}{\sqrt{K}} \tag{21}$$

$$=\int_0^{2\pi}\int_{C_2}\frac{d\phi\,dz_2}{(z_2-z_2')\,(z_3-z_3')} \tag{22}$$

$$=\int\int_{C_3}\frac{d\phi\,dz_2}{(z_2-z_2')\,(z_3-z_3')}. \tag{23}$$

§ 4. Extension to the general scattering amplitude

We consider and treat the two-body scattering matrix element A_{eo} as perturbationally constructed. The variable s is limited to be positive. Picking up one of the two-body intermediate states with its masses m_{1i}, m_{2i}, we can write

$$A_{eo}(s,t)=\frac{i}{(2\pi)^4}\int A_{ei1}^{*}(s,t',P_1^2,P_2^2)\,\frac{1}{(P_1^2-m_{1i}^2)\,(P_2^2-m_{2i}^2)}$$

$$\times A_{i1o}(s,t'',P_1^2,P_2^2)\,d^4P_1\,d^4P_2\,\delta(P_1+P_2-P_{1e}-P_{2e})+A_{eo}'. \tag{24}$$

P_{1e} and P_{2e} are the 4-momenta of initial particles and we define P_{1o} and P_{2o} as those of outgoing ones.

$$\begin{aligned} t' &= (P_{1e}-P_1)^2, & u' &= (P_{1e}-P_2)^2,\\ t'' &= (P_{1o}-P_1)^2, & u'' &= (P_{1o}-P_2)^2,\\ s &= (P_{1e}+P_{2e})^2,\\ t &= (P_{1e}-P_{1o})^2,\\ u &= (P_{1e}-P_{2o})^2. \end{aligned} \tag{25}$$

A_{eo}' is the term which does not contain any such two-body intermediate state.

For A_{ei1}^{*} and A_{i1o}, etc., the absorptive part with s is defined in the physical region, we further assume the following expansion

$$A^*_{ei1}=\int_{t_0'(s)}^{\infty}\frac{a^*_{ei1}(s,\,t_0',\,P_1^2,\,P_2^2)}{(t'-t_0')}dt_0'+\int_{u_0'(s)}^{\infty}\frac{a^*_{ei1}(s,\,u_0',\,P_1^2,\,P_2^2)}{(u'-u_0')}du_0', \quad (26)$$

and similar relation for A_{i1o}.

The quantities a^*_{ei1}, a_{i1o} may be complex. Rewriting (24) with (26), and taking the absorptive part, we obtain

$$\begin{aligned}AbA_{eo}(s,\,t)=&\frac{i}{(2\pi)^4}\iiint F(t',\,t_0',\,\cdots,\,u'',\,u_0'')\,\{Aba^*_{ei1}a_{i1o}\}\\&\times\frac{1}{(P_1^2-m_{1i}^2)\,(P_2^2-m_{2i}^2)}d^4P_1d^4P_2\delta(P_1+P_2-P_{1e}-P_{2e})\\&+\frac{1}{(2\pi)^4}\iiint a^*_{ei1}a_{i1o}\,Ab\Big\{iF\frac{1}{(P_1^2-m_{1i}^2)\,(P_2^2-m_{2i}^2)}\\&\times d^4P_1d^4P_2\delta(P_1+P_2-P_{1e}-P_{2e})\Big\}+AbA'_{eo}, \qquad (27)\end{aligned}$$

$$\begin{aligned}F\approx\Big[&\frac{dt_0'\,dt_0''}{(t'-t_0')\,(t''-t_0'')}+\frac{du_0'\,du_0''}{(u'-u_0')\,(u''-u_0'')}+\frac{du_0'\,dt_0''}{(u'-u_0')\,(t''-t_0'')}\\&+\frac{dt_0'\,du_0''}{(t'-t_0')\,(u''-u_0'')}\Big].\end{aligned}$$

We can explicitly integrate the second term of (27), because when s, t and u are in the physical region, P_1^2 and P_2^2 equal m_{1i}^2 and m_{2i}^2 respectively.[13]

Performing the integration by the same method as in perturbation theory in the physical region putting intermediate masses on the mass shell, we get parameter representation, which well defines the absorptive part in the unphysical region.

The other two terms are considered to correspond to the other intermediate states.

Comparing with the unitarity expression of AbA_{eo} in the physical region, we finally obtain

$$\begin{aligned}Ab\,A_{eo}(s,\,t)=&\sum_i\iiint\Big[Ab\Big\{\frac{i}{(2\pi)^4}F\frac{d^4P_1d^4P_2\delta(P_1+P_2-P_{1e}-P_{2e})}{(P_1^2-m_{1i}^2)\,(P_2^2-m_{2i}^2)}\Big\}\Big]\\&\times a^*_{ei}(s,\,u_0',\,t_0')\,a_{io}(s,\,u_0'',\,t_0'')\\&+\sum_j\int b^*_{ej}(s,\,M_j^2)\,b_{jo}(s,\,M_j^2)\,Ab\Big[\frac{1}{(P_1^2-M_j^2)}d^4P_1\delta(P_1-P_{1e}-P_{1o})\Big]\\&+AbA''_{eo}(s,\,t). \qquad (28)\end{aligned}$$

The first term is calculable if we have sufficient knowledge about a_{ei} and a_{io}. For the direct term the pair t_0' and t_0'' or u_0' and u_0'' corresponds to m_2 and m_4 in Fig. 2. For the cross term the pair t_0' and u_0'' or u_0' and t_0'' also corresponds to m_2 and m_4 if we interchange t and u.

The second term gives pole at $s=M_j^2$.

The third term which arises from many-particle intermediate state is also calculable in principle. But in practice they very often have only normal threshold, because their total mass is comparatively large.

From the formula (28) we can decide whether anomalous threshold exists or not. For instance, in case of π-meson-nucleon scattering or nucleon-nucleon scattering, if minimum of t_0', etc., is positive, thresholds are normal.

§ 5. Practical application to the problem of π-meson deuteron scattering

It is enough to consider only those absorptive parts which arise from two-nucleon intermediate state.

Other many-particle intermediate states, for instance, π-meson and deuteron or two nucleons and one meson give only normal singularities because their total masses are as large as the threshold of physical region.[12]

Absorptive part arising from two-nucleon intermediate state is related to the scattering amplitude of π-meson and deuteron to two nucleons. Minimum t_0' or u_0' is m_N^2, single pole, and the branch point is at t_0' (or u_0') $=(m_N+m_\pi)^2-\delta_t > m_N^2$ ($\delta_s=0.001\times 2m^2$, $\delta_t=0.85\times 2m\mu$). The value δ_t is fixed by the same method as the following proof for δ_s. This branch point (and of course larger t_0') does not matter with the anomalous singularities.

Thus a line singularity and two curve singularities are obtained from the integration of the momenta of intermediate two-nucleon state.

Other s-singularities occur from a_{ei}, etc. They are decided from analysing the amplitude of π-meson and deuteron to two nucleons. We obtain, besides normal singularities, only one anomalous line singularity $4m_N^2-\delta_s$ which agrees with the above one.

Proof:

We now take the l-th order scattering amplitude A_l of π and D to $2N$ in the perturbational sense, and write (m and n mean the order of amplitudes, $\pi+D$ to $2N$ and $2N$ to $2N$ respectively),

$$Ab\,A_l=\sum_{m+n=l}\iint (a^*_{\pi D,2N})_m\,(a_{2N,2N})_n\,Ab\left\{F\cdot\frac{i}{(2\pi)^4}\right.$$
$$\left.\times\frac{d^4P_1\,d^4P_2\,\delta(P_1+P_2-P_{1e}-P_{2e})}{(P_1^2-m_N^2)\,(P_2^2-m_N^2)}\right\}+Ab\,A_l'. \qquad (29)$$

$Ab\,A_l'$ which does not contain two-nucleon absorptive part is normal. We can neglect this term.

From the integration of intermediate state of the first term, we again obtain two line singularities, $s=4m_N^2$ and $4m_N^2-\delta_s$.

If $(a^*_{\pi D,2N})_m$ or $(a_{2N,2N})_n$ is singular with s, we may further analyse the m-th (or n-th) order matrix element. In fact $(a_{2N,2N})_n$ does not give anomalous singularity, and the analysis of $(a^*_{\pi D,2N})_m$ is quite the same as above. So it is sufficient to analyse with the lowest l at which the amplitude is singular with s. This also gives the same line singularity.

§ 6. Calculation of the absorptive part of scattering diagram by parameter integration

We restrict external masses M and μ, which correspond to deuteron and π-meson, and internal masses $m_N=m$. The diagram is shown in Fig. 12. We first transform the ordinary parameters α_i to (a, ξ, η) by (30),

$$\begin{aligned}\alpha_1&=(1-a)(1-\eta),\\ \alpha_2&=a\xi,\\ \alpha_3&=(1-a)\eta,\\ \alpha_4&=a(1-\xi).\end{aligned} \tag{30}$$

By the usual method we obtain

$$\mathrm{Im}\,A=\mathrm{Im}\left[-\frac{g^4}{16\pi^2}\int_0^1\int_0^1\int_0^1\frac{a(1-a)\,da\,d\xi\,d\eta}{(f-i\epsilon)^2}\right] \tag{31}$$

$$=\frac{g^4}{16\pi}\int_0^1\int_0^1\int_0^1\frac{d\delta(f+\epsilon)}{d\epsilon}\cdot a(1-a)\,da\,d\xi\,d\eta, \tag{32}$$

$$\begin{aligned}f=&m^2-a(1-a)(1-\eta)M^2-a(1-a)\eta\mu^2\\&-a^2\xi(1-\xi)t-(1-a)^2\eta(1-\eta)s.\end{aligned} \tag{33}$$

We assume $t<0$, and perform the integration of η.

$$\mathrm{Im}\,A=\frac{g^4}{8\pi}\iint a\,da\,d\xi\frac{d}{d\epsilon}\left(\frac{1}{\sqrt{\Delta-4s\epsilon}}\right), \tag{34}$$

$$\begin{aligned}\Delta=&((1-a)s+a(\mu^2-M^2))^2\\&-4s(m^2-a(1-a)M^2-a^2\xi(1-\xi)t).\end{aligned} \tag{35}$$

The range of a and ξ's integration is restricted by

$$\begin{aligned}&0\leq a\leq 1,\qquad 0\leq\xi\leq 1,\\&\Delta-4s\epsilon\geq 0.\end{aligned} \tag{36}$$

Case 1) $2(M^2+\mu^2)>s\geq 4m^2$

$$0\leq\xi\leq 1,\quad 0\leq a\leq a_0,\quad \Delta(a_0)=0. \tag{37}$$

$$\mathrm{Im}\,A=-\frac{g^4}{8\pi}\int_0^1\frac{d\xi\cdot 2\sqrt{s}\sqrt{s-4m^2}}{(-s+\mu^2+M^2)^2-4(s-4m^2)((s+M^2+\mu^2)^2-4sM^2+4s\xi(1-\xi)t)}$$

$$=\frac{g^4}{16\pi}\frac{\sqrt{s(s-4m^2)}}{\sqrt{\alpha(\alpha-4\beta)}}\left\{2i\tan^{-1}\frac{\sqrt{\alpha(\alpha-4\beta)}}{\alpha-2\beta}\right\},\tag{38}$$

$\alpha=st(s-4m^2)$,

$\beta=m^2s^2-2sm^2(M^2+\mu^2)+M^2\mu^2s+(M^2-\mu^2)^2m^2$.

Case 2) $4m^2>s\geq s_{a3}$

The range of a and ξ's integration is (shown in Fig. 13)

$$a_0'\leq a\leq a_0,\qquad 0\leq\xi\leq 1,$$

$$\Delta(a_0)=\Delta(a_0')=0.\tag{39}$$

$$\mathrm{Im}\,A=\frac{g^4}{8\pi}\frac{d}{d\epsilon}\int_0^1\frac{d\xi\cdot\pi s(-s+\mu^2+M^2)}{(4sM^2-4s\xi(1-\xi)t-(s+M^2-\mu^2))^{3/2}}$$

$$=\frac{g^4}{8\pi}\frac{d}{d\epsilon}\int_0^1 d\xi\,g(\xi)=0.\tag{40}$$

Case 3) $s_{a3}>s>s_{a1}=s_{a2}$

The range of ξ differs from Case 2), which is

$$0\leq\xi\leq\xi_-\quad\text{and}\quad\xi_+\leq\xi\leq 1,$$

$$\xi_\pm=\frac{1}{2}\pm\frac{\sqrt{(s-4m^2-4\epsilon)st\{(s-4m^2-4\epsilon)ts-4D\}}}{2st(s-4m^2-4\epsilon)},\tag{41}$$

$$D=m^2s^2-2(M^2+\mu^2)m^2s+M^2\mu^2s+(M^2-\mu^2)^2m^2$$
$$+\epsilon(s^2-2(M^2+\mu^2)s+(M^2-\mu^2)^2).\tag{42}$$

$$\mathrm{Im}\,A=\frac{g^4}{8\pi}\frac{d}{d\epsilon}\left\{\int_0^{\xi_-}d\xi\,g(\xi)+\int_{\xi_+}^1 d\xi\,g(\xi)\right\}=\frac{g^4}{8\pi}\left\{g(\xi_-)\frac{d\xi_-}{d\epsilon}-g(\xi_+)\frac{d\xi_+}{d\epsilon}\right\}$$

$$=g^4/4\sqrt{-t\{st(s-4m^2)-4s^2m^2+8sm^2(M^2+\mu^2)-4M^2\mu^2s-4m^2(M^2-\mu^2)^2\}}$$

$$=g^4/32\pi\sqrt{s}\,p_{1e}^2p_1\sqrt{K}.\tag{43}$$

Curve singularities are shown in Figs. 14 and 15. Fig. 15 corresponds to the diagram of Fig. 16.

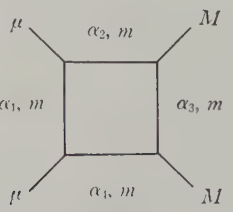


Fig. 12

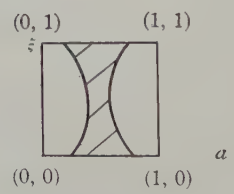


Fig. 13

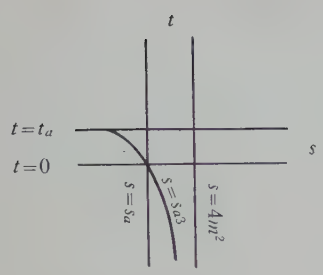


Fig. 14

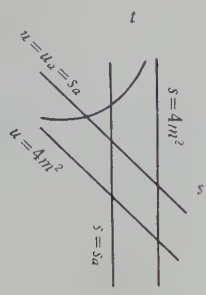


Fig. 15

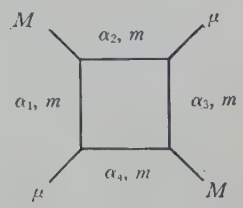


Fig. 16

Acknowledgments

The author is indebted to many colleagues for helpful discussions, and in particular to Dr. H. Miyazawa.

The author is also grateful for the deep interest and kind advices of Dr. N. Nakanishi.

Note added in Proof : After submission of this paper, the author received a paper containing similar results of those of § 3 in this paper by a different method of analytic continuation, but the singularity s_{a3} of this paper is not considered : R. E. Cutkosky, Journ. Math. Phys. **1** (1960), 429.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica tipe) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

Non-Adiabatic Effects on the Bound States of the μ-Mesonic Proton-Deuteron System*

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and

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November 2, 1960

In catalytic nuclear fusion reaction processes by μ^--mesons at liquid hydrogen temperature, as already pointed out,[1)-4)] the existence of the bound states of the μ^--mesonic proton-deuteron system plays an important role. Apart from the practical problem, it is a purpose of our research note to reveal the reality of the bound states of the present system by the perturbational treatment from the non-adiabatic point of view due to the largeness of the meson mass compared with the electron mass.

From this standpoint, the total Hamiltonian of a three-particle system composed of a muon and two nuclei with mass M_1 and M_2 can be expressed as follows in μ-mesic atom units:

$$
\begin{aligned}
H=&-\frac{1}{2}\Delta_{\alpha\beta\gamma}+U(\alpha,\beta,\gamma,\eta_0)\\
&+\kappa U^{(1)}-\frac{1}{2(M_1+M_2+1)}\Delta_{XYZ}\\
&+\kappa^2\left(-\frac{1}{2}\bar{\mu}\frac{\partial^2}{\partial\zeta^2}+U^{(2)}\right)\\
&+\kappa^3\left(-\frac{1}{2}\frac{2\bar{\mu}}{\eta_0}\frac{\partial}{\partial\zeta}+U^{(3)}\right)\\
&+\kappa^4\Bigg[-\frac{1}{2}\Delta_{\alpha\beta\gamma}-\frac{1}{2}\frac{\bar{\mu}}{\eta_0^2}\\
&\times\left\{\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}-iL_\alpha\right)\sin\theta\left(\frac{\partial}{\partial\theta}-iL_\alpha\right)\right.\\
&\left.+\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\phi}-iL_\beta\sin\theta-iL_\gamma\cos\theta\right)^2\right\}\\
&+\frac{1}{2}\frac{2\bar{\mu}}{\eta_0^2}\zeta\frac{\partial}{\partial\zeta}+U^{(4)}\Bigg],
\end{aligned}
\tag{1}
$$

where Cartesian coordinates (α, β, γ) of a light particle are referred to the coordinate system fixed on the two heavy particles with equilibrium internuclear distance η_0, and $\kappa=\sqrt[4]{1/(M_1+M_2)}$, $\bar{\mu}=(M_1+M_2)^2/M_1M_2$, $\eta=\eta_0+\kappa\zeta$, $\kappa\zeta$ being the displacement coordinates from the equilibrium position. $U^{(i)}$ is κ^i term of the Coulomb interaction between light and heavy particles. Besides, $\Delta_{\alpha\beta\gamma}$ and Δ_{XYZ} represent Laplacians with respect to coordinates of a light particle and

* Read at the annual meeting of the Physical Society of Japan (1957).

the center of mass of the total system, respectively. Further the polar angles θ and ϕ are referred to the direction of the internuclear axis; L_α, L_β and L_γ mean components of total orbital angular momentum referred to this heavy-particle coordinate-system.

Now if the wave function of the total system can be written

$$\Psi=\psi_\mu(\alpha,\beta,\gamma;X_0,Y_0,Z_0;\eta_0+\kappa\zeta)\times\sigma_{\mu v}(\zeta)\varphi_{\mu vr}(\theta,\phi,\eta_0)\chi_{kin}(X,Y,Z), \tag{2}$$

where X_0, Y_0, Z_0 are the center of mass coordinates of heavy particles only, μ, v and r being light particle state, vibrational, and rotational quantum numbers, respectively, the wave function of a light particle satisfies the following wave equation:

$$\left[-\frac{1}{2}\Delta_{\alpha\beta\gamma}+U(\alpha,\beta,\gamma,\eta_0)+\sum_{i=1}^{\infty}\kappa^i U^{(i)}\right]\psi_\mu=\left(W_\mu^{(0)}+\sum_{i=2}^{\infty}\kappa^i W_\mu^{(1)}\right)\psi_\mu. \tag{3}$$

Then the wave equation with respect to the heavy particles is given by

$$\left[\kappa^2\left\{-\frac{1}{2}\bar{\mu}\frac{\partial^2}{\partial\zeta^2}+W_\mu^{(2)}\right\}\right. \tag{4a}$$

$$+\kappa^4\iiint\tilde{\psi}_\mu(\eta)\left\{-\frac{1}{2}\bar{\mu}\frac{\partial^2}{\partial\eta^2}\right\}\times\psi_\mu(\eta)\,d\alpha\,d\beta\,d\gamma+\kappa^4\frac{1}{2}\frac{\bar{\mu}}{\eta^2}$$

$$\left.\times\iiint\tilde{\psi}_\mu(\eta)\{L_\alpha+L_\beta\}\psi_\mu(\eta)\,d\alpha\,d\beta\,d\gamma\right\} \tag{4b}$$

$$+\kappa^4\iiint\tilde{\psi}_\mu(\eta)\left\{-\frac{1}{2}\Delta_{\alpha\beta\gamma}\right\}\times\psi_\mu(\eta)\,d\alpha\,d\beta\,d\gamma$$

$$+\kappa^3\left\{-\frac{1}{2}\frac{2\bar{\mu}}{\eta_0^2}\frac{\partial}{\partial\zeta}\right\}+\kappa^4\left\{\frac{1}{2}\frac{2\bar{\mu}}{\eta_0^2}\zeta\frac{\partial}{\partial\zeta}\right\}+\cdots+\kappa^3W_\mu^{(3)}+\kappa^4W_\mu^{(4)}+\cdots \tag{4c}$$

$$+\kappa^4\left\{-\frac{1}{2}\frac{\bar{\mu}}{\eta_0^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}-i\Lambda\cos\theta\right)^2\right]\right\} \tag{4d}$$

$$\left.+\frac{1}{2(M_1+M_2+1)}\Delta_{XYZ}\right]\sigma_{\mu v}\varphi_{\mu vr}\chi_{kin}=(E-W_\mu^{(0)})\sigma_{\mu v}\varphi_{\mu vr}\chi_{kin},$$

where Λ is the quantum number of the component of the orbital angular momentum of a light particle along the internuclear axis, and U_{na} (4b) and U_{anh} (4c) represent non-adiabatic and anharmonic terms, respectively.

The contribution from $\Sigma_i\kappa^i W_\mu^{(i)}$ to energy levels is given by the expression reduced from the Hylleraas formula[8] to the muonic system, while the non-adiabatic term U_{na} can be calculated by using the James function[9]

$$\psi_\mu=c_1\exp(-\delta\lambda)+c_2\nu^2\exp(-\delta\lambda) \tag{5}$$

for a μ-meson, where $\lambda=(r_1+r_2)/\eta$ and $\nu=(r_1-r_2)/\eta$ ($\eta=\eta_0+\kappa\zeta$, $\eta_0=2.00\mu$.a.u.)

Table I. Eigenvalues (in ev) for various bound states

$v=0$	$(pp)_\mu$	$(pd)_\mu$	$(dd)_\mu$
$J=0$	2742 (2771)	2886 (2878)	2948 (2986), 2840* (2845)
$J=1$	2571 (2623)	2742 (2754)	2853 (2887)
$J=2$			2706 (2746)

* This state corresponds only to $v=1$ ($J=0$).

represent elliptic coordinates, and δ and $c_i (i=1, 2)$ are variational parameters. This wave function can give the eigenvalues which is very close to the values obtained by Bates et al.[10] by variating δ and c_i as functions of ζ.

In this way we get the vibrational energy levels in the approximation up to the order of κ^{10}, and also rotational levels by taking into account the change of moment of inertia due to the vibration and non-rigidity of the nuclear system. As the results, energy level structures of μ-mesic systems are shown in the table in comparison with the values in parentheses calculated by variational treatments.[11] All other results[5)-7)] are not so accurate as to be able to compare with ours.

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Excited State in Λ-π System and K^--p Collision

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November 11, 1960

It seems that the existence of an excited state Λ^* in Λ-π system* has recently been established by measurement of both π^+ and π^- momentum spectra in the reaction $K^- + p \to \Lambda^0 + \pi^+ + \pi^-$ at 1.15 Bev/c and that the mass of Λ^* is of about (1370$\sim$1380) Mev.[1)] Then the other K^--meson reactions will be affected by the existence of Λ^*. Since the mass of Λ^* is nearly equal to the energy of 1430 Mev which corresponds to the lowest energy of K^--p scattering, it may be expected that the Λ^* plays an important role in K^--p collision at low energy.

According to the experimental results,[2)] the values of $\sigma(K^-\text{-}p)$ in low energy region are much larger than those of $\sigma(K^+\text{-}p)$. Then we are inclined to regard the largeness of $\sigma(K^-\text{-}p)$ as an effect of Λ^*. But it is pointed out in this note that such an interpretation as this may not be appropriate. If the Λ^* is mainly responsible to the large cross section for K^--p collision and if there is no other excited state, the isotopic spin $I=1$ state must play the most important role. We denote the reaction amplitudes for isotopic spin $I=0$ and $I=1$ states in $K^- + p \to \Sigma + \pi$

* We use the notation Λ^* in order to discriminate this excited state in Λ-π system from an excited state Σ^* in Σ-π system.

as A_0 and A_1 respectively, and the reaction amplitude for $I=1$ state in $K^-+p\to\Lambda^0+\pi^0$ as A_1', then the magnitude of $|A_1'|$ should of course be much larger than that of $|A_0|$ and should not be smaller than that of $|A_1|$, because the Λ^* is an excited state in Λ-π system.

$$|A_1'|\gg|A_0|,\quad |A_1'|\geq|A_1|. \qquad (1)$$

Cross sections for K^--p collision are expressed as follows:

$$\sigma(K^-+p\to\Sigma^\pm+\pi^\mp)=(1/2)\,|A_1|^2 + (1/3)\,|A_0|^2,$$

$$\sigma(K^-+p\to\Sigma^0+\pi^0)=(1/6)\,|A_0|^2,$$

$$\text{and } \sigma(K^-+p\to\Lambda^0+\pi^0)=(1/2)\,|A_1'|^2. \qquad (2)$$

Now it must be noted that the experimental results[3] for the absorption of K^- by proton are

$$\Sigma^-:\Sigma^+:\Sigma^0:\Lambda^0=4:2:2:1/2. \qquad (3)$$

Considering these results* together with those of (1) and (2), we may say that the large cross section for K^--p collision at low energy may not be regarded as an effect due to the existence of Λ^*.

It is said from recent experimental data for $K^-+\mathrm{He}^4$ reaction that there may also exist an excited state Σ^* in Σ-π system and that the mass of Σ^* is of about 1440 Mev.** With regard to reliability for this experimental result we have not any information. But, if this is confirmed, the mass difference between Σ^* and Λ^* is nearly equal to that between Σ and Λ. Moreover the excitation energies both of Σ^* and of Λ^* are about 120 Mev and are approximately equal to that of the well-known (3/2, 3/2)-state N^* in π-N system. This may be regarded as one of favorable results for the Global symmetry. As will be discussed below, however, such an interpretation as this may not necessarily be correct. If Σ^* and Λ^* correspond to N^* in π-N system, spin of those particles must be equal to 3/2. The mass of Σ^* corresponds nearly to the total energy of Thomson Limit for K^--p scattering. If the Σ^* is responsible to the largeness of $\sigma(K^-$-$p)$ in low energy region, spin of Σ^* must be equal to 1/2 because the angular distributions both for K^--p scattering and for K^--p reactions at low energy are isotropic.[2] Therefore the existence of Σ^* may not necessarily be regarded as an evidence of Global symmetry if Σ^* plays the most important role in K^--p collision at low energy.*

In order to re-examine the existence of Λ^*-state and to determine the value of its spin we want to have the reliable experimental data for $\pi^-+p\to\Lambda^0+\pi^-+K^+$ reaction at high energy. The momentum spectrum of K^+ in this reaction ought to have a strong peak in the total center-of-mass system. And the angular distribution of π^- in the Λ-π system will give us some knowledge about spin of Λ^*.

---

* If these results are modified remarkably, our conclusion may turn out to be invalid.

** We heard from Dr. S. Iwao that such results have been obtained by preliminary experiment at Syracuse.

* If Σ^* has no part in the largeness of $\sigma(K^-$-$p)$ at low energy, the assumption of Global symmetry is not inconsistent with the experimental results.

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The Effect of Pion-Pion Interaction to the Pion-Nucleon Small *P*-Wave Phase Shifts

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November 15, 1960

Chew, Goldberger, Low and Nambu (CGLN)[1] have neglected the effect of pion-pion interaction in their dispersion-theoretic treatment of the pion-nucleon scattering.[2),3)] However, recently, Bowcock, Cottingham and Lurie have shown that the effect of pion-pion *P*-wave interaction well explains the relative *P*-wave phase shifts, $\alpha_{11}-\alpha_{31}$, $\alpha_{13}-\alpha_{31}$.[4),5)] The author and his co-worker have shown that the effect of pion-pion *S*-wave interaction can explain the behavior of α_{13} which becomes positive at intermediate energy.[6)] These have not been explained within the CGLN theory. In this letter, we will show that the contributions from *S*- and *P*-wave pion-pion interactions can well explain the behavior of all the small *P*-wave phase shifts. The pion-pion interaction will contribute to the pion-nucleon scattering amplitudes $A^{(\pm)}$ and $B^{(\pm)}$ in the CGLN's representation by the amount of $\delta A^{(\pm)}$ and $\delta B^{(\pm)}$, which are written as[4),6),7)]

$$\delta A^{(+)}=A^{(+)}_{\pi-\pi(s)},\quad \delta A^{(-)}=A^{(-)}_{\pi-\pi(p)},$$
$$\delta B^{(+)}=0 \text{ and } \delta B^{(-)}=B^{(-)}_{\pi-\pi(p)}, \quad (1)$$

where the subscripts π-$\pi(s)$ and π-$\pi(p)$ represent the *S*- and *P*-wave pion-pion interactions, respectively. Following the procedure of CGLN, one can write the contributions to the *P*-wave pion-nucleon phase shifts, in the low energy limit, as

$$\delta(\alpha_{33}/q^3)=a-\epsilon,\quad \delta(\alpha_{13}/q^3)=a+2\epsilon,$$
$$\delta(\alpha_{31}/q^3)=a-(b+\epsilon),$$
$$\delta(\alpha_{11}/q^3)=a+2(b+\epsilon),$$
$$a\equiv\frac{2}{3}\frac{1}{1+1/M}A^{(+)\prime}_{\pi-\pi(s)}/4\pi, \quad (2)$$
$$b\equiv\frac{1}{2M}B^{(-)}_{\pi-\pi(p)}/4\pi,$$
$$\epsilon\equiv\frac{2}{3}\frac{1}{1+1/M}\frac{(A^{(-)\prime}_{\pi-\pi(p)}+B^{(-)\prime}_{\pi-\pi(p)})}{4\pi},$$

where the superscript $'$ represents the derivative with respect to $t(\equiv-4\kappa^2)$ at $t=0$. Now, CGLN's expressions for the small *P*-wave phase shifts predict that

$$\alpha^{(s)}_{13}=\alpha^{(s)}_{31}\simeq-0.04q^3,\quad \alpha^{(s)}_{11}\simeq-0.11q^3, \quad (3)$$

for $f^2\simeq0.08$ in the static approximation. Then, if we choose a, b and ϵ in (2) as

$$a\simeq0.05,\quad b\simeq0.025,\quad \epsilon\simeq0, \quad (4)$$

the pion-pion contributions change the small *P*-wave phase shifts of CGLN (3) for

* Leave from the Faculty of Liberal Arts and Science, Yamagata University, Yamagata.

$$\alpha_{13} \cong +0.01\,q^3, \quad \alpha_{31} \cong -0.015\,q^3,$$
$$\alpha_{11} \cong -0.01\,q^3. \qquad (5)$$

These are all consistent with Chiu-Lomon's analysis of the experiments, which shows

$$\alpha_{13} = +0.016\,q^3, \quad \alpha_{31} = -0.016\,q^3,$$
$$\alpha_{11} = -0.014\,q^3. \text{ [8]} \qquad (6)$$

Here, it will be worthwhile to stress that the very small ϵ is also the theoretical consequence. Really, we have derived that the ϵ becomes zero when one uses the static approximation on the course of the calculation of the pion-pion effect.[6] As has been made by Bowcock et al., the relative phase shifts $\alpha_{11}-\alpha_{31}$, $\alpha_{13}-\alpha_{31}$ can be explained by the contribution from P-wave pion-pion interaction, but the phase shifts themselves will not be explained without a coming from S-wave pion-pion interaction. Especially, the positive α_{13}, at least, in the intermediate energy will be well explained by introducing a, then, from (2), the other phase shifts become consistent with the experiment.

We then have tested, *without the static approximation* but leaving the assumption of the 3–3 state dominance in the dispersion integrals, whether the CGLN's relations may be consistent with the experiment (7) at low energy limit or not, when the pion-pion interaction is neglected. The conclusion is following: The CGLN's relations which describe $f^{(+)}_{p1/2}-f^{(+)}_{p3/2}$, $f^{(-)}_{p3/2}$, $f^{(+)}_{p3/2}$ and $f^{(-)}_{p1/2}-f^{(-)}_{p3/2}$ become in the order named more inconsistent with the experiment. This is seen from the calculations of the coupling constant f^2 by using these relations. The obtained f^2 are in the order 0.068, 0.060, 0.140 and 0.215, respectively. Only the first one which essentially comes from the dispersion relation of $B^{(+)}$ is close to the usual $f^2=0.08$, which is consistent with $\delta B^{(+)}=0$ in (1). On using $f^2=0.068$, the a, b and ε in (2) are calculated so as to be consistent with the experiment as

$$a=0.048, \quad b=0.022,$$
$$\epsilon \cong 0.003\,(\pm 0.01). \qquad (7)$$

These are close to (4), thus our previous conclusion will not be changed. Bowcock et al. have obtained $b \cong 0.03$ and $\epsilon \cong -0.007$ which are consistent with (7).

The contribution to α_{33} from the mainly S-wave pion-pion interaction might exist from (2). Really, if one would neglect this contribution, the coupling constant $f^2=0.10$ would be necessary in order to be consistent with the experimental α_{33} at low energy limit.

Finally, we mention that the similar calculation with the use of the data reported at 1958 Annual International Conference on High Energy Physics at CERN gives

$$a=0.028\pm 0.015, \quad b=0.041\pm 0.027,$$
$$\epsilon=-0.014\pm 0.02 \text{ and } f^2=0.072\pm 0.01.$$

The author wishes to thank Prof. I. Sato and Prof. A. Takahashi for their kind interest.

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Validity of the Integral Representations for the Vertex Part in PerturbationTheory

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November 17, 1960

Deser, Gilbert and Sudarshan,[1] and Ida[2] independently derived an integral representation for the vertex part as a function of two mass variables. But, recently, Oehme, Araki and Symanzik,[3] and Minguzzi and Streater[4] have pointed out that Dyson's integral representation for the vacuum expectation value of a double commutator[5] is not correct in general, therefore the Deser-Gilbert-Sudarshan-Ida integral representation is not justified since their derivations are based on the above Dyson' integral representation. In the present note, we shall point out that the DGSI integral representation is valid in every order of perturbation theory.

We denote the three external lines of the vertex part by A, B and C, and the corresponding mass variables by s, t and u. We set u on the mass shell $u=M^2$. Then the DGSI integral representation is written as

$$M(s,t)=\int_0^1 dz \int_0^\infty d\alpha \frac{\varphi(z,\alpha)}{\alpha-zs-(1-z)t-i\varepsilon}. \tag{1}$$

On the other hand, when all particles are scalar, the general term of perturbation theory is given by the following parametric integral :[6]

$$M_G(s,t) = \text{const.} \int_0^1 \cdots \int_0^1 \frac{\delta\left(1-\sum_{i=1}^N x_i\right) \prod_{i=1}^N dx_i}{U^2 \cdot (V-i\varepsilon)^{N-2n}} \tag{2}$$

with

$$V \equiv \sum_{i=1}^N x_i m_i^2 - (\eta_1 s+\eta_2 t+\eta_3 u). \tag{3}$$

Here, m_i is an internal mass, and U and η_j are *non-negative definite*[6),7)] functions of parameters x_i. We multiply (2) by

$$1=\int_0^1 dz \cdot \delta\left(z-\frac{\eta_1}{\eta_1+\eta_2}\right) \int_{-\infty}^\infty d\alpha \times \delta\left(\alpha-\frac{\sum_i x_i m_i^2-\eta_3 u}{\eta_1+\eta_2}\right), \tag{4}$$

and $N-2n-1$ times integrate by parts with respect to α after V is rewritten in terms of z and α. We can thus see that $M_G(s,t)$ can be represented as

$$M_G(s,t)=\int_0^1 dz \int_{-\infty}^\infty d\alpha \times \frac{\varphi_G(z,\alpha)}{\alpha-zs-(1-z)t-i\varepsilon}, \tag{5}$$

provided that convergence condition is satisfied.

Now, the most important problem is

the restriction for the carrier of $\varphi_G(z, \alpha)$. We denote the lowest thresholds of the self-energies of three external particles A, B and C by M_A, M_B and M_C, respectively. M_A, M_B and M_C satisfy the triangle inequalities if we require the stability conditions. Then we can rigorously prove, by using the method presented in our previous work,[7] that

$$\sum_i x_i m_i^2 - \{c^2\eta_1 + (1-c)^2\eta_2 + \eta_3\} \times M_C^2 \geqq 0, \quad (6)$$

where c is an arbitrary constant but must satisfy

$$M_A/M_C \geqq c \geqq 1 - M_B/M_C. \quad (7)$$

Therefore, from (4) and (6) with $u = M^2 \leqq M_C^2$, we obtain

$$\alpha \geqq [c^2 z + (1-c)^2 (1-z)] M_C^2. \quad (8)$$

The right-hand side of (8) becomes the maximum at either $c = M_A/M_C$ or $c = 1 - M_B/M_C$ in the domain (7). We thus obtain as the carrier of $\varphi_G(z, \alpha)$ that

$$\alpha \geqq \mathrm{Max}[M_A^2 z + (M_C - M_A)^2 (1-z), \; (M_C - M_B)^2 z + M_B^2 (1-z)] \geqq 0, \quad (9)^*$$

which exactly coincides with the carrier derived by DGS and Ida.

Finally, we remark that the following integral representation for the vertex part as a function of three mass variables likewise holds in every order of perturbation theory:

$$M_G(s, t, u) = \int_0^1 \int_0^1 \int_0^1 \delta(1 - \sum_{j=1}^3 z_j) \, dz_1 dz_2 dz_3 \int_0^\infty d\alpha \times \frac{\varphi_G(z_1, z_2, z_3, \alpha)}{\alpha - z_1 s - z_2 t - z_3 u - i\varepsilon} \quad (10)$$

with

$$\alpha \geqq \mathrm{Max}[M_A^2 z_1 + M_B^2 z_2 + (M_A - M_B)^2 z_3, \; M_A^2 z_1 + (M_A - M_C)^2 z_2 + M_C^2 z_3, \; (M_B - M_C)^2 z_1 + M_B^2 z_2 + M_C^2 z_3]. \quad (11)$$

Detailed account will be reported in a Supplement of this journal.

The author is indebted to Dr. H. Araki for a helpful suggestion in the proof of (6).

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* This carrier is realized already in the non-trivial lowest order, hence no further improvement is possible.

Motion of a Charged Particle in a Static Magnetic Field with Axial Symmetry*

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The motion of charged particles in a

* The essential points of this work have been read at the Physical Society Meetings in April and October, 1960, by S. Fukui, S. Hayakawa, H. Nishimura and H. Obayashi as part of their experimental work with a mirror device.

given electromagnetic field has been investigated concerning various problems, such as the behavior of cosmic-ray particles, particle orbits in magnetic machines for fusion or acceleration devices. In many cases the investigation is due to enormous numerical calculations, so that the characteristic features of the problem are not too clear. Here we intend to see how the characteristic behavior of the particle is derived from the general dynamical treatment. As a simple and useful case we consider a particle in an axially symmetric static magnetic field.

In a cylindrical coordinate system (r, θ, z) with the axis as z direction, the magnetic field $\boldsymbol{H}=(H_r, 0, H_z)$ is expressed with a vector potential $\boldsymbol{A}=(0, A_\theta, 0)=(0, r^{-1}\psi(r, z), 0)$ as $\boldsymbol{H}=\mathrm{rot}\,\boldsymbol{A}=(-r^{-1}\partial\psi/\partial z, 0, r^{-1}\partial\psi/\partial r)$. Here $\psi(r, z)$ is a flux function with which the equation for a magnetic line of force is written $\psi=\mathrm{const.}$

The motion of a particle with charge Ze and mass m is described by the Hamiltonian as

$$\mathcal{H}=c[m^2c^2+\pi_r^2+(1/r^2)\times\{\pi_\theta-(Ze/c)\psi\}^2+\pi_z]^{1/2}, \quad (1)$$

where π_r, π_θ, and π_z are canonical momenta conjugate to r, θ, and z.

In our problem the particle motion should be expected to consist of three characteristic parts:[1] (i) rapid gyration in the plane normal to $\boldsymbol{H}$, (ii) longitudinal and rather fast drift along $\boldsymbol{H}$, and (iii) perpendicular and slow drift around the axis. Therefore we transform the variables to more suitable ones. Now because of the non-existence of external current, our magnetic field has a potential function $s(r, z)$ such as $\nabla s=\boldsymbol{H}$. The set of variables (ψ, θ, s) makes an orthogonal coordinate system as well as (r, θ, z), and is suitable for the discrimination of (ii) from (i) and (iii). To make further discrimination between (i) and (iii), we introduce a canonical transformation $(r, \theta, z; \pi_r, \pi_B, \pi_z)\rightarrow(Q_1, Q_2, Q_3; P_1, P_2, P_3)$, which is similar to Gardner's.[2] The new variables are expressed as

$$\begin{aligned}
P_1&=(c/Ze)\,(1/rH^2)\,(\pi_r H_z-\pi_z H_r),\\
Q_1&=(Ze/c)\psi-P_3,\\
P_2&=(c/Ze)\,(1/H^2)\,(\pi_r H_r+\pi_z H_z),\\
Q_2&=(Ze/c)\,s, \qquad (2)\\
P_3&=\pi_\theta, \quad Q_3=\theta-P_1,
\end{aligned}$$

the Hamiltonian becomes

$$\mathcal{H}=c[m^2c^2+(Q_1/r)^2+(Ze/c)^2\,(rHP_1)^2+(Ze/c)^2\,(HP_2)^2]^{1/2}. \quad (3)$$

As $\mathcal{H}$ does not contain Q_3 because of axial symmetry, we have an angular momentum integral, $P_3=L$, in addition to an obvious constant of motion, the magnitude of total momentum p, according to the absence of electric field.

In (3) there still remains the coupled motion in (Q_1, P_1) and (Q_2, P_2). But if we confine ourselves to those regions near the axis which are of practical interest in problems of mirror devices, etc., the motion can be separated. In a current-free space an axially symmetric field as a solution of the Maxwell equation is given in power series of r.

$$\psi=r^2h(z)/2-r^4h''(z)/16+\cdots,$$

$$s=\int h(z)\,dz-r^2h'(z)/4+\cdots, \quad (4)$$

$$H=h(z)+r^2\{h'(z)^2 -2h(z)h''(z)\}/8h(z)+\cdots,$$

where $h(z)$ denotes the field distribution on the axis. If we take the lowest order approximation, $r^2H=2\psi=2(c/Ze)(Q_1+P_3)=2(c/Ze)(Q_1+L)$ and $H=h=H(Q_2)$. Thus (3) reduces to

$$p^2=\frac{Ze}{c}H(Q_2)\left\{\frac{Q_1^2}{2(Q_1+L)} +2(Q_1+L)P_1^2\right\} +\left(\frac{Ze}{c}\right)^2\{H(Q_2)P_2\}^2. \quad (5)$$

As variables (Q_1, P_1) appear only in the first bracket, we can separate it with a separation constant A.

$$\frac{Q_1^2}{2(Q_1+L)}+2(Q_1+L)P_1^2=A \quad (6)$$

$$p^2=\frac{Ze}{c}H(Q_2)A+\left(\frac{Ze}{c}\right)^2\{H(Q_2)P_2\}^2. \quad (7)$$

With the aid of the Hamilton-Jacobi equations which separately hold for two sets of the variables, the orbit of the particle is obtained:

$$Q_1=A+A^{1/2}(A+2L)^{1/2} \times\sin\left[\int dz\frac{Ze}{c}h(z) \times\left\{p^2-A\frac{Ze}{c}h(z)\right\}^{-1/2}+C\right]. \quad (8)$$

Q_1 oscillates about A, and this corresponds to the gyration (i), while Q_2 gives the longitudinal motion (ii) and Q_3 (iii). Putting $h(z)$ as quadratic form, we can compare our method with the results by Solovjev[3] who has solved the drift equations.

Now in our scheme the adiabatic invariants can be obtained straightforwardly. From (6) and (7) respectively we have

$$\oint P_1\,dQ_1=\pi A,$$

$$\oint P_2\,dQ_2=2\int_{z_1}^{z_2}\left\{p^2-A\frac{Ze}{c}h(z)\right\}^{1/2}dz, \quad (9)$$

where $h(z)$ is assumed to have the turning points at z_1 and z_2. These are the well-known invariants concerning the gyration and the longitudinal motions respectively.[4]

The author wishes to express his thanks to Professor S. Hayakawa for suggesting and discussing the problem. He is also indebted to Professor S. Fukui and Mr. H. Nishimura who are now working with a mirror field of the geometrical conditions satisfying (4).

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Very High Energy Nuclear Reactions and Alpha-Particle Model

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December 8, 1960

When a heavy primary cosmic-ray particle interacts with emulsion nuclei, the incident nucleus does not always break up completely. Rather, the heavy fragments or the α-particles often remain. This suggests that at very high energies (a few Gev per nucleon) the light nuclei behave as if they had some substructure. In the case of incident carbon or oxygen nuclei which are regarded as $4n$-nuclei, the appearance of high energy α-particles is more predominant.

We suppose that the alpha-particle model of the carbon and oxygen nuclei can explain this phenomenon. The incident nucleus will be stripped off one or more constituent alpha-particles during the collision, and the remainder will hit the target nucleus giving rise to pion production. A simple geometrical consideration gives us the probabilities $P(n)$ for stripping n alpha particles from an incident nucleus.

The fractional volume v of the incident nucleus, which hits the target will determine n, the number of emergent α-particles. In the case of incident carbon, for $1 \geqq v > 2/3$ no α-particles appear, for $2/3 \geqq v > 1/3$, one α-particle emerges and for $1/3 \geqq v > 0$, two α-particles emerge. The fractional volume v depends on the impact parameter ρ. Denoting the values of ρ by ρ_1 and ρ_2 for which $v=2/3$ and $1/3$ respectively, we obtain $P(0) \sim \pi\rho_1^2$, and $P(1) \sim \pi(\rho_2^2 - \rho_1^2)$. Because of the transparency of the nucleus for glancing collisions, the maximum impact parameter ρ_m for which the collisions occur is not r(carbon) + r(target) but r(carbon) + r(target) $- 2\Delta R$, where $r = 1.45 \times 10^{-13} \times A^{1/3}$ cm, (A, the mass number) and ΔR is taken to be 0.85×10^{-13} cm.[1] Then we obtain $P(2) \sim \pi(\rho_m^2 - \rho_2^2)$. The present model cannot be applied to the C-3α reaction which may be due to the Coulomb interaction with the target. Similar calculation is also possible for incident oxygen.

We shall confine ourselves to the case of heavy target nuclei in emulsion. Neglecting the curvature of the surface of a target nucleus in order to simplify the calculation, we get the relative values for $P(n)$, which are shown in the Table.

In the Table the experimental values of $P(n)$ are also given, and we deduced their values from the existing data[2)-4)] as follows. The elements contained in emulsion are classified into 3 groups, i.e. heavy group (Ag and Br), light group (C, N, O) and hydrogen group. For each group the collision probability P is proportional to the product of the collision cross section σ and the number density N of the target nuclei. The cross sections for collisions with the heavy and light groups are calculated by the semi-empirical formula $\sigma = \pi\rho_m^2$

Incident nucleus		no α emission	1α emission	2α emission	3α emission
C	calculated	50	30	20	—
	experimental	60	26	14	—
O	calculated	39	21	25	15
	experimental	53	24	16	7

of Bradt and Peters.[1] For hydrogen collisions the cross section is taken to be the geometrical one of the incident nucleus, i.e. $\sigma=\pi(1.45\times10^{-13}A^{1/3})^2$ cm^2. Then we get $P_{heavy}: P_{light}: P_{hydrogen}=57:28:15$ for incident carbon, and $55:29:16$ for incident oxygen. Out of 259 events for incident carbon available from the data, 97 events have $N_h\geqq8$, 98 events have $7\geqq N_h\geqq2$, and 64 events have $N_h=0$ or 1. Considering that all the events having $N_h\geqq8$ are collisions with heavy target nuclei and that the events due to the collisions with the hydrogen nuclei have $N_h=0$ or 1 and using the above figures, we estimated the total number of events due to collisions with heavy target nuclei.

In spite of our crude assumptions the agreement between the calculated and experimental values is fairly good. Therefore it seems reasonable to consider that alpha-particle model of $4n$-nuclei holds in the very high energy regions and that fast α-particle emission occurs in stripping reactions.

Finally, we shall mention the order of magnitude of the energy of the events concerned here. The geomagnetic cut-off energy at the locations where the emulsions were exposed is 1–1.5 Gev per nucleon. Of course the above-mentioned model will not hold for the extremely high energy events. In order to exclude them from the experimental data, we choose for our analysis only those events in which the number of produced charged pions per absorbed nucleon does not exceed one. Considering the high energy machine data and the energy spectrum of heavy primaries, the average energy of the events concerned is about 2–3 Gev per nucleon.

The authors would like to express their sincere thanks to Prof. S. Nakagawa for his continual encouragement.

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On the G_A/G_V Ratio in β-Decay. II

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December 10, 1960

Previously we investigated the G_A/G_V

ratio in β-decay from the standpoint of the composite model for the pion[1] and showed that it was possible to improve the result obtained by the meson theory $G_A/G_V<1$ if we considered the contributions from the lowest order process. However, it is necessary to take into account the contributions from the second order process at least in order to reproduce the experimental value $G_A/G_V=1.25$ or to compare the result on this model with one from the meson theory. Therefore here we calculate them and study the possibility of explaining this phenomenon on the basis of the composite model.

The following calculation and approximation are same as those used in the previous paper.[1] At first if we recognize that the weak V-A interaction has the universality and the vector coupling constant is not affected by the renormalization on the assumption of the conserved current,[2] we may calculate the G_A/G_A^0 ratio instead of the G_A/G_V ratio where $G_{V,A}$ and $G_{V,A}^0$ are the observed and bare coupling constants respectively. The fundamental interaction Hamiltonians are given by

$$H_w=\left\{\bar{\psi}\gamma_\mu(G_V^0+G_A^0\gamma_5)\frac{\tau_1+i\tau_2}{2}\psi\right\}\times\left(\bar{e}\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\nu\right)+\text{H. C.} \quad (1)$$

$$H_{st}=\sum_k\sum_{i=1}^{3}\frac{g_{1k}}{M^2}(\bar{\psi}O_k\tau_i\psi)(\bar{\psi}O_k\tau_i\psi)+\sum_k\frac{g_{0k}}{M^2}(\bar{\psi}O_k\psi)(\bar{\psi}O_k\psi) \quad (2)$$

where $O_s=1$ $O_v=i\gamma_\mu$

$$O_t=\frac{1}{2\sqrt{2}\,i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$$

$O_a=i\gamma_5\gamma_\mu$ $O_p=\gamma_5$

and the diagrams contributing to the G_A/G_A^0 ratio are illustrated in Fig. 1.

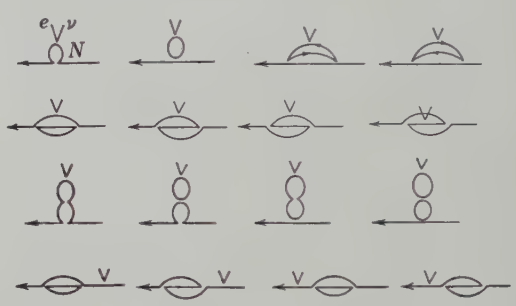


Fig. 1

As the coupling constants were already determined,[3] hence the effective pion-

Table I

g_{1p}	g_{0p}	g_{1s}	g_{0s}	g_{1t}	g_{0t}	G_A/G_A^0-1	
						cut off M	cut off 1.7 M
2.18	−38.15	2.18	−38.15	−9.17	−2.82	0.01	0.24

Table II

g_{1v}	g_{0v}	g_{1a}	g_{0a}	G_A/G_A^0-1	
				cut off 0.78 M	cut off M
7.5	−15	7.5	0	0.25	1.13

nucleon interaction, the anomalous magnetic moment and the charge distribution of the nucleon were consistent with the experiments, we use them and obtain the calculation values in the following Table I (s, t, p) and Table II (v, a). The ambiguity in the correction of electric charge[4] is eliminated in Case I (s, t, p) because $g_{1p}=g_{1s}$ and $g_{0p}=g_{0s}$, but in Case II (v, a) we do not consider it.

Thus it seems to us that it is possible to explain the G_A/G_V ratio from the same viewpoint that we accounted for the other phenomena consistently if we cut off the momentum integral adequately.

Previously we studied the pion-nucleon interaction on the composite model[5] where the Fermi coupling type was only pseudoscalar and the result obtained there was equivalent to the meson theory. Now if we adopt only pseudoscalar coupling, the G_A/G_A^0 ratio is less than 1 like the meson theory.

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Errata

On Hugenholtz-Pines' Theory about Ground State Energy of Interacting Bosons

Setsuo MISAWA

Prog. Theor. Phys. **24** (1960), 1224

After publication of this article, we have learned from Dr. K. Sawada that $\mu N'(N_0, \mu)$ must be added to Eq. (3·16) (being E_0' not E_0) in order to obtain E_0 and then Hugenholtz-Pines' method gives correct result.

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Unified Gravitational and Electromagnetic Waves

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(Received September 8, 1960)

Starting with a very general form of the nonsymmetric tensor g_{ik} expressed in a coordinate system suitably chosen to obtain wave solutions, a scheme is developed to derive rigorous solutions of the field equations of Einstein which describe the flow of unified gravitational and electromagnetic radiation. Several such solutions are derived giving waves with two dimensional symmetry. It is found that solutions describing gravitational and electromagnetic waves obtained in the general theory of relativity with the help of an energy momentum tensor, can be derived in exactly the same form from the geometrical theory of the unified law of inertia enunciated by Hlavaty.

§ 1. Introduction

The geometrical structure of Einstein's unified field theory has been presented by Hlavaty[1] in a well ordered form. But the physical structure of the theory is not properly understood as yet. In spite of numerous efforts, mathematical solutions of the field equations are yet to be found which will gain recognized physical interpretation. The only known solutions of the equations with a plausible physical interpretation are (1) the spherically symmetric static solutions worked out by Papapetrou[2] Wyman,[3] Bonner[4]; and (2) the plane wave solutions of Hlavaty,[1] Takeno.[5] The spherically symmetric solution has failed to give a simple model of either a point charge or a point mass particle (distinct from Schwarzschild's model of general relativity).

An effort was made to generalize the spherically symmetric solutions of Papapetrou, for the case of a nonstatic radiating mass. But it was found that though spherically symmetric gravitational field of a radiating star can be described by the field equations of general relativity, no such solutions are permitted in the scheme of the unified field theory (Vaidya[6]).

In the present paper an attempt is made to obtain general solutions of the second type giving propagation of gravitational and electromagnetic waves in the scheme of the unified field theory. Throughout the paper we shall use the notations of Hlavaty.[1]

§ 2. The nonsymmetric g_{ik} field

We write

$$g_{\alpha\beta}=h_{\alpha\beta}+k_{\alpha\beta}$$

where $$h_{\alpha\beta}=h_{\beta\alpha},\quad k_{\alpha\beta}=-k_{\beta\alpha}.$$

Since electromagnetic as well as gravitational waves are propagated in directions lying along the null cone, we shall search for solutions of the field equations which give the $k_{\alpha\beta}$ field being propagated along the null cones of the corresponding $h_{\alpha\beta}$ field. Now whenever a metric field permits the existence of a family of null surfaces in the field, (as in the case of wave propagation of a skew field along null cones), coordinates can be so chosen that h_{11}, h_{12}, h_{13}, h_{23} vanish and $h_{14}\neq 0$ (Vaidya and Pandya[8]). This is done by stipulating that the null vector v^μ giving the direction of the propagation of the skew field have, in the new coordinates, the components $(v^1, 0, 0, 0)$. Again orienting the dx^2 and the dx^3 directions in a proper manner one can take $h_{22}=h_{33}$. Thus we choose coordinates in such a way that $h_{\alpha\beta}$ takes the form

$$h_{\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & B & 0 & \lambda \\ 0 & 0 & B & \mu \\ a & \lambda & \mu & -y \end{pmatrix}. \tag{2·1}$$

One can form the tensor $h^{\alpha\beta}$ by the relation $h_{\alpha\mu}h^{\mu\beta}=\delta_\alpha{}^\beta$ and then use these two tensors for raising and lowering of suffices.

We now turn to the skew tensor field $k_{\alpha\beta}$. Since known electromagnetic wave fields can be characterised by a skew tensor which belongs to the class III of Hlavaty we take the $k_{\alpha\beta}$ field to be class III, i.e. we take it to be what Synge[9] calls a null field. Then

$$\det|k_{\alpha\beta}|=0,\quad k_{\alpha\beta}k^{\alpha\beta}=0. \tag{2·2}$$

Since in the coordinate system which we have chosen the direction of propagation of the $k_{\alpha\beta}$ field is given by the null vector $(v^1, 0, 0, 0)$, we must have

$$k_{14}=k_{23}=0. \tag{2·3}$$

The two equations (2·2) can now be satisfied by simply taking

$$k_{12}=k_{13}=0.$$

Thus in the problem at hand we are taking the $k_{\alpha\beta}$ field to be of the form

$$k_{\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & p \\ 0 & -q & -p & 0 \end{pmatrix}. \tag{2·4}$$

The final form of the nonsymmetric $g_{\alpha\beta}$ for our problem becomes

$$g_{\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & B & 0 & \lambda+q \\ 0 & 0 & B & \mu+p \\ a & \lambda-q & \mu-p & -y \end{pmatrix}, \tag{2·5}$$

the various components being functions of the four variables x^1, x^2, x^3, x^4.

Our experience with similar problems in the scheme of the general theory of relativity points to one simplifying assumption. The Schwarzschild's exterior solution, the solution for a radiating star (Vaidya[10]), the field of axially symmetric radiation flow, (Vaidya and Pandya[8]), and the field of gravitational radiation flow (Vaidya and Pandya[11]), all when expressed in terms of $h_{\alpha\beta}$ of the form (2·1) require that $a=1$. Therefore in the present investigation we shall be working under the simplifying assumption

$$g_{14}=g_{41}=a=1.$$

§ 3. The field equations

The field equations of Einstein's unified field theory are

$$\partial_\omega g_{\lambda\mu}-\Gamma^\alpha_{\lambda\omega}g_{\alpha\mu}-\Gamma^\alpha_{\omega\mu}g_{\lambda\alpha}=0, \tag{3·1}$$

$$\Gamma^\mu_{[\alpha\mu]}=\Gamma_\alpha=0, \tag{3·2}$$

$$R_{\mu\lambda}=-\partial_\alpha\Gamma^\alpha_{\mu\lambda}+\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\beta\lambda}+\partial_\mu\Gamma^\alpha_{\alpha\lambda}-\Gamma^\alpha_{\mu\lambda}\Gamma^\beta_{\alpha\beta}, \tag{3·3}$$

$$R_{(\mu\lambda)}=0, \tag{3·4}$$

$$R_{[\mu\nu,\lambda]}=0. \tag{3·5}$$

For solutions belonging to the class III, Hlavaty[1] has suggested the replacement of the 4 equations (3·5) by the equations

$$D_\lambda Q^{\lambda\mu}=U^\mu_{\alpha\beta}H^{\alpha\beta}. \tag{3·6}$$

Defining $^*g^{\alpha\beta}$ by the relation

$$g_{\lambda\mu}{}^*g^{\lambda\nu}=\delta_\mu{}^\nu \tag{3·7}$$

and writing the skew part of $^*g^{\alpha\beta}$ as $^*k^{\alpha\beta}$, one can write Eq. (3·2) in the form (Hlavaty[7])

$$\frac{\partial}{\partial x^\alpha}(^*k^{\mu\alpha}\sqrt{-g})=0. \tag{3·8}$$

We take the $g_{\alpha\beta}$ field as

$$g_{\alpha\beta}=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & B & 0 & \lambda+q\\ 0 & 0 & B & \mu+p\\ 1 & \lambda-q & \mu-p & -\gamma\end{pmatrix}. \tag{3·9}$$

Then $g=\det|g_{\alpha\beta}|=-B^2$ so that $\sqrt{-g}=B$. The four equations (3·8) can be immediately handled. They give

$$p_{,1}=0,\quad q_{,1}=0,\quad q_{,2}+p_{,3}=0. \tag{3·10}$$

Here and in the rest of the paper we adopt the convention that suffices attached

to functional symbols following a comma denote the derivatives of the functions with respect to the corresponding variables, e.g. $q_{,2}=\partial q/\partial x^2$, $p_{,23}=\partial^2 p/\partial x^2\partial x^3$, etc.

We shall now solve the algebraic equation (3·1) for the 64 connections $\Gamma^{\alpha}_{\lambda\omega}$. Instead of using the general algebraic solution developed by Hlavaty, we find it more convenient to replace (3·1) by the two equations,

$$g_{(ls)}\Gamma^{s}_{(ik)}=-\frac{1}{2}g_{(ik),l}+\frac{1}{2}g_{(kl),i}+\frac{1}{2}g_{(li),k}+g_{[is]}\Gamma^{s}_{[lk]}+g_{[sk]}\Gamma^{s}_{[il]}, \tag{3·11}$$

$$g_{(ls)}\Gamma^{s}_{[ik]}=\frac{1}{2}g_{[ik],l}-\frac{1}{2}g_{[k,l],i}-\frac{1}{2}g_{[li],k}-g_{[is]}\Gamma^{s}_{(lk)}-g_{[sk]}\Gamma^{s}_{(il)}. \tag{3·12}$$

For the present form (3·9) of $g_{\alpha\beta}$ we have found that if we use (3·11) and (3·12) to evaluate the 64 connections in a particular order, on no occasion have we ever to solve even two equations together and therefore as far as the present work is concerned (3·11) and (3·12) serve the purpose of explicit solutions of the algebraic equation (3·1).

Of the 64 connections, the 28 connections $\Gamma_{1\lambda}{}^{\mu}$ or $\Gamma_{\lambda 1}{}^{\mu}$ are particularly simple in form, the surviving ones being

$$\Gamma^{1}_{(12)}=\frac{\lambda}{2}\left(\frac{\lambda_{,1}}{\lambda}-\frac{B_{,1}}{B}\right),\quad \Gamma^{1}_{(13)}=\frac{\mu}{2}\left(\frac{\mu_{,1}}{\mu}-\frac{B_{,1}}{B}\right),\quad \Gamma^{2}_{(12)}=\Gamma^{3}_{(13)}=\frac{B_{,1}}{2B},$$

$$\Gamma^{1}_{(14)}=-\frac{y_{,1}}{2}-\frac{(\lambda^2+\mu^2)_{,1}}{4B}-\frac{(p^2+q^2)B_{,1}}{2B^2},$$

$$\Gamma^{1}_{[14]}=(\lambda q+\mu p)\frac{B_{,1}}{2B^2}-\frac{(\lambda q+\mu p)_{,1}}{2B}, \tag{3·13}$$

$$\Gamma^{2}_{(14)}=\frac{\lambda_{,1}}{2B},\quad \Gamma^{2}_{[14]}=-\frac{q}{B}\frac{B_{,1}}{2B},$$

$$\Gamma^{3}_{(14)}=\frac{\mu_{,1}}{2B},\quad \Gamma^{3}_{[14]}=-\frac{p}{B}\frac{B_{,1}}{2B}.$$

Before proceeding further with calculations of the remaining 36 connections, we shall take Eq. (3·3). We are now in a position to calculate R_{11}. We find that the equation,

$$R_{11}=0,$$

gives the following simple differential equation for B,

$$\frac{B_{,11}}{B}-\frac{1}{2}\left(\frac{B_{,1}}{B}\right)^2=0 \quad \text{or} \quad (\sqrt{B})_{,11}=0, \tag{3·14}$$

the general solution of which is

$$B=(\alpha x^1+\beta)^2,$$

α and β being functions of the other three variables x^2, x^3, x^4. Two simple

cases are immediately suggested. They are case I, $B=\beta(x^2, x^3, x^4)$; case II, $B=(x^1)^2\alpha(x^2, x^3, x^4)$. The second case can be easily recognized as leading to solutions with spherical and rotational symmetry while the first case will give us solutions with plane or two-dimensional symmetry. In this paper we are dealing with case I only.

§ 4. Field equations in case I

In this case we have taken $B_{,1}=0$. The connections (3·13) are much simpler now. They become $\Gamma_{(12)}{}^1=(1/2)\lambda_{,1}$, $\Gamma_{(13)}{}^1=(1/2)\mu_{,1}$.

$$\Gamma^1_{(14)}=-\frac{y_{,1}}{2}-\frac{(\lambda^2+\mu^2)_{,1}}{4B},\quad \Gamma^1_{[14]}=-\frac{(\lambda q+\mu p)_{,1}}{2B},\quad \Gamma^2_{(14)}=\frac{\lambda_{,1}}{2B},\quad \Gamma^3_{(14)}=\frac{\mu_{,1}}{2B}. \tag{4·1}$$

These give

$$R_{11}=0. \tag{4·2}$$

In order to calculate R_{12} and R_{13} we need to calculate only $\Gamma_{22}{}^4$, $\Gamma_{33}{}^4$ and $\Gamma_{23}{}^4$. We find that all these three connections vanish. One then easily calculates R_{12} and R_{13} to obtain

$$R_{12}=R_{21}=-\frac{1}{2}\lambda_{,11},\quad R_{13}=R_{31}=-\frac{1}{2}\mu_{,11} \tag{4·3}$$

$$R_{(\mu\lambda)}=0 \text{ requires } \lambda_{,11}=0,\ \mu_{,11}=0. \tag{4·4}$$

These equations are of the same form as Eq. (3·14) for $\sqrt{B}$. Thus consistent with the symmetry imposed on our solution by our assumption for the form of the function $\sqrt{B}$, we shall take in this case

$$\lambda=\lambda(x^2, x^3, x^4),\ \mu=\mu(x^2, x^3, x^4).$$

Most of the connections (4·1) now vanish and we have at this stage

$$\Gamma^1_{(14)}=-\frac{1}{2}y_{,1},$$

as the only surviving component of $\Gamma_{1\lambda}{}^\mu$ and this enables us to calculate R_{14} without any knowledge of the remaining connections. We find that $R_{(14)}=(1/2)y_{,11}$ so that $R_{(14)}=0$ gives $y_{,11}=0$. In order to get the simplest possible solution consistent with our assumption characterising this case I, we shall choose $y=y(x^2, x^3, x^4)$. We have thus completed the first stage of solution. The results are: (1) All the functional symbols occurring in the matrix (3·9) of $g_{\alpha\beta}$ are functions of x^2, x^3, x^4 only. (2) Then $\Gamma_{1\lambda}{}^\mu=0$ and $R_{11}=R_{12}=R_{21}=R_{13}=R_{31}=R_{14}=R_{41}=0$.

We now take the second stage of solution where we work out $\Gamma_{2\lambda}{}^\mu$, $\Gamma_{3\lambda}{}^\mu$ and R_{22}, R_{33}, R_{23}. 32 components of the type $\Gamma_{\alpha\beta}{}^\mu$ where α or β is 2 or 3 are yet to be evaluated. Of these 12 are of the type $\Gamma_{24}{}^i$, $\Gamma_{42}{}^i$, $\Gamma_{34}{}^i$, $\Gamma_{43}{}^i$, $i=1, 2, 3$. We do not need to know their values in order to get R_{22}, R_{33} and R_{23}. Of the remaining 20 components the surviving ones are

$$\Gamma^1_{22}=-\lambda\frac{B_{,2}}{2B}+\mu\frac{B_{,3}}{2B}-\frac{1}{2}B_{,4}+\lambda_{,2},\quad \Gamma^1_{33}=-\mu\frac{B_{,3}}{2B}+\lambda\frac{B_{,2}}{2B}-\frac{1}{2}B_{,4}+\mu_{,3},$$

$$\Gamma^2_{22}=\frac{B_{,2}}{2B},\quad \Gamma^3_{22}=-\frac{B_{,3}}{2B},\quad \Gamma^2_{33}=-\frac{B_{,2}}{2B},\quad \Gamma^3_{33}=\frac{B_{,3}}{2B},$$

$$\Gamma^2_{(23)}=\frac{B_{,3}}{2B},\quad \Gamma^3_{(23)}=\frac{B_{,2}}{2B},\tag{4·5}$$

$$\Gamma^1_{(23)}=-\lambda\frac{B_{,3}}{2B}-\mu\frac{B_{,2}}{2B}+\frac{1}{2}(\lambda_{,3}+\mu_{,2}),\quad \Gamma^1_{[23]}=\frac{1}{2}(q_{,3}-p_{,2}).$$

We are now ready to calculate R_{22}, R_{33}, R_{23}. We find that

$$R_{22}=R_{33}=\frac{B_{,22}}{2B}-\frac{1}{2}\left(\frac{B_{,2}}{B}\right)^2+\frac{B_{,33}}{2B}-\frac{1}{2}\left(\frac{B_{,3}}{B}\right)^2 \tag{4·6}$$

$$R_{23}=R_{32}=0. \tag{4·7}$$

The field equations $R_{(\alpha\beta)}=0$ now give us

$$(\log B)_{,22}+(\log B)_{,33}=0.$$

One could take B as expressed in terms of the general solution of this Laplace's equation; but for our present purposes we shall choose $B=B(x^4)$, so that $B_{,1}=B_{,2}=B_{,3}=0$. The three-index symbols in (4·5) are much simplified and we have

$$\Gamma^1_{22}=-\frac{1}{2}B_{,4}+\lambda_{,2},\quad \Gamma^1_{33}=-\frac{1}{2}B_{,4}+\mu_{,3},$$

$$\Gamma^1_{(23)}=\frac{1}{2}(\lambda_{,3}+\mu_{,2}),\quad \Gamma^1_{[23]}=\frac{1}{2}(q_{,3}-p_{,2}). \tag{4·8}$$

This completes the second stage in the solution of the field equations. We have worked out 48 components of the connection and found only 4 surviving components given in (4·8) above. We now work out the last 16 components. They are found to be

$$\Gamma^1_{(24)}=-\frac{\lambda B_{,4}}{2B}-\frac{\mu}{2B}(\mu_{,2}-\lambda_{,3})-\frac{1}{2}y_{,2}+\frac{qq_{,2}}{B}+\frac{p}{2B}(q_{,3}+p_{,2}),$$

$$\Gamma^1_{[24]}=-\lambda\frac{q_{,2}}{B}-\frac{\mu}{2B}(q_{,3}+p_{,2})+q_{,4}-q\frac{B_{,4}}{2B}-\frac{p}{2B}(\mu_{,2}-\lambda_{,3}),$$

$$\Gamma^2_{(24)}=\frac{B_{,4}}{2B},\quad \Gamma^2_{[24]}=\frac{q_{,2}}{B},\quad \Gamma^3_{(24)}=\frac{\mu_{,2}-\lambda_{,3}}{2B},\quad \Gamma^3_{[24]}=\frac{q_{,3}+p_{,2}}{2B},$$

$$\Gamma^1_{(34)}=-\mu\frac{B_{,4}}{2B}-\frac{\lambda}{2B}(\lambda_{,3}-\mu_{,2})-\frac{1}{2}y_{,3}+p\frac{p_{,3}}{B}+\frac{q}{2B}(p_{,2}+q_{,3}),$$

$$\Gamma^1_{[34]}=-\mu\frac{p_{,3}}{B}-\frac{\lambda}{2B}(p_{,2}+q_{,3})+p_{,4}-p\frac{B_{,4}}{2B}-\frac{q}{2B}(\lambda_{,3}-\mu_{,2}),$$

$$\Gamma^{2}_{(34)}=\frac{1}{2B}(\lambda_{,3}-\mu_{,2}),\quad \Gamma^{2}_{[34]}=\frac{1}{2B}(p_{,2}+q_{,3}),\quad \Gamma^{3}_{(34)}=\frac{B_{,4}}{2B},\quad \Gamma^{3}_{[34]}=\frac{p_{,3}}{B}$$

$$\Gamma^{2}_{44}=\frac{y_{,2}}{2B}+\frac{\lambda_{,4}}{B}-\frac{2qq_{,2}}{B^2}-\frac{p}{B^2}(q_{,3}+p_{,2}),$$

$$\Gamma^{3}_{44}=\frac{y_{,3}}{2B}+\frac{\mu_{,4}}{B}-\frac{2pp_{,3}}{B^2}-\frac{q}{B^2}(q_{,3}+p_{,2}),$$

$$\Gamma^{1}_{44}=-\lambda\Gamma^{2}_{44}-\mu\Gamma^{3}_{44}-\frac{1}{2}y_{,4},\quad \Gamma^{4}_{44}=0. \tag{4·9}$$

Finally then, (4·8) and (4·9) give the 19 surviving components of the connection $\Gamma_{\alpha\beta}{}^{\mu}$. The surviving components of the tensor $R_{\alpha\beta}$ are the following:

$$R_{(24)}=\frac{1}{2B}(\lambda_{,33}-\mu_{,23}),\quad R_{(34)}=\frac{1}{2B}(\mu_{,22}-\lambda_{,32}),$$

$$R_{44}=-\frac{1}{2B}(y_{,22}+y_{,33})-\frac{1}{2B}(\lambda_{,24}+\mu_{,34})-\frac{1}{2B^2}(\mu_{,2}-\lambda_{,3})^2+\frac{B_{,44}}{B}$$

$$-\frac{1}{2}\left(\frac{B_{,4}}{B}\right)^2+\frac{q^2}{B^2}\left(\frac{2q_{,22}}{q}+\frac{q_{,33}}{q}+\left(\frac{q_{,2}}{q}\right)^2+\left(\frac{q_{,3}}{q}\right)^2\right) \tag{4·10}$$

$$+\frac{p^2}{B^2}\left(\frac{2p_{,33}}{p}+\frac{p_{,22}}{p}+\left(\frac{p_{,2}}{p}\right)^2+\left(\frac{p_{,3}}{p}\right)^2\right)+\frac{2pq}{B^2}\left(\frac{q_{,32}}{q}+\frac{p_{,23}}{p}+\frac{2p_{,2}q_{,3}}{pq}\right),$$

$$R_{[24]}=-\frac{1}{2B}(q_{,22}+q_{,33}),\quad R_{[34]}=-\frac{1}{2B}(p_{,22}+p_{,33}).$$

The restrictions that we have imposed on the components (3·9) of $g_{\alpha\beta}$ are

$$g_{\alpha\beta,1}=0,\quad B_{,2}=B_{,3}=0,\quad p_{,3}+q_{,2}=0. \tag{4·11}$$

§ 5. Wave solutions of the field equations

We shall solve the remaining field equations viz.

$$R_{(\mu\nu)}=0,\quad R_{[\mu\nu,\lambda]}=0, \tag{E}$$

$$R_{(\mu\nu)}=0,\quad D_{\lambda}Q^{\lambda\mu}=U^{\mu}_{\alpha\beta}H^{\alpha\beta}, \tag{H}$$

and obtain several particular solutions describing propagation of unified waves. It can be seen from (4·10) and (4·11) that the field equations (E) or (H) present rather easy differential equations and therefore quite a large number of mathematical solutions will be available. We are giving below some solutions which lead to a better understanding of the physical nature of the field theory.

Solution I:

$$p=\cos\alpha,\quad q=\sin\alpha,\quad \alpha=\alpha(x^4). \tag{5·1}$$

$$B=y=1,\quad \lambda=\mu=0.$$

We then find that $R_{(\mu\nu)}=0$, $R_{[\mu\nu]}=0$, $H^{\alpha\beta}=0$, $Q^{\mu\lambda}=0$ so that the equations (E) or (H) are satisfied. This is the plane wave solution described by Hlavaty.[1] Since $h_{\alpha\beta}$ are Galilean there is no gravitational field described by this solution.

Solution II:

$$\begin{gathered} p=\cos\alpha,\quad q=\sin\alpha,\quad \alpha=\alpha(x^4) \\ B=1,\quad y=1+f(x^2, x^3, x^4),\quad f_{,22}+f_{,33}=0 \\ \lambda=\mu=0. \end{gathered} \tag{5·2}$$

Once again $R_{\mu\nu}=0$, $H^{\alpha\beta}=0$, $Q^{\mu\lambda}=0$ so that both (E) and (H) are satisfied. This solution has the same $k_{\alpha\beta}$ field as the solution I and therefore describes the propagation of plane electromagnetic waves. But these plane waves are now associated with a gravitational field. One can verify that the curvature tensor $H_{\beta\gamma\delta}{}^{\alpha}$ of the gravitational field is nonzero. As a matter of fact, this field has a metric recorded earlier (Vaidya and Pandya[11]). There it is shown that this metric satisfies the conditions recently formulated by Pirani[12] *for propagation of gravitational waves.* The unified field theory associates these gravitational waves with plane electromagnetic waves. Thus with $g_{\alpha\beta}$ given by

$$g_{\alpha\beta}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \sin\alpha \\ 0 & 0 & 1 & \cos\alpha \\ 1 & -\sin\alpha & -\cos\alpha & -(1+f) \end{pmatrix}, \tag{5·3}$$

$$\alpha=\alpha(x^4),\ f=f(x^2, x^3, x^4) \text{ such that } f_{,22}+f_{,33}=0,$$

we have a very simple rigorous solution giving propagation of unified waves.

Solution III: This is an immediate generalization of the above solution. We do not put λ and μ as zero but choose them as two conjugate functions of x^2 and x^3

$$\lambda_{,3}-\mu_{,2}=0,\quad \lambda_{,2}+\mu_{,3}=0,\quad \lambda_{,22}+\lambda_{,33}=0,\quad \mu_{,22}+\mu_{,33}=0.$$

B, y, p, q have the same form as in solution II.

These three solutions satisfy both the equations (E) and (H). As a matter of fact, in these solutions $R_{\alpha\beta}=H_{\alpha\beta}$ and so they give a linear superposition of gravitational and electromagnetic waves without any unifying interaction between them. Many more solutions of this type can be readily derived. The existence of such solutions in the scheme of the unified field theory clearly suggests that sources emitting electromagnetic waves which ultimately become plane waves at large distances, also emit gravitational waves and that in the regions in which the electromagnetic waves become plane, the gravitational waves become cylindrical of the type given in the above solutions.

We shall now give a solution exhibitting the unified interaction of electromagnetic and gravitational waves.

Solution IV:

$$q=0,\quad p=p(x^3, x^4)$$

$$B=1, \quad y=1+f,$$

with

$$f_{,22}+f_{,33}=2p^2\left(\frac{p_{,22}}{p}+\left(\frac{p_{,2}}{p}\right)^2\right),$$

so that

$$(f_{,22}+f_{,33})_{,3}=0.$$

We then have $R_{(\alpha\beta)}=0$. It can be verified that Hlavaty's equation $D_\lambda Q^{\lambda\mu}=U_{\alpha\beta}{}^{\mu}H^{\alpha\beta}$ is identically satisfied, while Einstein's equation $R_{[\mu\nu,\lambda]}=0$ puts a restriction on the function p: $p_{,222}=0$.

§ 6. Conclusions

The above solution IV can be compared with a similar wave solution in general relativity. In the general theory of relativity, we have a symmetrical tensor $h_{\alpha\beta}$ as the metric tensor and then the 3-index symbols $\left\{{k \atop ij}\right\}$ are defined as the Christoffel symbols

$$\left\{{k \atop ij}\right\}=\frac{1}{2}h^{k\lambda}(h_{i\lambda,j}+h_{j\lambda,i}-h_{ij,\lambda}).$$

With these symbols the contracted curvature tensor $H_{\alpha\beta}$ is obtained. The field equations of general relativity state that

$$H_{\alpha\beta}-\frac{1}{2}h_{\alpha\beta}H=-kT_{\alpha\beta}, \tag{6·1}$$

where $T_{\alpha\beta}$ is a tensor of energy and momentum specified by the physical condition of the gravitating system. Thus the left-hand member of (6·1) is a tensor derived from the geometry of the space-time specified by the metric tensor $h_{\alpha\beta}$ while the right-hand member is a purely hypothetical tensor to be approximated from the physical nature of the gravitating matter. This disparity in the nature of the members on both sides was very much criticised by Einstein[13] himself and one could compare the geometrical left-hand member with a pillar of marble while the physical right-hand member of (6·1) with a pillar of straw. One of the many reasons for producing a unified field theory was to replace this pillar of straw by a pillar of marble, i.e. to derive the physical nature of gravitating matter from geometrical consideration. In solution IV above we have a concrete illustration of how a gravitational situation described in general relativity by a certain physical energy tensor $T_{\alpha\beta}$ can be described in identical terms by the geometrical nonsymmetric tensor $g_{\alpha\beta}$.

Taking the energy-momentum tensor for an electromagnetic field $F_{\mu\nu}$ to be of the form

$$T_\mu{}^\nu=-F_{\mu\alpha}F^{\nu\alpha}+\frac{1}{4}\delta_\mu{}^\nu F_{\alpha\beta}F^{\alpha\beta},$$

Vaidya and Pandya[11] have obtained certain wave solutions of the field equations (6·1) of general relativity. In the present paper we have our solution IV which is derived from the non-symmetric tensor $g_{\alpha\beta}$ which is completely a tensor of a geometric nature. And this geometric solution IV with $p=p(x^2, x^4)$ unspecified, describes the same gravitational situation as described by the geometrico-physical solution of Vaidya and Pandya referred to above. But with $p=p(x^2, x^4)$ unspecified, our solution IV satisfies field equations (H) and not (E). Thus we have found that certain wave solutions describing gravitational and electromagnetic waves obtained in the general theory of relativity with the help of an energy-momentum tensor can be derived in exactly the same form from the geometrical theory of the unified law of inertia enunciated by Hlavaty.

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Quantum Statistics of Interacting Electron Gas in a Strong Magnetic Field

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The quantum statistical mechanical propagator which has been introduced by Matsubara is calculated by using the free electron eigenfunction in a magnetic field. This form of the propagator seems to be appropriate for the treatment of the oscillatory behaviours of magnetic properties of the interacting electron gas. As a tentative application of this method, the effect of the Coulomb interaction between electrons on the de Haas-van Alphen effect is discussed.

§ 1. Introduction

In recent years, remarkable advances have been brought about in the theory of many-body problems. Among various investigators, Matsubara[1] has introduced a method from the quantum statistical mechanical viewpoint, in which similar techniques as in the theory of field reveal their powers. In his theory and other theories that have appeared since then, the plane wave functions are mostly used in the actual calculations. For the investigation of the electron gas under strong magnetic field, however, the plane wave expansion is inconvenient as is well known. To avoid this difficulty, we want to propose in this paper the use of the eigenfunction of an electron in a uniform magnetic field for the construction of the statistical mechanical propagator which plays the principal role in the method of Matsubara. The summation which are required in the calculation of this propagator can be performed by a technique analogous to that of Sondheimer and Wilson[2]* in their treatment of free electron gas in a magnetic field. In our results, the propagator is decomposed into three parts which will show their roles in the actual calculations rather clearly. As a tentative application of our results, the effect of Coulomb interaction between electrons on the de Haas-van Alphen effect of electron gas—oscillatory behaviours of the diamagnetic susceptibility—is calculated. To the first approximation in the energy, it is shown that the new oscillatory terms due to the interaction has different magnetic field dependence in its amplitude and also different phase constant of the oscillation from those of the free electron theory or of the modified theory by Dingle.[3]

§ 2. Calculation of statistical mechanical propagator

The propagator $S(x', x)$ which plays important roles in the expansion tech-

* Hereafter we mention this reference as S.W.

nique mentioned above is defined as

$$\begin{aligned} S(x', x) &= \sum_r f(\varepsilon_r) u_r(\boldsymbol{x}') u_r^*(\boldsymbol{x}) e^{-\varepsilon_r(t'-t)} \qquad t > t' \\ &= -\sum_r (1-f(\varepsilon_r)) u_r(\boldsymbol{x}') u_r^*(\boldsymbol{x}) e^{-\varepsilon_r(t'-t)} \quad t < t' \end{aligned} \tag{2·1}$$

where x stands for $\boldsymbol{x}$, t in which $\boldsymbol{x}$ is the space coordinate of the electron and t is the fourth variable corresponding to the inverse temperature $1/\kappa T$, $u_r(\boldsymbol{x})$ is the eigenfunction of the eigenvalue ε_r, $f(\varepsilon_r)=(1+\exp\beta(\varepsilon_r-\zeta))^{-1}$, where $\beta=1/\kappa T$ and κ is Boltzmann constant and ζ is the chemical potential of the electron under consideration.

The grand partition function Ξ of the interacting electron gas can be given in the following form as has been discussed by several authors.[1)4)5)]

$$\Xi = \Xi_0 \exp(\langle S(\beta) \rangle_c) \tag{2·2}$$

$$\Xi_0 = \prod_r (1+\exp[\beta(\zeta-\varepsilon_r)]) \tag{2·3}$$

$$\langle S(\beta) \rangle_c = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta dx_1 \int_0^\beta dx_1' \cdots \int_0^\beta dx_n' \langle P[\mathcal{H}_I(x_1, x_1') \cdots \mathcal{H}_I(x_n, x_n')] \rangle_c \tag{2·4}$$

where Ξ_0 is the grand partition function for the free electron gas and $\langle S(\beta) \rangle_c$ represents the effect of the interparticle interaction $\mathcal{H}_I(x, x')$.*

The analysis of the statistical average of the well-known P-product in (2·4) has been given by Matsubara[1)] and by Bloch et de Dominicis[5)] in a form named the connected diagram expansion.

$$\langle S(\beta) \rangle_c = \sum_G \tilde{S}(\beta, G),$$

where G means the type of the connected diagram and $\tilde{S}(\beta, G)$ the contribution from the diagram G. The first term of this expansion is given by

x  x' $-\frac{1}{2}\int_0^\beta dx \int_0^\beta dx' S(x', x)\, S(x, x')\, v(x, x')$ (2·5)*

where $S(x', x)$ is the statistical mechanical propagator defined in (2·1).

Now our task is the evaluation of the summation in (2·1), where the free electron eigenfunction in a magnetic field is introduced as complete set $\{u_r(\boldsymbol{x})\}$. For the time being, spin magnetic moment of the electron is disregarded. For our complete set $\{u_r(\boldsymbol{x})\}$, the summation

$$K(\sigma, \boldsymbol{x}', \boldsymbol{x}) = \sum_r e^{-\sigma\varepsilon_r} u_r(\boldsymbol{x}') u_r^*(\boldsymbol{x})$$

which is in fact the density matrix of our system under the assumption of the Boltzmann statistics, has been evaluated by S. W. Their result can be written as

* We use the notation which is used in the paper by Hubbard.[6)]

$$K(\sigma, \boldsymbol{x}', \boldsymbol{x}) = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{1}{\sigma^{3/2}} \frac{\mu_0 H\sigma}{\sinh(\mu_0 H\sigma)} \exp\left[-\frac{1}{2} i\alpha^2 (x'+x)(y'-y)\right]$$
$$\times \exp\left(-\frac{C}{\sigma} - BL(\mu_0 H\sigma)\right) \tag{2·6}$$

$$L(x) = \coth x - 1/x, \tag{2·7}$$

$$\alpha^2 = eH/\hbar c, \qquad \zeta_0 = \hbar^2 k_0^2/2m, \qquad k_0 = 2\pi(3N/8\pi V)^{1/3},$$
$$C = k_0^2 R^2/4\zeta_0, \qquad R^2 = |\boldsymbol{x}'-\boldsymbol{x}|^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2, \tag{2·8}$$
$$B = k_0^2 \rho^2/4\zeta_0, \qquad \rho^2 = (x'-x)^2 + (y'-y)^2,$$

where μ_0 is the Bohr magneton and m is the effective mass of the electron. The constant magnetic field is assumed in the z-direction.

The propagator $S(x', x)$ can be obtained from this $K(\sigma, \boldsymbol{x}', \boldsymbol{x})$ with the aid of Laplace transformation by a similar procedure to that of S. W. For the case

(a) $t > t'$, we have

$$S(\beta, \tau, \boldsymbol{x}', x) = S(x', x) = \sum f(\varepsilon_r) e^{-\tau\varepsilon_r} u_r(\boldsymbol{x}') u_r^*(\boldsymbol{x}), \qquad \tau = t'-t$$
$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \sum_r e^{\varepsilon_r s} \phi(s) K(\tau, \boldsymbol{x}', \boldsymbol{x})$$
$$= \int_0^\infty \gamma(\varepsilon) \frac{\partial f}{\partial \varepsilon} d\varepsilon \tag{2·9}$$

where $\phi(s)$ has the following relation to $f(\varepsilon)$,

$$f(\varepsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\varepsilon s} \phi(s)\, ds, \qquad c > 0 \tag{2·10}$$

and $\gamma(\varepsilon)$ is defined by

$$\gamma(\varepsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s+\tau, \boldsymbol{x}', \boldsymbol{x}) \frac{e^{\varepsilon s}}{-s} ds\,. \tag{2·11}$$

In obtaining the final expression in (2·9), we have utilized the well-known convolution theorem in the Laplace transformation as in S. W. Accordingly, the main problem reduces to the evaluation of $\gamma(\varepsilon)$. We obtain

$$\gamma(\varepsilon, \tau, \boldsymbol{x}', \boldsymbol{x}) = -e(\boldsymbol{x}', \boldsymbol{x}) \Gamma(\varepsilon, \tau, \boldsymbol{x}', \boldsymbol{x}) \tag{2·12}$$

$$e(\boldsymbol{x}', \boldsymbol{x}) = \left(\frac{2\pi m}{h^2}\right)^{3/2} \exp\left(-\frac{1}{2} i\alpha^2 (x'+x)(y'-y)\right) \tag{2·13}$$

$$\Gamma(\varepsilon, \tau, \boldsymbol{x}', \boldsymbol{x}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M\, d\sigma$$
$$M = \frac{e^{-\varepsilon\tau}}{(\sigma-\tau)\sigma^{3/2}} \frac{\mu_0 H\sigma}{\sinh(\mu_0 H\sigma)} \exp\left(\varepsilon\sigma - \frac{C}{\sigma} - BL(\mu_0 H\sigma)\right) \tag{2·14}$$

where we have changed the integration variable s to $\sigma=s+\tau$. (cf. Fig. 1 and Fig. 2)

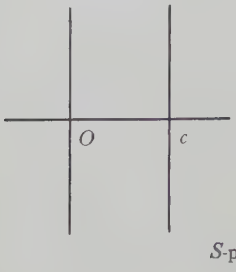


Fig. 1.

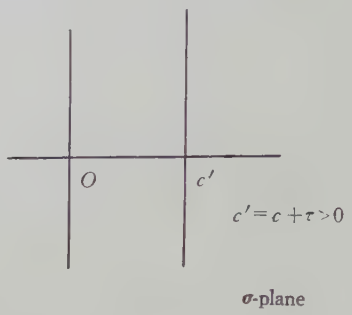


Fig. 2.

The function M has singularities at $\sigma=\tau$, $\sigma=0$, and $\sigma=\sigma_r$ which are defined by $\mu_0 H\sigma_r=r\pi i(r=\pm1, \pm2, \cdots)$. By the application of the Cauchy theorem on the contour given in Fig. 3, it is shown that Γ is equivalent to the sum of integrals along the contour given in Fig. 4.

$$\Gamma(\varepsilon, \tau, \boldsymbol{x}', \boldsymbol{x})=\Gamma_0+\sum_{r=1}^{\infty}(\Gamma_r+\Gamma_{-r}) \quad (2\cdot15)$$

$$\Gamma_r=\frac{1}{2\pi i}\int_{C_r} M d\sigma \,.$$

(i) *Evaluation of* Γ_0

The exact evaluation of this integral seems to be very difficult. To investigate the dependence in the magnetic field, however, it will be convenient to divide up this integral into several parts, one which is independent of magnetic field and others which vanish with H. Changing integration variable σ to $u=\varepsilon\sigma$, we obtain

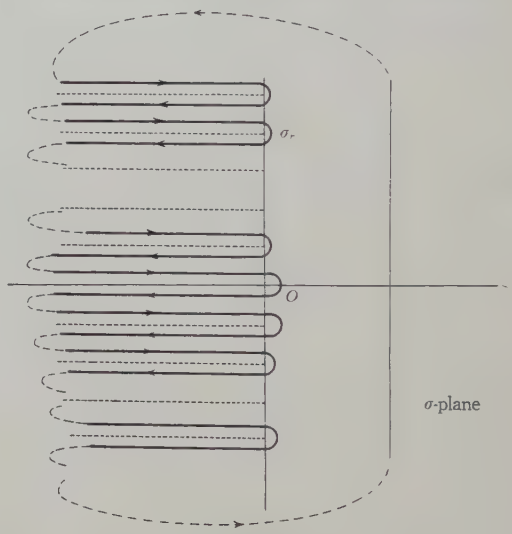
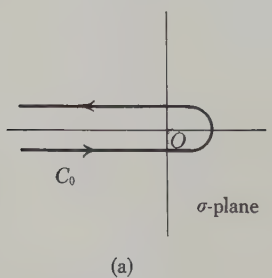
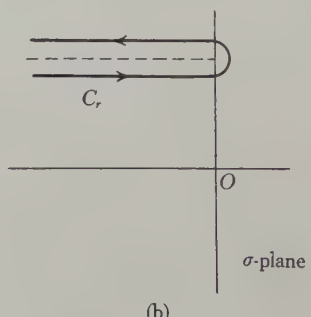


Fig. 3.



(a)



(b)

Fig. 4.

$$\Gamma_0=e^{-\varepsilon\tau}\varepsilon^{3/2}(\tilde{J}(\tau, D)+\tilde{L}(\tau, \eta)+\tilde{C}(\tau, \eta)), \tag{2·16}$$

$$\tilde{J}(\tau, D)=\frac{1}{2\pi i}\int_{c_0}\frac{du}{u-\varepsilon\tau}\frac{1}{u^{3/2}}e^{u-D/u} \tag{2·17}$$

$$\tilde{L}(\tau, \eta)=\frac{1}{2\pi i}\int_{c_0}\frac{du}{u-\varepsilon\tau}\frac{1}{u^{3/2}}e^{u-D/u}(e^{-BL(\eta u)}-1), \tag{2·18}$$

$$\tilde{C}(\tau, \eta)=\frac{1}{2\pi i}\int_{c_0}\frac{du}{u-\varepsilon\tau}\frac{1}{u^{3/2}}e^{u-D/u}\left(\frac{\eta u}{\sinh(\eta u)}-1\right)e^{-BL(\eta u)}, \tag{2·19}$$

$$D=\varepsilon C=\varepsilon k_0^2R^2/4\zeta_0, \quad \eta=\mu_0H/\varepsilon\,. \tag{2·20}$$

$\tilde{J}(\tau, D)$ is independent of H and can be given in the form

$$\tilde{J}(\tau, D)=\sum_{\lambda=0}^{\infty}(\varepsilon\tau)^{\lambda}\left(\frac{2}{k_0R}\left(\frac{\zeta_0}{\varepsilon}\right)^{1/2}\right)^{\lambda+3/2}J_{\lambda+3/2}\left(k_0R\left(\frac{\varepsilon}{\zeta_0}\right)^{1/2}\right), \tag{2·21}$$

$$J_\nu(x)=\left(\frac{2}{x}\right)^{\nu}\frac{1}{2\pi i}\int_{c_0}\frac{1}{u^{1+\nu}}e^{u-x^2/4u}du \tag{2·22}$$

where $J_\nu(x)$ is the Bessel function. $\tilde{L}$ and $\tilde{C}$ vanish as H becomes zero.

(ii) *Evaluation of* Γ_r.

Changing the variable σ to $t=\sigma-\sigma_r$ we obtain

$$\Gamma_r=\frac{\mu_0H}{2\pi i}\int_{c_0}\frac{1}{(\sigma_r+t)^{1/2}}\frac{1}{\sigma_r+t-\tau}\frac{1}{\sinh(\mu_0H(\sigma_r+t))}$$
$$\times\exp\left[\varepsilon(\sigma_r+t)-\frac{C}{\sigma_r+t}-BL(\mu_0H(\sigma_r+t))\right]dt. \tag{2·23}$$

We can also divide up this integrand into two parts, one which does not contain H and the other which vanishes with H. Since this Γ_r already has H as its factor, we retain only those parts of the integrand which does not vanish with H. Considering $\sigma_r=r\pi i/\mu_0H$ and the properties of sinh, we get

$$\Gamma_r=(\mu_0H)^{3/2}\frac{(-1)^r}{(r\pi)^{3/2}}\exp\left\{i\left(\frac{r\pi\varepsilon}{\mu_0H}+\frac{\mu_0HA}{r\pi}-\frac{3}{4}\pi\right)\right\}$$
$$\times\frac{e^{-\varepsilon\tau}}{1+i\frac{\mu_0H\tau}{r\pi}}J_0\left(k_0\rho\left(\frac{\varepsilon}{\zeta_0}\right)^{1/2}\right), \tag{2·24}$$
$$A=k_0^2(z'-z)^2/4\zeta_0\,.$$

Similarly we obtain also

$$\Gamma_{-r}=(\mu_0H)^{3/2}\frac{(-1)^r}{(r\pi)^{3/2}}\exp\left\{-i\left(\frac{r\pi\varepsilon}{\mu_0H}+\frac{\mu_0HA}{r\pi}-\frac{3}{4}\pi\right)\right\}$$
$$\times\frac{e^{-\varepsilon\tau}}{1-i\frac{\mu_0H\tau}{r\pi}}J_0\left(k_0\rho\left(\frac{\varepsilon}{\zeta_0}\right)^{1/2}\right). \tag{2·25}$$

Combining these two, we have

$$\Gamma_\tau+\Gamma_{-\tau}=2(\mu_0H)^{3/2}J_0\left(k_0\rho\left(\frac{\varepsilon}{\zeta_0}\right)^{1/2}\right)\frac{(-1)^\tau}{(\tau\pi)^{3/2}}\frac{e^{-\varepsilon\tau}}{1+\left(\frac{\mu_0H\tau}{\tau\pi}\right)^2}$$

$$\times\left\{\sin\left(\frac{\tau\pi\varepsilon}{\mu_0H}+\frac{\mu_0HA}{\tau\pi}-\frac{\pi}{4}\right)-\left(\frac{\mu_0H\tau}{\tau\pi}\right)\cos\left(\frac{\tau\pi\varepsilon}{\mu_0H}+\frac{\mu_0HA}{\tau\pi}-\frac{\pi}{4}\right)\right\}. \tag{2·26}$$

For the case

(b) $t<t'$, $(\tau>0)$ we must start with

$$S(x'\ x)=\sum_\tau(f(\varepsilon_\tau)-1)u_\tau(\boldsymbol{x}')u_\tau^*(\boldsymbol{x})e^{-\tau\varepsilon_\tau}$$

$$=\sum_\tau f(\varepsilon_\tau)u_\tau(\boldsymbol{x}')u_\tau^*(\boldsymbol{x})e^{-\tau\varepsilon_\tau}-K(\tau,\boldsymbol{x}',\boldsymbol{x}). \tag{2·27}$$

Proceeding in a similar way to that in the case (a), we obtain

$$\Gamma(\varepsilon,\tau,\boldsymbol{x}',\boldsymbol{x})=\Gamma_0'+\sum_\tau(\Gamma_\tau+\Gamma_{-\tau}) \tag{2·28}$$

corresponding to (2·15). The second term is exactly the same as that of (2·15). Γ_0' is different from Γ_0 in (2·15) in the point that a pole at $\sigma=\tau$ is enclosed in the contour of integration as shown in Fig. 5(a). Splitting this contour as in Fig. 5(b), we obtain

$$\Gamma_0'=\frac{1}{2\pi i}\int_{c_0}Md\sigma+R_\tau \tag{2·29}$$

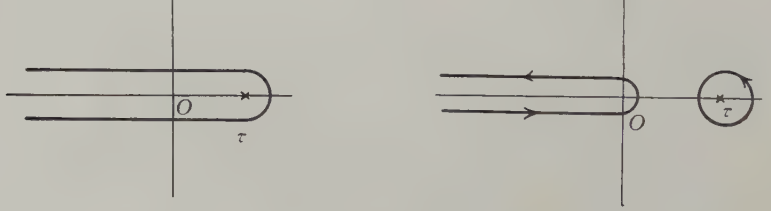
where R_τ is a residue at the pole $\sigma=\tau$. This residue can be given immediately as

$$R_\tau=\frac{1}{\tau^{3/2}}\frac{\mu_0H\tau}{\sinh(\mu_0H\tau)}e^{-C/\tau-BL(\mu_0H)}. \tag{2·30}$$

The contribution of R_τ to $S(x',x)$ exactly cancels $K(\tau,\boldsymbol{x}',\boldsymbol{x})$ in (2·27), hence we obtain

$$S(x',x)=e(\boldsymbol{x}',\boldsymbol{x})\int_0^\infty\left(-\frac{\partial f}{\partial\varepsilon}\right)(\Gamma_0+\sum_\tau(\Gamma_\tau+\Gamma_{-\tau}))d\varepsilon \tag{2·31}$$

for both cases $\tau\lesseqgtr0$.



(a) (b)

Fig. 5.

Now we want to arrange our result in the following form.

$$S(x', x)=S^{(1)}(x', x)+S^{(2)}(x', x)+S^{(3)}(x', x) \tag{2·32}$$

$$S^{(1)}(x', x)=e(\boldsymbol{x}', \boldsymbol{x})\int_0^\infty\left(-\frac{\partial f}{\partial\varepsilon}\right)\varepsilon^{3/2}e^{-\varepsilon\tau}\tilde{J}(\tau, D)d\varepsilon \tag{2·33}$$

$$S^{(2)}(x', x)=e(\boldsymbol{x}', \boldsymbol{x})\int_0^\infty\left(-\frac{\partial f}{\partial\varepsilon}\right)\varepsilon^{3/2}e^{-\varepsilon\tau}(\tilde{L}(\tau, \eta)+\tilde{C}(\tau, \eta))d\varepsilon \tag{2·34}$$

$$S^{(3)}(x', x)=e(\boldsymbol{x}', \boldsymbol{x})\int_0^\infty\left(-\frac{\partial f}{\partial\varepsilon}\right)\sum_\tau(\Gamma_\tau+\Gamma_{-\tau})d\varepsilon. \tag{2·35}$$

The evaluation of the integral over ε can be done easily by using the properties of the Fermi-Dirac function $f(\varepsilon)$.

$S^{(1)}(x', x)$ has almost the same characters as the free electron propagator based on the plane waves. $S^{(2)}(x', x)$ vanishes with H. $S^{(3)}(x', x)$ shows the oscillatory behaviours as a function of $1/H$. At first sight, these forms seem to be very complicated but in the actual calculations suitable simplifications will be rather easily introduced.

§ 3. Effect of Coulomb interaction on orbital diamagnetism of electron

As an application of our propagator $S(x', x)$ obtained in the previous section, we now take up the effect of the Coulomb interaction on the orbital diamagnetism of the electron gas. Since the monumental work by Landau[7] on the diamagnetism of the free electron gas, the effect of the Coulomb interaction on this diamagnetism have recently been discussed by Schafroth,[8] Wentzel,[9] Dresselhaus[10] and Kanazawa and Matsudaira.[11] Among them, Kanazawa and Matsudaira have made a detailed calculation concerning the effect of the Coulomb interaction on the so-called Landau diamagnetism—the constant part of the diamagnetic susceptibility—, and have given the result up to the correlation of the interaction. Dresselhaus has obtained the expression for the exchange energy in a strong magnetic field and discussed the effect of the exchange interaction on the de Haas-van Alphen oscillation. Here we also want to treat the effect of the interaction on the de Haas-van Alphen effect to the order of exchange effect. Our results, we think, might give clearer insight in these problems.

As is well known, the propagator $S(x', x)$ is represented by the electron line $x\longrightarrow x'$ in the diagram shown in (2·5). Now corresponding to our division of $S(x', x)$ into $S^{(1)}(x', x)$, $S^{(2)}(x', x)$ and $S^{(3)}(x', x)$, we decompose the electron line—we want to represent it here by $x\leftarrow M\rightarrow x'$ to show that the electron is quantized in the magnetic field—, into three parts as shown in Fig. 6. Consequently, various new diagrams will appear from one original diagram as shown, for example, in Fig. 7.

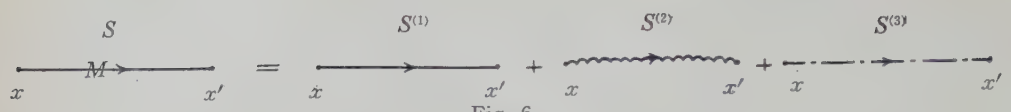


Fig. 6.

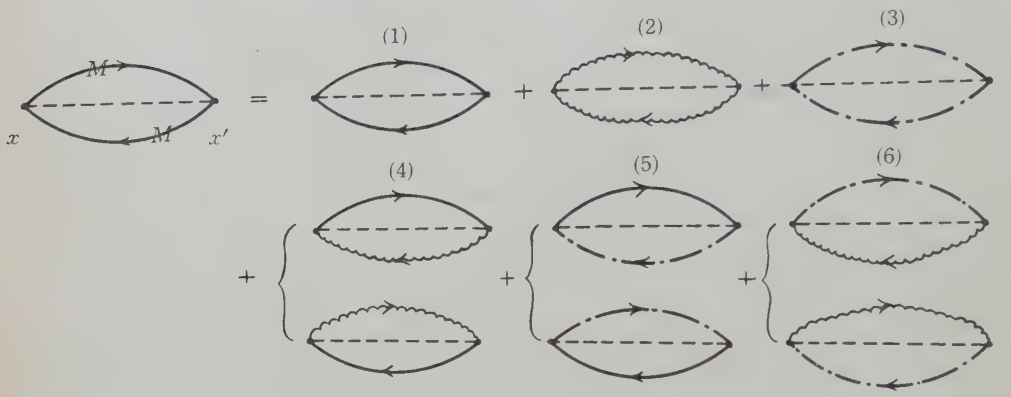


Fig. 7.

From the diagram (1) in the right-hand side in Fig. 7, we get a contribution to the free energy which corresponds to the exchange interaction and also a part of the free energy shift which originates from the shift in ζ due to magnetic field and makes some contribution to the modification of the Landau diamagnetism. The diagram (4) gives also a contribution to the modification of Landau diamagnetism. But if we expand it in terms of H, divergence difficulties will appear. To obtain a definite result, we must take account of the higher order diagrams such as ring diagrams. Since the problem concerning the higher order correction to the Landau diamagnetism has been treated in detail by Kanazawa and Matsudaira with the method of Green function, we do not treat it here. The diagram which we want to take up here is (5) which will give a contribution to the modification of the de Haas-van Alphen term. The lowest order terms in H which come from this diagram (5) do not show any divergence in contrast to the terms from the diagram (4). The reason of this non-divergence is the presence of cosine and sine factors in these terms. Owing to their presence, the lowest order terms in H in the contribution from (5) to the magnetic susceptibility have similar characters as the exchange energy in the electron gas and they give non-divergent result, whereas the terms from (4) have similar characters as the exchange energy contribution to the specific heat and give divergent results. The contributions from the diagram (2), (3) and (6) give higher order terms in H.

Evaluations of the contributions from the diagram (5).

Starting from the expression given in (2·5), we can immediately arrive at the following form for the contribution from the diagram (5) :

$$-2\beta\int dx\int dx' S^{(1)}(\beta, 0, x', x)\, S^{(3)}(\beta, 0, x, x')\, v(x'-x) \qquad (3\cdot1)$$

$$v(x'-x)=e^2/|x'-x|$$

where the spin summation is taken into account.

$S^{(1)}(\beta, 0, x', x)$ and $S^{(3)}(\beta, 0, x', x)$ can be given in an explicit form as

$$S^{(1)}(\beta, 0, x', x)=e(x', x)\zeta^{3/2}\left(\frac{2}{k_0R}\left(\frac{\zeta_0}{\zeta}\right)^{1/2}\right)^{3/2} J_{3/2}\left(k_0R\left(\frac{\zeta}{\zeta_0}\right)^{1/2}\right)+O(\kappa T/\zeta_0) \qquad (3\cdot2)$$

$$\begin{aligned} S^{(3)}(\beta, 0, x', x) &= e(x', x)\int_0^\infty\left(-\frac{\partial f}{\partial\varepsilon}\right)\sum_r(\Gamma_r+\Gamma_{-r})\,d\varepsilon \\ &= e(x', x)\,2(\mu_0H)^{3/2}\sqrt{\pi}\,(\kappa T/\mu_0H)\,J_0(k_0\rho(\zeta/\zeta_0)^{1/2}) \\ &\times\sum_{r=1}^\infty\frac{(-1)^r}{r^{1/2}}\frac{\sin(r\pi\zeta/\mu_0H+\mu_0HA/r\pi-\pi/4)}{\sinh(r\pi^2\kappa T/\mu_0H)}. \end{aligned} \qquad (3\cdot3)$$

For the evaluation of the integral over ε in $S^{(3)}(x', x)$ we cannot use Sommerfeld's technique, and the same direct integration as in the S. W.'s theory is performed to give these temperature dependence. Then we have for (3·1)

$$\begin{aligned} &-e^2(2\pi m/h^2)^3(4/\pi)(\mu_0H)\zeta^{3/2}((2/k_0)(\zeta_0/\zeta))^{3/2}V \\ &\times\sum_{r=1}^\infty\frac{(-1)^r}{r^{3/2}}\frac{r\pi^2\kappa T/\mu_0H}{\sinh(r\pi^2\kappa T/\mu_0H)}\int dR\frac{1}{R^{5/2}}J_{3/2}(k_0R(\zeta/\zeta_0)^{1/2}) \\ &\times J_0(k_0\rho(\zeta/\zeta_0)^{1/2})\sin\left(\frac{r\pi\zeta}{\mu_0H}+\frac{\mu_0HA}{r\pi}-\frac{\pi}{4}\right) \\ &=-2^{9/2}\pi^{-1/2}(\zeta/\zeta_0)^{1/2}(\mu_0H/\zeta_0)^{3/2}e^2(2\pi m/h^2)^3V\zeta_0^{\,3}k_0^{-2} \\ &\times\sum_{r=1}^\infty\frac{(-1)^r}{r^{3/2}}\frac{r\pi^2\kappa T/\mu_0H}{\sinh(r\pi^2\kappa T/\mu_0H)}\int_0^\infty dx\frac{J_{3/2}(x)}{x^{1/2}}\int_{-1}^1 dz\,J_0(x\sqrt{1-z^2}) \\ &\times\sin\left(\frac{r\pi\zeta}{\mu_0H}+\frac{\mu_0H}{4\pi r\zeta}x^2z^2-\frac{\pi}{4}\right) \end{aligned} \qquad (3\cdot4)$$

where in the last expression we have used the spherical polar coordinate.

Now we use the formulas in the statistical mechanics to obtain the magnetic moment M of our system.*

$$\begin{aligned} &F=N\zeta-\kappa T\log\Xi, \qquad (F: \text{Helmholtz free energy}) \\ &N=\kappa T\frac{\partial}{\partial\zeta}\log\Xi, \\ &M=\kappa T\left(\frac{\partial}{\partial H}\log\Xi\right)_{\zeta,T}, \end{aligned} \qquad (3\cdot5)$$

and for the magnetic susceptibility χ

* We take up only oscillatory part here.

$$\chi = M/H. \tag{3·6}$$

Retaining the lowest order term in H, and T, we get

$$\chi = \chi^{(0)} + \chi^{(I)} \tag{3·7}$$

$$\begin{aligned}\chi^{(I)} &= 2^{9/2}\pi^{-1/2}(\zeta/\zeta_0)^{1/2}(\mu_0 H/\zeta_0)^{3/2} e^2 (2\pi m/h^2)^3 V\zeta_0^{\,3} k_0^{-2} \\ &\times \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{3/2}} \frac{r\pi^2\kappa T/\mu_0 H}{\sinh(r\pi^2\kappa T/\mu_0 H)} \frac{\pi\zeta}{\mu_0 H^3} Q(p) \\ &= N(\mu_0^2/\zeta_0)(2\pi)^{1/2}(\zeta_0/\mu_0 H)^{3/2}(\varepsilon_{ex}/\zeta_0) \\ &\times \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{3/2}} \frac{r\pi^2\kappa T/\mu_0 H}{\sinh(r\pi^2\kappa T/\mu_0 H)} Q(p)\end{aligned} \tag{3·8}$$

$$Q(p) = \cos\left(\frac{r\pi\zeta}{\mu_0 H} - \frac{\pi}{4}\right) C(p) - \sin\left(\frac{r\pi\zeta}{\mu_0 H} - \frac{\pi}{4}\right) S(p), \tag{3·9}$$

$$\left.\begin{matrix} C(p) \\ S(p) \end{matrix}\right\} = \int_0^{\infty} dx \frac{J_{3/2}(x)}{x^{1/2}} \int_{-1}^{1} dz\, J_0(x\sqrt{1-z^2}) \begin{cases} \cos(px^2z^2) \\ \sin(px^2z^2), \end{cases} \tag{3·10}$$

$$p = \mu_0 H/4\pi\zeta, \quad \varepsilon_{ex} = (3/4\pi)e^2 k_0.$$

This can be compared to the free electron value

$$\begin{aligned}\chi^{(0)} &= -N(\mu_0^2/\zeta_0)(3/2\pi)(\zeta_0/\mu_0 H)^{1/2} \sum_{r=1}^{\infty} \frac{(-1)^r}{r^{3/2}} \frac{r\pi^2\kappa T/\mu_0 H}{\sinh(r\pi^2\kappa T/\mu_0 H)} \\ &\times \sin\left(\frac{r\pi\zeta}{\mu_0 H} - \frac{\pi}{4}\right).\end{aligned} \tag{3·11}$$

If we go to the limiting value at 0°K and take the term of $r=1$, we obtain

$$\begin{aligned}\chi^{(0)} &= N(\mu_0^2/\zeta_0)(3/2\pi)(\zeta_0/\mu_0 H)^{1/2} \sin(\pi\zeta_0/\mu_0 H - \pi/4) \\ &= (m_0/m)\nu^{2/3} r_s^{-2} \times 7.673 \times 10^{-2} \times H^{-2} \sin(\pi\zeta_0/\mu_0 H - \pi/4)\end{aligned} \tag{3·12}$$

$$r_s = (3/4\pi)^{1/3}(V_A/N_0)^{1/3}(1/a_0), \quad a_0 = \hbar^2/e^2 m_0$$

and

$$\begin{aligned}\chi^{(I)} &= -N(\mu_0^2/\zeta_0)(2\pi)^{1/2}(\zeta_0/\mu_0 H)^{3/2}(\varepsilon_{ex}/\zeta_0) Q(p_0) \\ &= -\nu r_s^{-3} \times 2.757 \times 10^8 \times H^{-3/2} Q(p_0)\end{aligned} \tag{3·13}$$

$$p_0 = \mu_0 H/4\pi\zeta_0.$$

In these last expressions, the ratio of the effective mass of the electron and the true electronic mass m_0 is important in the interpretation of the de Haas-van Alphen effect of real metallic electron. It is to be noted here that this ratio has disappeared from the 0°K form of $\chi^{(I)}$ in (3·13). As may be seen from the results in (3·12) and (3·13), the comparison of the free electron term $\chi^{(0)}$ and the newly appearing term $\chi^{(I)}$ is a very delicate problem because the presence of the factor m_0/m might drasticaly alter the magnitude of $\chi^{(0)}$ and the behaviour

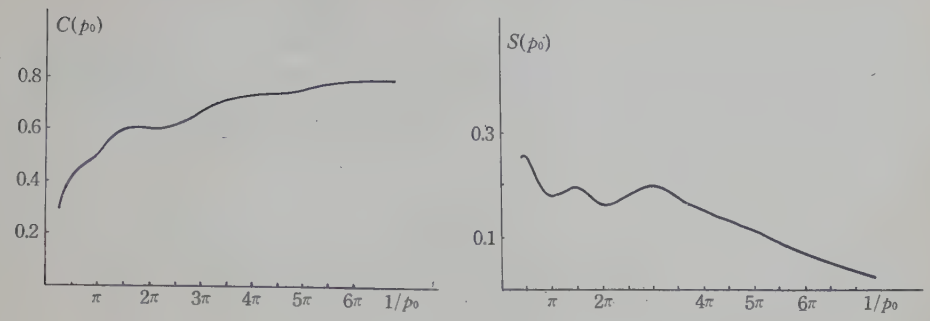


Fig. 8.

of the function $S(p)$, and $C(p)$ with H is very complicated. We have numerically examined the behaviour of these functions for small p as shown in Fig. 8. Considering these behaviours, we may conclude that for fairly large values of $1/H$, the contribution from $S(p_0)$ term in $Q(p_0)$ can be neglected and $C(p_0)$ may be replaced by a constant value $C(0)=\sqrt{2/\pi}$. Then we have

$$\chi^{(I)} \propto \frac{1}{H^{3/2}} \sin\left(\frac{\pi\zeta}{\mu_0 H} - \frac{3}{4}\right). \tag{3·14}$$

So, if in some metals this $\chi^{(I)}$ term might surpass the free electron term $\chi^{(0)}$, there might be a possibility to observe the oscillation of χ which has different field dependence in its amplitude and a different phase constant from those predicted by free electron theory or by modified theory by Dingle.[8]

But, of course, it will be dangerous to draw definite conclusions merely from these results, because the correlation effect* will have the same or larger effect and the phase constant might suffer drastic changes. Here we shall be contented with the proposition that there are some possibility that the different oscillatory behaviours of χ from those predicted by the free electron theory which have been observed in some metals[12] can be interpreted in terms of the Coulomb interaction between electrons.

§ 4. Conclusions

The form of the statistical mechanical propagator which has been derived in this paper will be useful, we think, also for the treatment of the effect of the interelectronic interaction on the other behaviours of electron gas, such as transport process in a strong magnetic field. Although the form in (2·32) will seem very complicated at first sight, suitable simplification will be rather easily introduced by actual applications.

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* Some approximate estimation of the correlation effect may be made by assuming "screened Coulomb potential" e^2e^{-ar}/r. The introduction of screened Coulomb potential does not cause any essential change in the discussion given above, apart from the change of $C(0)$ to

$$C(a)=C(0)\,(1-(a/2)^2\log(a/2)^2-a\tan^{-1}(2/a)+(a/2)^2\log(1+(a/2)^2)),$$

e.g. $C(0,\ 0.6)=0.45C(0)$ and $C(01)=0.29C(0)$.

Finally, we express our sincere thanks to our colleagues in our Institute, especially to Dr. K. Hiroike for his helpful discussions and criticisms. We also express our gratitude to the Scientific Research Foundation of the Ministry of Education for the financial aid.

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Dynamical Theory of Ultrasonic Attenuation in Metals

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A detailed investigation is given of the ultrasonic attenuation governed by dynamical properties of conduction electrons. It is pointed out that the ultrasonic attenuation arises from the same mechanism that is also responsible for the attenuation of collective oscillations in a conduction electron system, known as Landau's attenuation in some case. Applying the electro-magnetic dispersion relations in metals to the problem in hand, the attenuation constants are derived, and at the same time, the effects of conduction electrons on sound velocities are investigated. Some discussions are given of the relation between the results obtained and those of Pippard. Then the parallelism of the present phenomenon with the anomalous skin effect is pointed out and the attenuation constants are derived for the case where the energy of a conduction electron is an arbitrary function of the wave vector. As a by-product, the most general expression for Landau's diamagnetic susceptibility is derived with some applications.

§ 1. Introduction

Ultrasonics has for the last few years been noticed as a useful experimental tool in solid physics.[1] In the first place, it enables us to investigate the character of an interaction between a sound wave and other system (such as conduction electron, magnetization, etc.) as a function of the frequency of the sound wave. In other words, it offers us a knowledge about a dispersion of the interaction. In the second place, it realizes a new phenomenon, a sort of a resonance, often referred to later as a "spatial resonance", which is characterized by the non-uniformity of the electric field generated by a sound wave, and which is, therefore, absent in the ordinary resonance phenomena where uniform electro-magnetic fields are used.[2]

For a particle moving in the non-uniform field, the phase of the field changes in two ways. One is due to the proper vibration of the sound wave with a given frequency, and the other to the Doppler effect. We define as usual that a motion of a particle is coherent or incoherent according as the field has a non-vanishing or vanishing effect on the particle when averaged along the path of its motion. However, for most conduction electrons in metals, the Doppler effect is a dominant factor in determining the phase change; namely, the phase change is determined in the main by the motion of a conduction electron through the non-uniform field which may be considered to be fixed in time compared with the

rapidness of the motion. Therefore, the coherence of the motion in this case is not asked for in connection with the variation of the field in time, as is the case, for example, in the cyclotron resonance, but it is asked for in connection with the variation in space. In this sense, it may be called a "spatial coherence".

Through the spatial coherence between dynamical motions of some perturbed units and the non-uniform electric field, ultrasonics offers also interesting information about the dynamical properties of the perturbed system. The experimental condition for the "spatial resonance" is more easily realizable than it would be if the resonance were of the ordinary nature.

In the present article, our interest is centred in ultrasonic dispersions and attenuations determined by the last-mentioned spatial coherence between conduction electrons and sound waves. For the illustration, we take up an example which is already known as having such a nature and which has an intimate relation with the present phenomenon. This is Landau's attenuation of plasma oscillations in a non-degenerate free electron gas.[3] Actually, any attenuation known so far in collective oscillations of the electro-magnetic origin may be called Landau's attenuation, for they arise from the same mechanism as in Landau's case except when a quantum effect described later plays an essential role.

Landau's work is rather mathematical in nature, and the origin of the attenuation is also mathematically clear. He solved the Boltzmann equation coupled with Maxwell's equations and obtained from the requirement of a self-consistency a dispersion relation for the waves of longitudinal electric field (the plasma oscillations) possible in a non-degenerate free electron gas of infinite dimension. The attenuation arose from a pole in complex p-plane:

$$p+i\boldsymbol{\kappa}\boldsymbol{v}=0,$$

where $\boldsymbol{\kappa}$ is the wave vector of the electric field and $\boldsymbol{v}$ is the velocity of an electron. Since ip signifies a frequency in Landau's notation, the above equation is rewritten as

$$\omega-\boldsymbol{\kappa}\boldsymbol{v}=0, \qquad (1\cdot1)$$

where ω may be considered to be the frequency of the electric field. As is easily seen, $-\boldsymbol{\kappa}\boldsymbol{v}$ represents a frequency shift due to the Doppler effect, so that $\omega-\boldsymbol{\kappa}\boldsymbol{v}$ represents the frequency of the electric field which an electron running with velocity $\boldsymbol{v}$ feels. (1·1) then implies that the attenuation is determined only by those electrons for which the perturbing field happens to be steady as a result of the Doppler effect. It is obvious that those for which (1·1) does not hold do not participate the attenuation, for their velocity-changes due to the electric field are found to be out of phase with the field, so that they gain no energy on the average. Those satisfying (1·1) do not continue to be accelerated steadily,*

* In addition to the steady acceleration, there occurs also a steady deceleration in the electron system. However, the fact that a wave propagates in a specified direction destroys the time reversal symmetry of the system, and it never happens that the acceleration and the deceleration cancel each other to give rise to no attenuation. We may use the description as if there were the steady acceleration alone.

because they are accelerated to change their velocities and (1·1) ceases to be valid for changed velocities. Thus a finite attenuation is present even in the absence of a collision. The mechanism for the attenuation is not limited to the longitudinal waves*.

From the above argument, we can say that the non-uniformity of the field casts a spot light on a particular group of electrons satisfying (1·1). The same can be said also in a more realistic case where electrons have more complex dynamical properties. The artificial variation of the frequency as well as of the wave vector of the field would make it possible to analyze in detail the dynamical properties. This is just the case in the ultrasonic attenuation, both ω and $\boldsymbol{\kappa}$ being now easily controllable. If the electric field in the above argument is taken to be that generated by sound waves, we can proceed with the same argument as above. It will be clear that there exists a finite ultrasonic attenuation due to conduction electrons of infinite mean free time.

For degenerate conduction electrons in metals, we have, instead of (1·1),

$$v_0 \cos\theta = u, \tag{1·2}$$

where v_0 is a Fermi velocity, while u is the sound velocity and θ is the angle between the propagation direction of the sound wave and the Fermi velocity. Since $v_0/u \gtrsim 10^2$ in metals, θ is nearly $\pi/2$. Therefore one can say that the attenuation is determined only by those electrons whose velocities are perpendicular to the propagation direction. The situation strongly resembles that of the anomalous skin effect.[4] It is expected, therefore, that there exists an intimate relation between the ultrasonic attenuation and the surface resistance, a quantity indicating a magnitude of power loss of the microwave cast on a surface of a metallic specimen. It is also expected that we can determine the Fermi surface of a metal with ultrasonics just as Pippard could of copper with microwave.[5]

The quantum mechanical counterpart of (1·1) is the energy conservation:

$$E_{\boldsymbol{k}} - E_{\boldsymbol{k}+\boldsymbol{\kappa}} + \hbar\omega = 0, \tag{1·3}$$

where $E_{\boldsymbol{k}}$ is the energy of an electron with wave vector $\boldsymbol{k}$. In a free electron case, (1·3) is reduced to

$$\omega(1 - L^2\kappa^2) - \boldsymbol{v}\boldsymbol{\kappa} = 0, \tag{1·4}$$

where

$$L^2 = \frac{\hbar}{2m\omega}.$$

L is a quantity having the dimension of length and indicates a quantum effect. If the quantum effect is negligible (this is the case when $\omega \ll 10^{11}$ in metals), (1·4)

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* Prof. S. Nakajima kindly pointed out to the author that such an interpretation can be seen also from Bohm and Gross's work: Phys. Rev. **75** (1949), 1864.

coincides with (1·1), and the use of the Boltzmann equation is justified.

The first three sections of the present article are the preliminaries for the later sections. In § 2, some physical quantities are defined. § 3 is devoted to the determination of electro-magnetic fields in a deformable medium in the presence of sound waves. The content there is partly the same as those of some previous authors' works, especially that of Nozières and Pines.[6] In § 4, the results of § 3 are applied to a degenerate free electron case. In § 5, the simplest hydrodynamical description which is originally due to Steinberg[7] is made on the motions of sound waves in a medium, and the propagation equations for them are obtained. In § 6, our program is performed by the use of the Boltzmann equation comparing the results with those of Pippard.[8] In § 7, detailed investigation is given of the parallelism between the present phenomenon and the anomalous skin effect mentioned above. In Appendix I, diamagnetic susceptibility of conduction electrons is derived with two simple applications. In appendix II, a brief discussion is given on the collective oscillations possible in a free electron system of infinite dimension and Landau's attenuations are derived by our formulation.

§ 2. Definitions

Suppose, in a medium of a perfect crystal, distributions of test charge $\rho^0(\boldsymbol{x}_0, t)$ and test current $\boldsymbol{j}^0(\boldsymbol{x}_0, t)$ with, for the time being, prescribed space-time dependence satisfying the continuity equation

$$\operatorname{div} \boldsymbol{j}^0+\frac{\partial}{\partial t}\rho^0=0, \tag{2·1}$$

where the space derivative is performed with respect to $\boldsymbol{x}_0$.

The medium is affected by these test sources giving rise to resultant charge and current densities which we denote by $\rho(\boldsymbol{x}_0, t)$ and $\boldsymbol{j}(\boldsymbol{x}_0, t)$ respectively. Let the scalar and the vector potentials be denoted by $\phi(\boldsymbol{x}_0, t)$ and $\boldsymbol{A}(\boldsymbol{x}_0, t)$ respectively, which are determined by solving the Poisson equations

$$\Box\phi=-4\pi\rho \tag{2·2a}$$

and

$$\Box\boldsymbol{A}=-\frac{4\pi}{c}\boldsymbol{j} \tag{2·2b}$$

together with the Lorentz condition

$$\operatorname{div}\boldsymbol{A}+\frac{1}{c}\frac{\partial}{\partial t}\phi=0. \tag{2·3}$$

The electro-magnetic fields $\boldsymbol{E}(\boldsymbol{x}_0, t)$ and $\boldsymbol{B}(\boldsymbol{x}_0, t)$ in the medium are given by

$$\boldsymbol{E}=-\operatorname{grad}\phi-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}, \tag{2·4a}$$

and

$$\boldsymbol{B}=\mathrm{rot}\boldsymbol{A}. \tag{2·4b}$$

If we put

$$\rho=\rho^0+\rho^i \tag{2·5a}$$

and

$$\boldsymbol{j}=\boldsymbol{j}^0+\boldsymbol{j}^i, \tag{2·5b}$$

then $\rho^i(\boldsymbol{x}_0, t)$ and $\boldsymbol{j}^i(\boldsymbol{x}_0, t)$ mean respectively the induced charge and current distributions also satisfying the continuity equation

$$\mathrm{div}\,\boldsymbol{j}^i+\frac{\partial}{\partial t}\rho^i=0. \tag{2·6}$$

Hereafter, the Fourier components $X(\boldsymbol{\kappa}, \omega)$ of an arbitrary space-time function $X(\boldsymbol{x}_0, t)$ will be introduced through the transformation

$$X(\boldsymbol{x}_0, t)=\sum_{\boldsymbol{\kappa},\omega}X(\boldsymbol{\kappa}, \omega)e^{i(\boldsymbol{\kappa}\boldsymbol{x}_0-\omega t)}. \tag{2·7}$$

We will also use an expression like $X(\boldsymbol{\kappa}, t)$ which is the transformation of $X(\boldsymbol{x}_0, t)$ in the space coordinates only.

§ 3. Electro-magnetic dispersions

The density matrix of our system, a function of time $f(t)$, is determined by the equation of motion together with the initial condition that the system had been in thermal equilibrium in the absence of a perturbation which was introduced adiabatically at $t=-\infty$. The solution in a matrix element is[5]

$$\langle m|f(t)|n\rangle=f_0(E_n)\delta_{mn}+\langle m|f_1(t)|n\rangle+\cdots, \tag{3·1}$$

where

$$\langle m|f_1(t)|n\rangle=\sum_{\omega}\{f_0(E_n)-f_0(E_m)\}\langle m|H'(\omega)|n\rangle D_{nm}(\omega)e^{-i\omega t} \tag{3·2}$$

with

$$D_{nm}(\omega)=\frac{P}{E_n-E_m+\hbar\omega}-i\pi\delta(E_n-E_m+\hbar\omega). \tag{3·3}$$

(P means to take the principal value.)

$f_0(E_n)$ is the equilibrium distribution function, and E_n is the eigen-value in the eigen-state $|n\rangle$ of the unperturbed Hamiltonian which is assumed in this section to have the periodicity of the crystal. $H'(\omega)$ is a Fourier component of the perturbation $H'(t)$:

$$H'(t)=\sum_{\omega}H'(\omega)e^{-i\omega t} \tag{3·4}$$

with the property

$$H'(-\omega)=H'(\omega)^* \quad (*\text{means the conjugate complex}).$$

In the present case,

$$H'(t)=\sum_{\kappa,\omega}\left[\frac{e}{2mc}\{p_\nu e^{i\kappa x}+e^{i\kappa x}p_\nu\}A_\nu(\kappa,\omega)e^{-i\omega t}-e\phi(\kappa,\omega)e^{i(\kappa x-\omega t)}\right]$$
$$=\sum_{\kappa,\omega}\{-\delta_\nu(\kappa)A_\nu(\kappa,\omega)+\delta_0(\kappa)\phi(\kappa,\omega)\}e^{-i\omega t} \qquad (3\cdot5)$$

with

$$\left.\begin{aligned}\delta_0(\kappa)&=-ee^{i\kappa x},\\ \delta_\nu(\kappa)&=-\frac{e}{2mc}\{p_\nu e^{i\kappa x}+e^{i\kappa x}p_\nu\},\end{aligned}\right\} \qquad (3\cdot6)$$

and where electron charge is taken to be $-e$. The repeated indices are the summation convention.

To be calculated are the charge and current densities which are respectively the statistical averages of the operators

$$\left.\begin{aligned}&-e\delta(x-x_0)=\sum_\kappa\delta_0(\kappa)e^{i\kappa x_0}\\ \text{and}\qquad&\\ &\frac{1}{c}j_\nu(x-x_0)=\sum_\kappa\delta_\nu(\kappa)e^{i\kappa x_0}-\frac{e^2}{mc^2}\sum_{\kappa,\omega}A_\nu(\kappa,\omega)e^{i(\kappa x_0-\omega t)}.\end{aligned}\right\} \qquad (3\cdot7)$$

Namely,

$$\left.\begin{aligned}\rho^i(x_0,t)&=\sum_{\kappa,\omega}\rho^i(\kappa,\omega)e^{i(\kappa x_0-\omega t)}=-e\mathrm{Tr}\{f(t),\delta(x-x_0)\},\\ j_\nu^{\,i}(x_0,t)&=\sum_{\kappa,\omega}j_\nu^{\,i}(\kappa,\omega)e^{i(\kappa x_0-\omega t)}=\mathrm{Tr}\{f(t),j_\nu(x-x_0)\},\end{aligned}\right\} \qquad (3\cdot8)$$

where the system is assumed for simplicity to have a unit volume.

After elementary manipulations, we obtain the linear responses:

$$\rho^i(\kappa,\omega)=-\sum_{n,m}f_0(E_n)\{\langle n|\delta_0(-\kappa)|m\rangle\langle m|\delta_\nu(\kappa)|n\rangle D_{nm}(\omega)$$
$$+\langle n|\delta_\nu(\kappa)|m\rangle\langle m|\delta_0(-\kappa)|n\rangle D_{nm}^*(-\omega)\}A_\nu(\kappa,\omega)$$
$$+\sum_{n,m}f_0(E_n)\{|\langle n|\delta_0(-\kappa)|m\rangle|^2D_{nm}(\omega)$$
$$+|\langle m|\delta_0(\kappa)|n\rangle|^2D_{nm}^*(-\omega)\}\phi(\kappa,\omega), \qquad (3\cdot9a)$$

and

$$\frac{1}{c}j_\mu^{\,i}(\kappa,\omega)=\left[-\frac{ne^2}{mc^2}\delta_{\mu\nu}-\sum_{n,m}f_0(E_n)\{\langle n|\delta_\mu(-\kappa)|m\rangle\langle m|\delta_\nu(\kappa)|n\rangle D_{nm}(\omega)\right.$$
$$\left.+\langle n|\delta_\nu(\kappa)|m\rangle\langle m|\delta_\mu(-\kappa)|n\rangle D_{nm}^*(-\omega)\}\right]A_\nu(\kappa,\omega)$$

$$+\sum_{n,m} f_0(E_n)\{\langle n|\delta_\mu(-\boldsymbol{\kappa})|m\rangle\langle m|\delta_0(\boldsymbol{\kappa})|n\rangle D_{nm}(\omega)$$

$$+\langle n|\delta_0(\boldsymbol{\kappa})|m\rangle\langle m|\delta_\mu(-\boldsymbol{\kappa})|n\rangle D_{nm}^*(-\omega)\}\phi(\boldsymbol{\kappa},\omega). \quad (3\cdot 9b)$$

In the expression of the current, n denotes the electron density. It is shown that the relations (3·9a, b) are gauge invariant and the continuity equation (2·6) holds between ρ^i and $\boldsymbol{j}^i$.* In the simplest case where the system is isotropic, these are reduced to

$$\rho^i(\boldsymbol{\kappa},\omega) = -\frac{1}{\kappa^2}\left(\kappa^2 - \frac{\omega^2}{c^2}\right)\Gamma(\boldsymbol{\kappa},\omega)\ \phi(\boldsymbol{\kappa},\omega), \quad (3\cdot 10)$$

$$\frac{1}{c} j_t^i(\boldsymbol{\kappa},\omega) = -\Delta(\boldsymbol{\kappa},\omega) A_t(\boldsymbol{\kappa},\omega) \quad (3\cdot 11)$$

and

$$\frac{1}{c} j_l^i(\boldsymbol{\kappa},\omega) = \frac{\omega}{\kappa c}\rho^i(\boldsymbol{\kappa},\omega), \quad (3\cdot 11)'$$

where t and l mean the transverse and the longitudinal components respectively, and

$$\Gamma(\boldsymbol{\kappa},\omega) = -\sum_{n,m} f_0(E_n)\{|\langle n|\delta_0(-\boldsymbol{\kappa})|m\rangle|^2 D_{nm}(\omega)$$

$$+|\langle n|\delta_0(\boldsymbol{\kappa})|m\rangle|^2 D_{nm}^*(-\omega)\}, \quad (3\cdot 12)$$

$$\Delta(\boldsymbol{\kappa},\omega) = \frac{ne^2}{mc^2} + \sum_{n,m} f_0(E_n)\{|\langle n|\delta_t(-\boldsymbol{\kappa})|m\rangle|^2 D_{nm}(\omega)$$

$$+|\langle n|\delta_t(\boldsymbol{\kappa})|m\rangle|^2 D_{nm}^*(-\omega)\}. \quad (3\cdot 13)$$

In this reduction, some sum-rule type identities together with (2·3) are employed. In the forms of (3·10) and (3·11), the gauge invariance is obvious.

Now, the Poisson equations (2·2) are written as

$$\left(\kappa^2 - \frac{\omega^2}{c^2}\right)\left(1 + \frac{4\pi}{\kappa^2}\Gamma(\boldsymbol{\kappa},\omega)\right)\begin{pmatrix}\phi(\boldsymbol{\kappa},\omega)\\ A_l(\boldsymbol{\kappa},\omega)\end{pmatrix} = 4\pi\begin{pmatrix}\rho^0(\boldsymbol{\kappa},\omega)\\ (1/c) j_l^0(\boldsymbol{\kappa},\omega)\end{pmatrix} \quad (3\cdot 14a)$$

and

$$\left(\kappa^2 - \frac{\omega^2}{c^2} + 4\pi\Delta(\boldsymbol{\kappa},\omega)\right) A_t(\boldsymbol{\kappa},\omega) = \frac{4\pi}{c} j_t^0(\boldsymbol{\kappa},\omega). \quad (3\cdot 14b)$$

It is seen that

$$\Gamma^*(\boldsymbol{\kappa},\omega) = \Gamma(-\boldsymbol{\kappa},-\omega)$$

and

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* Accordingly, diffusion as well as ohmic current is automatically included in our $\boldsymbol{j}^i(\boldsymbol{\kappa},\omega)$. See ref. 9).

$$\Delta^*(\kappa, \omega) = \Delta(-\kappa, -\omega),$$

so that Γ and Δ as space-time functions are real as it should be.[10] From (3·9b), the most general expression for Landau's diamagnetic susceptibility can be derived. This will be done in Appendix I.

§ 4. Calculations in a degenerate free electron case

When E_n consists only of the kinetic energy of free electrons, and when they are degenerate, $\Gamma(\kappa, \omega)$ and $\Delta(\kappa, \omega)$ are easily calculated. The results are written down.

[1] *The longitudinal case*

$$4\pi\Gamma(\kappa, \omega) = q_l^2\left[\frac{1}{4\lambda}\{F(\lambda_+) + F(\lambda_-)\} + \pi i\gamma\right], \tag{4·1}$$

where

$$\left.\begin{aligned} F(x) &= x + \frac{1}{2}(1-x^2)\log\left|\frac{1+x}{1-x}\right|, \\ \lambda_\pm &= \frac{\kappa}{2k_0} \pm \frac{\omega}{\kappa v_0}, \\ \lambda &= \frac{\kappa}{2k_0}, \end{aligned}\right\} \tag{4·2}$$

and

$$q_l^2 = 4\pi e^2 n(\zeta). \tag{4·3}$$

In these expressions, k_0, v_0 and $n(\zeta)$ are respectively the Fermi wave number, the Fermi velocity and the state density at the Fermi level ζ. q_l^2 is the screening constant appearing in the Thomas-Fermi model. $F(x)$ has the following expansion:

$$F(x) = \begin{cases} 2\left(x - \dfrac{x^3}{3} - \dfrac{x^5}{15} - \cdots\right), & \text{for } x \ll 1, \\ \\ 2\left(\dfrac{1}{3x} + \dfrac{1}{15x^3} + \dfrac{1}{35x^5} + \cdots\right), & \text{for } x \gg 1. \end{cases} \tag{4·4}$$

γ is a function defined by

$$\begin{aligned} \gamma &= 0, && \text{when } \lambda_+ > 1 \text{ and } |\lambda_-| > 1, \\ &= \gamma_-, && \text{when } \lambda_+ > 1 \text{ and } |\lambda_-| < 1, \\ &= \gamma_- - \gamma_+, && \text{when } \lambda_+ < 1 \text{ and } |\lambda_-| < 1, \end{aligned} \tag{4·5}$$

where

$$\gamma_{\pm}=\frac{1}{8\lambda}(1-\lambda_{\pm}^{2}). \tag{4·6}$$

[2] *The transverse case*

$$4\pi\Delta(\boldsymbol{\kappa},\,\omega)=q_t^{2}\left[1-\frac{1}{2\lambda}\{G(\lambda_{+})+G(\lambda_{-})\}-\pi i\eta\right], \tag{4·7}$$

where

$$G(x)=\frac{3}{16}\{(1-x^{2})^{2}\log\left|\frac{1+x}{1-x}\right|+x(-2x^{2}+\frac{10}{3})\} \tag{4·8}$$

and

$$q_t^{2}=4\pi ne^{2}/mc^{2}. \tag{4·9}$$

q_t^{2} is the screening factor appearing in the simplest theory of the superconductivity. $G(x)$ has the following expansion:

$$G(x)=\begin{cases} x-x^{3}-\dfrac{1}{5}x^{5}-\cdots & \text{for } x\ll 1, \\ \dfrac{1}{5x}+\dfrac{1}{35x^{3}}+\cdots & \text{for } x\gg 1. \end{cases} \tag{4·10}$$

η is a function defined by

$$\begin{aligned} \eta&=0, && \text{when } \lambda_{+}>1 \text{ and } |\lambda_{-}|>1, \\ &=\eta_{-}, && \text{when } \lambda_{+}>1 \text{ and } |\lambda_{-}|<1, \\ &=\eta_{-}-\eta_{+}, && \text{when } \lambda_{+}<1 \text{ and } |\lambda_{-}|<1, \end{aligned} \tag{4·11}$$

where

$$\eta_{\pm}=\frac{3}{32}\,\frac{1}{\lambda}(1-\lambda_{\pm}^{2})^{2}. \tag{4·12}$$

γ or η is nothing but the quantum mechanical expression of Landau's attenuation mentioned in § 1. In Fig. 1, the functions γ and η are plotted schematically against κ. In the figure, $\kappa_1^{\pm}$ and $\kappa_2^{\pm}$ are defined by

$$\begin{aligned} \kappa_1^{\pm}&=k_0(\sqrt{1+(\hbar\omega/\zeta)}\pm 1), \\ \kappa_2^{\pm}&=k_0\,(1\pm\sqrt{1-(\hbar\omega/\zeta)}). \end{aligned} \tag{4·13}$$

There is no attenuation in the shaded regions. The critical values $\kappa_1^{\pm}$ are essentially the same as those of Ferrell,[11] with the difference that ω in Ferrell's

case is a function of $\kappa_1^{\pm}$ (see Appendix II), so that $\kappa_1^{\pm}$ must be solved. The ultrasonic attenuation arises from the region III of case B.

In Appendix II, a brief discussion is given of the collective oscillations in a free electron system of infinite dimension.

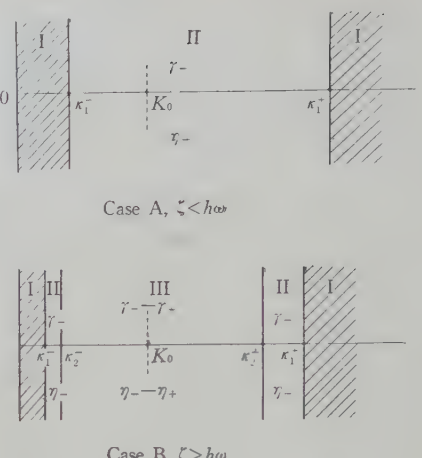


Fig. 1. Schematic representation of functions γ and η.

§ 5. Ultrasonic dispersions and attenuations

In this section, $\rho^0(\boldsymbol{x}_0, t)$ and $\boldsymbol{j}^0(\boldsymbol{x}_0, t)$ are taken to be generated by sound waves in a medium. The linear hydrodynamical equation of motion for a sound wave is

$$\frac{M}{e}\frac{\partial}{\partial t}\boldsymbol{j}_\mu{}^0(\boldsymbol{x}_0, t) = ne\boldsymbol{E}_\mu(\boldsymbol{x}_0, t) - \frac{\partial}{\partial x_\nu}\mathrm{T}_{\mu\nu}(\boldsymbol{x}_0, t), \tag{5·1}$$

where M is the mass of an ion core (positively charged),* and $\mathrm{T}_{\mu\nu}(\boldsymbol{x}_0, t)$ is the pressure tensor arising from that part of interactions between ion cores which is not described in the continuity model. It is assumed that there is one electron per one ion core. The first term on the right side is the volume force exerted on the positive charges and $\boldsymbol{E}(\boldsymbol{x}_0, t)$ in it is given by (2·4a) together with (3·14a). The effect of the magnetic field, $\boldsymbol{B}(\boldsymbol{x}_0, t)$, is shown to be non-linear.

For simplicity, we assume the simplest form of Hook's law:

$$\begin{pmatrix} \mathrm{T}_{xx}(\boldsymbol{\kappa}, \omega) & & \\ & \mathrm{T}_{yy}(\boldsymbol{\kappa}, \omega) & \\ & & \mathrm{T}_{zz}(\boldsymbol{\kappa}, \omega) \end{pmatrix} = \frac{M\kappa}{e\omega}\begin{pmatrix} (u_t{}^0)^2 & & \\ & (u_t{}^0)^2 & \\ & & (u_l{}^0)^2 \end{pmatrix}\begin{pmatrix} j_x{}^0(\boldsymbol{\kappa}, \omega) \\ j_y{}^0(\boldsymbol{\kappa}, \omega) \\ j_z{}^0(\boldsymbol{\kappa}, \omega) \end{pmatrix} \tag{5·2}$$

where z-direction is taken to be the propagation direction of the sound wave, and $u_l{}^0$ and $u_t{}^0$ are parameters in respective directions having the dimension of velocity.

In the Fourier components, (5·1) is rewritten as follows.

[1] *The longitudinal case*

Replacing $j_z{}^0(\boldsymbol{\kappa}, \omega)$ by $\rho^0(\boldsymbol{\kappa}, \omega)$ by the use of (2·1), we have

$$-\frac{M}{e}\frac{i\omega^2}{\kappa}\rho^0(\boldsymbol{\kappa}, \omega) = ne\left\{\frac{i\omega}{c}A_z(\boldsymbol{\kappa}, \omega) - i\kappa\phi(\boldsymbol{\kappa}, \omega)\right\}$$

$$-\frac{i\kappa}{e}M(u_l{}^0)^2\rho^0(\boldsymbol{\kappa}, \omega),$$

or from (2·3) and (3·14a)

* The positive charges are also considered to be smeared out. M/e is the factor which converts the current density into the momentum density.

$$\omega^2\rho^0(\boldsymbol{\kappa},\ \omega)=\left\{\frac{4\pi ne^2}{M}\ \frac{\kappa^2}{\kappa^2+4\pi\Gamma(\boldsymbol{\kappa},\ \omega)}+(u_l^0)^2\kappa^2\right\}\rho^0(\boldsymbol{\kappa},\ \omega),$$

whence follows the dispersion relation

$$\omega^2=\omega_p{}^2\frac{m}{M}\ \frac{\kappa^2}{\kappa^2+4\pi\Gamma(\boldsymbol{\kappa},\ \omega)}+(u_l^0)^2\kappa^2, \tag{5·3a}$$

where ω_p is the electron plasma frequency.

[2] *The transverse case*

The equation of motion is

$$-\frac{M}{e}i\omega j_t^0(\boldsymbol{\kappa},\ \omega)=ne\frac{i\omega}{c}A_t(\boldsymbol{\kappa},\ \omega)-\frac{i\kappa^2}{e\omega}M(u_t^0)^2j_t^0(\boldsymbol{\kappa},\ \omega),$$

or from (3·14b)

$$\omega^2j_t^0(\boldsymbol{\kappa},\ \omega)=\left\{-\frac{4\pi ne^2}{Mc^2}\ \frac{\omega^2}{\kappa^2-(\omega^2/c^2)+4\pi\Delta(\boldsymbol{\kappa},\ \omega)}+(u_t^0)^2\kappa^2\right\}j_t^0(\boldsymbol{\kappa},\ \omega),$$

whence follows the dispersion relation

$$\omega^2=-\omega^2\frac{m}{M}\ \frac{\omega_p{}^2/c^2}{\kappa^2-(\omega^2/c^2)+4\pi\Delta(\boldsymbol{\kappa},\ \omega)}+(u_t^0)^2\kappa^2. \tag{5·3b}$$

For a given ω, the dispersion relations are solved approximately. Using the expansion formulae (4·1) and (4·7) for $x\ll1$, we have

$$4\pi\Gamma(\boldsymbol{\kappa},\ \omega)=q_l^2\left(1+\pi i\frac{\omega}{2\kappa v_0}\right) \tag{5·4a}$$

and

$$4\pi\Delta(\boldsymbol{\kappa},\ \omega)=q_t^2\left\{3\left(\frac{\omega}{\kappa v_0}\right)^2-\frac{3}{4}\pi i\frac{\omega}{\kappa v_0}\right\}. \tag{5·4b}$$

Since $|\lambda_\pm|$ are much less than unity for practical values of κ, the last expressions of (4·5) and (4·11) have been employed. From (5·3a, b) and (5·4a, b), we finally get: the sound velocities (adiabatic velocities)

$$u_l=\left[(u_l^0)^2+\frac{m}{3M}v_0^2\right]^{1/2}, \tag{5·5a}$$

$$u_t=u_t^0\left[1+\frac{16m}{3\pi^2M}\left\{1+\frac{1}{3}\left(\frac{\omega v_0}{q_t(u_t^0)^2}\right)^2\right\}\right]^{-1/2}\fallingdotseq u_t^0, \tag{5·5b}$$

and the attenuation constants

$$\alpha_l=\frac{\pi mv_0\omega}{12Mu_l^2}, \tag{5·6a}$$

$$\alpha_t=\frac{2mv_0\omega}{3\pi Mu_t^2}. \tag{5·6b}$$

The conditions for (5·5a) and (5·6a) are

$$\left(\left(\frac{\kappa}{q_t}\right)^2,\ \frac{\kappa}{2k_0}\right) \ll \frac{\omega}{\kappa v_0} \ll 1, \tag{5·7a}$$

which are satisfied in metals when $\omega \ll 10^{10}$, and those for (5·5b) and (5·6b) are

$$\left(\left(\frac{\kappa}{q_t}\right)^2,\ \frac{\kappa}{2k_0}\right) \ll \frac{\omega}{\kappa v_0} \ll 1, \tag{5·7b}$$

which are satisfied when $\omega \ll 10^8$.

In case

$$\frac{\kappa}{2k_0} \ll \frac{\omega}{\kappa v_0} \ll \left(\frac{\kappa}{q_t}\right)^2, \tag{5·8}$$

we have

$$\begin{aligned} u_t &= u_t^0\left[1+\frac{m}{M}\frac{q_t^2(u_t^0)^2}{\omega^2}\right]^{-1/2} \\ \alpha_t &= \frac{3\pi}{8}\frac{m}{M}\frac{q_t^2 u_t^2}{\omega v_0}. \end{aligned} \tag{5·9}$$

α_t decreases as ω increases. We cannot expect a similar behaviour in the longitudinal case, for inequalities corresponding to (5·8) do not hold for practical values of q_t and other parameters. (Note that $q_l^2 > 10^2 q_t^2$.) The second inequality in (5·8) is the condition for the polarization of the electrons to be negligible (see (3·11)):

$$\frac{4\pi \Delta(\kappa, \omega)}{\kappa^2} \ll 1. \tag{5·10}$$

The quantity on the left side signifies the extent to which the transverse sound wave is affected by the electrons. As ω increases, κ increases, while $\Delta(\kappa, \omega)$ is kept nearly constant for a considerably wide range of ω. It follows, therefore, that sound waves with frequencies high enough propagate almost unaffectedly by electrons.

If there is any connection between our results and those of Pippard, it is expected that this must be with Pippard's case in which $\kappa l \gg 1$ and $\omega\tau \gg 1$, where τ and $l(=v_0\tau)$ are respectively the mean free time and the mean free path of the Fermi electrons, because in our treatment these have been assumed to be infinite. However, contrary to this expectation (5·6a, b) coincide with Pippard's formulae[8] claimed by himself to be applicable to the case in which $\kappa l \gg 1$ but $\omega\tau \ll 1$.*

In the next section our calculations will be repeated with the use of the Boltzmann equation. As was pointed out in § 1, its use is conditioned by

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* At present the case $\omega\tau \gg 1$ is far from being realized.

$$(L\kappa)^2 \ll 1,$$

which holds practically in our problem.*

Judging from the results of the next section together with the fact that Pippard's formulae in question are in agreement with experiments in their applicable regions supposed, it may be concluded that the condition for a collision to be negligible is not $\omega\tau \gg 1$ (note that it holds $\kappa l \gg \omega\tau$ in metals) as we have first considered, but it is $\kappa l \gg 1$ irrespectively of the value of $\omega\tau$. This is in support of our statement in § 1 that the attenuation in this case is due to the spatial resonance of a group of electrons with the non-uniform electric field generated by sound waves. Moreover, the condition $\kappa l \gg 1$ is more easily realizable than $\omega\tau \gg 1$.

§ 6. The Boltzmann equation method

The existing concept is that if the thermal contact between the electron system and the surroundings (the interaction systems) in the medium is strong, the electron system will always be nearly in local thermal equilibrium under the electro-magnetic fields due to sound waves. This concept is displayed in the Boltzmann equation method by putting the collision term in a form

$$-\frac{f(t)-f_e(t)}{\tau}, \qquad (6\cdot1)$$

where $f_e(t)$ is the local equilibrium distribution, $f(t)$ is the actual distribution and τ denotes a collision time. What sort of a collision is responsible for setting up the local equilibrium is not *a priori* obvious. However, the experimentally observed parallelism between the attenuation and the electric resistance is suggestive enough to show that collisions causing the electric resistance are also of such a nature as $f(t)$ approaches directly or indirectly to $f_e(t)$ in consequence.**

Now in this method, the program is straightforward. The solution of the Boltzmann equation enables us to calculate $\rho^i(\kappa, \omega)$ and $j^i(\kappa, \omega)$, hence $\Gamma(\kappa, \omega)$ and $\Delta(\kappa, \omega)$ successively, and (5·3a,b) then give the desired formulae. The basic equation is

$$\frac{\partial f}{\partial t} + \{f(t), H_0\} + \{f_0, H'(t)\} = -\frac{f(t)-f_e(t)}{\tau},$$

where H_0 is the kinetic energy of a free electron, and $H'(t)$ is the perturbation given by (3·5). The curly brackets are the Poisson brackets. In the Fourier components, it is rewritten as

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* Through a private communication with Pippard, the author is recently informed of the fact that in his new paper: Proc. Roy. Soc. **257** (1960), 165, he extended his mothod to the case in which $\omega\tau \gg 1$, and obtained the results which are essentially the same as those of the present article.

** Strictly speaking, $f_e(t)$ should be conjectured from this fact!

$$(-i\omega+i\boldsymbol{\kappa v})f(\boldsymbol{\kappa},\omega)-i(\boldsymbol{\kappa v})\left\{-e\phi(\boldsymbol{\kappa},\omega)+\frac{e}{c}(\boldsymbol{A}(\boldsymbol{\kappa},\omega)\cdot\boldsymbol{v})\right\}\frac{\partial f_0}{\partial E}$$

$$=-\frac{f(\boldsymbol{\kappa},\omega)-f_e(\boldsymbol{\kappa},\omega)}{\tau}, \tag{6·2}$$

where $\boldsymbol{v}$ is the velocity of a free electron. For $\tau\to\infty$, we have

$$f(\boldsymbol{\kappa},\omega)=-\left\{\frac{P}{\omega-\boldsymbol{\kappa v}}-\pi i\delta(\omega-\boldsymbol{\kappa v})\right\}(\boldsymbol{\kappa v})\left\{-e\phi(\boldsymbol{\kappa},\omega)\right.$$

$$\left.+\frac{e}{c}(\boldsymbol{A}(\boldsymbol{\kappa},\omega)\cdot\boldsymbol{v})\right\}\frac{\partial f_0}{\partial E}. \tag{6·3}$$

After simple calculations along our program, we obtain for $\omega/v_0\kappa<1$*

$$4\pi\Gamma(\kappa,\omega)=q_i^2\left\{1-\frac{\omega}{2\kappa v_0}\log\left|\frac{\kappa v_0+\omega}{\kappa v_0-\omega}\right|+\pi i\frac{\omega}{2\kappa v_0}\right\}, \tag{6·4a}$$

$$4\pi\Delta(\kappa,\omega)=q_i^2\left\{\frac{3}{2}\left(\frac{\omega}{\kappa v_0}\right)^2+\frac{3\omega}{4\kappa v_0}\left(1-\left(\frac{\omega}{\kappa v_0}\right)^2\right)\log\left|\frac{\kappa v_0+\omega}{\kappa v_0-\omega}\right|\right. \tag{6·4b}$$

$$\left.-\frac{3}{4}\pi i\frac{\omega}{\kappa v_0}\left(1-\left(\frac{\omega}{\kappa v_0}\right)^2\right)\right\}.$$

For $\omega/\kappa v_0>1$, the imaginary terms vanish.

Comparison of (6·4a, b) with (5·4a, b) shows that both results are the same up to the terms of order of $(\omega/\kappa v_0)^2$, so that we have the same dispersion relations up to this order. Note that the condition for (6·3) is

$$\kappa l\gg 1, \tag{6·5}$$

as is verified easily.

Now, it is shown that a simply extrapolative application of Pippard's method to the case of a very long τ seems unjustifiable. Pippard's original calculations were based on the assumption that regardless of the values of τ, the attenuation at a time t is determined by $\Delta v'(t)=v'(t)-v_e'(t)$, where $v'(t)$ is a velocity distribution of the electron system deformed by sound waves, and $v_e'(t)$ is the local equilibrium one under the same deformation. This leads to the following. If the electron system is supposed to be in local equilibrium at an instant, then sound waves lose no energy at least at that instant, because, as is involved in the assumption, there occurs no net energy transfer from the electron system to the interaction systems.

Indeed, this assumption may be justified when τ is short enough, for the attenuation in this case is determined by the way of energy transfer mentioned above. However, the local equilibrium becomes a redundant concept as τ tends

* Some authors overlooked $A(\kappa,\omega)$-dependent term in (3·7). This makes the legitimacy of their results doubtful.

to be infinite, because the attenuation is now determined by dynamical properties of the electron system independently of collision mechanisms. That is to say, it participates now only with those electrons for which the perturbing fields happen to be steady as a result of the Doppler effect. So long as v_0 is larger than u_l or u_t, which is the case in metals, such electrons always exist and are accelerated steadily by the field even when the electron system happens to be in local equilibrium. Thus we may conclude that the above assumption is unjustifiable in the case of a very long mean free time.*

In case $\kappa l \ll 1$, $f_e(t)$ is an essential quantity. We have tried some calculations assuming various forms of $f_e(t)$. To begin with, Pippard's assumption of the collision drag[8] led to some difficulties in our program. For instance, we found that it led to a distribution which seems inconsistent in that charge and current densities calculated are not gauge invariant, nor they satisfy the continuity equation. Other forms of $f_e(t)$ assumed contained likewise more or less difficulties, though we could reach some formulae closely resembling those of Pippard after some unconvincing procedures.**

Although Pippard's formulae for $\kappa l \ll 1$ experience experimental confirmation, they need some revisions from a theoretical standpoint, which will lead us necessarily to the substantial problems of irreversible processes. To be calculated is, for instance,

$$\Gamma(\boldsymbol{\kappa}, \omega) = -\frac{1}{i\hbar}\int_0^\infty dt \langle [e^{(iH_0 t/\hbar)}\delta_0(-\boldsymbol{\kappa})\, e^{(-iH_0 t/\hbar)},\ \delta_0(\boldsymbol{\kappa})]\rangle e^{i\omega t}, \tag{6·6}$$

where

$$\langle X\rangle = \mathrm{Tr}\{f_0(H_0),\ X\} \tag{6·7}$$

with $f_0(H_0)$ denoting the equilibrium distribution of a system with Hamiltonian H_0, and the square bracket means a commutator. If necessary, all the variables are understood to be second quantized. It is easy to see that (6·6) reduces to (3·12) in the simplest case (a suitable convergence factor is involved in ω). Our interest is how to reduce (6·6) when, for instance, the electron-thermal-phonon interaction or an impurity potential is included in H_0.

§ 7. Connection with anomalous skin effect

In the general case where the energy of a conduction electron is an arbitrary function of the wave vector $\boldsymbol{k}$, (6·3) gives

$$4\pi\Gamma(\boldsymbol{\kappa}, \omega) = \left[q_l^2 - 4\pi^2 ie^2\omega \sum_{\boldsymbol{k}} \frac{\partial f_0}{\partial E_k} \cdot \delta(\omega - \boldsymbol{\kappa}\boldsymbol{v}) \right], \tag{7·1a}$$

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* See, however, the footnote of p. 13.

** Yet, Pippard's results should be noticeable, because his results agree fairly well with experiment even quantitatively.[1] In this respect, his intuition overwhelms our foolish carefulness.

and

$$4\pi \Delta_{\mu\nu}(\boldsymbol{\kappa}, \omega) = -4\pi^2 i \frac{e^2}{c^2} \omega \sum_{k} v_\mu v_\nu \frac{\partial f_0}{\partial E_k} \delta(\omega - \boldsymbol{\kappa}\boldsymbol{v}) \qquad (7\cdot 1b)$$

$$(\mu, \nu = x, y)$$

where

$$v_\mu = \frac{1}{\hbar} \frac{\partial E_k}{\partial k_\mu}, \qquad (7\cdot 2)$$

and the sum is taken over the $\boldsymbol{k}$-space. By writing (7·1a, b), we have neglected some terms of order of u_l/v_0 or u_t/u_0. The real parts of Γ and Δ are influenced only to a negligible amount by the detailed structure of E_k. This is a very fortunate fact, because this enables us to analize the Fermi surface with the attenuation experiment even though sound velocities have anisotropies caused from some other origins.

Application of a simple solid geometry gives

$$-\sum_{k} \frac{\partial f_0}{\partial E_k} \delta(\omega - \boldsymbol{\kappa}\boldsymbol{v}) = \frac{\hbar}{4\pi^3 \kappa} \int_C \frac{\rho \sin\theta ds}{|\mathrm{grad}\, E_k|^2}, \qquad (7\cdot 3a)$$

$$-\sum_{k} v_x^2 \frac{\partial f_0}{\partial E_k} \delta(\omega - \boldsymbol{\kappa}\boldsymbol{v}) = \frac{1}{4\pi^3 \kappa \hbar} \int_C \frac{(\partial E/\partial k_x)^2}{|\mathrm{grad} E_k|^2} \rho \sin\theta ds. \qquad (7\cdot 3b)$$

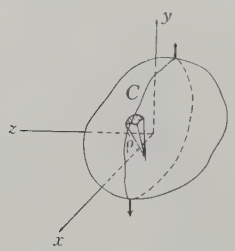


Fig. 2a

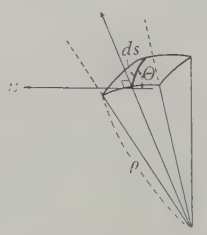


Fig. 2b

The integration is carried out along the path (not necessarily a simple, closed one) on the Fermi surface having the property that it holds at any point of the path that

$$\cos\beta = \frac{\omega}{|\kappa| v_0}, \qquad (7\cdot 4)$$

where β is the angle between z-direction (the propagation direction) and the normal at that point. In the integrand, ρ is the radius of curvature and θ denotes an angle between z-direction and a line element ds. These are illustrated in Fig. 2. Fig. 2b is the magnification of a sphere element (the shaded part) of Fig. 2a. It is a simple geometrical problem to see that the integrand of (7·3b) is identical with

$$\int \rho_y \, dy \tag{7·5}$$

in Pippard's notation.[5]* Without losing the generality, we can take x- and y-directions to be the principal axes of tensor $\Delta_{\mu\nu}(\boldsymbol{\kappa}, \omega)$. Then we have

$$4\pi\Gamma(\boldsymbol{\kappa}, \omega) = q_l^2\left[1+\frac{i\hbar\omega}{4\pi n(\zeta)|\kappa|}\int_C \frac{\rho \sin\theta ds}{|\mathrm{grad}\, E_k|^2}\right], \tag{7·6a}$$

and

$$4\pi\Delta_\mu(\boldsymbol{\kappa}, \omega) = -iq_t^2\frac{m\omega}{4\pi^2 n\hbar|\kappa|}\int_C \frac{(\partial E/\partial k_\mu)^2}{|\mathrm{grad} E_k|^2}\rho \sin\theta \, ds. \tag{7·6b}$$

$$(\mu = x, y.)$$

Assuming again for simplicity that u_l^0 and u_t^0 in (5·2) are isotropic, we obtain from (5·3a, b)

$$u_l = \left[(u_l^0)^2 + \frac{m}{M}\left(\frac{\omega_p}{q_l}\right)^2\right]^{1/2}, \tag{7·7a}$$

$$u_t \doteqdot u_t^0, \tag{7·7b}$$

and

$$\alpha_l = \omega\frac{m\hbar e^2\omega_p^2}{2Mu_l^2 q_l^4\pi}\int_C \frac{\rho \sin\theta ds}{|\mathrm{grad}\, E_k|^2}, \tag{7·8a}$$

$$(\alpha_t)_\mu = \frac{2\pi^2 n\hbar}{Mu_t^2}\left[\int_C \frac{(\partial E/\partial k_\mu)^2}{|\mathrm{grad}\, E_k|^2}\rho \sin\theta ds\right]^{-1}. \tag{7·8b}$$

The validity conditions for (7·7a, b) and (7·8a, b) are the same as before, that is, (5·7a, b) and (6·5) with due interpretation of k_0 and v_0.

From (7·8b), we obtain an important formula

$$\frac{2\pi^2 n\hbar\omega}{Mu_t^2(\alpha_t)_\mu} = \int_C \frac{(\partial E/\partial k_\mu)^2}{|\mathrm{grad}\, E_k|^2}\rho \sin\theta ds, \tag{7·9}$$

or

$$\frac{2\pi^2 n\hbar\omega}{Mu_t^2}\left[\frac{1}{(\alpha_t)_x}+\frac{1}{(\alpha_t)_y}\right] = \int_C \rho \sin\theta ds, \tag{7·9}'$$

which should be compared with the surface resistance in the anomalous skin effect. The general expression for the surface impedance can be derived by Nakajima's technique.[12] Suppose that an electromagnetic field with a given frequency propagates in z-direction with the electric field vector in x-direction (one of the prin-

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* Note that our ρ differs from Pippard's ρ in the dimension.

cipal axes) and that the medium is divided into two parts in such a way that the region $z>0$ is occupied by a metal and the region $z<0$ is empty. Assumption of a specular boundary scattering leads to the equation

$$\left[\kappa^2-\frac{\omega^2}{c^2}+4\pi\Delta(\kappa,\ \omega)\right]A_x(\kappa,\ \omega)=2\left(\frac{\partial}{\partial z}A_x(z,\ \omega)\right)_{z=0},$$

whence follows the surface impedance $Z_x(\omega)$

$$Z_x(\omega)=R_x(\omega)+iX_x(\omega)=4\pi\left(-\frac{1}{c^2}\frac{\partial}{\partial t}A_x(z,\ \omega)\Big/\frac{\partial}{\partial z}A_x(z,\ \omega)\right)_{z=0}$$

$$=\frac{8\pi i\omega}{c^2}\sum_{\kappa}\frac{1}{\kappa^2-(\omega^2/c^2)+4\pi\Delta(\kappa,\ \omega)}, \tag{7·10}$$

where $R_x(\omega)$ and $X_x(\omega)$ are the surface resistance and reactance respectively, and are real quantities.

Neglecting ω^2/c^2 in the denominator and putting

$$4\pi\Delta(\kappa,\ \omega)\sim\frac{-ig^3}{|\kappa|}, \tag{7·11}$$

where g is defined by identifying (7·11) with (7·6b), the summation of (7·10) can be carried out by a contour integration to give

$$Z_x(\omega)=\frac{8\pi\omega}{3c^2}\frac{((1/\sqrt{3})-i)}{g}$$

or

$$\left[\frac{3c^2\sqrt{3}}{8\pi\omega}R_x(\omega)\right]^{-3}\frac{4\pi^2n\hbar}{m\omega q_t^2}=\int_C\frac{(\partial E/\partial k_x)^2}{|\mathrm{grad}\ E_k|^2}\rho\sin\Theta ds. \tag{7·12}$$

From (7·9) and (7·12), we get

$$\frac{(\alpha_t)_x}{(\alpha_t^0)_x}=\left[\frac{R_x(\omega)}{R_x^0(\omega)}\right]^3, \tag{7·13}$$

where $(\alpha_t^0)_x$ and $R_x^0(\omega)$ are the quantities which the system would have if the conduction electrons were supposed to be free electrons. $(\alpha_t^0)_x$ is given by (5·6a), while

$$R_x^0(\omega)=\frac{8}{9}\left(\frac{\sqrt{3}\pi\omega^2mv_0}{ne^2c^4}\right)^{1/3}. \tag{7·14}$$

Assumption of a diffuse surface scattering, on the other hand, replaces the right side of (7·13) by[13]

$$\left[\frac{R_x(\omega)}{R_x^{0\prime}(\omega)}\right]^3,$$

where

$$R_x^{0\prime}(\omega)=\left(\frac{\sqrt{3}\pi\omega^2 m v_0}{ne^2c^4}\right)^{1/3}. \tag{7·15}$$

The conditions for (7·14) and (7·15) are:

$$\frac{\omega}{gv_0}\ll 1, \tag{7·16a}$$

for the asymptotic formula (7·11) to be applicable,

$$\frac{g}{2k_0}\ll\frac{\omega}{gv_0}, \tag{7·16b}$$

for the classical description (the use of the Boltzmann equation) to be valid, and

$$gl\gg 1, \tag{7·16c}$$

for scattering effects to be negligible. Neglect of the displacement current is justified by (7·16a). Since g is a quantity of order of $(q_t^2\omega/v_0)^{1/3}$, it is $\sim 10^4$ in metals for $\omega=10^{10}$, so that (7·16a, b) are fulfilled.

It is now clear that the measurement of the attenuation anisotropies enables us to analize the Fermi surface just as that of the surface resistance does. The anisotropies of u_t^0 or u_t^0, if present, do not give rise to any serious difficulties in the analysis unless there exist anisotropies in the attenuation itself caused from other mechanisms than what we have been considering. The apprehension for such undesirable excess anisotropies will be a great drawback in the experiment with ultrasonics which is absent in the anomalous skin effect case, but we have one advantage over the previous experiment in the fact that the surface treatment of a specimen will not have to be so severe this time.

It is desirable to derive a mathematical method which reduces experimental data of the attenuation anisotropy directly to a Fermi surface. This is the inverse transformation to (7·8a) or (7·9). Such a transformation is known in other case.[14]

§ 8. Summary and discussions

The hydrodynamical equations of motion coupled with Maxwell's equations give the propagation equations for sound waves in a medium:

$$\omega^2=\omega_p^2\frac{m}{M}\frac{\kappa^2}{\kappa^2+4\pi\Gamma(\boldsymbol{\kappa},\omega)}+(u_t^0)^2\kappa^2, \tag{8·1a}$$

$$\omega^2=-\omega^2\frac{m}{M}\frac{\omega_p^2/c^2}{\kappa^2-(\omega^2/c^2)+4\pi\Delta(\boldsymbol{\kappa},\omega)}+(u_t^0)^2\kappa^2, \tag{8·1b}$$

with

$$\Gamma(\boldsymbol{\kappa},\omega)=-\frac{1}{i\hbar}\int_0^\infty dt\, e^{i\omega t}\langle[e^{(iH_0t/\hbar)}\,\delta_0(-\boldsymbol{\kappa})\,e^{(-iH_0t/\hbar)},\,\delta_0(\boldsymbol{\kappa})]\rangle, \tag{8·2a}$$

$$\Delta(\boldsymbol{\kappa}, \omega) = \frac{ne^2}{mc^2} + \frac{1}{i\hbar}\int_0^\infty dt\, e^{i\omega t}\langle [e^{(iH_0 t/\hbar)}\delta_t(-\boldsymbol{\kappa})e^{(-iH_0 t/\hbar)}, \delta_t(\boldsymbol{\kappa})]\rangle, \quad (8\cdot 2\mathrm{b})$$

where H_0 is the total Hamiltonian of the medium, and where the isotropy of the medium and dispersion-freedom of ${u_l}^0$ and ${u_t}^0$ are assumed. When a degenerate free electron system is the medium, and when quantum effects are negligible (the condition for this is (7·16b)), (8·1a, b) lead to the attenuation constants (5·6a, b) agreeing with Pippard's results with corrected restrictive condition (6·5).

In case where interactions are included in H_0, our formulation is of no practical use at present, for we have no prescription how to treat interaction terms in (8·2a, b). Careless use of a damping theoretical method, for instance, will be likely to lead us to an error.

It is interesting to note that, when ${u_l}^0$ in (8·1a) or ${u_t}^0$ in (8·1b) is put to zero, we have dispersion relations for the collective oscillations in a system of heavy free positive charges the interaction between which is screened by light electrons. If the screening effect ($\Gamma(\boldsymbol{\kappa}, \omega)$) were to be removed, (8·1a) would give a plasma oscillation, but the screening changes it into a sound wave type oscillation. This is already known.[15] Similar behaviour is not observed in the transverse case, the screening being weak this time. The weakness of the screening is also the reason why the attenuation of a transverse sound wave is of the same order as that of a longitudinal one, though the original interaction with the electrons is weaker in the transverse case.

The argument in the last paragraph puts forth a question that one might have to examine the effects of transverse lattice vibrations on the electric resistance. Although there can be no direct effects as is generally recognized, there can be effects through the induction force as is just the case in the ultrasonic attenuation.

In this question, an important problem will be the screening due to conduction electrons. The author investigated it in another connection.[16] The results of § 3 of the present article also will be the solution in a sense. Our inference is then that if the screening is determined dynamically ($\kappa l \gg 1$), and if $|\lambda_\pm|$ difined by (4·2) are much less than unity, the screening is negligible,* so that transverse lattice vibrations would have some effects on the electric resistance which

* An objection seems possible as follows. Every charged particle in metals has its own correlation hole around it, as a result of which the interaction between any such particles is strongly screened. Hence a motion of the particle does not necessarily mean an electric current, because it is followed by the hole. However, the objection is incorrect, otherwize, it leads, first of all, to the absurdity that motions of conduction electrons in metals do not give rise to any effects on the outside (as would be the case in a superconductor), each electron being accompanied by the correlation hole of Bohm-Pines.[17] Indeed, the hole screens the static interaction, but not the effect of the motion of the particle (the electric current), for the same hole does not always localize around it. The screening of the electric current arises from other mechanism which we have investigated in detail in § 3 or in ref. 16).

might be comparable with those of longitudinal ones. In case where the collision drag effect might determine the screening, as is expected to be the case when $\kappa l \ll 1$, we cannot say anything definite. Anyhow, further investigations are required.

Finally, one conjecture is given of what happens if the medium becomes a superconducting state. In this state, we can put

$$4\pi \Delta(\boldsymbol{\kappa}, \omega) \simeq q_t^2 \tag{8·3}$$

in the lowest approximation.[16] (We are not in a position to discuss the imaginary part of $\Delta(\boldsymbol{\kappa}, \omega)$.) Then we get from (5·3a) the sound velocity

$$u_t' = u_t^0 \left[1+\frac{m}{M}\right]^{-1/2}, \tag{8·4}$$

and comparison of this with (5·5b) gives

$$\frac{u_t - u_t'}{u_t} = \frac{m}{2M}\left[1-\frac{16}{3\pi^2}\left\{1+\frac{1}{3}\left(\frac{\omega v_0}{q_t (u_t^0)^2}\right)^2\right\}\right]. \tag{8·5}$$

The quantity on the right side is of order 10^{-5} for a typical superconductor. Thus it is possible in principle that the difference in the screening of electromagnetic interactions in the normal and the superconducting states yields also the difference in sound velocities.

The author wishes to express his sincere thanks to Prof. Z. Mikura and other members of a group of solid state physics in Tohoku University for stimulating discussions, and to Dr. T. Kamigaki for discussing the present experimental situation of the phenomenon.

Appendix I

Landau's diamagnetism

When $\omega=0$ and $\phi(\boldsymbol{\kappa}, \omega)=0$, (3·9b) is simplified to

$$\frac{1}{c} j_\mu^{\,i}(\boldsymbol{\kappa}, 0) = -\mathrm{T}_{\mu\nu}(\boldsymbol{\kappa}) A_\nu(\boldsymbol{\kappa}, 0), \tag{AI·1}$$

where

$$\mathrm{T}_{\mu\nu}(\boldsymbol{\kappa}) = \frac{ne^2}{mc^2}\delta_{\mu\nu} + \sum_{n,m} f_0(E_n)\left[\frac{\langle n|\delta_\mu(-\boldsymbol{\kappa})|m\rangle\langle m|\delta_\nu(\boldsymbol{\kappa})|n\rangle}{E_n - E_m} + \frac{\langle n|\delta_\nu(\boldsymbol{\kappa})|m\rangle\langle m|\delta_\mu(-\boldsymbol{\kappa})|n\rangle}{E_n - E_m}\right]. \tag{AI·2}$$

Let the induced magnetization be denoted by $\boldsymbol{M}^i(\boldsymbol{x}_0)$, which is connected with $\boldsymbol{j}^i(\boldsymbol{x}_0)$ by the relation

$$\frac{1}{c}\boldsymbol{j}^i(\boldsymbol{x}_0) = \mathrm{rot}\, \boldsymbol{M}^i(\boldsymbol{x}_0). \tag{AI·3}$$

The magnetic susceptibility tensor $\boldsymbol{\chi}(\boldsymbol{\kappa})$ is defined by the algebraic equation

$$M_{\mu}^{i}(\boldsymbol{\kappa}) = \chi_{\mu\nu}(\boldsymbol{\kappa}) B_{\nu}(\boldsymbol{\kappa}) \tag{AI·4}$$

or from (2·4b), (AI·1) and (AI·3),

$$\mathbf{T}(\boldsymbol{\kappa}) = \boldsymbol{\kappa} \times \boldsymbol{\chi}(\boldsymbol{\kappa}) \times \boldsymbol{\kappa}, \tag{AI·5}$$

whose solution is*

$$\boldsymbol{\chi}(\boldsymbol{\kappa}) = \frac{\boldsymbol{\kappa} \times \mathbf{T}(\boldsymbol{\kappa}) \times \boldsymbol{\kappa}}{\kappa^4}. \tag{AI·6}$$

There exists an ambiguity in the solution, for we can add to $\boldsymbol{\chi}(\boldsymbol{\kappa})$ any tensor $\boldsymbol{\chi}'(\boldsymbol{\kappa})$ having the property that $\boldsymbol{\kappa} \times \boldsymbol{\chi}'(\boldsymbol{\kappa})$ and/or $\boldsymbol{\chi}'(\boldsymbol{\kappa}) \times \boldsymbol{\kappa}$ vanish. For $\boldsymbol{\kappa} \neq 0$, the ambiguity is suppressed by the requirements: div $\boldsymbol{M}(\boldsymbol{x}_0) = 0$ and div $\boldsymbol{B}(\boldsymbol{x}_0) = 0$. However, they remain for $\boldsymbol{\kappa} = 0$. It follows, therefore, that once a direction of $\boldsymbol{\kappa}$ is fixed, some components of $\boldsymbol{\chi}(0)$ cannot be determined. In order to obtain a full expression of $\boldsymbol{\chi}(0)$, we must perform the calculations of $\mathbf{T}(\boldsymbol{\kappa})$ at least three times changing the directions of $\boldsymbol{\kappa}$'s. The diamagnetic susceptibility in a uniform magnetic field is

$$\boldsymbol{\chi} = \lim_{\boldsymbol{\kappa} \to 0} \boldsymbol{\chi}(\boldsymbol{\kappa}) = \lim_{\boldsymbol{\kappa} \to 0} \frac{\boldsymbol{\kappa} \times \mathbf{T}(\boldsymbol{\kappa}) \times \boldsymbol{\kappa}}{\kappa^4}. \tag{AI·7}$$

For a free electron system, taking $\boldsymbol{\kappa}$ in z-direction, (4·1) gives

$$\mathbf{T}(\boldsymbol{\kappa}) = -\begin{pmatrix} \chi_L & 0 & \Delta \\ 0 & \chi_L & \Delta \\ \Delta & \Delta & \Delta \end{pmatrix} \kappa^2$$

with

$$\chi_L = -\frac{ne^2}{mc^2} \left(\frac{1}{2k_0}\right)^2 \tag{AI·8}$$

being Landau's susceptibility.[18] The components marked with Δ in $\mathbf{T}(\boldsymbol{\kappa})$ are undetermined by the above reasoning. The results when $\boldsymbol{\kappa}$ is taken to be x- and y-direction are obvious, and these three results supplement each other to give

$$\boldsymbol{\chi} = \chi_L \mathbf{I} \tag{AI·9}$$

with $\mathbf{I}$ a unit matrix.

In a model case where the energy of an electron is an arbitrary function of the wave vector, $E(\boldsymbol{k})$, while the wave functions are the plane waves, we must reformulate our basic formulae based on the Hamiltonian $E(\boldsymbol{p} + (e/c)\boldsymbol{A})$. We obtain after all (AI·1) with the first term of (AI·2) replaced by

---

* For example, $(\boldsymbol{\kappa} \times \chi(\boldsymbol{\kappa}))_{xy} = \kappa_y \chi_{zy}(\boldsymbol{\kappa}) - \kappa_z \chi_{yy}(\boldsymbol{\kappa})$.

$$\frac{e^2}{\hbar^2 c^2}\sum_k f_0(E_k)\frac{\partial^2 E_k}{\partial k_\mu \partial k_\nu},$$

and $\delta_\mu(\boldsymbol{\kappa})$ there defined by

$$\delta_\mu(\boldsymbol{\kappa}) = -\frac{e}{2c}\left\{\frac{\partial E(\boldsymbol{p})}{\partial p_\mu}e^{i\kappa z} + e^{i\kappa z}\frac{\partial E(\boldsymbol{p})}{\partial p_\mu}\right\}$$

where $\mu, \nu = x, y$ and $\boldsymbol{\kappa} = (0, 0, \kappa)$.

After elementary but lengthy calculations, we get

$$\mathrm{T}_{\mu\nu}(\boldsymbol{\kappa}) = -\frac{e^2}{12c^2\hbar^2}\sum_k \kappa^2 \frac{\partial f_0}{\partial E_k}\left\{\frac{\partial^2 E_k}{\partial k_\mu \partial k_\nu}\frac{\partial^2 E_k}{\partial k_z^2} - \frac{\partial^2 E_k}{\partial k_\mu \partial k_z}\frac{\partial^2 E_k}{\partial k_\nu \partial k_z}\right\}$$

$$-\frac{e^2}{4c^2\hbar^2}\sum_k \kappa^2 f_0 \frac{\partial^4 E_k}{\partial k_\mu \partial k_\nu \partial k_z^2}, \qquad \text{(AI·10)}$$

whence

$$\chi_{xy} = \frac{-e^2}{12c^2\hbar^2}\sum_k\left[\frac{\partial f_0}{\partial E_k}\left\{\frac{\partial^2 E_k}{\partial k_x \partial k_y}\frac{\partial^2 E_k}{\partial k_z^2} - \frac{\partial^2 E_k}{\partial k_x \partial k_z}\frac{\partial^2 E_k}{\partial k_y \partial k_z}\right\}\right.$$

$$\left. + 3f_0\frac{\partial^4 E_k}{\partial k_x \partial k_y \partial k_z^2}\right], \qquad \text{(AI·11)}$$

$$\chi_{xx} = \frac{e^2}{12c^2\hbar^2}\sum_k\left[\frac{\partial f_0}{\partial E_k}\left\{\frac{\partial^2 E_k}{\partial k_y^2}\frac{\partial^2 E_k}{\partial k_z^2} - \left(\frac{\partial^2 E_k}{\partial k_y \partial k_z}\right)^2\right\}\right.$$

$$\left. + 3f_0\frac{\partial^4 E_k}{\partial k_x \partial k_y \partial k_z^2}\right]. \qquad \text{(AI·12)}$$

The other components can be obtained by obvious permutations of the suffices. The first two terms in curly brackets of (AI·12) were derived by Peierls, Wilson and other authors, and the last term by Hebborn and Sondheimer.[19] (AI·7) is a generalization of Schafroth's method.[20]

Appendix II

Collective oscillations in an electron sytem

If we define a four-dimensional current vector $\mathbf{J}$ by

$$\frac{1}{c}\mathbf{J} \equiv \frac{1}{c}(j_0, j_1, j_2, j_3) = (i\rho, \frac{1}{c}\boldsymbol{j}),$$

and similarly a vector potential $\mathbf{A}$ by

$$\mathbf{A} \equiv (A_0, A_1, A_2, A_3) = (i\phi, \boldsymbol{A}),$$

then (3·9a, b) are collected into a single algebraic equation

$$\frac{1}{c}\mathrm{J}_i(\boldsymbol{\kappa}, \omega) = \mathrm{K}_{ij}(\boldsymbol{\kappa}, \omega)A_j(\boldsymbol{\kappa}, \omega), \qquad \text{(AII·1)}$$

where

$$K_{ij}(\boldsymbol{\kappa}, \omega) = -\frac{ne^2}{mc^2}\delta_{\mu\nu} - \sum_{n,m} f_0(E_n)\{\langle n|\delta_i(-\boldsymbol{\kappa})|m\rangle\langle m|\delta_j(\boldsymbol{\kappa})|n\rangle D_{nm}(\omega) + \langle n|\delta_j(\boldsymbol{\kappa})|m\rangle\langle m|\delta_i(-\boldsymbol{\kappa})|n\rangle D_{nm}^*(-\omega)\} \quad \text{(AII·2)}$$

with

$$\delta(\boldsymbol{\kappa}) = (i\delta_0(\boldsymbol{\kappa}),\ \boldsymbol{\delta}(\boldsymbol{\kappa})).$$

Greek indices run from 0 to 3, while Latin indices from 1 to 3.*

Poisson equations are written as

$$\left(\kappa^2 - \frac{\omega^2}{c^2}\right) A_i(\boldsymbol{\kappa}, \omega) = -4\pi K_{ij}(\boldsymbol{\kappa}, \omega) A_j(\boldsymbol{\kappa}, \omega) + \frac{4\pi}{c} J_i^0(\boldsymbol{\kappa}, \omega).$$

The necessary condition that there exist non-vanishing solutions of $\mathbf{A}(\boldsymbol{\kappa}, \omega)$ in the absence of $\mathbf{J}^0(\boldsymbol{\kappa}, \omega)$ is

$$\det\left|\kappa^2 - \frac{\omega^2}{c^2} + 4\pi K_{ij}(\boldsymbol{\kappa}, \omega)\right| = 0, \quad \text{(AII·3)}$$

which is nothing but the dispersion relation for the collective oscillations in the medium, or, what amounts to the same, for the propagation of an electromagnetic wave in the medium.[21] In the isotropic case, (AII·3) is divided into two equations, which are from (3·14a, b) :

$$\kappa^2 + 4\pi\Gamma(\boldsymbol{\kappa}, \omega) = 0, \quad \text{(AII·4)}$$

for the longitudinal oscillation, and

$$\kappa^2 - \frac{\omega^2}{c^2} + 4\pi\Delta(\boldsymbol{\kappa}, \omega) = 0, \quad \text{(AII·5)}$$

for the transverse one, each of which contains two equations, for $\Gamma(\boldsymbol{\kappa}, \omega)$ and $\Delta(\boldsymbol{\kappa}, \omega)$ are complex.

For degenerate free electrons, the real part of (AII·4) is reduced to

$$\omega^2(\kappa) = \omega_p^2\left\{1 + \frac{3}{5}\left(\frac{\kappa}{q_l}\right)^2 + \left(\frac{612}{175}\right)\cdot\left(\frac{\kappa}{q_l}\right)^4 + L^4\kappa^4 + \cdots\right\}, \quad \text{(AII·6)}$$

---

* In the notation of § 6,

$$K_{ij}(\boldsymbol{\kappa}, \omega) = \frac{ne^2}{mc^2}\delta_{\mu\nu} - \frac{1}{i\hbar}\int_0^\infty dt\ e^{i\omega t}\langle[e^{(iH_0t/\hbar)}\ \delta_i(-\kappa)\, e^{(-iH_0t/\hbar)},\ \delta_j(\boldsymbol{\kappa})]\rangle.$$

Sum-rules are

$$\frac{1}{\pi}\int_{-\infty}^{\infty} K_{0\mu}(\boldsymbol{\kappa}, \omega)\, d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} K_{\mu 0}(\boldsymbol{\kappa}, \omega) d\omega = -\kappa_\mu \frac{ne^2}{mc},$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} K_{00}(\boldsymbol{\kappa}, \omega) d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{K_{\mu\nu}(\boldsymbol{\kappa}, \omega) - \frac{ne^2}{mc^2}\delta_{\mu\nu}\right\} d\omega = 0.$$

and that of (AII·5) to

$$\omega^2(\kappa)=\omega_p{}^2+\left(c^2-\frac{v_0{}^2}{5}\right)\kappa^2+\frac{v_0{}^2}{5\omega_p{}^2}\left(c^2-\frac{22}{35}v_0{}^2\right)\kappa^4+\cdots, \qquad \text{(AII·7)}$$

both for small κ's. L is a parameter having the dimension of length and indicating a quantum effect:

$$L^2=\frac{\hbar}{2m\omega_p}. \qquad \text{(AII·8)}$$

The imaginary parts of (AII·4 and 5) yield the inequalities:

$$\kappa<\kappa_1{}^- \text{ or } \kappa>\kappa_1{}^+, \qquad \text{(AII·9)}$$

where

$$\kappa_1{}^\pm=k_0\left(\sqrt{1+\frac{\hbar\omega(\kappa_1{}^\pm)}{\zeta}}\pm 1\right). \qquad \text{(AII·10)}$$

(AII·9) is the requirement that κ should belong to the region I of Fig. 1, where there is no attenuation at all (see also reference 11)).

When the medium is not isotropic, there is a coupling between the longitudinal and transverse oscillations. One can say, for instance, that an excited plasma of a frequency in the medium radiates an electromagnetic wave of the same frequency, or that an incident electromagnetic wave attenuates in the medium by exciting not only the transverse oscillations, but also the plasmas. When the coupling is too strong, the collective modes have not so simple structure as to be called "longitudinal" or "transverse" even approximately.

For non-degenerate free electrons, elementary calculations give

$$4\pi\Gamma(\kappa,\omega)=\kappa^2\left[\frac{1}{\sqrt{\pi}(a\kappa)^2}\int_0^\infty d\xi e^{-\xi^2}\frac{\mu_+\mu_--\xi^2}{(\mu_-{}^2-\xi^2)(\mu_+{}^2-\xi^2)}\right.$$

$$\left.+i\sqrt{\frac{\pi}{2}}\frac{1}{(a\kappa)^3}\frac{\sinh(L^2\omega/a^2\omega_p)}{L^2/a^2}\exp\left\{\frac{-1}{2(a\kappa)^2}\left(\left(\frac{\omega}{\omega_p}\right)^2+(L\kappa)^4\right)\right\}\right], \qquad \text{(AII·11a)}$$

and

$$4\pi\Delta(\kappa,\omega)=\frac{\omega_p{}^2}{c^2}\left[\frac{2}{\sqrt{\pi}}\int_0^\infty d\xi e^{-\xi^2}\frac{(\mu_+\mu_-)^2-\mu_+\mu_--\xi^2(\mu_+{}^2+\mu_-{}^2)+\xi^2(\xi^2+1)}{(\mu_+{}^2-\xi^2)(\mu_-{}^2-\xi^2)}\right.$$

$$\left.-i\sqrt{\frac{\pi}{2}}\frac{1}{(a\kappa)}\frac{\sinh(L^2\omega/a^2\omega_p)}{L^2/a^2}\exp\left\{\frac{-1}{2(a\kappa)^2}\left(\left(\frac{\omega}{\omega_p}\right)^2+(L\kappa)^4\right)\right\}\right], \qquad \text{(AII·11b)}$$

in which a is the well-known Debye length:

$$a^2=\frac{kT}{4\pi ne^2}, \qquad \text{(AII·12)}$$

and

$$\mu_\pm=\frac{1}{\sqrt{2}a\kappa}\left\{\pm\left(\frac{\omega}{\omega_p}\right)+L^2\kappa^2\right\}. \qquad \text{(AII·13)}$$

For $a\kappa \ll 1$ and $L \ll a$, we get from (AII·4) and (AII·5) the dispersion relations

$$\omega_t^2 = \omega_p^2\left[\left\{1+\frac{3}{2}(a\kappa)^2+\cdots\right\} -2i\left\{\sqrt{\frac{\pi}{8}}\,\frac{1}{(a\kappa)^3}\exp(-2(\kappa a)^2)+\cdots\right\}\right], \qquad \text{(AII·14a)}$$

$$\omega_t^2 = c^2\kappa^2+\omega_p^2\left[\left\{1-(a\kappa)^2-\frac{3}{2}(a\kappa)^4-\cdots\right\} -2i\left\{\sqrt{\frac{\pi}{8}}\,\frac{1}{(a\kappa)}\exp(-2(\kappa a)^2)+\cdots\right\}\right]. \qquad \text{(AII·14b)}$$

The first imaginary term in ω_t is Landau's attenuation (Eq. (17) of ref. 3)). The propagation of a light in an ionosphere is governed by (AII·14b).

Note added in proof : A Russian paper has come to the author's notice : A. B. Migdal, J. E. T. P. **34** (1958), 1438, in which (5·3a) is obtained with the use of the Green function method.

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The Non-leptonic Decays of Hyperons

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The non-leptonic decays of hyperons are investigated based on the Sakata model and the V-A theory of weak interactions. The Λ-decay and the Σ-decay are treated in a unified way. In the latter case, a possibility of interpreting the experimental evidences is given provided that the structure of Σ-hyperon satisfies some suitable conditions.

§ 1. Introduction and summary

In this article, we shall discuss the non-leptonic decays of hyperons on the basis of the Sakata model.[1] The starting points of our approach are as follows:

(1) *The assumption of (extended) universal V-A interaction*[2]

According to the Sakata model, the proton (P), the neutron (N) and the lambda (Λ) are regarded to be the basic fields of the strongly interacting particles from which the other mesons and hyperons should be constructed. In this model, the extended form of universal V-A coupling of weak interaction including strange particles is rather uniquely determined by introducing the so-called "strangeness-violating current" $(\bar{P}v_\lambda\Lambda)$, where $v_\lambda \equiv \gamma_\lambda(1+\gamma_5)$. The non-leptonic weak processes of strange particles are then induced by the Hamiltonian:[2]

$$H_w = f:(\bar{N}v_\lambda P)(\bar{P}v_\lambda\Lambda): + \text{herm. conj.} \qquad (1)$$

where f is the Fermi coupling constant $\approx 1\times10^{-49}$ erg cm^3.

It should be remarked that thus extended Hamiltonian (1) satisfies the symmetry property of weak interactions which was first pointed out by Gamba, Marshak and Okubo (G.M.O.).[3] In fact, according to G.M.O., (1) is to be regarded as the counter part of the μ-e decay interaction and necessary to exist in nature. Sakata, Nakagawa and the present authors[4] have proposed a unified model of elementary particles in which the basic particles of the Sakata model (P, N and Λ) themselves are further considered to be some kind of compound systems composed of ν, e^- and μ^- and a new sort of matter B^+ respectively. This model also supports the G. M. O.-symmetry.

(2) *The lowest configuration approximation is used for the composite particles.*

In dealing with the non-leptonic decays of strange particles, we are inevitably confronted with a complexity of treating the two- or many-body composite systems. It is doubtful whether or not the structure of such a system is to be determined

as a quantum mechanical system. In spite of this difficulty, we assume that the wave functions representing the composite particles are not at all meaningless and that, in Fock and Tamm-Dancoff's sense, the lowest configuration amplitudes for these particles play a dominant role. This approximation will be formulated in the subsequent sections in a covariant form. We shall not, however, discuss the strong interaction dynamics in any explicit way, although it plays a vital role for the Sakata model.

(3) *Finally, we shall neglect the effective two-fermion interaction such as* $\lambda(\bar{\Lambda}N)$ *which will enter our problem through the virtual processes indicated, for example, by the diagram Fig. 1.*

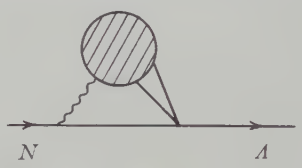


Fig. 1.

The origin of this effect is essentially the same as that of the self-energy divergence and the physical reality of such a transition is more or less doubtful. This point of view may be contrasted with that of Oneda and others,[5] for they supposed that the above-mentioned interaction should be predominant in the strangeness violating weak processes and the value of this coupling was estimated to be $\lambda \sim 10^{-8}\, m^2$.* It is worthwhile to notice here that the introduction of two-body interaction is not preferable in the pionic decay of Σ-hyperons in so far as this interaction is a dominant one.[6]

Starting from these assumptions, we shall treat in the next section (§ 2) the pionic decay of Λ-hyperons. It will be seen that the results obtained are essentially equivalent with those of the theory developed by Okubo, Marshak and Sudarshan (O. M. S.).[7] § 3 is devoted to the discussion for the Σ-decays and a possible way of explaining the experimental information is given under certain conditions imposed on the structure of the composite Σ-hyperons. The non-leptonic decay of Ξ is also discussed. According to our theory, it is found that the decay rate of the mode $\Xi^- \to \Lambda + \pi^-$ is twice that of the mode $\Xi^0 \to \Lambda + \pi^0$.

Not discussed are the K-meson decays. It seems to us that the straightfoward application of our method will fail to account for the fast decay of K_1^0 into two pions, whereas the other modes of kaon decays might be well explained.[8]

§ 2. The non-leptonic decay of Λ

The general definition for the S-matrix elements of weak processes is given, in the lowest order perturbation for weak interaction, as

$$S_{fi} = i(2\pi)^4 \delta(P^f - P^i) \langle f\,; \text{out} \,|H_w(0)|\, i\,; \text{in} \rangle, \qquad (2\cdot1)$$

where the state $|i\,; \text{in}\rangle$ (or $|f\,; \text{out}\rangle$) means the initial (or final) state with outgoing (or incoming) wave boundary condition. P^i (or P^f) is the energy-

* m denotes the mass of nucleons. Throughout this paper we use the natural unit: $\hbar = c = 1$.

momentum eigenvalue belonging to the state $|i\rangle$ (or $|f\rangle$). $H_w(0)$ is the weak interaction Hamiltonian density at the origin.

We will now start our discussion with the matrix element of the decay mode $\Lambda \to P + \pi^-$:

$$\mathfrak{M}_\Lambda^- \equiv \langle \pi^-,\ P\ ;\ \text{out}\ |H_w(0)|\Lambda\rangle. \tag{2·2}$$

Then, $\mathfrak{M}_\Lambda^-$ can be easily transformed as follows :[9]

$$\mathfrak{M}_\Lambda^- = f\left(\frac{m}{p_0}\right)^{1/2}\left(\frac{M}{p_{\Lambda 0}}\right)^{1/2}(Z_P Z_\Lambda)^{-1/2}\int d^4x d^4y e^{-ip_P x + ip_\Lambda y}\,\bar{u}(p)$$
$$\times(\gamma\partial_x + m)(\gamma^T\partial_y - M)G_\Lambda(x, 0, y)\,u(p_\Lambda), \tag{2·3}$$

where

$$G_\Lambda(x, 0, y) \equiv \langle \pi^-|T[P(x)(\bar{N}v_\lambda P)(\bar{P}v_\lambda \Lambda(0))\bar{\Lambda}(y)]|\Omega\rangle. \tag{2·4}$$

In these expressions, p and p_Λ denote the 4-momenta of the nucleon and the Λ respectively, M the mass of Λ, $|\pi^-\rangle$ and $|\Omega\rangle$ are the one-pion and the vacuum state, $u(p)$ is the Dirac wave function of the nucleon (or Λ) normalized as $\bar{u}(p)u(p) = u^*(p)\gamma_4 u(p) = 1$, finally Z_P and Z_Λ represent the renormalization factors of propagators for the proton and the Λ respectively.

As is well-known, $G_\Lambda(x, 0, y)$ is nothing but the Bethe-Salpeter amplitude for the pion. We then rewrite this amplitude by the sum of the Feynman-amplitudes which is defined through the N-product of field operators.* Neglecting all higher order configurations of the Feynman-amplitude, we have an approximate expression for G_Λ :

$$G_\Lambda(x, 0, y) \simeq \langle \Omega|T[P(x)(\bar{P}v_\lambda \Lambda(0))\bar{\Lambda}(y)]|\Omega\rangle\langle \pi^-|T[\bar{N}v_\lambda P(0)]|\Omega\rangle. \tag{2·5}$$

This corresponds to the diagram Fig. 2.** If we omit the vertex-part correction, we obtain the matrix element in the form :

$$\mathfrak{M}_\Lambda^- \simeq \left(\frac{m}{p_0}\right)^{1/2}\left(\frac{M}{p_{\Lambda 0}}\right)^{1/2} f(\bar{u}_P v_\lambda u_\Lambda)\,\mathrm{Tr}[v_\lambda \tilde{\varphi}_{\pi^-}(0)] \tag{2·6}$$

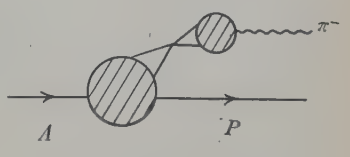


Fig. 2.

with

$$\tilde{\varphi}_{\pi^-}(0) \equiv \langle \pi^-|T[P(x)\bar{N}(y)]|\Omega\rangle|_{x=y=0}. \tag{2·7}$$

For simplicity, we put $f \approx g_V \approx g_A$. ($g_V$ and g_A are the observed vector and axial

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* The T-product of the Heisenberg operators is decomposed into the sum of N-products and vacuum functions by the formula : [11]

$$T(AB\cdots Z) = N(AB\cdots Z) + \sum\langle \Omega|T(AB\cdots K)|\Omega\rangle N(L\cdots Z).$$

The sum is to be taken over all possible decompositions.

** As was mentioned in §1, we have discarded the diagram including the effective two-body interaction of the form $(\bar{\Lambda}N)$.

vector interaction constants.) The second factor in (2·6) may be evaluated by the method of Goldberger and Treiman,[10] and from the experimental decay rate of $\pi\to\mu+\nu$ it reads

$$f\mathrm{Tr}[v_\lambda\tilde{\varphi}_{\pi^-}(0)]\cong(-i)k_\lambda\frac{1}{\sqrt{2k_0}}(-0.13)\sqrt{2}\,Gmg_A/2\pi^2, \tag{2·8}$$

where $G^2/4\pi\approx 14$ is the renormalized Yukawa interaction constant and k_λ is the four-momenta of π^-.

In the same way, the matrix element for the process $\Lambda\to N+\pi^0$ can be evaluated under the same approximations to give

$$\mathfrak{M}_\Lambda^{\,0}\cong\left(\frac{m}{p_0}\right)^{1/2}\left(\frac{M}{p_{\Lambda 0}}\right)^{1/2}f(\bar{u}_N v_\lambda u_\Lambda)\frac{1}{\sqrt{2}}\mathrm{Tr}[v_\lambda\tilde{\varphi}_{\pi^0}(0)] \tag{2·9}$$

with

$$\tilde{\varphi}_{\pi^0}(0)\equiv\frac{1}{\sqrt{2}}\langle\pi^0|T[P(x)\bar{P}(y)-N(x)\bar{N}(y)]|\Omega\rangle_{x=y=0}. \tag{2·10}$$

(2·6) and (2·9) are essentially the "Born approximation" discussed in detail by O. M. S.

§ 3. The decay of Σ-hyperons

Let us now turn to the Σ-decays. The three pionic modes

(a) $\Sigma^+\to N+\pi^+$

(b) $\Sigma^+\to P+\pi^0$

(c) $\Sigma^-\to N+\pi^-$

are now fairly well investigated by several experiments.[12] The up-down asymmetry of pion in each decay mode has been measured: $\alpha^+\bar{P}\approx 0.02\pm 0.07$, $\alpha^0\bar{P}\approx 0.70\pm 0.30$, $\alpha^-\bar{P}\approx 0.02\pm 0.05$ in which $\bar{P}$ is the average value of polarization of parent hyperons, and the decay rates (a) : (b) : (c) is found to be nearly equal to 1 : 1 : 1.

Decay mode (a)

The matrix element for this process $\mathfrak{M}_{\Sigma^+}$ is

$$\mathfrak{M}_{\Sigma^+}=if(Z_N Z_\pi)^{-1/2}\left(\frac{m}{p_0}\right)^{1/2}\int d^4x d^4y\,\bar{u}(p)\frac{1}{\sqrt{2k_0}}e^{ipx-iky}(\gamma\partial_x+m)$$
$$\times(\Box_y-\mu^2)G_{\Sigma^+}(y,x,0), \tag{3·1}$$

where

$$G_{\Sigma^+}(y,x,0)\equiv\langle\Omega|T[\varphi(y)\bar{N}(x)(\bar{N}v_\lambda P)(\bar{P}v_\lambda\Lambda(0))]|\Sigma^+\rangle. \tag{3·2}$$

Here k is the momentum of the pion φ (of mass μ) and Z_π means the renormalization constant of pion propagator. G_{Σ^+} may be approximated along the similar way to the Λ's case and we obtain

$$G_{\Sigma}^{+}(y,x,0)\cong\langle\Omega|T[\varphi(y)N(x)\bar{P}(0)]|\Omega\rangle v_{\lambda}$$
$$\times\langle\Omega|T[\Lambda(\bar{N}v_{\lambda}P(0))]|\Sigma^{+}\rangle, \qquad (3\cdot3)$$

from which the matrix element $\mathfrak{M}_{\Sigma}^{+}$ is expressed as

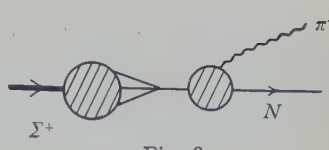


Fig. 3.

$$\mathfrak{M}_{\Sigma}^{+}\cong f\langle\pi^{+},\ N;\ \text{out}\,|\bar{P}(0)|\Omega\rangle v_{\lambda}\langle\Omega|T[\Lambda(\bar{N}v_{\lambda}P(0))]|\Sigma^{+}\rangle. \qquad (3\cdot4)$$

This corresponds to the diagram in Fig. 3. When we suppose the spin of the Σ-hyperons to be 1/2, the second factor in (3·4) generally takes the form:

$$\gamma_{\lambda}[\alpha(M'^{2})+\beta(M'^{2})\gamma_{5}]u_{\Sigma}(p_{\Sigma}) \qquad (3\cdot5)$$

where p_{Σ} is the energy-momentum of the Σ (of mass M') and u_{Σ} is a properly normalized spinor. At first sight, it is inconsistent to neglect the contribution of $(\bar{\Lambda}N)$-transition since Fig. 3 contains a similar two-fermion transition like $(\bar{\Sigma}^{+}P)$.

We should remark, however, that this objection may fail, since the Σ and the Λ are not regarded as the members of the same hyperonic multiplet but as the particles of quite different structures, according to the present composite model. The first factor of (3·4) is the pion-nucleon interaction. If we suppose that the pion interacts mainly through the P-wave to the nucleon as has been well established in low energy meso-dynamics, the non-relativistic version for $\mathfrak{M}_{\Sigma}^{+}$ is (in the rest system of Σ)

$$\mathfrak{M}_{\Sigma}^{+}\approx f Z_{p}^{1/2}\sqrt{2}\,G\frac{1}{\sqrt{2k_{0}}}\left(\frac{1}{2}\right)\left[1-\frac{1}{2}\left(\frac{\mu}{M'-m}\right)^{2}+\cdots\right]$$
$$\times\frac{4(\alpha(M'^{2})-\beta(M'^{2}))}{M'}\,\bar{u}_{n}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{|\boldsymbol{k}|}u_{\Sigma}. \qquad (3\cdot6)$$

It is evident from (3·6) that the P-wave character of emitted pion is preserved irrespective of the relative parity of Σ to the nucleon.

Mode (c)

In a similar way, the corresponding function G_{Σ}^{-} for this mode is evaluated to be

$$G_{\Sigma}^{-}\cong\langle\Omega|T[\varphi^{*}(y)(\bar{N}v_{\lambda}P)]|\Omega\rangle\langle\Omega|T[N(x)(\bar{P}(x)v_{\lambda}\Lambda(0))]|\Sigma^{-}\rangle, \qquad (3\cdot7)$$

and the matrix element $\mathfrak{M}_{\Sigma}^{-}$ is

$$\mathfrak{M}_{\Sigma}^{-}\cong f\,\mathrm{Tr}[v_{\lambda}\tilde{\varphi}_{\pi^{-}}(0)]\langle N|\bar{P}v_{\lambda}\Lambda(0)|\Sigma^{-}\rangle. \qquad (3\cdot8)$$

(See Fig. 4.)

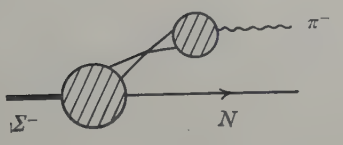


Fig. 4.

The second factor in (3·8) is determined, in principle, when we know the wave function for the composite Σ-particles. We here, however, introduce an assumption on the structure of Σ. Namely, we suppose that there occurs no spin-flip transition between Σ^{-} and N. Then we can write

$$\langle N|\bar{P}v_\lambda \Lambda(0)|\Sigma^-\rangle \approx \zeta(\bar{u}\gamma_\lambda u_\Sigma), \tag{3·9}$$

where ζ is a constant to be determined later. The meaning of this assumption is clarified as follows. Suppose that the Σ, say Σ^-, is essentially a bound state of N and K^--meson, then the weak transition from Σ^- to N may be caused via the transition $\langle\pi^0|\bar{P}v_\lambda\Lambda(0)|K^-\rangle$. If the K^--meson is pseudo-scalar, this matrix element is equivalent to $\langle\pi^0|\bar{P}\gamma_\lambda\Lambda(0)|K^-\rangle$ and thus the relation (3·9) can be reproduced. Another way of interpreting (3·9) is to treat Σ as a 3-body bound state of static baryons. The configuration of Σ^- is represented in general as

$$\Sigma^- \propto \alpha_1 N(\bar{P}\Lambda) + \alpha_2 \boldsymbol{\sigma} N(\bar{P}\boldsymbol{\sigma}\Lambda), \tag{3·10}$$

provided that all the constituent particles are bound in the S-states. The first term is the singlet spin state of $\bar{P}$ and Λ, whereas the second corresponds to the spin-triplet state. If we assume that $\alpha_1 \gg \alpha_2$, then the relation (3·9) is again satisfied . Under the assumption (3·9), the pion is emitted in the S-state relative to the nucleon. The matrix element then, in the rest system for Σ, takes the form :

$$\mathfrak{M}_\Sigma^- \approx \frac{fk_0}{\sqrt{2k_0}}(-0.13)\sqrt{2}\,G\frac{m\zeta}{2\pi^2}(\bar{u}u_\Sigma). \tag{3·11}$$

Mode (b)

We are now in a position to discuss the mode (b). By using the results of preceding investigations, it is easy to see that the matrix element $\mathfrak{M}_\Sigma^0$ for this mode can be expressed as

$$\mathfrak{M}_\Sigma^0 = \mathfrak{M}_\Sigma^{(1)} + \mathfrak{M}_\Sigma^{(2)}$$

with

$$\left.\begin{aligned}\mathfrak{M}_\Sigma^{(1)} &\cong f\langle \pi^0,\ P;\ \text{out}\,|\,\bar{P}(0)\,|\,\Omega\rangle v_\lambda \langle \Omega|\,T[P(0)\,(\bar{N}v_\lambda\Lambda(0))]\,|\,\Sigma^+\rangle,\\ \mathfrak{M}_\Sigma^{(2)} &\cong f\frac{1}{\sqrt{2}}\mathrm{Tr}[v_\lambda\tilde{\varphi}_{\pi^0}(0)]\langle P|\,\bar{N}v_\lambda\Lambda(0)\,|\Sigma^+\rangle.\end{aligned}\right\} \tag{3·12}$$

The first factor in $\mathfrak{M}_\Sigma^{(1)}$ is apparently $1/\sqrt{2}$ times of (3·4). Therefore, if the decay rate of (a) and (b) is equal, then we have (a) : (b) : (c) $\cong$ 1 : 1 : 1 in accordance with experiments. In terms of the experimental branching ratio (a) : (c) $\cong$ 1 : 1, we obtain the second condition for the structure of Σ particles, that is, we find the relation

$$|\alpha-\beta| \approx 6\times10^{-4}\zeta m^2 \tag{3·13}$$

by using the ratio $|\mathfrak{M}_\Sigma^-|^2/|\mathfrak{M}_\Sigma^+|^2 \approx 1$. Further, by comparing $\mathfrak{M}_\Sigma^-$ with $\mathfrak{M}_\Lambda^-$, we obtain the value $|\zeta|^2 \approx 1$ from the experimental ratio.* As is seen in (3·12), $\mathfrak{M}_\Sigma^0$ is a superposition of P- and S-wave amplitude of equal magnitude with respect to the emitted pions. The large asymmetry parameter ($|\alpha^0| > 0.7$) is a natural

* More precisely, we have $0.95 < |\zeta| < 1.4$, taking the experimental errors into account.

consequence of our discussion, although the sign of α^0 cannot be determined.

Thus we have described here a possible way of explaining the non-leptonic decay of Σ on the basis of the composite model and the extended V-A interaction (1). The validity of the assumptions made on the structure of Σ should be further investigated from a wider point of view. In this respect, it would be interesting to discuss the decay of Ξ-hyperons. Two modes are possible in this case:

$$\Xi^- \to \Lambda + \pi^-$$

$$\Xi^0 \to \Lambda + \pi^0.$$

The straightfoward application of our method gives a decay amplitude of the similar structure to that of the mode $\Sigma^- \to N + \pi^-$. In place of the matrix element $\langle N|\bar{P} v_\lambda \Lambda|\Sigma^-\rangle$, there appear the following expressions for each process:

$$\langle \Lambda|\bar{P} v_\lambda \Lambda|\Xi^-\rangle$$

and

$$\langle \Lambda|\bar{N} v_\lambda \Lambda|\Xi^0\rangle.$$

Since these two matrix elements have equal magnitudes, we can expect that the decay rate of the former process is twice that of the latter. This is the same result as the case of $|\Delta I| = 1/2$ rule. In contrast to the case of Σ^--decay, it is not clear whether one can introduce an assumption on the structure of Ξ such as (3·9) owing to the Pauli principle operating on the constituent Λ-particles. Therefore, we may rather expect, in general, an asymmetric distribution of emitted pions.

§ 4. Discussions

In concluding our discussion, we here give a few remarks.

(1) The treatment of Σ-decay developed in the preceding section is the one which may be regarded as a direct generalization of the method used in the case of Λ-decays. There is, however, another method of treating this problem in which the Σ is supposed to be a bound state of a Λ and a pion. In this method the weak interaction Hamiltonian is replaced by the effective $\Lambda \to N + \pi$ decay interaction which was determined in the second section (§ 2), and the lowest configuration approximation is made for the amplitude of Σ. In this method, too, several conditions have to be imposed upon the structure of Σ-hyperons in order to reproduce the experimentally established features of this process.

The arguments proceed essentially in the same way as those of the recent work of Barshay and Schwartz.[13] We shall not develope the detailed descussions here.

(2) The condition (3·9) seems somewhat contradicting with the full-symmetry hypothesis[14] developed by Ikeda, Ogawa and one of the present anthors (Y.O.), (I.O.O.). Namely, under the I.O.O. symmetry the value of the ratio

α_1/α_2 in (3·10) will be 3 or -1. In the former case the asymmetry parameter α^- is 3/5 and the latter gives -1. Therefore, if the polarization of the produced Σ^- is verified to be large, the I.O.O. symmetry and the present treatment are not consistent in each other. This point should be further investigated carefully from both the theoretical and experimental sides.

(3) We have not taken into account the possible inequality of the values of the coupling constant f in (1) to those of the other group (ordinary beta-interactions or μ-capture processes) of V-A couplings.[15] The apparent slowness of the leptonic decay of hyperons is left as an open question to be explained by future investigations.

Acknowledgement

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Integral Representation of Absorptive Part of Vertex Function

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On the basis of the Lorentz invariance, local commutativity and mass spectral conditions, it is shown that the absorptive part of the vertex function $A(z_1, z_2, \sigma^2)$ has the integral representation in the form

$$A(z_1, z_2, \sigma^2) = \int dm_1 dm_2 dm_3 \varphi(\sigma, m_1, m_2, m_3) A^p(z_1, z_2, \sigma^2; m_1, m_2, m_3),$$

provided that z_1 and z_2 are real negative, where A^p is that of the lowest order perturbation theory and m_i is the mass of the virtual particle. The vanishing region of the weight function φ is determined by the mass spectral conditions. As an immediate consequence of this representation, the usual proof of the dispersion relation of the vertex function is given. If we add the information derivable from the perturbation theory to this representation, we can say that the dispersion relation always holds and the threshold is not lower than the lowest threshold of the vertex function in the lowest order perturbation theory which satisfies the mass spectral condition. It seems to us that Jost's example has not this integral representation. Finally it is conjectured that the non-vanishing region of the weight function is narrowed by introducing the conservation of the nucleon number.

§ 1. Introduction and main results

During the last few years, many efforts have been devoted to the exploration of the general properties of the vertex function, the scattering amplitude, and so on. There are two ways for the exploration. One of them is the way based on the axioms, that is, the Lorentz invariance, local commutativity, mass spectral conditions and asymptotic condition. The other is the one based on the perturbation theory. Of course the former is excellent than the latter. Sometimes we are obliged to use the perturbation theory. However, as is well known,[1] so far as we explore the analytic properties of the vertex function by the former way, we cannot obtain such a large analyticity domain as we can get by the perturbation theory. The large domain found in the perturbation theory may be merely the consequence of the very special interaction used there. There is also the possibility that this is the consequence of the expansion used. Therefore it is desirable to find the relation between above mentioned two ways. The result of this paper is, in some sense, an answer to this desire. We shall confine ourselves to the vertex function.

The vertex function $F(z_1, z_2, z_3)$ is defined by[2]

$$F[k_1^2,\ k_2^2,\ (k_1+k_2)^2]=\int dx_1 dx_2 \exp i(k_1x_1+k_2x_2)G(x_1,\ x_2), \tag{1}$$

where
$$G(x_1,\ x_2)=-\theta(-x_1)\langle 0|\theta(-x_2)[[C(0),\ B(x_2)],\ A(x_1)] +\theta(x_1-x_2)[C(0),\ [A(x_1),\ B(x_2)]]|0\rangle. \tag{2}$$

Here, $A(x)$, $B(x)$ and $C(x)$ are current operators. The spectral conditions are

$$\begin{aligned}&\langle 0|A|n\rangle=0 \text{ unless } p_n^2>a^2,\\ &\langle 0|B|n\rangle=0 \text{ unless } p_n^2>b^2\\ \text{and}\quad &\langle 0|C|n\rangle=0 \text{ unless } p_n^2>c^2.\end{aligned} \tag{3}$$

If z_1 and z_2 are real and negative, we can write the following dispersion relation[2]

$$F(z_1,\ z_2,\ z_3)=\frac{1}{\pi}\int_{\sigma^2}^{\infty} d\sigma^2\ \frac{A(z_1,\ z_2,\ \sigma^2)}{\sigma^2-z_3}, \tag{4}$$

assuming that $F(z_1,\ z_2,\ z_3)$ is sufficiently bounded for $z_3\to\infty$. In the region of interest, that is, for z_1 and z_2 real negative, the absorptive part is given by[2]

$$A[k_1^2,\ k_2^2,\ (k_1+k_2)^2]=\frac{i}{2}\int dx\ dy \exp i\left[\frac{k_1-k_2}{2}y+(k_1+k_2)x\right] \times\theta(y)\langle 0|[C(-x),\ [A(y/2),\ B(-y/2)]]|0\rangle. \tag{5}$$

The main result of this paper is that the absorptive part defined by (5) can be represented in the form

$$A(z_1,\ z_2,\ \sigma^2)=\int dm_1 dm_2 dm_3 \varphi(\sigma,\ m_1,\ m_2,\ m_3)A^p(z_1,\ z_2,\ \sigma^2\ ;\ m_1,\ m_2,\ m_3) \tag{6}$$

provided z_1 and z_2 are real negative. $A^p(z_1,\ z_2,\ \sigma^2\ ;\ m_1,\ m_2,\ m_3)$ is the absorptive part of the vertex function $F^p(z_1,\ z_2,\ z_3\ ;\ m_1,\ m_2,\ m_3)$ in the lowest order perturbation theory given by Fig. 1, that is,

$$F^p(z_1,\ z_2,\ z_3\ ;m_1,\ m_2,\ m_3)=\pi^{-1}\int_0^1 d\alpha d\beta d\gamma\ \delta(1-\alpha-\beta-\gamma) \times(\alpha m_1^2+\beta m_2^2+\gamma m_3^2-z_1\beta\gamma-z_2\gamma\alpha-z_3\alpha\beta)^{-1}. \tag{7}$$

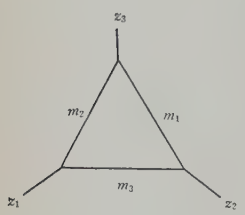


Fig. 1.

The vanishing region of the weight function φ is determined by the spectral conditions (3) and the property of A^p:

$$\varphi=0 \text{ unless } \begin{cases} m_1\geqq 0,\ m_2\geqq 0,\ m_3\geqq 0,\ m_1+m_2\leqq\sigma, \\ m_2+m_3\geqq a,\ m_3+m_1\geqq b. \end{cases} \tag{8}$$

Sometimes it is convenient to rewrite as

$$\varphi=0 \text{ unless } \begin{cases} m_1\geqq 0,\ m_2\geqq 0,\ m_1+m_2\leqq\sigma, \\ m_3\geqq \mathrm{Max}(0,\ a-m_2,\ b-m_1). \end{cases} \tag{8'}$$

It should be noted that there is no restriction like $m_1+m_2 \geqq c$. The proof of the representation (6) will be made in § 3.

§ 2. Discussions

The difference between the absorptive part of the vertex function in the lowest order perturbation theory and that of the general theory appears only in the σ dependence of the weight function φ. As will be seen in Appendix B, A^p for $m_1+m_2<\sigma$ can be analytically continued to any real values of z_1 and z_2 except for $z_1>(m_2+m_3)^2$, $z_2<(m_1-m_3)^2$ or $z_1<(m_2-m_3)^2$, $z_2>(m_1+m_3)^2$, and A^p for $\sigma=m_1+m_2$ is analytic except for $m_1z_1+m_2z_2=(m_1+m_2)(m_1m_2+m_3^2)$. Therefore, A^p for $m_1+m_2<c$ gives no unfavourable influence to the validity of the dispersion relation, and if, for all F^p's for $m_1+m_2 \geqq c$, $m_3+m_1 \geqq b$ and $m_2+m_3 \geqq a$, the dispersion relation with normal threshold holds, the dispersion relation (4) is correct, provided $z_1 \leqq a^2$ and $z_2 \leqq b^2$. When the anomalous threshold appears for some of F^p for $m_1+m_2 \geqq c$, $m_3+m_1 \geqq b$ and $m_2+m_3 \geqq a$, the proof of the dispersion relation is difficult, because the analytic properties of the weight function φ in the neighbourhood of the real axis as a function of σ is unknown. In the perturbation theory there will be no singularity and we can say that in the perturbation theory the dispersion relation always holds. Hence the threshold is not lower than the lowest threshold of F^p given by Karplus, Sommerfield and Wichmann.[3]

For arbitrary weight function φ which satisfies (8), $F(z_1, z_2, z_3)$ does not always satisfy the analytic properties discussed by Källén and Wightman[1] (KW). It is easy to construct an example which contradicts this analytic properties. Further, it seems to us that the absorptive part of the functions of three variables which satisfy the analytic properties of KW have not always the representation in the form (6). Jost's example[4] of the vertex function, which satisfies the analytic properties of KW, is

$$F^J(z_1, z_2, z_3)=[(1+\alpha^2-z_1m^{-2})^{1/2}+(1+\alpha^2-z_2m^{-2})^{1/2} + (1+\alpha^2-z_3m^{-2})^{1/2}-\beta-i\gamma]^{-1}. \tag{9}$$

The conditions on the parameters are $\alpha>0$, $\beta>0$, $\gamma>0$, $\beta>2\alpha$ and $1+\alpha^2-(\beta+\gamma)^2>0$. The absorptive part of F^J is

$$A^J(z_1, z_2, \sigma^2)=(\sigma^2m^{-2}-\alpha^2-1)^{1/2}[\{(1+\alpha^2-z_1m^{-2})^{1/2} + (1+\alpha^2-z_2m^{-2})^{1/2}-\beta-i\gamma\}^2+\sigma^2m^{-2}-\alpha^2-1]^{-1}. \tag{10}$$

Therefore, if $\gamma^2=\sigma^2m^{-2}-\alpha^2-1$, A^J is singular at

$$(1+\alpha^2-z_1m^{-2})^{1/2}+(1+\alpha^2-z_2m^{-2})^{1/2}=\beta. \tag{11}$$

This singularity is not disconnected from the real axis, even though the value of σ^2 changes by the infinitesimal imaginary value. It should be noted that the

existence of the singularity at (11) is important for showing that $F^J(m^2, m^2, z_3)$ contradicts the dispersion relation of the usual form. On the other hand, as will be seen in Appendix B, if stability conditions hold, all singular points of A^p in (z_1, z_2) plane are disconnected from the real axis by the value of σ^2 changing by the infinitesimal imaginary value. The analytic properties of A^J are obviously different from those of A^p. Accordingly it seems that Jost's example has not the representation in the form (6), although we cannot conclude, because the weight function φ is not defined at the complex value of σ^2. It is desirable to find the general properties of the functions of three variables which satisfy the analyticity of KW and have the representation in the form (6). However, this problem remains unsolved (see Notes added in proof).

The representation (6) is, as will be seen in the next section, derived by applying Dyson's representation[5] to the causal commutator in the right-hand side of (5). The weight function φ and the integral variables m_1, m_2 and m_3 are essentially those of Dyson's one. The vanishing region of the weight function φ exactly agrees with the corresponding region of Dyson's one. Therefore, if it is true that we can prove the dispersion relation with normal threshold of the nucleon form factor and the absorptive part for $z_1=z_2=m_N^2$ is given by (5) by introducing the conservation of the nucleon number, we can expect that the non-vanishing region of the weight function of Dyson's representation of the commutator in the right-hand side of (5) is narrowed by introducing the conservation of the nucleon number.

The representation (6) may be the special example of the general feature of the local field theory. For example, the absorptive part of the scattering amplitude may be written by the integral representation like (6) (see Notes added in proof).

§ 3. Proof

In this section, we shall prove the integral representation (6). We shall choose a special Lorentz frame such that $k_1+k_2=(\sigma, 0)$, $\sigma>0$. Because of $\sigma>0$, the first term in the commutator in (5) gives no contribution, hence we have

$$A(k_1^2, k_2^2, \sigma^2)=\frac{(2\pi)^4}{2i}\sum_n\int dy \exp\left[\frac{i}{2}(k_1-k_2)y\right]\theta(y)$$
$$\times\langle 0|[A(y/2), B(-y/2)]|n\rangle\langle n|C(0)|0\rangle\delta(k_1+k_2-p_n). \qquad (12)$$

For the commutator in the right-hand side of (12), we shall use Dyson's representation :[5]

$$(2i)^{-1}\int dy \exp iqy\langle 0|[A(y/2), B(-y/2)]|n\rangle\theta(y)$$
$$=\int du\int d\kappa\psi(\kappa, u, \sigma^2)[\kappa^2-(q-u)^2]^{-1}, \qquad (13)$$

where ψ vanishes except for

$$|u_0|+|\boldsymbol{u}|<\sigma/2,\ \kappa>\kappa_0 \tag{14}$$

with $$\kappa_0=\mathrm{Max}\{0,\ a-[(\sigma/2+u_0)^2-\boldsymbol{u}^2]^{1/2},\ b-[(\sigma/2-u_0)^2-\boldsymbol{u}^2]^{1/2}\}. \tag{15}$$

Insert the representation (13) into (12) and introduce, as a new weight function, the result of the summation over $|n\rangle$ with $p_n=(\sigma,\ 0)$. Then we obtain a representation[2]

$$A(k_1^2,\ k_2^2,\ \sigma^2)=\int du\int d\kappa\frac{\chi(\kappa,\ u_0\ |\boldsymbol{u}|,\ \sigma^2)}{\kappa^2-(q-u)^2}, \tag{16}$$

where $q=(k_1-k_2)/2$. The non-vanishing region of the new weight function χ is exactly the same as that of ψ. The new weight function χ has no dependence on the direction of the vector $\boldsymbol{u}$. Performing the angle integration in (16), we get

$$\begin{aligned}A(k_1^2,\ k_2^2,\ \sigma^2)=&\pi\int d|\boldsymbol{u}|\int du_0\int d\kappa\chi(\kappa,\ u_0,\ |\boldsymbol{u}|,\ \sigma^2)\frac{|\boldsymbol{u}|}{|\boldsymbol{q}|}\\&\times\log\frac{\kappa^2-(q_0-u_0)^2+\boldsymbol{q}^2+\boldsymbol{u}^2+2|\boldsymbol{q}||\boldsymbol{u}|}{\kappa^2-(q_0-u_0)^2+\boldsymbol{q}^2+\boldsymbol{u}^2-2|\boldsymbol{q}||\boldsymbol{u}|}.\end{aligned} \tag{17}$$

In order to rewrite (17) as (6), we shall calculate the absorptive part A^p of the vertex function F^p given by (7). From (7), we obtain

$$\begin{aligned}A^p(z_1,\ z_2,\ \sigma^2\ ;\ m_1,\ m_2,\ m_3)=&\int_0^1 d\alpha d\beta d\gamma\delta(1-\alpha-\beta-\gamma)\\&\times\delta(\alpha m_1^2+\beta m_2^2+\gamma m_3^2-z_1\beta\gamma-z_2\gamma\alpha-\sigma^2\alpha\beta)\end{aligned} \tag{18}$$

for real negative z_1 and z_2. From (18), we have

$$\left.\begin{aligned}&A^p(z_1,\ z_2,\ \sigma^2\ ;\ m_1,\ m_2,\ m_3)=\frac{\theta(\sigma-m_1-m_2)}{[\lambda(z_1,\ z_2,\ \sigma^2)]^{1/2}}\log\frac{X+Y^{1/2}}{X-Y^{1/2}},\\&\text{where}\quad X=2m_3^2+\sigma^2-(z_1+z_2+m_1^2+m_2^2)-(z_1-z_2)(m_1^2-m_2^2)/\sigma^2\\&\text{and}\quad Y=\lambda(z_1,\ z_2,\ \sigma^2)\lambda(m_1^2,\ m_2^2,\ \sigma^2)/\sigma^4\end{aligned}\right\} \tag{19}$$

with the definition $$\lambda(x,\ y,\ z)=x^2+y^2+z^2-2(xy+yz+zx), \tag{20}$$

provided that z_1 and z_2 are real negative. A special example ($z_1=z_2$, $m_1=m_2$) of (19) is (13) in Oehme's paper.[6] The derivation (19) from (18) will be outlined in Appendix A.

Now q_0 and $\boldsymbol{q}^2$ in (17) may be taken as complex. They are given in terms of the covariant variables $z_1=k_1^2$, $z_2=k_2^2$, $\sigma^2=(k_1+k_2)^2$ by

$$\left.\begin{aligned}&q_0=(z_1-z_2)/(2\sigma)\\&\text{and}\quad \boldsymbol{q}^2=\lambda(z_1,\ z_2,\ \sigma^2)/(2\sigma)^2.\end{aligned}\right\} \tag{21}$$

We introduce the new variable m_1, m_2 and m_3 instead of u_0, $|\boldsymbol{u}|$ and κ:

$$\begin{aligned} &m_1=[(\sigma/2-u_0)^2-\boldsymbol{u}^2]^{1/2}, \\ &m_2=[(\sigma/2+u_0)^2-\boldsymbol{u}^2]^{1/2}, \quad m_3=\kappa: \\ &u_0=(m_2^2-m_1^2)/(2\sigma), \\ &\boldsymbol{u}^2=\lambda(m_1^2, m_2^2, \sigma^2)/(2\sigma)^2. \end{aligned} \tag{22}$$

As is easily seen, (14), (15), (17), (19), (21) and (22) are sufficient to complete the proof of the integral representation (6) and the non-vanishing region (8) of the weight function.

Appendix A

Derivation of A^p

The derivation of (19) from (18) is elementary. However, we shall outline this derivation here, since (19) is one of the most important equation in this paper. From (18), we have

$$A^p=\int_0^1 dt \int_0^t d\beta\, \delta[\sigma^2\beta^2-B(t)\beta+C(t)], \tag{23}$$

where $t=\alpha+\beta$, $B(t)=\sigma^2 t+(1-t)(z_1-z_2)+m_1^2-m_2^2$, and $C(t)=tm_1^2+(1-t)m_3^2-(1-t)tz_2$. Recalling $z_1<0$, $z_2<0$ and $\sigma>m_1+m_2$, we can easily see that if

$$\left.\begin{aligned} &(\sigma^2+z_1-z_2)t>z_1-z_2+m_1^2-m_2^2, \\ &(\sigma^2+z_2-z_1)t>z_2-z_1+m_2^2-m_1^2, \\ &[B(t)]^2-4\sigma^2C(t)>0 \text{ and } 1>t>0, \end{aligned}\right\} \tag{24}$$

$\sigma^2\beta^2-B(t)\beta+C(t)=0$ has two roots between $\beta=t$ and $\beta=0$, and if (24) does not hold, $\sigma^2\beta^2-B(t)\beta+C(t)=0$ has no root between $\beta=t$ and $\beta=0$. Therefore, we get

$$A^p=2\int dt\{[B(t)]^2-4\sigma^2C(t)\}^{-1/2}, \tag{25}$$

where the region of the integration is given by (24). From (24), the region of the integration is $1>t>t_0$, where t_0 is the larger root of $[B(t)]^2-4\sigma^2C(t)=0$. So we have

$$A^p=(E)^{-1/2}\log\frac{E+F+[E(E+2F+G)]^{1/2}}{E+F-[E(E+2F+G)]^{1/2}}, \tag{26}$$

where E, F and G are defined by $[B(t)]^2-4\sigma^2C(t)=Et^2+2Ft+G$. Substituting the concrete expressions of E, F and G, we get (19).

Appendix B

Analyticity of A^p

From (19), we see that A^p is singular if and only if

$$m_1^2t_1^2+m_2^2t_2^2+(\sigma^2-m_1^2-m_2^2)t_1t_2+m_3^2\lambda(m_1^2,m_2^2,\sigma^2)=0, \tag{27}$$

where $t_1=z_1-m_2^2-m_3^2$ and $t_2=z_2-m_1^2-m_3^2$.

The condition (27) can be rewritten as

$$2m_1^2t_1=-(\sigma^2-m_1^2-m_2^2)t_2\pm[\lambda(z_2,m_1^2,m_3^2)\lambda(m_1^2,m_2^2,\sigma^2)]^{1/2}. \tag{28}$$

Therefore, if $\sigma>m_1+m_2$, the necessary condition for A^p having the singularity in real z_1 is $z_2>(m_1+m_3)^2$ or $z_2<(m_1-m_3)^2$. Further, in this case, if t_1 and t_2 are real and $t_1t_2>0$, A^p has no singularity. Considering the symmetry for suffix 1 and 2, we come to the conclusion that, if $\sigma>m_1+m_2$, the necessary condition for A^p having the singularity in real z_1 and z_2 is $z_1>(m_2+m_3)^2$, $z_2<(m_1-m_3)^2$ or $z_1<(m_2-m_3)^2$, $z_2>(m_1+m_3)^2$.

When $\sigma=m_1+m_2$, (27) becomes

$$m_1z_1+m_2z_2=(m_1+m_2)(m_1m_2+m_3^2). \tag{29}$$

Hence, if $m_1z_1+m_2z_2<(m_1+m_2)(m_1m_2+m_3^2)$, F^p has no anomalous threshold.[3]

When the value of σ^2 changes by the infinitesimal imaginary value, in order that the singular point of A^p in (z_1, z_2) complex plane is not disconnected from the real axis,

$$t_1t_2+2m_3^2(\sigma^2-m_1^2-m_2^2)=0 \tag{30}$$

and (27) must hold simultaneously. From (27) and (30), we have

$$t_1=\pm(\sigma^2-m_1^2-m_2^2)m_3/m_1,\ t_2=\mp 2m_1m_3$$

$$\text{or } t_1=\mp 2m_2m_3,\ t_2=\pm(\sigma^2-m_1^2-m_2^2)m_3/m_2.$$

Therefore, if stability conditions hold ($z_1<(m_2+m_3)^2$, $z_2<(m_3+m_1)^2$), all singular points in (z_1, z_2) plane are disconnected from the real axis by the value of σ^2 changing by the infinitesimal imginary value.

Notes added in proof :

A) Recently G. Konisi and the present author[7] have found the more elegant method of the proof of the integral representation (6). This method can be applied also for the absorptive parts of scattering amplitudes, in which cases, however, such a simple connection with the perturbation theory is not obtained as is the case of the vertex function.

B) Without changing the value σ^2, it has been shown that, the absorptive part of Jost's example does not satisfy an analytic property derivable from the integral representation (6). Hence we can conclude that Jost's example contradicts the local commutativity. The details will be published in the near future.

C) Using the integral representation (6), we have proved that, if one of z_i's is real negative, the vertex function $F(z_1, z_2, z_3)$ satisfy the double dispersion representation. When z_1 is real nega-

tive, the thresholds of this double dispersion representation are b^2 for z_2 and c^2 for z_3. The details will be published in the near future.

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Soluble Examples in Field Theory with Fermi Interactions

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A soluble example in field theory with Fermi interaction is proposed. The two fermion fields N and $\bar{N}$ are interacting as $N+\bar{N} \rightleftarrows N+\bar{N}$. It is shown that the limit of Machida's soluble model in which the probability amplitude for a bare meson identically vanishes coincides with our model with the separable form factors.

This model with general form factors and another model are discussed.

§ 1. Introduction

It has been expected sometimes that the limit of meson theory in which *the probability amplitudes for bare mesons identically vanish* may coincide with the Fermi-Yang-Sakata's composite model. Such an expectation may give a possibility to test meson theory and the composite model in the high-energy experiments. However, the composite model has the different structure from meson theory, and therefore it is doubtful whether such an expectation may be allowed. *It is the purpose of this work to obtain some clues for such a limit of meson theory and its connection with the composite model.*

We propose a soluble example in field theory with Fermi interaction. Corresponding to this model, we adopt the Machida model,[1] where one neutral meson π and the neutral fermion (nucleon) pair N and $\bar{N}$ interacting in the following scheme (shown in Fig 1) are treated:

$$\pi \rightleftarrows N+\bar{N}. \tag{1·1}$$

Now let us consider the limit of the Machida model in which the probability amplitude for a bare meson identically vanishes. Then it is seen that a soluble field theory with Fermi interaction allowing only the following process (shown in Fig. 2) is obtained:

$$N+\bar{N} \rightleftarrows N+\bar{N}. \tag{1·2}$$

In § 2, this soluble composite model which has the separable form factor for the interaction is discussed. In § 3, § 4 and § 5, it is shown that our model gives the identical results for the N-$\bar{N}$ bound state (the physical meson state) and the N-$\bar{N}$ scattering as the Machida model in the limit of zero probability amplitude for bare mesons. In § 6, the form factors for the interaction (1·2) are generally discussed.

In Appendix A, another model which corresponds to Goldstein's model is discussed.

In Appendix B, the meson propagator and the charge renormalization factor Z_3 is calculated.



Fig. 1.



Fig. 2.

§ 2. Hamiltonian and discrete eigenstate

Let us consider the Fermi interaction between two fermion (nucleon) fields N and $\bar{N}$. The Hamiltonian is

$$H=H_0+H', \tag{2·1}$$

where

$$H_0=\sum_p E(p)\,(A_p^{\dagger}A_p+B_p^{\dagger}B_p), \tag{2·2}$$

$$H'=-\sum_{pqk}\Omega^{-1}G(p,\,q,\,k)\,A_p^{\dagger}B_{k-p}^{\dagger}B_{k-q}A_q. \tag{2·3}$$

$E(p)$ is the bare energy of N and $\bar{N}$ for momentum p. $A_p^{\dagger}$ and A_p are the creation and annihilation operators for N, respectively. $B_p^{\dagger}$ and B_p are the similar operators for $\bar{N}$. These operators obey the usual anticommutation relations. $G(p,\,q,\,k)$ is the form factor for the interaction. For the simplest case $G(p,\,q,\,k)$ is identified with the coupling constant G. Ω is the normalization volume.

We consider a discrete stationary state. A meson state as a discrete eigenstate is given by

$$|\boldsymbol{\pi}_k\rangle=\sum_q \Omega^{-1/2}f_\pi(k,\,q)\,A_q^{\dagger}B_{k-q}^{\dagger}|\mathrm{vac}\rangle. \tag{2·4}$$

$|\mathrm{vac}\rangle$ represents the free-particle vacuum and also the physical vacuum. Introducing the energy eigenvalue $\bar{\omega}_k$ of a meson state with momentum k, we obtain

$$H|\boldsymbol{\pi}_k\rangle=\bar{\omega}_k|\boldsymbol{\pi}_k\rangle, \tag{2·5}$$

from which the integral equation for $f_\pi(k,\,q)$ is derived:

$$[E(q)+E(k-q)-\bar{\omega}_k]f_\pi(k,\,q)-\sum_{q'}\Omega^{-1}G(q,\,q',\,k)\,f_\pi(k,\,q')=0. \tag{2·6}$$

In particular, when the form factor is separable as

$$G(p,\,q,\,k)=\eta G v^{(a)}(p,\,k)\,v^{(b)}(q,\,k),\quad (G>0), \tag{2·7}$$

(2·6) is easily solved. The factor η which is either $+1$ or -1 will be determined in such a way that the state $|\boldsymbol{\pi}_k\rangle$ is stable. $v^{(a)}(p,\,k)$ and $v^{(b)}(p,\,k)$ are complex conjugate to each other, because of the hermitian property of the Hamiltonian. In the case*

* In the center-of-mass system $k=0$, this condition is equivalent to the assumption for the existence of a solution in one partial wave state. The general case is discussed in § 6.

$$\sum_q \Omega^{-1} v^{(b)}(q, k) f_\pi(k, q) \neq 0,$$

we get

$$1 - \sum_q \Omega^{-1}[E(q) + E(k-q) - \bar{\omega}_k]^{-1} G(q, q, k) = 0, \tag{2·8}$$

which determines the energy eigenvalue of the meson. From the stability of the meson, we find

$$\eta = 1. \tag{2·9}$$

Thus (2·7) is rewritten as

$$G(p, q, k) = G v(p, k) v^*(q, k), \tag{2·7a}$$

where $v(p, k) \equiv v^{(a)}(p, k)$, and the simplest case $G(p, q, k) = G$ is contained in the separable form factor.

The present model provides the conservation law for fermion (nucleon) number :

$$Q = \sum_p A_p^\dagger A_p - \sum_p B_p^\dagger B_p. \tag{2·10}$$

§ 3. The Machida model

Now we turn to study the Machida model. The Hamiltonian is

$$H = H_0 + H_1, \tag{3·1}$$

where

$$H_0 = \sum_p E(p)\,(A_p^\dagger A_p + B_p^\dagger B_p) + \sum_k \omega_k c_k^\dagger c_k, \tag{3·2}$$

$$H_1 = g \sum_{pk} (2\Omega\omega_k)^{-1/2}[v^*(p, k) c_k^\dagger B_{k-p} A_p + v(p, k) A_p^\dagger B_{k-p}^\dagger c_k]. \tag{3·3}$$*

ω_k is the bare energy of meson. The field operators for meson c_k^+ and c_k obey the usual commutation relation. $v(p, k)$ is the form factor for the interaction.

A physical meson state with energy $\bar{\omega}_k$ and momentum k is given by

$$|\boldsymbol{\pi}_k\rangle = \mathfrak{N}_3(k)[c_k^\dagger + \sum_q \Omega^{-1/2} f_3(k, q) A_q^\dagger B_{k-q}^\dagger]|\mathrm{vac}\rangle, \tag{3·4}$$

which satisfies the same equation as (2·5), where H is replaced by (3·1). Thus we obtain

$$\omega_k - \bar{\omega}_k + g(2\omega_k)^{-1/2} \sum_q \Omega^{-1/2} v^*(q, k) f_3(k, q) = 0, \tag{3·5}$$

$$[E(q) + E(k-q) - \bar{\omega}_k] f_3(k, q) + g(2\omega_k)^{-1/2} v(q, k) = 0. \tag{3·6}$$

Eliminating $f_3(k, q)$ from (3·5) and (3·6), we get the following eigenvalue equation for $\bar{\omega}_k$:

$$\omega_k - \bar{\omega}_k - g^2(2\omega_k)^{-1} \sum_q \Omega^{-1} v(q, k) v^*(q, k)[E(q) + E(k-q) - \bar{\omega}_k]^{-1} = 0. \tag{3·7}$$

---

* g is the unrenormalized coupling constant.

$\mathfrak{N}_3(k)$ in (3·4) is the normalization factor given by

$$[\mathfrak{N}_3(k)]^{-2}=1+g^2(2\omega_k)^{-1}\sum_q \Omega^{-1}v(q,k)v^*(q,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-2}. \quad (3\cdot8)$$

In the Machida model, we perform the following limiting process:

$$\left.\begin{aligned} &g\to\infty,\\ &\mu\to\infty,\\ &g/\mu\to\bar{g}, \end{aligned}\right\} \quad (3\cdot9)^*$$

where $\mu=\lim_{k\to0}\omega_k$ is the bare mass for meson. $\bar{g}$ is finite. It is clear that the limiting value of g/ω_k in the limit (3·9) is $\bar{g}$. Then *the probability amplitude for bare meson in a physical meson state* $\mathfrak{N}_3(k)$ *vanishes.*

The eigenvalue equation for $\bar{\omega}_k$ is converted to

$$1-\bar{g}^2\sum_q (2\Omega)^{-1}v(q,k)v^*(q,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-1}=0, \quad (3\cdot10)$$

which is just identified with (2·8). The corresponding form factor for the Fermi interaction is

$$G(p,q,k)=\frac{\bar{g}^2}{2}v(p,k)v^*(q,k). \quad (3\cdot11)$$

Thus it is verified that the Machida model in the limit (3·9) coincides with our model proposed in § 2 with the form factor (2·7) or (2·7a), without loss of generality.

§ 4. Coupling constant renormalization

In the Machida model, the renormalized coupling "constant" g_c is defined by

$$g_c=gZ_3^{1/2}. \quad (4\cdot1)$$

Z_3 is calculated in Appendix B. With the aid of (4·1), (B·11) and (3·6) we obtain

$$\mathfrak{N}_3(k)f_3(k,q)=-g_c v(q,k)(2\bar{\omega}_k)^{-1/2}[E(q)+E(k-q)-\bar{\omega}_k]^{-1}. \quad (4\cdot2)$$

Multiplying both sides of (3·8) by $[\mathfrak{N}_3(k)]^2$ and using (4·1) and (B·11), we get

$$[\mathfrak{N}_3(k)]^2=1-g_c^2(2\Omega\bar{\omega}_k)^{-1}\sum_q v(q,k)v^*(q,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-2}. \quad (4\cdot3)$$

Now we introduce $g_{c\,max}$ as follows:

$$[g_{c\,max}]^{-2}\equiv(2\Omega\bar{\omega}_k)^{-1}\sum_q v(q,k)v^*(q,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-2}, \quad (4\cdot4)$$

---

* In the Machida model, the physical mass for meson $\bar{\mu}$ and the renormalized coupling constant g_c are the functions of the bare mass for meson μ and the unrenormalized coupling constant g. We consider $\bar{\mu}$ and g_c as finite throughout our discussions. This assumption contains no inconsistency, as will be discussed in the author's forthcoming paper. This procedure has already been discussed by B. Jouvet, Nuovo Cimento **5** (1957), 1.

then (4·3) is rewritten as

$$[\mathfrak{N}_3(k)]^2=1-g_c^2/g_{c\,max}^2. \tag{4·5}$$

Because of $[\mathfrak{N}_3(k)]^2\geqq 0$, we can conclude

$$g_c^2\leqq g_{c\,max}^2. \tag{4·6}$$

In the limit (3·9), we get from (4·4)

$$\begin{aligned}\lim \omega_k^{-1}[\mathfrak{N}_3(k)]^{-2}&=\bar{g}^2\sum_q (2\Omega)^{-1}v(q,\,k)\,v^*(q,\,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-2}\\&=\bar{g}^2\bar{\omega}_k/g_{c\,max}^2.\end{aligned}$$

Using (B·11) and (4·1), the limiting value of g_c in the limit (3·9) is given by

$$\lim g_c=g_{c\,max}. \tag{4·7}$$

From (4·7), it is shown that our model proposed in § 2 with the form factor (2·7) is contained in the Machida model.

The Hamiltonian is rewritten by the renormalized field operator

$$c_k'=[\mathfrak{N}_3(k)]^{-1}c_k, \tag{4·8}$$

which satisfies the condition

$$\langle \mathrm{vac}|c_k'|\pi_k\rangle=1. \tag{4·9}$$

Thus

$$\left.\begin{aligned}H_0&=\sum_p E(p)\,(A_p^{\dagger}A_p+B_p^{\dagger}B_p)+\sum_k \bar{\omega}_k[\mathfrak{N}_3(k)]^2 c_k'^{\dagger}c_k'\\&\quad-\sum_k \delta\omega_k[\mathfrak{N}_3(k)]^2 c_k'^{\dagger}c_k',\\H_1&=\sum_{pk} g_c(2\Omega\bar{\omega}_k)^{-1/2}[v^*(p,\,k)\,c_k'^{\dagger}B_{k-p}A_p+v(p,\,k)\,A_p^{\dagger}B_{k-p}^{\dagger}c_k']\end{aligned}\right\} \tag{4·10}$$

where

$$\delta\omega_k=\bar{\omega}_k-\omega_k. \tag{4·11}$$

§ 5. Scattering state

We consider first the N-$\bar{N}$ scattering in our model in § 2. The scattering state is represented by

$$|N+\bar{N};\,k,\,q\rangle=\sum_p \Omega^{-1/2}\psi_q(k,\,p)\,A_p^{\dagger}B_{k-p}^{\dagger}|\mathrm{vac}\rangle+\phi_q(k)\,|\pi_k\rangle, \tag{5·1}$$

which satisfies the Schrödinger equation

$$H|N+\bar{N};\,k,\,q\rangle=[E(q)+E(k-q)]|N+\bar{N};\,k,\,q\rangle. \tag{5·2}$$

Using the expression for H and the orthogonality

$$\langle \boldsymbol{\pi}_k|N+\bar{N};\,k,\,q\rangle=0, \tag{5·3}$$

we obtain the integral equation

$$[E(p)+E(k-p)-E(q)-E(k-q)]\psi_q(k,p)$$
$$-\sum_{p'}\Omega^{-1}\{G(p,p',k)+[\bar{\omega}_k-E(q)-E(k-q)]f_\pi(k,p)f_\pi^*(k,p)\}\psi_q(k,p')=0. \tag{5·4}$$

(2·6) is easily solved in the case (2·7). $f_\pi(k,p)$ has the form

$$f_\pi(k,p)=C(k)G^{1/2}v(p,k)[E(p)+E(k-p)-\bar{\omega}_k]^{-1}, \tag{5·5}$$

where $C(k)$ is determined by the normalization condition and is given by

$$[C(k)]^{-2}=G\sum_p\Omega^{-1}v(p,k)v^*(p,k)[E(p)+E(k-p)-\bar{\omega}_k]^{-2}. \tag{5·6}$$

Thus after several calculations, the solution of (5·4) is obtained as follows:

$$\psi_q(k,p)=\delta(p,q)$$
$$-\{[E(p)+E(k-p)-E(q)-E(k-q)-i\epsilon]^{-1}-[E(p)+E(k-p)-\bar{\omega}_k]^{-1}\}$$
$$\times\Omega^{-1}Gv(p,k)v^*(q,k)[-1+GD(k,q)]^{-1}, \tag{5·7}$$

where

$$D(k,q)=\sum_p\Omega^{-1}v(p,k)v^*(p,k)[E(p)+E(k-p)-E(q)-E(k-q)-i\epsilon]^{-1}. \tag{5·8}$$

From (5·7) the scattering phase shift is easily calculated.

In the Machida model, the N-$\bar{N}$ scattering state is represented by the same equation as (5·1) and satisfies (5·2), where H is replaced by (3·1). The integral equations for $\psi_q(k,p)$ and $\phi_q(k)$ are

$$[E(p)+E(k-p)-E(q)-E(k-q)]\psi_q(k,p)$$
$$+[\bar{\omega}_k-E(q)-E(k-q)]\phi_q(k)\mathfrak{N}_3(k)f_3(k,p)=0, \tag{5·9}$$
$$[\bar{\omega}_k-E(q)-E(k-q)]\phi_q(k)\mathfrak{N}_3(k)$$
$$+g(2\omega_k)^{-1/2}\sum_{p'}\Omega^{-1}v^*(p',k)\psi_q(k,p')=0. \tag{5·10}$$

Using the orthogonality condition (5·3), we obtain

$$\psi_q(k,p)=\delta(q,p)$$
$$-\{[E(p)+E(k-p)-E(q)-E(k-q)-i\epsilon]^{-1}-[E(p)+E(k-p)-\bar{\omega}_k]^{-1}\}$$
$$\times g_c^2(2\bar{\omega}_k)^{-1}v(p,k)\sum_{p'}\Omega^{-1}v^*(p',k)[E(p')+E(k-p')-\omega_k]^{-1}\psi_q(k,p'). \tag{5·11}$$

(5·11) is easily solved and the solution coincides with Machida's result. Using (3·10) and (4·4), the solution for (5·11) is written as

$$\psi_q(k,p)=\delta(q,p)$$
$$-\{[E(p)+E(k-p)-E(q)-E(k-q)-i\epsilon]^{-1}-[E(p)+E(k-p)-\bar{\omega}_k]^{-1}\}$$

$$\times g_c^2(2\bar{\omega}_k\Omega)^{-1}v(p,k)v^*(q,k)[E(q)+E(k-q)-\bar{\omega}_k]^{-1}$$
$$\times\{1-(g_c^2/g_{c\,max}^2)+g_c^2(2\bar{\omega}_k)^{-1}[E(q)+E(k-q)-\bar{\omega}_k]^{-1}D(k,q)$$
$$+(g_c^2/g^2)(\omega_k-\bar{\omega}_k)[E(q)+E(k-q)-\bar{\omega}_k]^{-1}\}^{-1}. \tag{5·15}$$

In the limit (3·9), (5·12) is identified with (5·7), where G is replaced by $\bar{g}^2/2$. Thus *the Machida model and our model give the same result for the N-$\bar{N}$ scattering process in the limit* (3·9).

§ 6. Form factors in Fermi interaction

We have considered our example in § 2 only for the separable form factor (2·7). In this section the form factor in general case is investigated. Now we can assume

$$G(p,q,k)=\sum_{i=1}^{n}G\eta_i v_i(p,k)v_i^*(q,k), \tag{6·1}$$

where $\eta_i(i=1,2,\cdots n)$ are either $+1$ or -1 and all $v_i(p,k)$ are linearly independent. From (2·6) we obtain the coupled homogeneous linear equations

$$\sum_{i=1}^{n}\{\delta_{ij}-G\eta_i\sum_q\Omega^{-1}[E(q)+E(k-q)-\bar{\omega}_k]^{-1}v_i(q,k)v_j^*(q,k)\}F_i(k)=0, \tag{6·2}$$

where

$$F_i(k)=\sum_q\Omega^{-1}v_i^*(q,k)f_\pi(k,q). \tag{6·3}$$

Assuming that $F_i(k)$ does not all vanish, the determinant constructed by the coefficients of $F_i(k)$ $(i=1,2,\cdots n)$ in (6·2) is equal to zero, i.e.

$$\|\delta_{ij}-G\eta_i\sum_q\Omega^{-1}[E(q)+E(k-q)-\bar{\omega}_k]^{-1}v_i(q,k)v_j^*(q,k)\|=0. \tag{6·4}$$

It is clear that the eigenvalues of (6·4) $\bar{\omega}_k$ are real, because of the hermitian property of the kernel $G(p,q,k)$.

Now it is shown that our model with the form factor (6·1) corresponds to the model (the mixed meson theory), where N and $\bar{N}$ interact with n kinds of mesons. In this mixed meson theory, the Hamiltonian is given by

$$H=H_0+H_n, \tag{6·5}$$

$$H_0=\sum_p E(p)(A_p^\dagger A_p+B_p^\dagger B_p)+\sum_{i=1}^{n}\sum_k\omega_{ki}c_{ki}^\dagger c_{ki}, \tag{6·6}$$

$$H_n=\sum_{i=1}^{n}g_i\sum_{p,k}(2\Omega\omega_{ki})^{-1/2}[v_i^*(p,k)c_{ki}^\dagger B_{k-p}A_p+v_i(p,k)A_p^\dagger B_{k-p}^\dagger c_{ki}]. \tag{6·7}$$

Physical meson states satisfy the energy eigenvalue equations

$$H|\boldsymbol{\pi}_{ki}\rangle=\bar{\omega}_{ki}|\boldsymbol{\pi}_{ki}\rangle,\ (i=1,2,\cdots n). \tag{6·8}$$

The eigenstates can be written as

$$|\boldsymbol{\pi}_{ki}\rangle=\mathfrak{N}_{3i}(k)[c_{ki}^{\dagger}+\sum_{j(\neq i)}\alpha_j(k)c_{kj}^{\dagger}+\sum_{q}\Omega^{-1/2}f_3(k,\,q)A_q^{\dagger}B_{k-q}^{\dagger}]|\mathrm{vac}\rangle. \quad (6\cdot9)$$

From (6·8) and (6·9), we obtain

$$\omega_{ki}-\bar{\omega}_{ki}+g_i(2\omega_{ki})^{-1/2}\sum_{q}\Omega^{-1}v_i^*(q,\,k)f_{3i}(k,\,q)=0, \quad (6\cdot10)$$

$$(\omega_{kj}-\bar{\omega}_{ki})\alpha_j(k)+g_j(2\omega_{kj})^{-1/2}\sum_{q}\Omega^{-1}v_j^*(q,\,k)f_{3i}(k,\,q)=0,\ (j\neq i), \quad (6\cdot11)$$

$$[E(q)+E(k-q)-\bar{\omega}_{ki}]f_{3i}(k,\,q)+g_iv_i(q,\,k)(2\omega_{ki})^{-1/2}$$
$$+\sum_{j(\neq i)}g_jv_j(q,\,k)(2\omega_{kj})^{-1/2}\alpha_j(k)=0,\ \ (i,\,j=1,\,2,\cdots n). \quad (6\cdot12)$$

(6·11) and (6·12) are reduced to the following coupled linear equation:

$$F_{3j}^{(i)}(k)=\Gamma_j^{(i)}(k)+\sum_{j(\neq i)}\Lambda_{jj'}^{(i)}(k)F_{3j'}^{(i)}(k), \quad (6\cdot13)$$

where

$$F_{3j}^{(i)}(k)=\sum_{q}\Omega^{-1}v_j^*(q,\,k)f_{3i}(k,\,q), \quad (6\cdot14)$$

$$\Gamma_j^{(i)}(k)=-g_i(2\omega_{ki})^{-1/2}\sum_{q}\Omega^{-1}v_i(q,\,k)v_j^*(q,\,k)[E(q)+E(k-q)-\bar{\omega}_{ki}]^{-1}, \quad (6\cdot15)$$

$$\Lambda_{jj'}^{(i)}(k)=g_{j'}^2(\omega_{kj'}-\bar{\omega}_{ki})^{-1}(2\omega_{kj'})^{-1}$$
$$\times\sum_{q}\Omega^{-1}v_j^*(q,\,k)v_{j'}(q,\,k)[E(q)+E(k-q)-\bar{\omega}_{ki}]^{-1}. \quad (6\cdot16)$$

Thus we get

$$f_{3i}(k,\,q)=-g_iv_i(q,\,k)(2\omega_{ki})^{-1/2}[E(q)+E(k-q)-\bar{\omega}_{ki}]^{-1}$$
$$+\sum_{j(\neq i)}g_j^2(\omega_{kj}-\bar{\omega}_{ki})^{-1}v_j(q,\,k)(2\omega_{kj})^{-1}[E(q)+E(k-q)-\bar{\omega}_{ki}]^{-1}F_{3j}^{(i)}(k). \quad (6\cdot17)$$

From (6·10) and (6·17), we obtain

$$\omega_{ki}-\bar{\omega}_{ki}+g_i(2\omega_{ki})^{-1/2}\Gamma_i^{(i)}(k)+g_i(2\omega_{ki})^{-1/2}\sum_{j(\neq i)}\Lambda_{ij}^{(i)}(k)F_{3j}^{(i)}(k)=0. \quad (6\cdot18)$$

In the limit

$$\left.\begin{aligned}&g_i\to\infty,\\&\mu_i\to\infty,\\&g_i/\mu_i\to\bar{g},\end{aligned}\right\} \quad (6\cdot19)$$

where $\mu_i=\lim_{k\to0}\omega_{ki}$, it is easily shown that (6·18) coincides with (6·4), where G is replaced by $\bar{g}^2/2$.

In the above discussion we have investigated for finite n. Now what happens in the limit $n\to\infty$? In this case, the N-$\bar{N}$ bound states are, in general, infinitely possible. Then our model cannot be reduced to meson theory. Especially if we can put

$$G(p, q, k)=V(p-q), \tag{6·20}$$

(2·6) is rewitten as

$$[E(q)+E(k-q)-\bar{\omega}_k]f_\pi(k, q)-\sum_{q'}\Omega^{-1}V(q-q')f_\pi(k, q')=0, \tag{6·21}$$

which corresponds to the model that N and $\bar{N}$ are bound to each other by the appropriate static potential (the Fermi-Yang model).

§ 7. Discussion

In our model, the N-N-$\bar{N}$ bound state and meson-nucleon scattering are possible, where we cannot obtain the explicit solutions. Similarly, in the Machida model, the meson-nucleon bound state and scattering are possible (see Fig. 3). It is easily seen that the integral equation for these bound and scattering states in the Machida model is reduced to that in our model with the form factor (2·7a), where G is replaced by $(1/2)\bar{g}^2$. Therefore these circumstances for the correspondence of our model and the Machida model seem to be established in the other possible processes.

Thus, the problems to be investigated are

(i) whether the limiting process (3·9), which made the probability amplitude for a bare meson state Z_3 vanish in the Machida model, also derive $Z_3=0$ in pseudoscalar meson theory or not, and



Fig. 3.

(ii) whether the theory with Fermi interaction which is equivalent to the $Z_3=0$ pseudoscalar meson theory may be constructed or not, if (i) is possible.

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The author wishes to express his heartfelt thanks to Prof. Y. Munakata for valuable discussions.

Appendix A

Another soluble example in field theory with Fermi interactions is suggested from the Lee model[2] modified by Goldstein[3] in the limit that the probability amplitudes for bare θ-particle identically vanish. The Hamiltonian between four neutral fermion fields V, N, χ and $\bar{\chi}$ is

$$H=H_0+H'+H'', \tag{A·1}$$

where

$$H_0=\sum_p E_V(p)a_p^\dagger a_p+\sum_p E_N(p)b_p^\dagger b_p+\sum_p E_\chi(p)(A_p^\dagger A_p+B_p^\dagger B_p), \tag{A·2}$$

$$H'=\sum_{pqk}\Omega^{-1}G'(p, q, k)A_p^\dagger B_{k-p}^\dagger B_{k-q}A_q, \tag{A·3}$$

$$H''=\sum_{pqk}\Omega^{-1}G''(p,\,q,\,k)\,(A_q^{\dagger}B_{k-q}^{\dagger}b_{p-k}^{\dagger}a_p+a_p^{\dagger}b_{p-k}B_{k-q}A_q). \tag{A·4}$$

$E_V(p)$, $E_N(p)$ and $E_\chi(p)$ are the bare energy of V, N and χ, $\bar{\chi}$ for the momentum p, respectively. $a_p^{\dagger}$, a_p; $b_p^{\dagger}$, b_p; $A_q^{\dagger}$, A_q and $B_q^{\dagger}$, B_q are the field operators of V, N, χ and $\bar{\chi}$, respectively, and obey the usual anticommutation relations.

There are two discrete eigenstates of H in the separable form factors for $G'(p,\,q,\,k)$ and $G''(p,\,q,\,k)$:

$$|\boldsymbol{\theta}_k\rangle=\sum_q\Omega^{-1/2}f_\theta(k,\,q)\,A_q^{\dagger}B_{k-q}^{\dagger}|\mathrm{vac}\rangle \tag{A·5}$$

and

$$|\boldsymbol{V}_p\rangle=\mathfrak{N}(p)[a_p^{\dagger}+\sum_{p,q}\Omega^{-1}f_V(p\,;\,k,\,q)\,A_q^{\dagger}B_{k-q}^{\dagger}b_{p-k}^{\dagger}]|\mathrm{vac}\rangle, \tag{A·6}$$

which satisfy the eigenvalue equations :

$$H|\boldsymbol{\theta}_k\rangle=\bar{\omega}_k|\boldsymbol{\theta}_k\rangle \tag{A·7}$$

and

$$H|\boldsymbol{V}_p\rangle=\bar{E}_V(p)\,|\boldsymbol{V}_p\rangle. \tag{A·8}$$

The Goldstein model is given by the Hamiltonian as follows :

$$H=H_0+H_1+H_2, \tag{A·9}$$

$$H_0=\sum_p E_V(p)a_p^{\dagger}a_p+\sum_p E_N(p)b_p^{\dagger}b_p+\sum_k\omega_k c_k^{\dagger}c_k+\sum_q E_\chi(q)\,(A_q^{\dagger}A_q+B_q^{\dagger}B_q), \tag{A·10}$$

$$H_1=g_1\sum_{pk}(2\Omega\omega_k)^{-1/2}[v_1^{*}(p,\,k)a_p^{\dagger}b_{p-k}c_k+v_1(p,\,k)c_k^{\dagger}b_{p-k}^{\dagger}a_p], \tag{A·11}*$$

$$H_2=g_2\sum_{pk}(2\Omega\omega_k)^{-1/2}[v_2^{*}(p,\,k)c_k^{\dagger}B_{k-p}A_p+v_2(p,\,k)A_p^{\dagger}B_{k-p}^{\dagger}c_k]. \tag{A·12}*$$

The physical θ-particle and V-particle states are given by

$$|\boldsymbol{\theta}_k\rangle=\mathfrak{N}_3(k)\,[c_k^{\dagger}+\sum_q\Omega^{-1/2}f_3(k,\,q)\,A_q^{\dagger}B_{k-q}^{\dagger}]|\mathrm{vac}\rangle \tag{A·13}$$

and

$$|\boldsymbol{V}_p\rangle=\mathfrak{N}_2(p)[a_p^{\dagger}+\sum_k\Omega^{-1/2}f_1(p,\,k)c_k^{\dagger}b_{p-k}^{\dagger}+\sum_{kq}\Omega^{-1}f_2(p\,;\,k,\,q)\,A_q^{\dagger}B_{k-q}^{\dagger}b_{p-k}^{\dagger}]|\mathrm{vac}\rangle, \tag{A·14}$$

which satisfy (A·7) and (A·8), respectively, where H are replaced by (A·9).

In the Goldstein model, we perform the limiting process

$$\left.\begin{aligned}&g_1\to\infty,\\&g_2\to\infty,\\&\mu\to\infty,\\&g_1/\mu\to\bar{g}_1,\\&g_2/\mu\to\bar{g}_2,\end{aligned}\right\} \tag{A·15}$$

* g_1 and g_2 are the unrenormalized coupling constants.

as in § 3. Then the probability amplitudes for bare θ-particles, i.e. $\mathfrak{N}_3(k)$ defined by (A·13) and $\mathfrak{N}_2(p)$ defined by (A·14), vanish and the same results as our model with Fermi interactions (A·1)—(A·4) with separable form factors are obtained except the physical mass of V-particle. These circumstances are due to the fact that one of the self-energy contributions shown in Fig. 4 does not vanish in the limit (A·15).



Fig. 4.

In the Goldstein model, the vertex function Γ is identically equal to 1 as in the Lee model. It has been shown that the vertex function $\Gamma(p_1, p_2)$ in the usual quantum field theory (i.e. in which no abnormal states exist) has to satisfy a condition that it vanishes for large momenta.[4] The Goldstein model as well as the Lee model does not satisfy this condition. The above mentioned incomplete correspondence between our model and Goldstein's model seems to be due to such a special character of the models.

Appendix B

The meson propagator in the Machida model can be written as

$$\Delta_F(x, x') = \langle \mathrm{vac} | T(\phi(x), \phi(x')) | \mathrm{vac} \rangle \tag{B·1}$$

in the Heisenberg representation. In the Schrödinger representation, the field operator $\phi(x)$ is represented as*

$$\phi(x) = (2\pi)^{-3/2} \int d\boldsymbol{k} (2\omega_k)^{-1/2} (c_k e^{ik\cdot x} + c_k^{\dagger} e^{-ik\cdot x}). \tag{B·2}$$

Using the well-known relation between Schrödinger and Heisenberg representations, we get

$$\begin{aligned} \Delta_F'(x, x') &= \langle \mathrm{vac} | \phi^{(-)}(x)\, e^{-iH(t-t')} \phi^{(+)}(x') | \mathrm{vac} \rangle \quad \text{for } t > t' \\ &= \langle \mathrm{vac} | \phi^{(-)}(x')\, e^{-iH(t'-t)} \phi^{(+)}(x) | \mathrm{vac} \rangle \quad \text{for } t' > t, \end{aligned} \tag{B·3}$$

where

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x), \tag{B·4}$$

$$\left.\begin{aligned} \phi^{(+)}(x) &= (2\pi)^{-3/2} \int d\boldsymbol{k} (2\omega_k)^{-1/2} c_k^{\dagger} e^{-ik\cdot x}, \\ \phi^{(-)}(x) &= (2\pi)^{-3/2} \int d\boldsymbol{k} (2\omega_k)^{-1/2} c_k e^{ik\cdot x}. \end{aligned}\right\} \tag{B·5}$$

It is easily seen that $\Delta_F'(x, x')$ is expressed as a function of $\boldsymbol{x}-\boldsymbol{x}'$ and $t-t'$. The Fourier transform of (B·3) is

$$\begin{aligned} \Delta_F'(\boldsymbol{k}, \omega) &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \int d\boldsymbol{r}\, \Delta_F'(x, x') e^{-ik\cdot r} \\ &= i(2\omega_k)^{-1} \{ [\omega - \omega_k - \Pi_k(\omega) + i\epsilon]^{-1} - [\omega + \omega_k + \Pi_k(-\omega) - i\epsilon]^{-1} \}, \end{aligned} \tag{B·6}$$

* In Apprendix B, Ω is taken to be infinite.

where $\tau \equiv t-t'$, $\boldsymbol{r} \equiv \boldsymbol{x}-\boldsymbol{x}'$ and

$$\Pi_k(\omega) = g^2(2\pi)^{-3}\int d\boldsymbol{q}|v(\boldsymbol{q}, \boldsymbol{k})|^2(2\omega_k)^{-1}[\omega - E(\boldsymbol{q}) - E(\boldsymbol{k}-\boldsymbol{q}) + i\epsilon]^{-1}. \quad \text{(B·7)}$$

$\Delta_F'(\boldsymbol{k}, \omega)$ has poles at $\omega = \omega_k + \Pi_k(\omega)$ and $-\omega = \omega_k + \Pi_k(-\omega)$. Denoting such ω-values by $\omega = \bar{\omega}_k$ and $\omega = -\bar{\omega}_k$, we obtain

$$\bar{\omega}_k = \omega_k + \Pi_k(\bar{\omega}_k), \quad \text{(B·8)}$$

which coincides with (3·7). Thus (B·6) is rewritten as

$$\Delta_F'(\boldsymbol{k}, \omega) = \pi \frac{\bar{\omega}_k}{\omega_k}[\mathfrak{N}_3(k)]^2\delta(\omega^2 - \bar{\omega}_k^2) + i(2\omega_k)^{-1}\left\{P \frac{1}{\omega - \omega_k - \Pi_k(\omega)} - P\frac{1}{\omega + \omega_k + \Pi_k(-\omega)}\right\}. \quad \text{(B·9)}$$

The coupling constant renormalization factor Z_3 is defined by[5),6)]

$$Z_3 = [i\Delta_F'(\boldsymbol{k}, \omega)(\bar{\omega}_k^2 - \omega^2)]_{\omega^2 = \bar{\omega}_k^2}. \quad \text{(B·10)}$$

Using (B·9), (B·10) and (3·8), we obtain

$$Z_3 = \frac{\bar{\omega}_k}{\omega_k}\left[1 - \frac{\partial \Pi_k(\omega)}{\partial \omega}\bigg|_{\omega = \bar{\omega}_k}\right]^{-1} = \frac{\bar{\omega}_k}{\omega_k}[\mathfrak{N}_3(k)]^2. \quad \text{(B·11)}$$

Note added in proof: After sending this manuscript, I saw the similar work by J. C. Houard and B. Jouvet, Nuovo Cimento **18**(1960) 466, where the Lee model and its corresponding model are treated.

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Phase Shift and Identities in Quantized Field Theory

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We point out certain basic relations between the phase shifts and the energy spectrum which we illustrate in simple examples. And several applications of these relations are considered: A determinant form for the S-matrix is found; n-body forces in the static pair-coupling theory are evaluated; some properties of the g-derivative of the phase shift are discussed and a covariant method for the construction of nuclear force is proposed.

§ 1. Introduction

The purpose of this paper is to point out certain basic relations concerning the phase shift and the energy spectrum which follow from the formalism previously given by the present author,[1] which will hereafter be referred to as I, and to give some discussions about their application.

In quantum mechanical model in which the potential is spherically symmetric, if the system is enclosed by a spherical rigid wall, the levels become discrete and, as will be easily seen, the phase shift $\eta(E)$ and the energy shift ΔE of each level are connected by the relation

$$\Delta E = -(1/R)(dE/dk)\eta(E), \tag{1·1}$$

where E is the energy of the level, k is the radial component of the momentum variable and R is the radius of the wall. Meanwhile for the energy eigenvalue of a non-degenerate discrete level

$$dE/dg = d/dg\ \Delta E = (\Psi \mathcal{H} \Psi), \tag{1·2}$$

where g is the coupling constant and

$$\mathcal{H} = dH/dg, \tag{1·3}$$

H being the total Hamiltonian. Combining (1·1) and (1·2), one gets a relation between the phase shift and the potential,

$$d/dg\ \eta_{nE} = -R(dE/dk)^{-1}(\Psi \mathcal{H} \Psi), \tag{1·4}$$

where Ψ is normalized to unity.

The above reasoning holds only when the potential is spherically symmetric. A formal discussion on the relation as (1·4) was given by S. Okubo.[2] A general and rigorous discussion in field theory was given in I, according to which we have for two-body elastic scattering

$$d/dg\ \eta_{nE}=-(1/2)\,\delta^{-1}(E=0)\,k^{-1}(E)\,(\Psi_{nE}\bar{D}_g\Psi_{nE}),\qquad(1\cdot5)^*$$

and for a general case

$$\sum_n^j d/dg\ \eta_{nE}=-(1/2)\,\delta^{-1}(E=0)\,k^{-1}(E)\,S_p[P_E P_j \bar{D}_g],\qquad(1\cdot6)$$

where the subscript n, which specifies the diagonal elements of the S-matrix, is summed over all states belonging in an eigen-space of a set of constants of motion such as the total angular momentum, the eigenvalues of which shall be denoted by j; state vectors are normalized by

$$(\Psi_{nE}\Psi_{n'E'})=\delta_{nn'}k(E)\,\delta(E-E'),\qquad(1\cdot7)$$

and $\bar{D}_g$ is the quantity introduced in I which relates to the generating operators with respect to the variation of g. Under the assumption of the asymptotic condition (see I and also ref. 3))

$$\bar{D}_g=\int_{-\infty}^{\infty}d\tau\,\mathcal{K}(\tau),\qquad(1\cdot8)$$

which is, however, not rigorous when there are bound states. Under (1·8), (1·5) becomes

$$d/dg\ \eta_{nE}=-\pi k^{-1}(E)\,(\Psi_{nE}\mathcal{K}\Psi_{nE}).\qquad(1\cdot9)$$

We shall again take the case of a spherically symmetric potential which is enclosed by a spherical rigid wall and consider a partial wave component. The number of levels involved in a given energy region is dependent on the energy dependence of the energy shift of these levels. It will be easily seen that the number $N(E)dE$ of levels involved between E and $E+dE$ is decreased by an amount proportional to the gradient of ΔE:

$$N(E)\,dE=(1-\partial(\Delta E)/\partial E)\,N_0(E)\,dE,$$

or

$$\Delta N(E)=N(E)-N_0(E)=-N_0(E)\,\partial(\Delta E)/\partial E,\qquad(1\cdot10)$$

where $N(E)$ is the level density and $N_0(E)=N(E)|_{g=0}$. While $N(E)$ and $N_0(E)$ tend to infinity in the limit $R\to\infty$, ΔE tends to zero in this limit, thus $\Delta N(E)$ remains finite. Substituting from

$$N_0(E)=(1/\pi)\,R(dE/dk)^{-1},\qquad(1\cdot11)$$

and from (1·1) into (1·10) one obtains

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* The factor $\delta^{-1}(E=0)$ is understood as cancelling the factor $\delta(E-E')$ contained in $(\Psi_{nE}\bar{D}_g\Psi_{nE'})$ where $\bar{D}_g$ commutes with H. In the following the factors such as $\delta^{-1}(E=0)$ which always appear multiplied with an energy conserving matrix element have a similar meaning.

$$\Delta N(E) = (1/\pi)\,\partial\eta/\partial E. \qquad (1\cdot12)^*$$

As an illustration, in a simple model of § 2, we shall explicitly construct $\Delta N(E)$ and directly examine the relation (1·12). Also from this special example we are suggested a method for calculating the S-matrix in the general potential case, which may be called "the determinant method" (see § 2 and App. III).

In field theory the increment $\Delta N_{jE_1E_2}$ in the number of levels belonging to the eigenvalue j and to the energy region $E_1 \leqq E < E_2$ is defined by (see § 5 of I)

$$d/dg\ \Delta N_{jE_1E_2} = d/dg\ Sp[P_{jE_1E_2}], \qquad (1\cdot13)$$

where $P_{jE_1E_2}$ is the projection operator such that

$$P_{jE_1E_2}\Psi_{nj'E} = \begin{cases} \Psi_{njE} & \text{for } j'=j \text{ and } E_1 \leqq E < E_2, \\ 0 & \text{for any other case.} \end{cases} \qquad (1\cdot14)$$

Levinson's theorem was proved in I by substituting $A=P_j$ into the equation expressing the invariance of spur:

$$d/dg\ Sp[A] = Sp[dA/dg], \qquad (1\cdot15)$$

and on account of

$$d/dg\ P_j = 0, \qquad (1\cdot16)$$

which represents the quantum mechanical counter-part of the adiabatic invariance of the volume of the phase space in classical theory.

Also, if one puts $A=P_j\mathcal{H}$ in (1·15), one easily obtains the following relation,

$$d^2/dg^2\ E^{(j)} = -(1/\pi)\int_0^\infty dE \sum_a d^2/dg^2\ \eta_{jaE}, \qquad (1\cdot17)$$

where $E^{(j)}$ is the energy of the bound state belonging to the eigen-space denoted by j. (1·17) will be examined directly in the square well potential (see App. II). In field theory (1·17) can be integrated, yielding

$$\Delta E^{(j)} = -(1/\pi)\int_0^\infty dE \sum_a \eta_{jaE}. \qquad (1\cdot18)$$

As an application of the basic relations of this section we shall in § 3 and App. IV consider the static pair-coupling theory of a general type. It is shown there that an identity holds between the self energy of the sources and the phase shifts of mesons scattered by the sources. In § 5 we shall give a covariant method for the construction of the nuclear potential in field theory.

In § 4 an equation for the derivative of the phase shift which follows from

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* It will be interesting to note here that $\partial\eta/\partial E$ is equal to the time delay of the scattered wave packet in the time dependent treatment.[4]

(1·5) is given. In App. I a rule concerning the differentiation with respect to g under the asymptotic condition is given.

§ 2. Energy spectrum in the potential scattering

In order to illustrate the basic relation (1·12) between the energy spectrum and the phase shift, we shall first consider the simple example of the scattering by a potential of factorable type:

$$(\boldsymbol{r}|V|\boldsymbol{r}')=gv(r)v(r'), \tag{2·1}$$

where $v(r)$ is assumed to be a scalar function depending only on r. The levels become discrete and are decomposed into partial waves if we enclose the system in a spherical rigid wall of radius R. Let us consider only the S-waves. If ω_l and E_l are the free and the real energies and $\varphi_l(r)$ and $\psi_l(r)$ are the free and the real radial wave functions respectively, of the l-th normal mode, the eigenvalue equation for the l-th normal mode is

$$(\omega_{l'}-E_l)\psi_{l'l}+gv_{l'}\sum_{l''}v_{l''}^*\psi_{l''l}=0, \tag{2·2}$$

where

$$\psi_{l'l}=(\phi_{l'}\psi_l),\quad v_l=(\phi_l v). \tag{2·3}$$

The energy eigenvalue E_l may be determined by the secular equation

$$\det\left|\delta_{l'l}+g\frac{v_{l'}v_l^*}{\omega_{l'}-E}\right|=0. \tag{2·4}$$

Or one can make use of the equation

$$\Delta_1(E)=1+g\sum_l\frac{v_lv_l^*}{\omega_l-E}=0, \tag{2·5}$$

which is obtained by dividing the both sides of (2·2) by $\omega_{l'}-E_l$, multiplying $v_{l'}$ and then summing over l'. The change in the level density

$$\Delta N(E)=N(E)-N_0(E)$$

is evaluated according to Wentzel[5] as follows. Since $\Delta_1(E)$ has zero points at $E=E_l$ and poles at $E=\omega_l$,

$$d/dE\log\Delta_1(E)$$

has poles of residue 1 at $E=E_l$ and poles of -1 at $E=\omega_l$. Therefore the contour integral of this function along the contour C (see Fig. 1) in the complex E plane equals the difference between the numbers of interacting and free levels involved in the energy region limited by E and $E+dE$. Thus

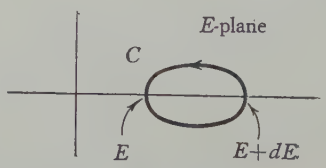


Fig. 1.

$$\Delta N(E)dE=(1/2\pi i)\int_C dE\, d/dE \log \Delta_1(E)$$

$$=(1/2\pi i)\int_E^{E+dE} dE\, d/dE \log(\Delta^*(E)/\Delta(E))=(1/2\pi i)[d/dE \log(\Delta^*(E)/\Delta(E))]dE,$$

or

$$\Delta N(E)=(1/2\pi i)\, d/dE \log(\Delta^*(E)/\Delta(E)), \tag{2·6}$$

where

$$\Delta(E)=\Delta_1(E+i\varepsilon), \tag{2·7}$$

ε being infinitesimal and positive. Meanwhile, as will be easily obtained, the S-matrix $S(E)$ for this model is

$$S(E)=\Delta^*(E)/\Delta(E). \tag{2·8}$$

Therefore comparing (2·6) with (2·8) one obtains

$$\Delta N(E)=(1/\pi)\, d/dE\, \eta(E). \tag{2·9}$$

Thus in this model (1·12) has been really confirmed.

The above result suggests an expectation that the form (2·8) for $S(E)$ will hold also in the general case of a potential not necessarily of factorable type, if $\Delta_1(E)$ is replaced by

$$\Delta_1(E)=\det\left|\delta_{\iota\iota'}+g\frac{V_{\iota\iota'}}{\omega_{\iota'}-E}\right|, \tag{2·10}$$

where $l=(Ejm)$ is a set of variables which are sufficient to specify the state vectors and

$$V_{\iota\iota'}=(\phi_{\iota}, V\phi_{\iota}), \tag{2·11}$$

because in the above discussion leading to (2·6) one can replace $\Delta_1(E)$ by that defined by (2·10), therefore, if the relation (2·9) is generally valid, one will be led to (2·8). This expectation is really confirmed in App. III. We understand here that in the limit $R\to\infty$ (2·10) is formally defined by the expansion of the determinant,

$$\Delta_1(E)=1+g\sum_{\iota}\frac{V_{\iota\iota}}{\omega_{\iota}-E}+g^2\sum_{\iota'}\sum_{\iota}\frac{V_{\iota\iota'}V_{\iota'\iota}}{(\omega_{\iota'}-E)(\omega_{\iota}-E)}+\cdots. \tag{2·12}$$

§ 3. Self energy of sources in the static pair theory

In the static pair theory of simple types it was shown by K. Sawada[6] that there is a certain relation between the self-energy of a source and the phase shifts of scattered mesons. This relation was made use of, without proof, by K. Hasegawa and the present author[7] to evaluate the interaction energy of two sources. It will be shown in App. IV that this relation holds generally in the pair theory. In this

section we shall first present the relation in question and then as an illustration consider the case of three sources.

Consider first the case in which the neutral scalar mesons interact with a source through the direct pair coupling:

$$H=H_0+H_1, \tag{3·1}$$

$$H_0=(1/2)\int d^3r(\pi^2(r)+(\nabla\varphi)^2+\mu^2\varphi^2(r)), \tag{3·2}$$

$$H_1=(g/2)(\int d^3r v(r)\varphi(r))^2, \tag{3·3}$$

where $v(r)$ is assumed to be a real scalar function depending only on r. In this case the meson field can explicitly be decomposed into normal modes by enclosing the system in a spherical rigid wall, so that the total Hamiltonian becomes (see App. IV)

$$H=(1/2)\sum_l E_l(a_l a_l^\dagger+a_l^\dagger a_l), \tag{3·4}$$

where a_l is an operator which annihilate a real quantum of the l-th normal mode. The minimum energy state (meson vacuum) Ψ_0 is defined by

$$a_l\Psi_0=0. \tag{3·5}$$

Thus the self energy of the source is given by

$$\Delta E=(\Psi_0 H\Psi_0)-(\Phi_0 H_0\Phi_0)=(1/2)\sum_l(E_l-\omega_l), \tag{3·6}$$

where Φ_0 is the free vacuum, and $\omega_l/2$ and $E_l/2$ are the zero point energies of the l-th normal mode of the free and the interacting field respectively. On account of (1·1) each $E_l-\omega_l$ is proportional to the phase shift η_l of the l-th normal mode. Therefore one obtains

$$\Delta E=-(1/2\pi)\int_0^\infty d\omega\sum_{l,m}\eta_{lm}(\omega), \tag{3·7}$$

where (l, m) denotes each spherical harmonics.

The above discussion decomposing the field into normal modes is not valid when the interaction is not spherically symmetric as is not so in the case of two sources. It will, however, be shown that the relation (3·7) still holds in general (see App. IV). Consider the general case in which the meson field may have components such as the iso-spinor components, the number of sources may be arbitrary and the interaction Hamiltonian is given by

$$H_1=(g/2)\sum_{\alpha,i}(\int d^3r\varphi_\alpha(\boldsymbol{r})v_i^\dagger(\boldsymbol{r}))(\int d^3r v_i(\boldsymbol{r})\varphi_\alpha(\boldsymbol{r})), \tag{3·8}$$

where α denotes, for example, the iso-spinor components, i denotes the number of the each source and $v_i(\boldsymbol{r})$ may be an operator defined on the iso-space and

may at the same time include the gradient coupling. For these cases (3·7) will be verified in App. IV. The integrand of the right-hand side will be evaluated (see App. IV), yielding

$$\sum_{l,m} \eta_{lm}(\omega) = (1/2i) \log (\Delta^*(\omega)/\Delta(\omega)), \tag{3·9}$$

where

$$\Delta(\omega) = \det |\delta_{ij} + g \int d^3k' v_i(\boldsymbol{k}') (\omega' - \omega - i\varepsilon)^{-1} v_j^\dagger(\boldsymbol{k}')|, \tag{3·10}$$

this determinant consisting also of the iso-spinor matrix elements. Therefore one generally obtains for the self energy of the sources

$$\Delta E = (1/4\pi i) \int_0^\infty d\omega \log (\Delta^*(\omega)/\Delta(\omega)). \tag{3·11}$$

As an illustration let us consider the interaction energy of three sources interacting with the neutral scalar meson field through direct pair coupling. For this case

$$v_i(\boldsymbol{r}) = v(\boldsymbol{r} - \boldsymbol{r}_i), \quad v_i(\boldsymbol{k}) = \exp(i\boldsymbol{k}\boldsymbol{r}_i) v(\boldsymbol{k}), \tag{3·12}$$

hence (3·10) becomes

$$\Delta(\omega) = \begin{vmatrix} 1+u_{11} & u_{12} & u_{13} \\ u_{21} & 1+u_{22} & u_{23} \\ u_{31} & u_{32} & 1+u_{33} \end{vmatrix}, \tag{3·13}$$

where

$$u_{ij} = g \int d^3k' v_i(\boldsymbol{k}') (\omega' - \omega - i\varepsilon)^{-1} v_j^*(\boldsymbol{k}')$$

$$= g \int d^3k' \exp(i\boldsymbol{k}' \boldsymbol{r}_{ij}) (\omega' - \omega - i\varepsilon)^{-1} |v(k')|^2, \tag{3·14}$$

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j. \tag{3·15}$$

We see that ΔE is decomposed into

$$\Delta E = 3\Delta E_{self} + V(r_{12}) + V(r_{23}) + V(r_{31}) + U(r_{12}, r_{23}, r_{31}), \tag{3·16}$$

where

$$\Delta E_{self} = (1/4\pi i) \int_0^\infty d\omega \log [(1+u_{11}^*)/(1+u_{11})] \tag{3·17}$$

is the self energy of a source,[6),7)]

$$V(r_{12}) = (1/4\pi i) \int_0^\infty d\omega \log (\Delta_{(1,2)}^*(\omega)/\Delta_{(1,2)}(\omega)) \to 0 \quad \text{for } r_{12} \to \infty, \tag{3·18}$$

is the interaction energy between sources 1 and 2,[7)] and

$$U(r_{12}, r_{23}, r_{31}) = (1/4\pi i) \int_0^{\infty} d\omega \log \left(\Delta^*_{(1,2,3)}(\omega) / \Delta_{(1,2,3)}(\omega) \right), \tag{3·19}$$

$$\Delta_{(1,2)}(\omega) = \begin{vmatrix} 1+u_{11} & u_{12} \\ u_{21} & 1+u_{22} \end{vmatrix}, \tag{3·20}$$

$$\Delta_{(1,2,3)}(\omega) = \frac{\Delta(\omega)\,(1+u_{11})\,(1+u_{22})\,(1+u_{33})}{\Delta_{(1,2)}(\omega)\,\Delta_{(2,3)}(\omega)\,\Delta_{(3,1)}(\omega)}. \tag{3·21}$$

If the source 1 is carried off far from the other two:

$$r_{12},\ r_{13} \to \infty,$$

then

$$u_{12},\ u_{31} \to 0,$$

hence

$$\Delta_{(1,2)} \to (1+u_{11})\,(1+u_{22}), \quad \Delta_{(3,1)} \to (1+u_{33})\,(1+u_{11}),$$
$$\Delta(\omega) \to (1+u_{11})\,\Delta_{(2,3)},$$

and

$$\Delta_{(1,2,3)}(\omega) \to 1, \quad \log \Delta_{(1,2,3)}(\omega) \to 0.$$

Therefore $U(r_{12}, r_{23}, r_{31})$ does not vanish only when the three sources are all at finite distances from each other. Thus $U(r_{12}, r_{23}, r_{31})$ is just the three-body part of the interaction. The generalization of this result to the other cases is straightforward.

§ 4. Derivative of the phase shift

We shall in this section obtain some information about the derivative of the phase shift which follow from (1·5) under the asymptotic condition. As an example, consider the case of the pion-nucleon scattering. The in-state Ψ^{in}_{pk} is normalized by

$$(\Psi^{in}_{p'k'}, \Psi^{in}_{pk}) = \delta(\boldsymbol{p}'-\boldsymbol{p})\,\delta(\boldsymbol{k}'-\boldsymbol{k}),$$

and in the center of mass system $\Psi^{in}_{q} = \Psi^{in}_{En} = \text{const} \times \Psi^{in}_{pk}$, where $\boldsymbol{q}$ is the relative momentum and $\boldsymbol{n} = \boldsymbol{q}/|\boldsymbol{q}|$, is normalized by

$$(\Psi^{in}_{q'}, \Psi^{in}_{q}) = \delta(\boldsymbol{q}'-\boldsymbol{q}) = k(E)\,\delta(E'-E)\,\delta(\boldsymbol{n}'-\boldsymbol{n}).$$

When an inelastic process does not occur, $d\eta_{nE}/dq$ is given, on account of (1·5), by a term in the partial wave expansion of

$$(\Psi^{in}_{En'}, \bar{D}_q \Psi^{in}_{En}),$$

or of

$$(\Psi^{out}_{En'}, \bar{D}_q \Psi^{out}_{En}),$$

or of the average

$$\{(\Psi^{in}_{En'}\bar{D}_g\Psi^{in}_{En})+(\Psi^{out}_{En'}\bar{D}_g\Psi^{out}_{En})\}/2.$$

For these quantities we see that (see App. V), excepting a trivial factor,

$$(\Psi^{in}_{p'k'}\bar{D}_g\Psi^{in}_{pk})=-i\int_{-\infty}^{\infty}dx'\,dx\{\exp(-ik'x'+ikx)+\exp(-ik'x+ikx')\}$$

$$\times(\Psi_{p'}[\mathcal{H}_\varphi(x'),H_\varphi(x)]\Psi_p)\theta(x'-x)-\int_{-\infty}^{\infty}dy\iint_{-\infty}^{\infty}dx'\,dx\{\exp(-ik'x'+ikx)$$

$$+\exp(-ik'x+ikx')\}(\Psi_{p'}[[\mathcal{H}(y),H_\varphi(x')],H_\varphi(x)]\Psi_p)\theta(y-x')\theta(x'-x), \tag{4·1}$$

where

$$H_\varphi(x)=(-\Box^2+\mu^2)\varphi(x)=-i\int d^3r'[H_1(x),\dot{\varphi}(\mathbf{r}',x_0)], \tag{4·2}$$

$$\mathcal{H}_\varphi(x)=-i\int d^3r'[\mathcal{H}(x),\dot{\varphi}(\mathbf{r}',x_0)]. \tag{4·3}$$

Also

$$(\Psi^{out}_{p'k'}\bar{D}_g\Psi^{out}_{pk})$$

is transformed into a form similar to (4·1).

In the right-hand side of (4·1), the 1-st and the 2-nd terms are the retarded and the advanced propagator respectively. Also the 3-rd and the 4-th terms are such quantities, since in these terms

$$A(x)=\int_{-\infty}^{\infty}dy[\mathcal{H}(y),H_\varphi(x)]\theta(y-x)$$

is a Heisenberg operator:

$$A(x+a)=\int_{-\infty}^{\infty}dy[\mathcal{H}(y),H_\varphi(x+a)]\theta(y-x-a)$$

$$=\int_{-\infty}^{\infty}dy[\mathcal{H}(y+a),H_\varphi(x+a)]\theta(y-x)$$

$$=\exp(-iPa)A(x)\exp(iPa). \tag{4·4}$$

These propagators have certain analyticities.[8),9)] Thus the matrix element

$$(\Psi^{in}_{p'k'}\bar{D}_g\Psi^{in}_{pk})$$

equals the sum of a part which has the analyticity analogous to the relativistic scattering matrix T and a part which has the analyticity analogous to $T^\dagger$. The

situation is similar to the fact that in a potential scattering, if $S(\omega)$ is a partial wave S-matrix,

$$d\eta(\omega)/dg=(1/2iS(\omega))\,dS(\omega)/dg$$

has a cut at the cut of $S(\omega)$ and has poles at poles and zero points of $S(\omega)$ (note that a zero point of $S(\omega)$ is a pole of $S^*(\omega)$), the residues of these two kinds of poles being $d\omega/dg$ and $-d\omega/dg$ respectively. Thus $d\eta(\omega)/dg$ has the combined singularities corresponding to those of $S(\omega)$ and corresponding to those of $S^*(\omega)$.

We further have the following relation (see App. V)

$$\begin{aligned}&(\Psi^{in}_{p'k'}\overline{D}_g\Psi^{in}_{pk})+(\Psi^{out}_{p'k'}\overline{D}_g\Psi^{out}_{pk})\\&=-i\int_{-\infty}^{\infty}\!\!\int dx'\,dx[\exp(-ik'x'+ikx)]\,d/dg(\Psi_{p'}\{P[H_\varphi(x')H_\varphi(x)]\\&-P^{-1}[H_\varphi(x')H_\varphi(x)]\}\Psi_p)+\int_{-\infty}^{\infty}\!\!\int dx'\,dx[\exp(-ik'x'+ikx)]\\&\times(\Psi_{p'}\{H_\varphi(x')\overline{D}_gH_\varphi(x)+H_\varphi(x)\overline{D}_gH_\varphi(x')\}\Psi_p),\end{aligned}\tag{4·5}$$

where P is the time ordering and P^{-1} is the reversed time ordering operator respectively. The 1-st plus 2-nd term is a term of the principal part type and the 3-rd and the 4-th terms are terms of the type of the energy conserving part.

Taking a partial wave component of (4·5), one has

$$d\eta_{nE}/dg=(1/4)\,d/dg(T_{nE}+T^*_{nE})-(1/2)T^*_{nE}(d\eta_{nE}/dg)T_{nE},\tag{4·6}$$

where

$$iT_{nE}=S_{nE}-1=\exp(2i\eta_{nE})-1=2i\exp(i\eta_{nE})\sin 2\eta_{nE}.\tag{4·7}$$

(4·6) equals the identity

$$d\eta_{nE}/dg=(1/2)\,d/dg\sin 2\eta_{nE}-2\sin^2\eta_{nE}(d\eta_{nE}/dg).\tag{4·8}$$

If the perturbation approach is valid, one will be able to solve (4·5), in which the 3-rd and the 4-th terms are of the higher order, by iteration. On account of (4·8) one sees that

$$\left.\begin{aligned}&\eta_{nE}=(1/2)\sin 2\eta_{nE}\qquad\text{in the 1-st approximation,}\\&\text{and}\\&\eta_{nE}=(1/2)\sin 2\eta_{nE}-\sin^2\eta_{nE}\cdot\sin 2\eta_{nE}\quad\text{in the 2-nd approximation,}\end{aligned}\right\}\tag{4·9}$$

and so forth.

In the proper perturbation expansion, as is seen from (4·5),

$$\eta_{nE}-(1/2)\sin\eta_{nE}$$

begins by a term of the 6-th order, therefore the 1-st approximation of (4·9) is

exact in the 4-th order perturbation. When η_{nE} is not so small compared with unity this iteration procedure and, therefore, the perturbation approach is not good.

§ 5. On the construction of potential

In this section we shall obtain a method of constructing the nuclear potential in field theory by the help of the basic equation of § 1. We first consider the adiabatic case and then the non-adiabatic cases.

In the adiabatic limit the potential V may be regarded as the energy eigenvalue of the minimum energy state of the meson field interacting with two static nucleon sources. Hence from (1·2) one has

$$dV/dg=(\Psi\mathcal{H}\Psi), \tag{5·1}$$

where Ψ is the minimum energy state. V is a function of the distance between two nucleons. Ψ is given by

$$\Psi=U(0, \mp\infty)Z^{1/2}\Phi, \tag{5·2}$$

where Φ is the free minimum energy state (see App. I). (Hereafter Z will be abbreviated.) Hence

$$dV/dg=[2\pi\delta(E=0)]^{-1}\int_{-\infty}^{\infty}d\tau(\Psi\mathcal{H}(\tau)\Psi)$$

$$=[\cdots]^{-1}\int_{-\infty}^{\infty}d\tau(\Phi U(\mp\infty, \tau)\mathcal{H}^{int}(\tau)U(\tau, \mp\infty)\Phi). \tag{5·3}*$$

On account of the hermiticity, we put

$$dV/dg=[\cdots]^{-1}(1/2)\int_{-\infty}^{\infty}d\tau(\Phi\{P[U(\infty, -\infty)\mathcal{H}^{int}(\tau)]$$
$$+P^{-1}[U(\infty, -\infty)\mathcal{H}^{int}(\tau)]\}\Phi)$$
$$=[\cdots]^{-1}(i/2)d/dg\{(\Phi U(\infty, -\infty)\Phi)-(\Phi U(-\infty, \infty)\Phi)\}.$$

Therefore

$$V=[\cdots]^{-1}(i/2)\{(\Phi U(\infty, -\infty)\Phi)-(\Phi U(-\infty, \infty)\Phi)\}. \tag{5·4}$$

We can apply the Feynman-Dyson method of calculation to the right-hand side of (5·4).

Let us next consider the non-adiabatic case. The potential can be defined as an equivalent potential which gives the same scattering matrix elements as the nucleon-nucleon part of the field theoretical S-matrix. We shall in the following

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* For the factor $[2\pi\delta(E=0)]^{-1}$, see the footnote on p. 382.

use small letters for quantities concerning the potential scattering. Then our condition is written either by

$$s_{II}=S_{II}, \tag{5·5}$$

or by

$$[\eta_{nE}]_{pot}=[\eta_{nE}]_{field}, \tag{5·6}$$

where $[\ ]_{pot}$ and $[\ ]_{field}$ mean the quantities which are calculated in the potential scattering and in the field theory respectively. In place of (5·6) we may consider

$$[d\eta_{nE}/dg]_{pot}=[d\eta_{nE}/dg]_{field}, \tag{5·7}$$

which is required for all g. Suppose in the following that the nucleons are Bose particles. On account of the identity (4·6) we have

$$[d\eta_{nE}/dg]_{pot}=-2\pi k^{-1}(E)\iint d\boldsymbol{n}\,d\boldsymbol{n}'\,u_n^*(\boldsymbol{n}')\,\{(1/4)\,d/dg[(\boldsymbol{p}'|t|\boldsymbol{p}) + (\boldsymbol{p}'|t^\dagger|\boldsymbol{p})]-(1/2)\,(\boldsymbol{p}'|t(d\eta/dg)\,t^\dagger|\boldsymbol{p})\}u_n(\boldsymbol{n}), \tag{5·8}$$

where

$$(\boldsymbol{p}'|t|\boldsymbol{p})=(\phi_{p'},v\psi_p^{in}), \tag{5·9}$$

and the last term of the right-hand side is defined by

$$-2\pi k^{-1}(E)\,(p'|t(d\eta/dg)\,t^\dagger|p)=\sum_n u_n(\boldsymbol{n}')[t_{nE}(d\eta_{nE}/dg)\,t_{nE}^*]u_n^*(\boldsymbol{n}). \tag{5·10}$$

Meanwhile, if one neglects inelastic processes,

$$d\eta_{nE}/dg=-(1/2)\,k^{-1}(E)[\delta(P_\mu=0)]^{-1}\iint d\boldsymbol{n}\,d\boldsymbol{n}'\,u_n^*(\boldsymbol{n}')\,(1/2)\,\{(\Psi^{in}_{p_1'p_2'},\overline{D}_g\,\Psi^{in}_{p_1p_2}) + (\Psi^{out}_{p_1'p_2'},\overline{D}_g\,\Psi^{out}_{p_1p_2})\}u_n(\boldsymbol{n}), \tag{5·11}$$

where the variables in the right-hand side are those in the center of mass system:

$$\boldsymbol{p}_1=-\boldsymbol{p}_2=\boldsymbol{p},\quad \boldsymbol{p}_1'=-\boldsymbol{p}_2'=\boldsymbol{p}'.$$

Further from (4·5) we have

$$2\pi[\delta(P_\mu=0)]^{-1}(1/2)\,\{(\Psi^{in}_{p_1'p_2'},\overline{D}_g\,\Psi^{in}_{p_1p_2})+(\Psi^{out}_{p_1'p_2'},\overline{D}_g\,\Psi^{out}_{p_1p_2})\} =(1/2)\,d/dg\{(\boldsymbol{p}'|T|\boldsymbol{p})+(\boldsymbol{p}'|T^\dagger|\boldsymbol{p})\}-(\boldsymbol{p}'|T(d\eta/dg)\,T^\dagger|\boldsymbol{p}) -(\boldsymbol{p}'|\{T(d\eta/dg)\,T^\dagger\}_{cross}|\boldsymbol{p}), \tag{5·12}$$

where excepting a trivial factor T is defined by[10]

$$-2\pi i\delta(p_\mu'-p_\mu)\,(\boldsymbol{p}'|T|\boldsymbol{p})=-i\iint dx_1'\,dx_1\exp(-ip_1'x_1'+ip_1x_1) \times(\Psi_{p_2'}p[H_\varphi(x_1')H_\varphi(x_1)]\Psi_{p_2})\equiv\iiiint dx_1\,dx_1'\,dx_2\,dx_2' \times\exp(-ip_1'x_1'-ip_2'x_2'+ip_1x_1+ip_2x_2)\,\delta^4S/\delta\varphi(x_1')\,\delta\varphi(x_2')\,\delta\varphi(x_1)\,\delta\varphi(x_2),^{9)} \tag{5·13}$$

and, $T(d\eta/dg)T^\dagger$ and $\{T(d\eta/dg)T^\dagger\}_{cross}$ are defined by

$$\left.\begin{aligned}
&-2\pi\delta(p_\mu'-p_\mu)\,(\boldsymbol{p}'|T(d\eta/dg)T^\dagger|\boldsymbol{p})\\
&=\iint dx_1'\,dx_1\exp(-ip_1'x_1'+\imath p_1x_1)\,(\Psi_{p_2'}H_\varphi(x_1')\,\overline{D}_g H_\varphi(x_1)\,\Psi_{p_2}),\\
&-2\pi\delta(p_\mu'-p_\mu)\,(\boldsymbol{p}'|\{T(d\eta/dg)T^\dagger\}_{cross}|\boldsymbol{p})\\
&=\iint dx_1'\,dx_1\exp(-ip_1'x_1'+ip_1x_1)\,(\Psi_{p_2'}H_\varphi(x_1)\,D_g H_\varphi(x_1')\,\Psi_{p_2}).
\end{aligned}\right\}\qquad(5\cdot14)$$

Therefore in order that (5·11) holds it is sufficient that

$$\begin{aligned}
&(1/2)\,d/dg\{(\boldsymbol{p}'|t|\boldsymbol{p})+(\boldsymbol{p}'|t^\dagger|\boldsymbol{p})\}-(\boldsymbol{p}'|t(d\eta/dg)t^\dagger|\boldsymbol{p})\\
&=(1/2)\,d/dg\{(\boldsymbol{p}'|T|\boldsymbol{p})+(\boldsymbol{p}'|T^\dagger|\boldsymbol{p})\}-(\boldsymbol{p}'|T(d\eta/dg)T^\dagger|\boldsymbol{p})\\
&-(\boldsymbol{p}'|\{T(d\eta/dg)T^\dagger\}_{cross}|\boldsymbol{p})
\end{aligned}\qquad(5\cdot15)$$

is satisfied on the energy shell.

We will for the potential v admit also the spacially non-local ones. If it is represented by $v(\boldsymbol{r}',\boldsymbol{r})$ in the co-ordinate space and by $v(\boldsymbol{p}',\boldsymbol{p})$ in the momentum space, it reduces to the conventional one when

$$v(\boldsymbol{r}',\boldsymbol{r})=\delta(\boldsymbol{r}'-\boldsymbol{r})\,v(\boldsymbol{r}),\quad v(\boldsymbol{p}',\boldsymbol{p})=v(\boldsymbol{p}'-\boldsymbol{p}).\qquad(5\cdot16)$$

v can be determined from the identity (see App. VI)

$$\begin{aligned}
v_{II}&=t_{II}-\sum_{III}t_{I\,III}(E_I-E_{III}-i\varepsilon)^{-1}t_{III}^\dagger=t_{II}^\dagger-\sum_{III}t_{I\,III}(E_I-E_{III}+i\varepsilon)^{-1}t_{III}^\dagger\\
&=(1/2)\,(t_{II}+t_{II}^\dagger)-(1/2)\sum_{III}t_{I\,III}\{(E_I-E_{III}+i\varepsilon)^{-1}+(E_I-E_{III}-i\varepsilon)^{-1}\}t_{III}^\dagger,
\end{aligned}\qquad(5\cdot17)$$

if t represents the correct scattering matrix, viz.

$$t=T\quad\text{on the energy shell,}\qquad(5\cdot18)$$

and further satisfies the correct orthonormalization condition:

$$t_{II}-t_{II}^\dagger+\sum_{III}t_{I\,III}^\dagger\{(E_I-E_{III}-i\varepsilon)^{-1}+(E_{III}-E_I-i\varepsilon)^{-1}\}t_{III}=0.\qquad(5\cdot19)$$

The condition (5·18) can not be extended off the energy shell, because the relativistic T-matrix does not generally satisfy (5·19). Meanwhile suppose that t is defined from (5·15) which has been extended also off the energy shell, in which the value of T off the energy shell is considered as being defined simply by substituting the off-energy shell values for the variables in the right-hand side of (5·13). $T(d\eta/dg)T^\dagger$ and $\{T(d\eta/dg)T^\dagger\}_{cross}$ also are defined by the expressions which are obtained by performing the further reductions[10] on the annihilation and creation operators in the right-hand sides of (5·14). If we have dropped the 2-nd terms from the both sides of (5·15) which coincide with each other on the energy shell and are of the higher order (see § 4), (5·15) will determine not t itself but only the real part $t_r=(t+t^\dagger)/2$. Therefore (5·15) does not con-

tradict with orthonormalization condition which will be satisfied by choosing the imaginary part $t_i=(1/2i)(t-t^\dagger)$ so as to satisfy (5·19). Thus the potential can be determined from (5·15), (5·17) and (5·19).

As an illustration let us determine this potential in the perturbation expansion. The expansion of t_r begins by a 2-nd order term, that of t_i which is determined successively by (5·19) begins by a 4-th order term and, as having been seen in the previous section, the 3-rd and the 4-th terms of (5·15) begin by 6-th order terms. Therefore:

[2-nd order] From (5·15) and (5·17) we have

$$v=t_r=T_{(2)r}, \tag{5·20}$$

where $T_{(2)r}$ means the real part of the 2-nd order term of T.

[4-th order] From (5·15)

$$t_r=T_{(2)r}+T_{(4)r},$$

and from (5·17)

$$v_{II}=t_{rII}-(1/2)\sum_{III} t_{(2)I III}\{E_I-E_{III}+i\varepsilon)^{-1}+(E_{II}-E_{III}-i\varepsilon)^{-1}\}t_{(2)III II},$$

hence

$$v_{II}=(T_{(2)r}+T_{(4)r})_{II}-(1/2)\sum_{III} T_{(2)r,I III}\{(E_I-E_{III}+i\varepsilon)^{-1}+(E_{II}-E_{III}-i\varepsilon)^{-1}\}T^\dagger_{(2)r,III II}. \tag{5·21}$$

Since, as will be easily seen,

$$Im[T_{(2)}]=0,$$

$$Im[T_{(4)II}]=\pi\sum_{III} T_{(2)I III}\delta(E_{III}-E_{II})T_{(2)III II},$$

(5·12) can also be written as follows,

$$v_{II}=T_{(2)II}+T_{(4)II}-(1/2)\sum_{III} T_{(2)I III}\times\{(E_I-E_{III}+i\varepsilon)^{-1}+(E_{II}-E_{III}+i\varepsilon)^{-1}\}T_{(2)III II}, \tag{5·22}$$

which essentially coincides with the 4-th order potential by the contact transformation.[11]

Appendix I

A rule for the g-differentiation consistent with the normalization

The connection of the formalism of the previous paper with the Feynman-Dyson S-matrix (see App. II of I) was not correct regarding the normalization of state vectors.* In this Appendix we present a rule under the asymptotic con-

* That is, $a^{in}(k)$'s obtained there are not supplied with the normalization factor.

dition for the differentiation with respect to g which is consistent with the normalization. We confine ourselves to the case of no bound state.

Under the asymptotic condition the infinitesimal variation operators $D_g^{in(out)}$ which are defined by

$$d/dg\ \Psi_I^{in}=iD_g^{in}\Psi_I^{in},\ (\text{in}\to\text{out}), \tag{AI·1}$$

are given by

$$D_g^{in(out)}=\int_0^{-\infty(+\infty)} d\tau\,\mathcal{H}(\tau), \tag{AI·2}$$

where we should regard the observed mass m to be independent of g and the mechanical mass m_0 to be dependent on g (see App. II of I). For the transformation function of the interaction representation it holds that

$$d/dg\ U(0,t)=i\int_0^t d\tau\,\mathcal{H}(\tau)\,U(0,t). \tag{AI·3}$$

However, we can not put, combining (A I·2) and (A I·3), for the solution of (A I·1)

$$\Psi_I^{in}=U(0,-\infty)\Phi_I,$$

because the unitarity of $U(0,t)$ is not preserved in the limit $t\to\mp\infty$. In the renormalization theory S-matrix elements are given by

$$\prod_{i\in I}\prod_{j\in II} Z_i^{1/2}Z_j^{1/2}(\Phi_I U(\infty,-\infty)\Phi_{II}),$$

where Z_i's are the normalization constants, therefore the unitarity is satisfied by $Z^{1/2}U(\infty,-\infty)Z^{1/2}$ rather than by $U(\infty,-\infty)$ itself, where

$$Z\Phi_I=\prod_{i\in I} Z_i\Phi_I. \tag{AI·4}$$

Accordingly we can under the asymptotic condition assume the rules

$$\Psi_I^{in}=U(0,-\infty)Z^{1/2}\Phi_I, \tag{AI·5}$$

$$U(0,-\infty)ZU(-\infty,0)=1, \tag{AI·6}$$

$$d/dg\{U(0,-\infty)Z^{1/2}\}=i\int_0^{-\infty} d\tau\,\mathcal{H}(\tau)\,U(0,-\infty)Z^{1/2}. \tag{AI·7}$$

(A I·5) is consistent with Gell-Mann and Low.[12]

Appendix II

Bound state and the phase shifts in the case of the square well potential

In the following the relation (1·16) between the energy of the bound state and the phase shifts is directly ascertained in the case of the square well potential.[13]

The radial wave equation for the S-wave is

$$d^2G/dr^2+[\omega-V(r)]G(r)=0, \tag{AII·1}$$

where $V(r)/2m$ is the potential and $\omega/2m$ is the energy eigenvalue. For the square well potential

$$V(r)=\begin{cases}-V & \text{for } r<a,\\ 0 & \text{for } r>a.\end{cases} \tag{AII·2}$$

We shall consider the case in which only one bound state exists:

$$\pi/2<V^{1/2}a<\pi. \tag{AII·3}$$

The wave function for the bound state is given by

$$G(r)=\begin{cases}A\sin k_0 r & \text{for } r<a,\\ B\exp(-\kappa r) & \text{for } r>a,\end{cases} \tag{AII·4}$$

$$\kappa^2=\omega_0=-\omega,\quad k_0^2=V-\omega_0, \tag{AII·5}$$

where κ is determined by

$$(1/k_0)\tan k_0 a=-(1/\kappa). \tag{AII·6}$$

The wave function of the continuous level is given by

$$G_k(r)=\begin{cases}C_k\sin k' r & \text{for } r<a,\\ \sin(kr+\eta(k)) & \text{for } r>a,\end{cases} \tag{AII·7}$$

$$k^2=\omega,\quad k'^2=\omega+V, \tag{AII·8}$$

where the phase shift $\eta(k)$ is determined by

$$(1/k')\tan k' a=(1/k)\tan(ka+\eta(k)). \tag{AII·9}$$

These wave functions satisfy the completeness relation

$$G(r)G(r')+(2/\pi)\int_0^\infty dk\,G_k(r)G_k(r')=\delta(r-r'). \tag{AII·10}$$

Now put

$$V(r)=Vv(r),\quad v(r)=\begin{cases}-1 & \text{for } r<a\\ 0 & \text{for } r>a,\end{cases} \tag{AII·11}$$

and regard V to be a variable parameter. Then from (A II·4)–(A II·9) we get the relations

$$d\omega_0/dV=(\kappa/V)(\kappa+Va)/(1+\kappa a)$$

$$=1-(B^2/2\kappa)\exp(-2\kappa a)=-\int_0^\infty dr\,G^2(r)v(r), \tag{AII·12}$$

$$d\eta(k)/dV=(aC_k^2/2k)\{1-(1/2ak)\sin 2k'a\}$$
$$=-(1/k)\int_0^\infty dr\,G_k^2(r)\,v(r). \qquad \text{(AII·13)}$$

Further from (A II·10)

$$d/dV\{\int_0^\infty dr\,G^2(r)\,v(r)+(2/\pi)\int_0^\infty dk\int_0^\infty dr\,G_k^2(r)\,v(r)\}=0. \qquad \text{(AII·14)}$$

Substituting (A II·12) and (A II·13) in (A II·14) one obtains

$$d^2\omega_0/dV^2+(2/\pi)\int_0^\infty dk\cdot k\,d^2\eta(k)/dV^2=0,$$

or

$$d^2\omega_0/dV^2=-(1/\pi)\int_0^\infty d\omega\,d^2\eta(k)/dV^2. \qquad \text{(AII·15)}$$

It will be interesting finally to compare the above relation with a relation which follows from the analyticity of the S-matrix element $S(\omega)$.

$$S(\omega)=\exp\{2i\eta(\omega)\}$$
$$=\exp(-2ika)\{1+i(k/k')\tan k'a\}/\{1-i(k/k')\tan k'a\} \qquad \text{(AII·16)}$$

has the analyticity such that $S(\omega)\exp(2ik\alpha)$ $(\alpha>a)$ is analytic on the upper half of the complex k-plane except at the point of infinity, from which one obtains

$$(1/2)\,d\omega_0/dV-i\sum_i \mathrm{Res}\{d\eta/dV-[d\eta/dV]_{V=0}\}_{\omega=\omega_i}$$
$$=-(1/\pi)\int_0^\infty d\omega\{d\eta(\omega)/dV-[d\eta(\omega)/dV]_{V=0}\}, \qquad \text{(AII·17)}$$

where $\omega=\omega_i$ are zero points of $S(\omega)$. The situation is similar to the fact that from the analyticity of $d\eta(\omega)/d\omega$ one obtains not Levinson's theorem but a relation between the number of bound states and the number of poles and zeros of $S(\omega)$.[14]

Appendix III

Determinant form for the S-matrix

In this Appendix the relation expected in § 2 for a general type potential

$$S(E)=\Delta^*(E)/\Delta(E) \qquad \text{(AIII·1)}$$

is verified. We have in § 2 considered the spherically symmetric potential, but the result is generalizable also to the spherically non-symmetric case, for which (A III·1) is modified to

$$2iSp[\eta(E)]=\log(\Delta^*(E)/\Delta(E)), \qquad \text{(AIII·2)}$$

where the spur and the determinat $\Delta(E)$ are defined on an eigen-space of a set of constants of motion.

The elements of the determinant $\Delta(E)$ are

$$\alpha_{\mu\mu}=\delta_{\mu\mu}+(\omega_\mu-E-i\varepsilon)^{-1}V_{\mu\mu}=[1+(H_0-E-i\varepsilon)^{-1}V]_{\mu\mu}. \qquad \text{(AIII·3)}$$

If the cofactor of $\alpha_{\mu\mu}$ is $A_{\mu\mu}$, then

$$\sum_\mu \alpha_{\nu\mu}A_{\mu\mu'}=\Delta(E)\delta_{\nu\nu'}, \qquad \text{(AIII·4)}$$

therefore

$$A_{\nu\nu'}=\Delta(E)[1+(H_0-E-i\varepsilon)^{-1}V]^{-1}_{\nu\nu'}. \qquad \text{(AIII·5)}$$

Meanwhile

$$d/dg\log(\Delta^*(E)/\Delta(E))=(1/\Delta^*(E))\,d/dg\,\Delta^*(E)-(1/\Delta(E))\,d/dg\,\Delta(E), \qquad \text{(AIII·6)}$$

$$\begin{aligned} d/dg\,\Delta(E)&=\sum_{\mu\mu}(d\alpha_{\mu\mu}/dg)A_{\mu\mu}\\ &=\Delta(E)Sp[(H_0-E-i\varepsilon)^{-1}(dV/dg)[1+(H_0-E-i\varepsilon)^{-1}V]^{-1}]\\ &=\Delta(E)Sp[[1+(H_0-E-i\varepsilon)^{-1}V]^{-1}(H_0-E-i\varepsilon)^{-1}dV/dg]\\ &=\Delta(E)Sp[(H-E-i\varepsilon)^{-1}dV/dg], \qquad \text{(AIII·7)}\end{aligned}$$

$$d/dg\,\Delta^*(E)=\Delta^*(E)Sp[(H-E-i\varepsilon)^{-1}dV/dg]. \qquad \text{(AIII·8)}$$

Therefore

$$(1/\Delta^*(E))\,d/dg\,\Delta^*(E)-(1/\Delta(E))\,d/dg\,\Delta(E)=-2\pi i\,Sp[\delta(H-E)\,dV/dg],$$

on account of (1·6) ($l=(E,n)=(ja E)$, j denotes the eigen-space),

$$=2i\sum_a d/dg\,\eta_{jaE}. \qquad \text{(AIII·9)}$$

Therefore from (A III·6) one obtains

$$2i\sum_a \eta_{jaE}=\log(\Delta^*(E)/\Delta(E)). \qquad \text{(AIII·10)}$$

Appendix IV

The static pair-coupling theory

We shall first consider the neutral meson field which interacts with a spherically symmetric static source (with its center at the origin) through the direct pair coupling (see §3). The equation of motion for the meson field $\varphi(x)$ becomes ($x=(\boldsymbol{r},t)$)

$$(-\Box^2+\mu^2)\varphi(x)=-gv(r)\varphi(x). \qquad \text{(AIV·1)}$$

If the interacting system is enclosed by a spherical rigid wall which has its center

at the origin, the meson field can be decomposed into the partial waves and into the normal modes,

$$\varphi(x)=\sum_{lmn} Y_{lm}(\theta\varphi)\,(2\omega_{lmn})^{-1/2}\{a_{lmn}\,r^{-1}\varphi_{lmn}(r)\exp(-i\omega_{lmn}t) + a^{\dagger}_{lmn}\,r^{-1}\varphi^{*}_{lmn}(r)\exp(i\omega_{lmn}t)\}, \tag{AIV·2}$$

where Y_{lm} are the spherical harmonics, n denotes each normal mode, ω_{lmn} is its energy, a_{lmn} is the annihilation operator for the real quantum of each normal mode:

$$[a_{lmn},\ a^{\dagger}_{l'm'n'}]=\delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{AIV·3}$$

and $\varphi_{lmn}(\boldsymbol{r})$ satisfy

$$d^2\varphi_{lmn}(\boldsymbol{r})/dr^2+[\omega^2_{lmn}-gv(r)-l(l+1)r^{-2}]\varphi_{lmn}(\boldsymbol{r})=0. \tag{AIV·4}$$

Substituting (A IV·2), we obtain for the total Hamiltonian

$$H=(1/2)\sum_{lmn}\omega_{lmn}\{a_{lmn}a^{\dagger}_{lmn}+a^{\dagger}_{lmn}a_{lmn}\}. \tag{AIV·5}$$

Such a decomposition is not possible when the source is not spherically symmetric.

We shall next consider the general case in which the number of sources, the type of meson field and the type of coupling except that it is of pair type are arbitrary, so that

$$H_0=(1/2)\sum_{\alpha}\int d^3r\,(\dot{\varphi}_{\alpha}{}^2(\boldsymbol{r})+(\nabla\varphi_{\alpha})^2+\mu^2\varphi_{\alpha}{}^2(\boldsymbol{r})), \tag{AIV·6}$$

$$H_1=(g/2)\sum_{\alpha}\sum_{i}\left(\int d^3r\,\varphi_{\alpha}(\boldsymbol{r})v^{\dagger}\ (\boldsymbol{r}-\boldsymbol{r}_i)\right)\left(\int d^3r'\,v(\boldsymbol{r}'-\boldsymbol{r}_i)\varphi_{\alpha}(\boldsymbol{r}')\right), \tag{AIV·7}$$

where α denotes the components of the meson field such as the iso-spinor components, i denotes the number of each of these sources and $v(\boldsymbol{r}-\boldsymbol{r}_i)$ may generally represents an operator on the iso-space and may at the same time include the gradient coupling. The equation of motion becomes

$$(-\Box^2+\mu^2)\varphi_{\alpha}(x)=-g\sum_{i}v^{\dagger}\ (\boldsymbol{r}-\boldsymbol{r}_i)\int d^3r'\,v(\boldsymbol{r}'-\boldsymbol{r}_i)\varphi_{\alpha}(\boldsymbol{r}'). \tag{AIV·8}$$

$\varphi_{\alpha}(x)$ can be decomposed as follows,

$$\varphi_{\alpha}(x)=\sum_{\beta}\int d^3k\,(2\omega_k)^{-1/2}\{a_{\beta}(\boldsymbol{k})\psi_{k\beta\alpha}(\boldsymbol{r})\exp(-i\omega_k t)+a^{\dagger}_{\beta}(\boldsymbol{k})\psi^{*}_{k\beta\alpha}(\boldsymbol{r})\exp(i\omega_k t)\} \tag{AIV·9}$$

(hereafter we use the matrix notation: $\psi_{k\alpha}=\{\psi_{k\beta\alpha},\ \beta\}$, etc.),

where

$$[a_{\alpha}(\boldsymbol{k}),\ a^{\dagger}_{\beta}(\boldsymbol{k}')]=\delta_{\alpha\beta}\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{AIV·10}$$

and

$$(\nabla^2+k^2)\psi_{k\alpha}(\boldsymbol{r})=g\sum_i v^\dagger\ (\boldsymbol{r}-\boldsymbol{r}_i)\int d^3r'\,v(\boldsymbol{r}'-\boldsymbol{r}_i)\,\psi_{k\alpha}(\boldsymbol{r}')\equiv\int d^3r'\,V(\boldsymbol{r},\boldsymbol{r}')\,\psi_{k\alpha}(\boldsymbol{r}'). \tag{AIV·11}$$

The minimum energy state Ψ_0 is defined by

$$a_\alpha(\boldsymbol{k})\,\Psi_0=0, \tag{AIV·12}$$

and the one meson in-state is given by

$$\Psi_{k\alpha}=a^\dagger_\alpha(\boldsymbol{k})\,\Psi_0, \tag{AIV·13}$$

provided that $\psi_{k\alpha}(\boldsymbol{r})$ satisfy the outgoing boundary condition. The outgoing wave solutions of (A IV·11) are

$$\psi_{k\alpha}(\boldsymbol{r})=\phi_{k\alpha}(\boldsymbol{r})+(\nabla^2+k^2+i\varepsilon)^{-1}g\sum_i v^\dagger\ (\boldsymbol{r}-\boldsymbol{r}_i)\int d^3r'\,v(\boldsymbol{r}'-\boldsymbol{r}_i)\,\psi_{k\alpha}(\boldsymbol{r}'), \tag{AIV·14}$$

where

$$(\nabla^2+k^2)\,\phi_{k\alpha}=0. \tag{AIV·15}$$

Multiplying $v(\boldsymbol{r}-\boldsymbol{r}_i)$ on the both sides of (A IV·14) and integrating over $\boldsymbol{r}$, we get

$$(v_i\psi_{k\alpha})=(v_i\phi_{k\alpha})+\sum_j u_{ij}(k)\,(v_j\psi_{k\alpha}), \tag{AIV·16}$$

where

$$\left.\begin{aligned}(v_i\psi_{k\alpha})&=\int d^3r\,v(\boldsymbol{r}-\boldsymbol{r}_i)\,\psi_{k\alpha}(\boldsymbol{r}),\\(v_i\phi_{k\alpha})&=\int d^3r\,v(\boldsymbol{r}-\boldsymbol{r}_i)\,\phi_{k\alpha}(\boldsymbol{r}),\\u_{ij}(k)&=g\int d^3r\,v(\boldsymbol{r}-\boldsymbol{r}_i)\,(\nabla^2+k^2+i\varepsilon)^{-1}v^\dagger\ (\boldsymbol{r}-\boldsymbol{r}_j).\end{aligned}\right\} \tag{AIV·17}$$

If we put

$$\Delta_{ij}(k)=\delta_{ij}-u_{ij}(k),$$
$$\Delta(k)=\det|\Delta_{ij}(k)|,$$

(where δ_{ij}, u_{ij} and Δ_{ij}, which are the elements concerning the i-th and the j-th sources, may at the same time be certain matrices in the iso-space and the determinat shall be understood as consisting of the elements of these matrices,) and if the cofactor of $\Delta_{ij}(k)$ is $A_{ji}(k)$ which satisfies

$$\sum_l \Delta_{il}(k)\,A_{lj}(k)=\Delta(k)\,\delta_{ij}, \tag{AIV·18}$$

then the solution of (AIV·16) for $(v_i\psi_{k\alpha})$ is

$$(v_i\psi_{k\alpha})=(1/\Delta(k))\sum_j A_{ij}(k)\,(v_j\phi_{k\alpha}). \tag{AIV·19}$$

From (A IV·14) and (A IV·19) we obtain for the S-matrix

$$(\boldsymbol{k}\alpha|S|\boldsymbol{k}'\beta)=\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta_{\alpha\beta}-2\pi i g\delta(w-w')\,(1/\Delta(k))\sum_{ij}(v_i\phi_{k\alpha})^{\dagger}\,A_{ij}(k)\,(v_j\phi_{k'\beta}). \tag{AIV·20}$$

The unitarity of S is represented by

$$-2\pi k^2(dw/dk)^{-1}g\sum_{\alpha}\int d\boldsymbol{n}\,(v_i\psi_{k\alpha})\,(v_j\psi_{k\alpha})^{\dagger}+iA_{ij}/\Delta-iA_{ij}^{\dagger}/\Delta^*=0,\quad(\boldsymbol{n}=\boldsymbol{k}/|\boldsymbol{k}|). \tag{AIV·21}$$

For the energy E of the minimum energy state Ψ_0

$$dE/dg=(\Psi_0\,dH_1/dg\,\Psi_0).$$

(A IV·7), (A IV·9) and (A IV·17) being substituted, it becomes

$$dE/dg=(\Psi_0\,dH_1/dg\,\Psi_0)=(1/2)\sum_{\alpha}\sum_{i}\int d^3k\,(2\omega_k)^{-1}(v_i\psi_{k\alpha})^{\dagger}\,(v_i\psi_{k\alpha}). \tag{AIV·22}$$

Meanwhile by virtue of the identity (1·9) which is applied to the scattering represented by (A IV·11) one has the relation for the phase shifts:

$$\begin{aligned}\sum_{n}d\eta_{nk}/dg&=-(\pi/2)\,k\sum_{n}(\psi_{nk}\,dV/dg\,\psi_{nk})\\&=-(\pi/2)\,k\sum_{\alpha}\int d\boldsymbol{n}\,(\psi_{k\alpha}\,dV/dg\,\psi_{k\alpha})\\&=-(\pi/2)\,k\sum_{\alpha}\sum_{i}\int d\boldsymbol{n}\,(v_i\psi_{k\alpha})^{\dagger}\,(v_i\psi_{k\alpha}).\end{aligned}$$

Therefore it follows that

$$dE/dg=-(1/2\pi)\int dk\,(k/\omega_k)\sum_{n}d\eta_{nk}/dg. \tag{AIV·23}$$

Further from (A IV·21) one has the relation

$$\begin{aligned}\sum_{n}d\eta_{nk}/dg&=-(\pi/2)\,k\sum_{\alpha}\sum_{i}\int d\boldsymbol{n}\,(v_i\psi_{k\alpha})^{\dagger}\,(v_i\psi_{k\alpha})\\&=-(1/2g)\sum_{i}\{iA_{ii}/\Delta-iA_{ii}^{\dagger}/\Delta^*\},\end{aligned} \tag{AIV·24}$$

where $\sum_i$ in the right-hand side represents also the spur on the iso-spinor space. One sees further

$$d/dg\,\log(\Delta^*/\Delta)=(1/\Delta^*)\,d\Delta^*/dg-(1/\Delta)\,d\Delta/dg,$$

$$d\Delta/dg=\sum_{ij}d\Delta_{ij}/dg\,A_{ji},$$

$$d\Delta_{ij}/dg=-du_{ij}/dg=-(1/g)\,u_{ij}=(1/g)\,(\Delta_{ij}-\delta_{ij}),$$

hence

$$d\Delta/dg=(1/g)\sum_{ij}(\Delta_{ij}-\delta_{ij})\,A_{ji}=(1/g)\sum_{i}(\delta_{ii}-A_{ii}).$$

Therefore

$$d/dg \log(\Delta^*/\Delta) = (1/g) \sum_i (A_{ii}/\Delta - A_{ii}^\dagger/\Delta^*). \qquad \text{(AIV·25)}$$

Comparing (A IV·24) with (A IV·25), one obtains

$$\sum_n d\eta_{nk}/dg = (1/2i)\, d/dg \log(\Delta^*/\Delta). \qquad \text{(AIV·26)}$$

For the case of a spherically symmetric potential we have from (A IV·26) for each partial wave

$$d\eta_k/dg = (1/2i)\, d/dg \log(\Delta^*/\Delta), \qquad \text{(AIV·27)}$$

or

$$S(E) = \Delta^*(E)/\Delta(E). \qquad \text{(AIV·28)}$$

The principal results of this Appendix are (A IV·23) and (A IV·26).

Appendix V

The relations (4·1) and (4·5) for $(\Psi^{in}_{p'k'} \bar{D}_g \Psi^{in}_{pk})$, etc., are obtained by applying the usual technique which transposes the in- or out-annihilation and creation operators from right to left or from left to right and by the help of the following relations:

$$(\Psi^{in}_{p'k'} \bar{D}_g \Psi^{in}_{pk}) = (\Phi_{p'k'} Z^{1/2} U(\infty, 0) \int_{-\infty}^{\infty} d\tau\, \mathcal{H}(\tau)\, U(0, -\infty) Z^{1/2} \Phi_{pk})$$

$$= (\Phi_{p'k'} Z^{1/2} \int_{-\infty}^{\infty} d\tau\, U(-\infty, \tau) \mathcal{H}^{int}(\tau)\, U(\tau, -\infty) Z^{1/2} \Phi_{pk}), \qquad \text{(AV·1)}$$

$$(\Phi_{p'} Z^{1/2} P^{-1}[U(-\infty, \tau) O_1(x_1)] \mathcal{H}^{int}(\tau) P[U(\tau, -\infty) O_2(x_2)] Z^{1/2} \Phi_p)$$

$$= (\Phi_{p'} O_1(x_1) \mathcal{H}(\tau) O_2(x_2) \Psi_p), \qquad \text{(AV·2)}$$

$$(\Phi_{p'} Z^{1/2} U(-\infty, \tau) \mathcal{H}^{int}(\tau) P[U(\tau, -\infty) O_1(x_1) O_2(x_2)] Z^{1/2} \Phi_p)$$

$$= (\Psi_{p'} \mathcal{H}(\tau) P[O_1(x_1) O_2(x_2)] \Psi_p), \qquad \text{(AV·3)}$$

$$O(\tau) A(t) O'(\tau') + O'(\tau') A(t) O(\tau) - A(t) P[O(\tau) O'(\tau')]$$

$$- P^{-1}[O(\tau) O'(\tau')] A(t) = -[[A(t), O(\tau)], O'(\tau')] \theta(t-\tau) \theta(\tau-\tau')$$

$$- [[A(t), O'(\tau')], O(\tau)] \theta(t-\tau') \theta(\tau'-\tau), \qquad \text{(AV·4)}$$

$$O(\tau) A(t) O'(\tau') + O'(\tau') A(t) O(\tau) - P[A(t) O(\tau) O'(\tau')]$$

$$- P^{-1}[A(t) O(\tau) O'(\tau')] = -\{[[A(t), O(\tau)], O'(\tau')]$$

$$+ [[A(t), O'(\tau')], O(\tau)]\} \{\theta(t-\tau) \theta(\tau-\tau') + \theta(\tau'-\tau) \theta(\tau-t)\},$$

$$\text{(AV·5)}$$

$$d/dg\,(\Psi_{p'}P[O_1(t_1)\,O_2(t_2)\,]\Psi_p)=-i\int_{-\infty}^{\infty}d\tau\,(\Psi_{p'}P[O_1(t_1)\,O_2(t_2)\,\mathcal{H}(\tau)\,]\Psi_p)$$
$$+(\Psi_{p'}\{P[dO_1/dg\,(t_1)\,O_2(t_2)\,]+P[O_1(t_1)\,dO_2/dg\,(t_2)\,]\}\Psi_p). \qquad (\mathrm{AV}\cdot 6)$$

Appendix VI

The equation

$$\psi_I^{in}=\phi_I+(\epsilon_I-h_0+i\varepsilon)^{-1}v\psi_I^{in} \qquad (\mathrm{AVI}\cdot 1)$$

is rewritten as follows,

$$\psi_I^{in}=\{1-(\epsilon_I-h_0+i\varepsilon)^{-1}v\}^{-1}\phi_I=\{1+(\epsilon_I-h+i\varepsilon)^{-1}v\}\phi_I. \qquad (\mathrm{AVI}\cdot 2)$$

Taking the scalar product with $\phi_{II}^{+}\,v$, one obtains

$$(\phi_{II}v\psi_I^{in})=(\phi_{II}v\phi_I)+(\phi_{II}v(\epsilon_I-h+i\varepsilon)^{-1}v\phi_I)$$

or

$$v_{II}=t_{II}-\sum_{III}t_{III}(\epsilon_I-\epsilon_{III}+i\varepsilon)^{-1}t_{III\,I}^{\dagger}. \qquad (\mathrm{AVI}\cdot 3)$$

(A VI·3) must be slightly modified when there is a bound state.

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Dispersion Relations and High Energy Limits in Quantum Field Theory. II

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The lower and upper bounds of the high energy limit of pion-nucleon forward scattering amplitudes are investigated. The former is studied using available experimental data and found that at least one subtraction is necessary in the dispersion relations. The latter can be determined under an additional requirement that the forward amplitude has no zero in the complex energy plane. Some discussions are given to the problems on the high-energy behaviours in quantum field theories.

§ 1. Introduction

It has been conjectured that the total cross section for the pion-nucleon scattering approaches a constant value and does not damp so rapidly at high energy limit. This conjecture comes from the experimental facts which show that at very high energies the cross section keeps a constant value, about 30mb.[1] The dispersion relations for pion-nucleon forward scattering amplitudes given by Goldberger[2] are derived from this supposition. However, no theoretical foundation which excludes the possibility of damping of cross section at high energy limit or excludes the validity of unsubtracted dispersion relations has been given except the work of Arnowitt and Feldman.[3]

In this paper we will show using available experimental data that at least one subtraction is necessary in the dispersion relations for pion-nucleon forward scattering amplitudes irrespective of the detailed structures of interactions. This is the same result as obtained by Arnowitt and Feldman who showed this, under the assumption of the existence of $\lambda\varphi^4$ term in the interaction Lagrangian, by constructing unsubtracted dispersion relation in the form derived by Goldberger and finally by leading to the inconsistency. However, their method seems to be incomplete and inconclusive. This point will be discussed in § 4. In the same section, some words shall be added to the work of Källén[4] on the magnitudes of the renormalization constants.

For the determination of the upper bound of high energy limit of total cross section, a method was proposed in the previous paper.[5] This method is based upon an assumption of uniqueness of the "solutions" of the dispersion relations, or equivalently, of the absence of zeros of the pion-nucleon forward scattering amplitudes in their complex energy planes. In I, the arguments have been given

on a model of the scattering of a neutral pion by a neutral scalar nucleon[6] in which there were some mistakes. In this paper, we will determine the upper bound of high energy limit of the total cross section for the scattering of $\pi^{\pm}$ mesons by proton by means of the same method as I and correct the conclusions drawn there.

§ 2. Lower bound of high energy limit of pion-nucleon scattering amplitude*

The forward scattering amplitude in which a pion with isotopic spin index β is scattered into the one with isotopic spin index α is written as

$$T_{\alpha\beta}(\omega)=\delta_{\alpha\beta}T^{(1)}(\omega)+\frac{1}{2}[\tau_\alpha,\tau_\beta]T^{(2)}(\omega). \tag{1}$$

Denoting $T_{\pm}(\omega)$ the forward scattering amplitude for the coherent scattering of $\pi^{\pm}$ mesons by protons, the following relations hold

$$\left.\begin{aligned} T^{(1)}(\omega)&=\frac{1}{2}[T_-(\omega)+T_+(\omega)],\\ T^{(2)}(\omega)&=\frac{1}{2}[T_-(\omega)-T_+(\omega)]. \end{aligned}\right\} \tag{2}$$

If it is assumed that $T^{(1)}$ and $T^{(2)}$ satisfy the unsubtracted dispersion relations, we shall have for $T^{(1)}(\omega)$

$$\mathrm{Re}T^{(1)}(\omega)=\frac{2}{\pi}\int_\mu^\infty \omega' d\omega' \frac{\mathrm{Im}\,T^{(1)}(\omega')}{\omega'^2-\omega^2}+\frac{f^2}{M}\frac{(\mu^2/2M)^2-\mu^2}{\omega^2-(\mu^2/2M)^2}. \tag{3}$$

Here, $\mathrm{Im}\,T^{(1)}(\omega)$ is related to the total $\pi^{\pm}$cross section, $\sigma_{\pm}(\omega)$, by the optical theorem:

$$\left.\begin{aligned} \mathrm{Im}T^{(1)}(\omega)&=\frac{k}{8\pi}[\sigma_-(\omega)+\sigma_+(\omega)],\\ k&=\sqrt{\omega^2-\mu^2}. \end{aligned}\right\} \tag{4}$$

Putting $\omega=\mu$ in Eq. (3) we have

$$\mathrm{Re}\,T^{(1)}(\mu)=\frac{1}{4\pi^2}\int_0^\infty dk\,[\sigma_-(\omega)+\sigma_+(\omega)]-\frac{f^2}{M}. \tag{5}$$

Writing $\mathrm{Re}T^{(1)}(\mu)=1/2[D_-(\mu)+D_+(\mu)]$, $D_{\pm}(\omega)$ the real parts of $T_{\pm}(\omega)$, Eq. (5) becomes

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* After submitting this paper for publication, a paper by H. Chew (H. Chew, Nuovo Cimento **17** (1960), 619) has appeared. He has obtained independently the same conclusion by the method same with that in this section.

$$\frac{1}{4\pi^2}\int_0^\infty dk[\sigma_-(\omega)+\sigma_+(\omega)]=\frac{1}{2}[D_-(\mu)+D_+(\mu)]+\frac{f^2}{M}. \tag{6}$$

We will check the validity of this equation by substituting the experimental results for $\sigma_\pm$, $D_\pm$ and using the value of coupling constant f^2 determined from the analyses of low energy pion phenomena.*

First we note that the left-hand side of Eq. (6) is positively definite and consequently for some finite pion momentum Λ the following inequality holds:

$$\frac{1}{2}\left[D_-(\mu)+D_+(\mu)\right]+\frac{f^2}{M}\geqq\frac{1}{4\pi^2}\int_0^\Lambda dk[\sigma_-(\omega)+\sigma_+(\omega)]\,. \tag{7}$$

On checking this inequality, we shall proceed in the following manner:

For $D_\pm(\mu)$, which are expressed in terms of the S-wave scattering lengths a_1 and a_3, we adopt the result obtained by Orear[7] from the low-energy phase shift analysis. Orear's result is that $a_1=+0.16$ and $a_3=-0.11$. Thus, $D_+(\mu)=-0.17$, $D_-(\mu)=+0.11$.

For f^2, the pion-nucleon coupling constant, we use the value $f^2=0.08$[8].

For the right-hand side of Eq. (7), it is sufficient for the present discussion to take $\Lambda=1.9$ Bev. The right-hand side of Eq. (7) is just the area under the curve of $[\sigma_-(\omega)+\sigma_+(\omega)]/2$ plotted versus k, so we can underestimate the integral of Eq. (7) replacing $[\sigma_-(\omega)+\sigma_+(\omega)]/2$ by a constant value 20mb.

In this way, evaluating the both sides of Eq. (7) we have

$$\begin{aligned} &-0.01\times10^{-13}\,\text{cm} && \text{for left-hand side,}\\ &+0.98\times10^{-13}\,\text{cm} && \text{for right-hand side.}\end{aligned} \tag{8}$$

These values do not satisfy the inequality (7) even if considerable errors exist in the experimental results. Namely, the dispersion relation in unsubtracted form, Eq. (3), contradicts the present experimental results. Therefore we should perform at least one subtraction in the dispersion relation for $T^{(1)}(\omega)$.

The same discussion as above, however, cannot be applied to the dispersion relation for $T^{(2)}(\omega)$ because Im $T^{(2)}(\omega)$ is not positively definite and the inequality such as Eq. (7) cannot be proved. Thus for $T^{(2)}(\omega)$, there is the possibility of the existence of the unsubtracted dispersion relation.

§ 3. Upper bound of high energy limit of pion-nucleon scattering amplitude

It was shown in I that the high energy limit of $T(\omega)$ can be determined if the pion-nucleon forward scattering amplitude has no zero on the complex energy plane. There, we had a conclusion that $T(\omega)$ cannot increase like ω^2 or more rapidly as $|\omega|\to\infty$, but this conclusion was incorrect and we will show now in

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* Essentially the same discussions in the case of the scattering of photons by free electrons have been made by Goldberger.[2]

this section that $T(\omega)$ cannot increase like ω or more rapidly as $|\omega|\to\infty$, that is, we obtain more stringent restriction on the increase of $T(\omega)$ at high energy. In the following discussions, we shall assume that $T(\omega)$ cannot increase more rapidly than some polynomial of ω, ω^n, as $|\omega|\to\infty$.

Defining $k(\omega)$ by the following relation:

$$k(\omega)=\mathrm{Im}\lim_{\epsilon\to 0}\frac{-1}{T^{(1)}(\omega+i\epsilon)} \quad \text{for real } \omega, \tag{9}$$

this quantity is positively definite; in fact

$$k(\omega)=\frac{\mathrm{Im}\,T^{(1)}(\omega)}{[\mathrm{Re}\,T^{(1)}(\omega)]^2+[\mathrm{Im}\,T^{(1)}(\omega)]^2}$$

$$=\frac{1/8\pi\cdot k[\sigma_-(\omega)+\sigma_+(\omega)]}{[\mathrm{Re}\,T^{(1)}(\omega)]^2+[\mathrm{Im}\,T^{(1)}(\omega)]^2}\geqq 0.$$

Therefore, if $T^{(1)}(\omega)$ increases like ω or more rapidly when ω tends to infinity, the following integral exists:*

$$\int_{\mu}^{\infty}\frac{\omega k(\omega)\,d\omega}{\omega^2-\omega_B{}^2}=-\frac{1}{4i}\int_C\frac{dz}{(z-\omega_B)T^{(1)}(z)}, \quad \text{for } n\geqq 1. \tag{10}$$

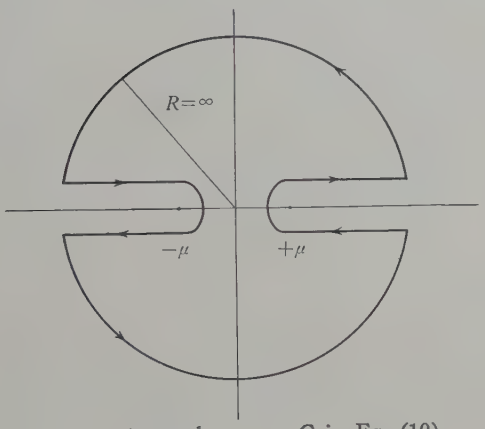


Fig. 1. Integral contour C in Eq. (10).

Here, ω_B is the position of bound state poles of $T^{(1)}(z)$, $\omega_B=\pm(\mu^2/2M)$, and the contour C in the right-hand side are shown in Fig. 1. In deriving this equation we used the symmetrical properties of $\mathrm{Re}\,T^{(1)}(\omega)$ and $\mathrm{Im}\,T^{(1)}(\omega)$ for real ω; $\mathrm{Re}\,T^{(1)}(-\omega)=\mathrm{Re}\,T^{(1)}(\omega)$, $\mathrm{Im}\,T^{(1)}(-\omega)=-\mathrm{Im}\,T^{(1)}(\omega)$.

Therefore, the integral of the right-hand side of Eq. (10) gives zero if $T^{(1)}(z)$ has no zero in the whole plane, because $T(z)$ has poles at $z=\omega_B$. Thus we shall obtain

$$\int_{\mu}^{\infty}\frac{\omega k(\omega)\,d\omega}{\omega^2-\omega_B{}^2}=0, \quad \text{for } n\geqq 1.$$

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* Eq. (11) of I, $\int_{\mu}^{\infty}k(\omega)d\omega=-\frac{1}{4i}\int_C\frac{dz}{T(z)}$, is not correct since $\int_C\frac{dz}{T(z)}$ is identically zero. It should be corrected as Eq. (10) in this paper.

This integral is positively definite and it can be concluded that $T^{(1)}(\omega)$ cannot increase like ω or more rapidly as $|\omega|\to\infty$. This result imposes more stringent restriction on the upper bound of high energy limit of the total cross section than that obtained by Symanzik.[9]

There is no proof in the relativistic field theory that the scattering amplitudes or propagators have no zeros on their complex planes; however, this problem has been solved for some specialized model, namely the Dyson model.[10] It can be proved that there is no zero if the unstable particles (with respect to the strong interactions) do not exist. The Lee model[11] in which no unstable particle exists has no zero, and the Dyson model in which arbitrary number of unstable particles exist has zeros. Therefore, if we apply this interpretation for realistic theory too, the assumption that there is no zero in the pion-nucleon forward amplitude is equivalent to the statement that there is no unstable particle capable to decay into mesons and baryons through the strong interactions.

From the discussions of this section, it follows that it is sufficient to perform one subtraction in the dispersion relation for $T^{(1)}(\omega)$. On the contrary any conclusion on the subtraction in the dispersion relation cannot be drawn for $T^{(2)}(\omega)$ because for this quantity we cannot construct the positively definite quantity such as Eq. (9). But the total cross sections $\sigma_{\pm}(\omega)$ should damp since $\mathrm{Im}\,T^{(1)}(\omega)=(k/8\pi)(\sigma_{+}(\omega)+\sigma_{-}(\omega))$ cannot increase like ω or more rapidly as $\omega\to\infty$, and it can be also concluded that $\mathrm{Im}\,T^{(2)}(\omega)$ cannot increase like ω or more rapidly when ω tends to infinity.

§ 4. Dispersion relations and high energy limits

I) Arnowitt and Feldman[3] used the technique of derivation by Goldberger[2] and obtained the following dispersion relation under the assumption that $\mathrm{Im}\,T^{(1)}(\omega)$ damps,

$$\mathrm{Re}\,T^{(1)}(\omega)-\lambda C-\frac{\mu^2}{\omega^2}(\mathrm{Re}\,T^{(1)}(\mu)-\lambda C)=\frac{2}{\pi}\frac{k^2}{\omega^2}\int_{\mu}^{\infty}\frac{d\omega'\omega'^3\mathrm{Im}T^{(1)}(\omega')}{(\omega'^2-\omega^2)(\omega'^2-\mu^2)}. \quad (11)$$

Here, we dropped the bound state term which is unessential for our discussions. λC represents a constant term which originates from the $\lambda\varphi^4$ term in the interaction Lagrangian.

Arnowitt and Feldman assumed that λ is not zero and insisted Eq. (11) is inconsistent, that is, the assumed existence $\int^{\infty}\sigma(\omega)\,d\omega<\infty$ is incorrect.* However, if we tend ω to infinity in Eq. (11), we shall obtain

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* In their paper[3] Arnowitt and Feldman insisted $\int^{\infty}d\omega\sigma(\omega)>\infty$, but this cannot necessarily be concluded, for the dispersion relation in the unsubtracted form can be derived only under the assumption that both $\mathrm{Re}T^{(1)}(\omega)$ and $\mathrm{Im}T^{(1)}(\omega)$ should damp. This is because we apply Cauchy's theorem and suppress the contributions on the large circle at infinity for the derivation of dispersion relations. However, we cannot exclude a possibility that $\mathrm{Re}T^{(1)}(\omega)$ increases at infinity.

$$\mathrm{Re}\, T^{(1)}(\infty) = \lambda C\,. \tag{12}$$*

Namely, $\mathrm{Re}\, T^{(1)}(\omega)$ does not damp at infinity as far as $\lambda \neq 0$. Therefore, in deriving the dispersion relation for $T^{(1)}(\omega)$ we must supply one more denominator in order to secure the convergence of Cauchy's integral at infinity. This forces us to perform one subtraction in constructing the dispersion relation and consequently the λC term disappears from the relation, thus any contradiction cannot be found. Just at this point their arguments are incomplete and inconclusive.

II) Next, we wish to remark Källén's proof[4] that at least one renormalization constant is infinite. Källén's proof consists in that the unrenormalized current operator between vacuum and one pair states approaches the Born approximation at very high energies when all the renormalization constants are finite, or in other words,

$$\left.\begin{aligned} &\Delta_F'(k) \to Z_3^{-1}\, \Delta_F(k), \\ &S_F'(k) \to Z_2^{-1} S_F(k), \\ &\Gamma_\mu(k) \to Z_2 \gamma_\mu \end{aligned}\right\} \quad \text{for } -k^2 \to \infty. \tag{13}$$

Here, Δ_F', S_F' and Γ_μ are renormalized photon propagator, electron propagator and proper vertex part, respectively. Whether Γ_μ really satisfies Eq. (13) is now an open question,[12] and here we will discuss the validity of Eq. (13).

It is very difficult to check Eq. (13) for Γ_μ, but it will be reasonable to infer the high energy behaviour of Γ_μ from the one of Δ_F' because Källén's Eq. (13) requires for both Γ_μ and Δ_F' that they approach the Born approximation limits. Namely, if the Born approximation is not valid for Δ_F', it is very plausible that the same is true for Γ_μ.

It is the imaginary parts that define the renormalization constants; for example, the charge renormalization constant Z_3^{-1} is defined as follows:[13]

$$Z_3^{-1} = \frac{1}{\pi}\int_0^\infty \mathrm{Im}\, \Delta_F'(k)\, d(-k^2). \tag{14}$$

Namely the finite renormalization constant Z_3^{-1} can be obtained if $\mathrm{Im}\, \Delta_F'$ rapidly damps. However, this does not necessarily mean that $\mathrm{Re}\, \Delta_F'$ also strongly damps. It may be even possible that $\mathrm{Re}\, \Delta_F'$ increases at infinity while $\mathrm{Im}\, \Delta_F'$ damps.

When this case happens, we must write down the spectral representation for Δ_F' as follows:[14]

$$\Delta_F'(k) = \frac{1}{k^2} + \sum_{j=2}^{n} C_j (k^2)^{j-2} + (k^2)^{n-1} \int_0^\infty \frac{\sigma(\lambda^2)\, d\lambda^2}{(-\lambda^2)^{n-1}(\lambda^2 + k^2)}. \tag{15}$$

* This can be obtained by comparing Eq. (11) and the one subtracted dispersion relation which certainly exists if Eq. (11) exists.

The case $n=1$ in this expression corresponds to the spectral representation derived by Källén and by Lehmann.

Making $-k^2$ approach ∞ in Eq. (15), we obtain

$$\Delta'_F(k) \to \frac{1}{k^2}\left[1+\int_0^\infty \sigma(\lambda^2)\,d\lambda^2\right] = Z_3^{-1}\Delta_F(k) \quad \text{for } n=1,$$

$$\Delta'_F(k) \to (k^2)^{n-2}\left[C_n+\int_0^\infty \frac{\sigma(\lambda^2)\,d\lambda^2}{(-\lambda^2)^{n-1}}\right] \quad \text{for } n\geqq 2.$$

In the case $n\geqq 2$, we cannot obtain Eq. (13) because the factors $\left[C_n+\int_0^\infty \frac{\sigma(\lambda^2)\,d\lambda^2}{(-\lambda^2)^{n-1}}\right]$, in general, are not ensured to vanish.

Eq. (13) physically implies the possible disappearance of the effects of clouds from unrenormalized quantities at high energy limit, but as seen above, there can be the possibility of the existence of the effects of clouds for propagators even if their imaginary parts do strongly damp. From this we infer that the same holds for Γ'_μ, that is, the effects of clouds do not disappear from $Z_2^{-1}\Gamma'_\mu$ and Eq. (13) does not hold for Γ'_μ even if the divergences do not exist in the theory.

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Effective Interaction in Nuclear Many-Body Problem

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General treatment of nuclear many-body problem will be given in terms of Green functions and the correlation property of nucleus will be exhibited by correlation factor. Effective interaction certifying the independent particle motion is introduced and proved to identify with Brueckner's K-matrix. And on the basis of this point, applying the method of the particle-hole pair to the system, a collective mode will be derived and the effective interaction contributing to collective motion will be studied. It will also be shown that this effective interaction has the property very different from electron plasma case.

§ 1. Introduction

In nuclear many-body problem, it is well known that the Hartree-Fock method may not be applied as it is, in the case of deriving the independent particle motion suggested from the shell model. This is because that the interparticle interaction has very singular part such as hard-core. Under the singular interaction for high-momentum part such as hard-core, Brueckner et al. succeeded in deriving the independent particle motion in terms of their K-matrix defined for two-particle state in the nucleus.[1] In the more detailed problem, in which low-momentum transfer plays an important role, i. e. collective motion, it may be considered as reasonable to take this results into account. So, in the case of considering the residual interaction, which is recently called in question, Brueckner's state is removed from the whole system at first and the remaining terms are considered as "residual interaction". If there is the so-called B-C-S state considered in the theory of super-fluidity in nucleus, this treatment may be unreasonable as it is. Here we assume that there is not such state, but if exists in nucleus, it seems to be able to treat, for example, in the Bogoliubov transformation firstly.

After such transformation it may be expected for the original interaction to suffer a drastic change. In the K-matrix method, it is known that long-range part of the interaction may not be altered but short-range part, i. e. hard core, is screened perfectly.[2] For long-range part of the interaction, the method of the particle-hole pair, considered by Sawada et al.[3] in electron plasma and applied by S. Takagi[4] for nuclear collective motion, may effect in some kind of screening of the interaction. It seems that the so-called "quadrupole-quadrupole interaction" may not be simply the part of $L=2$ in the Legendre expansion of the original

short-range force, but the rather long-range force formed by some kind of transformation. This situation will be considered in a later section (§ 6). Recently K. Sawada showed an interesting treatment of many-body system.[5] By the equation

$$[H_{total}, X]=\omega X+Y,$$

the excitation mode X of many-body system and the excitation energy ω are defined respectively, and the remaining term Y (the coupling term between mode X and the other) is rewritten in terms of K-matrix. Here, the particular dynamical mode is not defined from the begining but we will formulate in terms of the Green function generally[6] and make the behaviour of nuclear correlation clear. And we consider the treatment not only of the independent particle motion but of the other dynamical mode on the basis of independent particle motion.

§ 2. Formulation in terms of Green function

2·1. *Energy of the system*

Total Hamiltonian of the system is given as follows,*

$$H=\frac{1}{2m}\int d\boldsymbol{X}\cdot N\{\boldsymbol{\nabla}\psi^{+}(\boldsymbol{X})\boldsymbol{\nabla}\psi(\boldsymbol{X})\}$$
$$+\frac{1}{2}g\int d\boldsymbol{X}d\boldsymbol{X}'\cdot v(|\boldsymbol{X}-\boldsymbol{X}'|)N\{\psi^{+}(\boldsymbol{X}')\psi^{+}(\boldsymbol{X})\psi(\boldsymbol{X})\psi(\boldsymbol{X}')\}, \qquad (2\cdot1)$$

where N denotes the symbol of normal product and $\psi^{+}(\psi)$ is the field operator representing the particle creation (annihillation) and satisfies the anti-commutation rule. Only to simplify the situation the two-body interaction v is assumed to be central and not to have spin and isotopic spin dependence. Denote the ground state of nucleus with A nucleons as $|A\rangle$ and its energy as E_A, then

$$E_A=\langle A|H|A\rangle.$$

As is well known, differentiating both sides of the above with respect to the coupling constant g and integrating, we may obtain

$$E_A=E_A^{(0)}+\int_0^g dg'/g'\cdot\langle A|H'|A\rangle, \qquad (2\cdot2)$$

where H' represents the second term in the right-hand side of Eq. (2·1) and $E_A{}^{(0)}$ represents the energy of non-interacting system with A-nucleons, that is,

$$E_A^{(0)}=(3/5)AE_F, \qquad (2\cdot3)$$

where, E_F is Fermi energy.

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* We abopt the natural unit $\hbar=c=1$.

$$\langle A|H'|A\rangle = \frac{1}{2} g' \int dX\, dX'\, v(|X-X'|) \langle A|N\{\psi^+(X')\psi^+(X)\psi(X)\psi(X')\}|A\rangle . \tag{2·4}$$

Here, we define the Green function of N-fermions as follows,

$$G_n(x_1 x_2 \cdots ; x_1' \cdots x_n') = i^n \langle A|T\{\psi(x_1)\psi(x_2)\cdots\psi(x_n)\psi^+(x_n')\cdots\psi^+(x_1')\}|A\rangle, \tag{2·5}$$

where, T shows the chronological ordering symbol and x represents the coordinate vector with four components (X, t). Then we obtain

$$\langle A|H'|A\rangle = \frac{1}{2} g' (-i)^2 \int dX\, dX'\, v(|X-X'|) \lim_{t' \to t_+} G_2(Xt, X't; Xt', X't'). \tag{2·6}$$

We then obtain the total energy in terms of Eqs. (2·2), (2·4) and (2·6) as

$$E_A = E_A^{(0)} + \int_0^g \frac{dg'}{g'} (-i)^2 \frac{1}{2} g' \int dX\, dX'\, v(|X-X'|) \lim_{t' \to t_+} G_2(Xt, X't; Xt', X't'). \tag{2·7}$$

2·2. *Wave function of single particle*

The equation of motion of field operator $\psi(x)$ is

$$i\frac{\partial}{\partial t}\psi(x) = \frac{p^2}{2m}\psi(x) + g\int dX'\, v(|X-X'|) N\{\psi^+(X't)\psi(X't)\psi(x)\}. \tag{2·8}$$

Using the Green function defined by Eq. (2·5), we get

$$\left(-i\frac{\partial}{\partial t} + \frac{p^2}{2m}\right) G_1(x; x') + ig\int dX''\, v(|X''-X|) G_2(X''t, x; Xt, x') = \delta^4(x-x'). \tag{2·9}$$

For $t > t'$

$$G_1(x; x') = i\sum_n \varphi_n(x)\varphi_n^*(x'), \tag{2·10}$$

where,

$$\varphi_n(x) = \langle A|\psi(x)|A_n\rangle, \tag{2·11}$$

and

$$\varphi_n(x) = \varphi_n(X) e^{-iE_n t}, \tag{2·12}$$

where, $\varphi_n(x)$ represents the wave function of a single particle and $|A_n\rangle$ is eigenstate of $\psi(x)$ with eigenvalue $E_n - E_A$. Integrate Eq. (2·10) on X' multiplied $\varphi_\alpha(X't')$, we get

$$\int dX' G_1(x; X't')\varphi_\alpha(X't') = i\sum_n \varphi_n(x)\int \varphi_n^*(X')\varphi_\alpha(X')\, dX' \cdot e^{-i(E_\alpha - E_n)t'}.$$

In order to drop the oscillating term, the right-hand side may be averaged in

terms of the operator $(1/|t'|)\cdot\int_{2t'}^{t'}dt'$ (for $t'\to-\infty$).[7] Thus

$$\lim_{t'\to-\infty}\int dX' G_1(x;X't')\varphi_\alpha(X't')=i\varphi_\alpha(x)\int|\varphi_\alpha(X')|^2dX'.$$

Normalizing φ_α as follows,

$$\int|\varphi_\alpha(X')|^2dX'=1.$$

Then we obtain the equation of motion in integral form as follows,

$$\lim_{t'\to-\infty}\int dX' G_1(x;X't')\varphi_\alpha(X't')=i\varphi_\alpha(x). \tag{2·13}$$

Rewriting the above to wave equation, we operate $(-i\cdot\partial/\partial t+p^2/2m)$ to Eq. (2·13), we obtain from Eq. (2·9) as follows,

$$i\frac{\partial}{\partial t}\varphi_\alpha(x)=\frac{p^2}{2m}\varphi_\alpha(x)+\lim_{t'\to-\infty}g\int dX'\,dX''v(|X''-X|)$$
$$\times G_2(X''t,\ x;\ X''t,\ x')\varphi_\alpha(x'). \tag{2·14}$$

2·3. *Structure of Green function*

From the above formulation, we see that the second order Green function plays a crucial role. The second order Green function may be decomposed as follows,[8]

$$G_2(x_1x_2;\ x_1'x_2')=G^{(0)}(x_1;\ x_1')G_1(x_2;\ x_2')-G^{(0)}(x_1;\ x_2')G_1(x_2;\ x_1')$$
$$-i\int I(\xi,\ \xi')G^{(0)}(x_1;\ \xi)G_2(\xi x_2\xi';\ x_1'x_2'\xi')\,d\xi\,d\xi', \tag{2·15}$$

where

$$iI(x,\ x')=gv(|X-X'|)\delta(t-t'), \tag{2·16}$$

and $G^{(0)}(x;x')$ satisfies the following equation,

$$\left(-i\frac{\partial}{\partial t}+\frac{p^2}{2m}\right)G^{(0)}(x;\ x')=\delta^4(x-x').$$

Here, we will introduce the "correlation factor" defined by the following equation,

$$\begin{aligned}&G^{(0)}(x;\ x')=C_0(x;\ x'),\\ &G_1(x;\ x')=C_1(x;\ x'),\\ &G_2(x_1x_2;\ x_1'x_2')=C_1(x_1;\ x_1')C_1(x_2;\ x_2')-C_1(x_1;\ x_2')C_1(x_2;\ x_1')\\ &\qquad+C_2(x_1x_2;\ x_1'x_2'),\\ &\cdots\cdots\text{ and so on.}\end{aligned} \tag{2·17}$$

Substituting the above equation to Eq. (2·15), we will obtain the integral equation for the correlation factor to satisfy. Firstly, we get for C_1 as

$$C_1(x_1;x_1')=C_0(x_1;x_1')-i\int I(\xi,\xi')\,C_0(x_1;\xi)$$
$$\times[C_1(\xi;x_1')\,C_1(\xi';\xi')-C_1(\xi';x_1')\,C_1(\xi;\xi')+C_2(\xi\xi';x_1'\xi')]. \quad (2\cdot18)$$

In the above, as pointed out by Ch'un, the usual Hartree-Fock equation will be given at once if C_2 term is neglected. But this may not be allowed if the interaction is singular since the first order term diverges. Then, in order to obtain C_2 we will use the binary approximation,—it is the approximation in which C_3 is neglected in the expansion of G_3 in terms of correlation factor—and correct the vertex I to $\mathcal{I}$ to renormalize the self-energy in C_0.* Thus we get the equation satisfied by C_2 in closed form,

$$C_2(x_1x_2;x_1'x_2')=-i\int \mathcal{I}(\xi,\xi')\,C_1(x_1;\xi)$$
$$\times[C_2(\xi x_2;x_1'x_2')\,C_1(\xi';\xi')+C_2(x_2\xi';x_1'x_2')\,C_1(\xi;\xi')$$
$$-C_2(\xi\xi';x_1'x_2')\,C_1(x_2;\xi')+C_2(\xi x_2;x_2'\xi')\,C_1(\xi';x_1')$$
$$-C_2(\xi x_2;x_1'\xi')\,C_1(\xi';x_2')+C_2(x_2\xi';x_2'\xi')\,C_1(\xi;x_1')$$
$$-C_2(x_2\xi;x_1'\xi')\,C_1(\xi;x_2')+C_1(\xi;x_2')\,C_1(x_2;\xi')\,C_1(\xi';x_1')$$
$$-C_1(\xi;x_1')\,C_1(x_2;\xi')\,C_1(\xi';x_2')]\,d\xi\,d\xi'. \quad (2\cdot19)$$

Representing graphically the four important terms in the right-hand side of Eq. (2·19), we get Fig. 1.

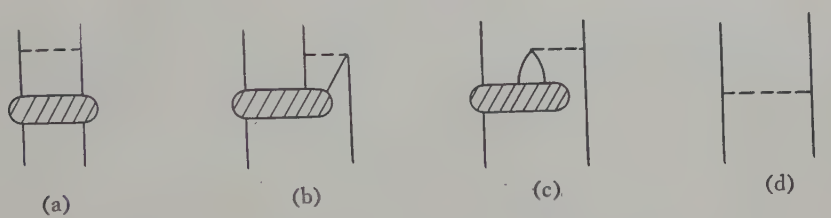


Fig. 1. Graphical representation contributing to C_2. ▨ denotes the correlation factor C_2 and ----represents interaction $\mathcal{I}$.

Other terms are the exchange terms, and so on. Diagrams are studied one by one in the following sections.

§ 3. Independent particle motion

As seen from Fig. 1 the graph which plays a dominant role is (a) when the interaction has a singular part, such as hard-core. In (a) almost the whole particles may participate even if the Pauli principle is taken into account since

* In this case $\mathcal{I}$ may not always depend only on two-particle coordinates but moreover on x_1. However, we here assume the interaction $\mathcal{I}$ to be local, that is, $\mathcal{I}(x,x')$.

the high momentum transfer is unrestricted. In the graphs (b) and (c) having the hole line,* however, the low momentum transfer may become important** because of the Pauli principle, and the particles being the neighbourhood of surface will mainly participate. Therefore, it may be a good approximation in order to obtain C_2 from Eq. (2·19), that Eq. (2·19) will be solved neglecting the contributions (b) and (c). As will be seen later, this may be the Brueckner term and the total energy be given.

Closing the integral equation for C_2 in the above approximation and writing an approximate solution of C_2 as $C_2^{(p)}$, then

$$C_2^{(p)}(x_1x_2;\, x_1'x_2') = i\int \mathcal{S}(\xi,\, \xi')\, C_1(x_1;\, \xi)\, C_1(x_2;\, \xi') \times [C_2^{(p)}(\xi\xi';\, x_1'x_2') + C_1(\xi;\, x_1')\, C_1(\xi';\, x_2') - C_1(\xi;\, x_2')\, C_1(\xi';\, x_1')]\, d\xi\, d\xi'. \tag{3·1}$$

Here we introduce K-matrix defined as

$$C_2^{(p)}(x_1x_2;\, x_1'x_2') \equiv i\int C_1(x_1;\, y)\, C_1(x_2;\, y')\, K(yy';\, zz') \times [C_1(z;\, x_1')\, C_1(z';\, x_2') - C_1(z;\, x_2')\, C_1(z';\, x_1')]\, dy\, dy'\, dz\, dz'. \tag{3·2}$$

Substituting the above in Eq. (3·1), we get the integral equation for K as follows,

$$K(x_1x_2;\, x_1'x_2') = \mathcal{S}(x_1,\, x_2)\,\delta(x_1-x_1')\,\delta(x_2-x_2') + \int d\xi\, d\xi' \cdot \mathcal{S}(x_1,\, x_2)\, iC_1(x_1;\, \xi)\, C_1(x_2;\, \xi')\, K(\xi\xi';\, x_1'x_2'). \tag{3·3}$$

This equation is identical with Eq. (12) of reference 6). We do not discuss the structure of this K in detail, but it may be easy to lead the equivalence to Brueckner's K-matrix in terms of Fourier transform of Eq. (3·3).[6]

Next, we will obtain the total energy and the single particle wave function using this $C_2^{(p)}$. From Eqs. (2·17) and (3·2), we get

$$G_2(x_1x_2;\, x_1'x_2') = C_1(x_1;\, x_1')\, C_1(x_2;\, x_2') - C_1(x_1;\, x_2')\, C_1(x_2;\, x_1') + i\int C_1(x_1;\, y)\, C_1(x_2;\, y')\, K(yy';\, zz') \times [C_1(z;\, x_1')\, C_1(z';\, x_2') - C_1(z;\, x_2')\, C_1(z';\, x_1')]\, dy\, dy'\, dz\, dz'.$$

Then,

$$\int d\boldsymbol{X}\, d\boldsymbol{X}' \cdot g v(|\boldsymbol{X}-\boldsymbol{X}'|)\, G_2(\boldsymbol{X}t,\, \boldsymbol{X}'t;\, \boldsymbol{X}t',\, \boldsymbol{X}'t') = \int d\boldsymbol{X}\, d\boldsymbol{X}' \int dz\, dz' \cdot K(\boldsymbol{X}t,\, \boldsymbol{X}'t;\, zz')$$

---

* In the case of low density, the contribution from the graph including any hole line may become higher order of density in comparison with the one including no hole line.

** Since the nuclear interaction is short range, there will not be such difficulty of divergence in low-momentum transfer as Coulomb interaction.

$$\times[C_1(z\,;\,\boldsymbol{X}t')C_1(z'\,;\,\boldsymbol{X}'t')-C_1(z\,;\,\boldsymbol{X}'t')C_1(z'\,;\,\boldsymbol{X}t')]. \tag{3·4}$$

Here, we will notice the following four situations:

1) As the interaction is time-local K-matrix has a factor $\delta(t_z-t_z')$.

$$K(\boldsymbol{X}t,\,\boldsymbol{X}'t\,;\,zz')=K(\boldsymbol{X}t,\,\boldsymbol{X}'t\,;\,\boldsymbol{Z}t_z,\,\boldsymbol{Z}'t_z')\delta(t_z-t_z'). \tag{3·5}$$

2) The integration on coupling constant g may be replaced by the limitation of integral region of t_z as $-\infty<t_z'<0$.[6]

$$\int_0^g dg'/g'\to\int_{-\infty}^0 dt_z'. \tag{3·6}$$

3) As $C_1(x\,;x')$ is one-particle Green function,

$$C_1(x\,;\,x')=-i\sum_k\varphi_k(\boldsymbol{X})\varphi_k^*(\boldsymbol{X}')\,e^{-iE_k(t-t')} \tag{3·7}$$

for the occupied state.

4) K-matrix can be Fourier-transformed as follows,

$$K(t-t')=\int K(E)\,e^{-iE(t-t')}\,dE/2\pi. \tag{3·8}$$

From the above arguments, the expression of energy can be given as follows,

$$E_A=E_A^{(0)}+\frac{1}{2}\int d\boldsymbol{X}\,d\boldsymbol{X}'\,d\boldsymbol{Z}\,d\boldsymbol{Z}'$$
$$\times\sum_{k,l}(\varphi_k^*(\boldsymbol{X})\varphi_l^*(\boldsymbol{X}')-\varphi_k^*(\boldsymbol{X}')\varphi_l^*(\boldsymbol{X}))K(E_k+E_l)\varphi_k(\boldsymbol{Z})\varphi_l(\boldsymbol{Z}'). \tag{3·9}$$

This is equivalent to Bethe's expression.

Next, will proceed to the expression of one-particle wave function. Returning to Eq. (3·4) and taking account of $t'\to-\infty$ and

$$G_2(\boldsymbol{X}''t,\,x\,;\,\boldsymbol{X}''t,\,x')=-G_2(\boldsymbol{X}''t,\,x\,;\,x',\,\boldsymbol{X}''t),$$

we get the wave equation of one particle by means of substituting Eq. (3·4) in Eq. (2·14),

$$i\frac{\partial}{\partial t}\varphi_\alpha(x)=\frac{p^2}{2m}\varphi_\alpha(x)-\lim_{t'\to-\infty}\int d\boldsymbol{X}'\,d\boldsymbol{X}''\,dz\,dz'$$
$$\times K(\boldsymbol{X}''t,\,x\,;\,zz')[C_1(z\,;\,x')C_1(z'\,;\,\boldsymbol{X}''t)-C_1(z\,;\,\boldsymbol{X}''t)C_1(z'\,;\,x')]\varphi_\alpha(\boldsymbol{X}').$$

Taking $\varphi_\alpha(x)=\varphi_\alpha(\boldsymbol{X})e^{-iE_\alpha t}$ into account and operating $\lim_{t'\to-\infty}(1/|t'|)\int_{2t'}^{t'}dt'$ in order to drop out the oscillating term, we obtain the wave equation as follows,

$$E_\alpha\varphi_\alpha(\boldsymbol{X})=\frac{p^2}{2m}\varphi_\alpha(\boldsymbol{X})-\sum_{k,l}\int\varphi_k^*(\boldsymbol{X})\varphi_l^*(\boldsymbol{X}')K_{E_\alpha+E_l}(\boldsymbol{X}'\boldsymbol{X}\,;\,\boldsymbol{Z}\boldsymbol{Z}')$$
$$\times\{\varphi_\alpha(\boldsymbol{Z})\varphi_l(\boldsymbol{Z}')-\varphi_\alpha(\boldsymbol{Z}')\varphi_l(\boldsymbol{Z})\}\,d\boldsymbol{X}'\,dz\,dz'\cdot\varphi_k(\boldsymbol{X}). \tag{3·10}$$

This may be considered as a more generalized Hartree-Fock equation.* And here, of course, there is not the difficulty of hard-core singularity.

§ 4. Exchange scattering—Contribution of diagram (b)

The lowest order of diagram (b) is exhibited in Fig. 2. Calculating C_2 or G_2 from the diagram (b) using the tensor force as interaction, and picking out the terms proportional to the spin σ_α of the particle in the state α, we have already shown that the above terms may give the order of magnitude of spin-orbit coupling in the shell-model. Detailed calculations may be referred to reference 9). Of course K gives also the terms proportional to $\sigma_\alpha \cdot \boldsymbol{l}_\alpha$, but in low energies this has a wrong sign.[9] Furthermore, if the interaction has $\boldsymbol{l}\cdot\boldsymbol{\sigma}$ part originally, we expect the larger contribution to the nuclear spin-orbit coupling. The study on this point is now proceeding. From diagram (b) there appears the central potential in addition to K, but this gives only the order of 10% to K[10].



Fig. 2. Lowest order of diagram (b).

§ 5. Effective interaction in reference to collective mode

In this section we consider the contribution of diagram (c). The lowest order of this diagram is shown in Fig. 2. In many-electron system it is known that the collection of this bubble type diagram gives the plasma mode.[3] As argued in the Introduction, several authors has considered this bubble type diagram in nuclear problem.[4] Here we will begin to discuss introducing the effective interaction in reference to collective mode.



Fig. 3. Lowest order of diagram (c).

5·1. *Introduction of effective interaction*

The basic equation may be led from Eq. (2·19) as follows,

$$C_2^{(c)}(x_1 x_2\,;\, x_1' x_2') = (-i)\int C_1^{(p)}(x_1\,;\,\xi)\,\mathcal{G}(\xi,\xi')$$
$$\times[C_2^{(c)}(x_2\xi'\,;\, x_2'\xi')C_1^{(p)}(\xi\,;\,x_1') - C_2^{(c)}(x_2\xi'\,;\, x_1'\xi')\;C_1^{(p)}(\xi\,;\,x_2')$$
$$+\{C_1^{(p)}(\xi\,;\,x_2')C_1^{(p)}(\xi'\,;\,x_1') - C_1^{(p)}(\xi\,;\,x_1')C_1^{(p)}(\xi'\,;\,x_2')\}C_1^{(p)}(x_2\,;\,\xi')]\,d\xi\,d\xi', \tag{5·1}$$

where $C_1^{(p)}$ represents the approximate one-particle propagator obtained by using K, and $C_2^{(c)}$ is the approximate solution of the above equation. Effective interaction $\mathcal{V}(x, x')$ may be introduced by the following definition.

---

* In the case of including hole-hole scattering, independent particle motion may no longer be guaranteed. This situation is now being studied.

$$C_2^{(c)}(x_1x_2;\,x_1'x_2')=i\int C_1^{(p)}(x_1;\,x_1'')C_1^{(p)}(x_2;\,x_2'')\,\mathcal{V}(x_1'',\,x_2'')$$
$$\times[C_1^{(p)}(x_1'';\,x_1')C_1^{(p)}(x_2'';\,x_2')-C_1^{(p)}(x_1'';\,x_2')C_1^{(p)}(x_2'';\,x_1')]\,dx_1''\,dx_2''. \tag{5·2}$$

Substituting Eq. (5·2) into Eq. (5·1) and arranging it, we may obtain the equation as,

$$L=R^{(i)}+R^{(ii)}, \tag{5·3}$$

where

$$L=i\int C_1^{(p)}(x_1;\,x_1'')C_1^{(p)}(x_2;\,x_2'')\,\mathcal{V}(x_1'',\,x_2'')$$
$$\times[C_1^{(p)}(x_1'';\,x_1')C_1^{(p)}(x_2'';\,x_2')-C_1^{(p)}(x_1'';\,x_2')C_1^{(p)}(x_2'';\,x_1')]\,dx_1''\,dx_2'', \tag{5·4}$$

$$R^{(i)}=\int C_1^{(p)}(x_1;\,\xi)C_1^{(p)}(x_2;\,\xi')\cdot i\mathcal{S}(\xi,\,\xi')$$
$$\times\{C_1^{(p)}(\xi;\,x_1')C_1^{(p)}(\xi';\,x_2')-C_1^{(p)}(\xi;\,x_2')C_1^{(p)}(\xi';\,x_1')\}\,d\xi\,d\xi'$$
$$-\int C_1^{(p)}(x_1;\,x_1'')C_1^{(p)}(x_2;\,x_2'')\cdot i\mathcal{S}(x_1'',\,\xi')\cdot iC_1^{(p)}(\xi;\,\xi')C_1^{(p)}(\xi';\,\xi)\,\mathcal{V}(\xi,\,x_2'')$$
$$\times\{C_1^{(p)}(x_1'';\,x_1')C_1^{(p)}(x_1'';\,x_2')-C_1^{(p)}(x_1'';\,x_2')C_1^{(p)}(x_2'';\,x_1')\}\,dx_1''\,dx_2''\,d\xi\,d\xi', \tag{5·5}$$

$$R^{(ii)}=\int C_1^{(p)}(x_1;\,x_1'')C_1^{(p)}(x_2;\,x_2'')\cdot i\mathcal{S}(x_1'',\,\xi')C_1^{(p)}(x_2'';\,\xi')C_1^{(p)}(\xi';\,\xi)$$
$$\times i\mathcal{V}(x_2'',\,\xi)\{C_1^{(p)}(x_1'';\,x_1')C_1^{(p)}(\xi;\,x_2')-C_1^{(p)}(x_1'';\,x_2')C_1^{(p)}(\xi;\,x_1')\}$$
$$\times dx_1''\,dx_2''\,d\xi\,d\xi'. \tag{5·6}$$

$R^{(ii)}$ is represented graphically as Fig. 4. If $R^{(ii)}$ term exists, the integral equation for $\mathcal{V}(x;\,x')$ is not closed, so we drop out $R^{(ii)}$ term. As seen in Fig. 4, $R^{(ii)}$ shows the vertex correction or exchange scattering and this is known as damping effect of plasma in many-electron gas.[11]



Fig. 4. Graphical representation of $R^{(ii)}$.

Thus we obtain the following integral equation of $\mathcal{V}$,

$$\mathcal{V}(x,\,x')=\mathcal{S}(x,\,x')-\int dx''\,dx'''\,\mathcal{S}(x,\,x'')[iC_1^{(p)}(x'';\,x''')C_1^{(p)}(x'''\,;\,x'')]$$
$$\times\mathcal{V}(x''',\,x'). \tag{5·7}$$

This effective interaction may be equivalent to the one introduced by J. Hubbard in many-electron case.[12]

5·2. *Expression of collective energy*

Denoting the collective energy or the contribution to energy of $C_2^{(c)}$ as $E_A^{(c)}$, we may write the basic equation for E_A as follows,

$$E_A^{(c)}=(-i)^2(1/2)\int_0^g dg'/g'\int dX\,dX'\,\mathcal{J}(|X-X'|)$$
$$\times\lim_{t'\to t_+}[C_1^{(p)}(Xt;Xt')C_1^{(p)}(X't;X't')-C_1^{(p)}(Xt;X't')C_1^{(p)}(X't;Xt')$$
$$+C_2^{(c)}(Xt,X't;Xt',X't')].$$

Substituting Eq. (5·2) in the above, after simple calculations we get

$$E_A^{(c)}=(1/2)\int_0^g dg'/g'\int dx\,dx'\,C_1^{(p)}(x;x')C_1^{(p)}(x';x)\,\mathcal{V}(x,x'). \tag{5·8}$$

In order to obtain the energy of phonon quanta we must derive the dispersion relation. Then we introduce the quantity $W(x,x')$ defined as

$$W(x,x')=-i\int\mathcal{J}(x,x'')C_1^{(p)}(x'';x')C_1^{(p)}(x';x'')dx''. \tag{5·9}$$

Then $\mathcal{V}(x,x')$ becomes

$$\mathcal{V}(x,x')=\mathcal{J}(x,x')+\int W(x,x'')\,\mathcal{V}(x'',x')dx''. \tag{5·10}$$

Assuming the translation invariant to take the Fourier transformation of Eq. (5·10), we get

$$\mathcal{V}(k,\omega)=\mathcal{J}(k)+W(k,\omega)\,\mathcal{V}(k,\omega).$$

Solving the above for $\mathcal{V}(k,\omega)$,

$$\mathcal{V}(k,\omega)=\mathcal{J}(k)/(1-W(k,\omega)). \tag{5·11}$$

Substituting this in Eq. (5·8), we may obtain

$$E_A^{(c)}=\frac{1}{2}\int_0^g\frac{dg'}{g'}\int\frac{dk}{(2\pi)^3}\frac{d\omega}{(2\pi)}\cdot\frac{iW(k,\omega)}{1-W(k,\omega)}. \tag{5·12}$$

Here, $W(k,\omega)$ may be separated in real and imaginary parts as

$$W(k,\omega)=D(k,\omega)+iS(k,\omega), \tag{5·13}$$

where

$$\text{real part of } W=D(k,\omega)$$

and

$$\text{imaginary part of } W=S(k,\omega). \tag{5·14}$$

Then Eq. (5·12) may be rewritten as follows,

$$E_A^{(c)}=\frac{1}{2}\int_0^g\frac{dg'}{g'}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\cdot\frac{-S}{(1-D)^2+S^2}. \tag{5·15}$$

As will be seen in later arguments, S may correspond to the scattering process, so it may be enough to take the limit as $-S\to 0$ for the estimation of the collective process. Then, taking account of

$$\lim_{-S\to 0}\frac{-S}{(1-D)^2+S^2}=\pi\delta(1-D), \tag{5·16}$$

Ep. (5·15) becomes

$$E_A^{(c)}=\frac{\pi}{2}\int_0^g\frac{dg'}{g'}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\cdot\delta(1-D). \tag{5·17}$$

$$1=D(k,\omega) \tag{5·18}$$

is the dispersion relation. Solving this, we get the energy of phonon quanta $\omega_c(k)$.* Then

$$1=D(k,\omega_c). \tag{5·19}$$

Expanding $D(k,\omega)$ about $\omega=\omega_c$,

$$D(k,\omega)=D(k,\omega_c)+\frac{\partial D}{\partial\omega}\Big|_{\omega=\omega_c}(\omega-\omega_c)+\cdots. \tag{5·20}$$

Then,

$$E_A^{(c)}=\frac{1}{2}\int_0^g\frac{dg'}{g'}\int\frac{d\boldsymbol{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\cdot\left[-\frac{\partial D}{\partial\omega}_{\omega=\omega_c}\right]^{-1}\delta(\omega-\omega_c). \tag{5·21}$$

And, as $D(k,\omega)$ depends linearly on g, as will be seen from the Appendix, differentiating the both sides of Eq. (5·19) on g, we get

$$\left[-\frac{\partial D}{\partial\omega}\right]^{-1}_{\omega=\omega_c}=g\frac{\partial\omega}{\partial g}\Big|_{\omega=\omega_c}. \tag{5·22}$$

Then the energy expression may be obtained in terms of phonon energy as follows,

$$E_A^{(c)}=\frac{1}{2}\int\frac{d\boldsymbol{k}}{(2\pi)^3}[\omega_c(g)-\omega_c(0)], \tag{5·23}$$

where, $\omega_c(g)$ and $\omega_c(0)$ represent the phonon energy in the existence of usual interaction and the limit of no-interaction respectively.

5·3. *The explicit form of effective interaction*

In order to study the role of particle-hole pair, we will consider the explicit form of effective interaction. But as effective interaction $\mathcal{V}(x,x')$ includes retardation, its form may be too complex to exhibit explicitly. So, here we will

* Hereafter, we write simply as ω_c instead of $\omega_c(k)$ as long as no confusion arises.

show, in rough estimation, the explicit form of effective interaction modified appropriately in configuration space.

Energy expression Eq. (5·8) would be rewritten as follows in terms of some functions defined by the Appendix.

$$E_A^{(c)} = \sum \int d\boldsymbol{X}\, d\boldsymbol{X}' \left[\frac{1}{2} \int \frac{dg'}{g'} \int \frac{d\omega}{2\pi} (A_{kl} + iB_{kl}) \cdot i\mathcal{V}(\boldsymbol{X}-\boldsymbol{X}', \omega) \right]. \tag{5·24}$$

Here we define the modified effective interaction $\overline{\mathcal{V}}_{kl}(|\boldsymbol{X}-\boldsymbol{X}'|)$ as follows,*

$$\overline{\mathcal{V}}_{kl}(|\boldsymbol{X}-\boldsymbol{X}'|) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{d\omega}{2\pi} \int \frac{1}{2} \frac{dg'}{g'} \cdot e^{i\boldsymbol{k}(\boldsymbol{X}-\boldsymbol{X}')}\, i\mathcal{V}(k, \omega)(A_{kl} + iB_{kl}). \tag{5·25}$$

$\mathcal{V}(k, \omega)$ is Fourier component of effective interaction, being given by Eq. (5·11). Taking the limit $(B_{kl} \to 0)$ similar to the case of energy expression and using the expansion of Eq. (5·20), we obtain

$$\begin{aligned} \overline{\mathcal{V}}_{kl}(|\boldsymbol{X}-\boldsymbol{X}'|) &= \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{d\omega}{2\pi} \int \frac{1}{2} \frac{dg'}{g'} \cdot e^{i\boldsymbol{k}(\boldsymbol{X}-\boldsymbol{X}')} \\ &\quad \times A_{kl} \cdot g' \frac{\partial \omega}{\partial g'}\bigg|_{\omega=\omega_c} \cdot \pi\delta(\omega-\omega_c)\, v(k),** \\ &= \int \frac{d\boldsymbol{k}}{(2\pi)^3} \int dg' \cdot e^{i\boldsymbol{k}(\boldsymbol{X}-\boldsymbol{X}')} \pi v(k) \left[A_{kl}|_{\omega=\omega_c} \cdot \frac{\partial \omega}{\partial g'}\bigg|_{\omega=\omega_c} \right]. \end{aligned} \tag{5·26}$$

In order to show the form of $\overline{\mathcal{V}}_{kl}$ for the lowest transition*** A_{kl} may be approximated as follows,

$$A_{kl}|_{\omega=\omega_c} = \frac{-2\Delta_{kl}}{\Delta_{kl}^2 - \omega_c^2} \cong \frac{-2\Delta}{\Delta^2 - \omega_c^2}, \tag{5·27}$$

where

$$\Delta = \min.|\Delta_{kl}|. \tag{5·28}$$

Then

$$\omega \cong \Delta\sqrt{1 + (2v/\Omega\Delta)}, \tag{5·29}$$

and the dispersion relation Eq. (A·7) may be rewritten as

$$\frac{1}{\Omega} \cdot \frac{-2\Delta v(k)}{\Delta^2 - \omega^2(k)} \cong 1. \tag{5·30}$$

---

* This means to define equivalent local interaction.

** Modified interaction $\mathcal{J}(|\boldsymbol{X}-\boldsymbol{X}'|)$ may be adopted here as the ordinary soft potential screening its hard-core. In this sense we write $v(q)$ instead of $\mathcal{J}(q)$.

*** Probably this may be considered to play the most important role and it is very complex to take other contributions into account. But the qualitative features may not be altered.

Now $\overline{v}_{kl}(|\boldsymbol{X}-\boldsymbol{X}'|)$ in this approximation denoted by $\overline{v}_0(|\boldsymbol{X}-\boldsymbol{X}'|)$ is obtained from Eq. (5·26) as follows,

$$\overline{v}_0(|\boldsymbol{X}-\boldsymbol{X}'|) \cong \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}(\boldsymbol{X}-\boldsymbol{X}')} v(k) F(k), \tag{5·31}$$

where

$$F(k)=\pi\left[1+\frac{2v}{\Omega\Delta}\right]^{-1/2}. \tag{5·32}$$

Here we used the dispersion relation Eq. (5·30) and treated $F(k)$ as the slow varying function on q. Adopting Fermi momentum k_F as unit, then Eq. (5·31) becomes

$$\overline{v}_0(\rho)=\frac{k_F}{2\pi^2}\cdot\frac{1}{\rho}\int dq\cdot\sin q\rho\cdot\overline{v}_0(q), \tag{5·33}$$

where

$$\overline{v}_0(q)=\frac{q\cdot v(q)}{\sqrt{1+(2/k_F{}^2\Omega\Delta)v(q)}}, \tag{5·34}$$

and

$$\rho=k_F|\boldsymbol{X}-\boldsymbol{X}'|,$$
$$q=k/k_F,$$
$$v(q)=k_F{}^2 v(k).$$

In order to estimate $\overline{v}_0(q)$, we assume the explicit form $v(q)$ as

$$v(q)=k_F{}^2\int V_0\frac{e^{-\kappa r}}{\kappa r}\cdot e^{ik_F qr}\,dr,$$

$$\therefore\quad v(q)=\frac{4\pi}{\kappa_F}\cdot\frac{V_0}{k_F}\cdot\frac{1}{\kappa_F{}^2+q^2}, \tag{5·35}$$

where $\kappa_F=\kappa/k_F$. The explicit form of $\overline{v}_0(q)$ is shown in Fig. 5.

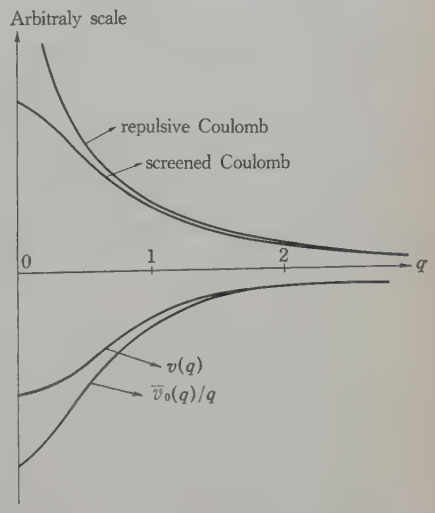


Fig. 5. Illustration of screening effect in cases of repulsive Coulomb and attractive nuclear short range interaction. All are normalized at $q=\infty$.

§ 6. Concluding remarks

Here we will mainly discuss the effective interaction v or $\overline{v}_0(q)$ obtained in § 5. First, we consider the difference of the mechanism in collective oscillation between electron plasma and nuclear collective vibration. (This problem was partly discussed in reference 13).) If we apply the method of electron plasma—such as the construction of normal mode of plasma by particle-hole pair—in infinite nuclear medium as it is, it is well known that the phonon energy may become

imaginary and the collective mode may be dissipative. As the above situation may be caused by the infinite property of nuclear medium and the attractive nuclear interaction, being different from repulsive Coulomb interaction in electron system, it is known that this difficulty may be avoided by taking account of finite size for nuclear medium,[4] that is, we must use the correlation factor $C_1^{(p)}$ instead of C_0. Then we see that the finite property of nucleus may play an essential role in nuclear collective motion. There exists another difference. That is screening effect about original interaction. As seen by Eq. (5·34), in the case of repulsive Coulomb interaction $v(q)=+\text{const.}/q^2$, then $q\cdot v(q)$ has a singularity $1/q$ at $q=0$. But the denominator, which represents the effect of particle-hole pair, has also a singularity $1/q$ at $q=0$. So, particle-hole pair screens the singularity of Fourier component of Coulomb interaction at $q=0$. Nuclear many-body system, however, is not the case, that is, $v(q)$ is attractive and unsingular at low-momentum side when the hard-core singularity has already been screened by Brueckner's method. Eq. (5·34) then shows us that the denominator, or the effect of the particle-hole pair increases the ratio of the low-momentum component of the interaction to the high-momentum component one in comparison with the original interaction.* Thus we may conclude that in nuclear many-body system, the effect of particle-hole pair produces the effective interaction in which the probability of low-momentum transfer is more favourable than that of the original one. In other words, the construction of particle-hole pair may transform the short-range force to the long-range one in contrast with the screening effect of Coulomb interaction in many-electron case.

Next we will comment on the quadrupole-quadrupole interaction introduced by J. P. Elliott[14] in order to explain the nuclear collective motion in particle picture. Explicit form of quadrupole-quadrupole force is given as

$$v(\boldsymbol{r}_1, \boldsymbol{r}_2)=G r_1^2 r_2^2 P_2(\cos\theta_{12}), \qquad (6\cdot1)$$

where P_2 represents Legendre polynomial.

The construction of particle-hole pair originating in terms of this force derives the nuclear collective mode.[4] Then, it may be remarked that the structure of radial part of this interaction is very essential in deriving the collective motion. As this is residual interaction, this may be weak. Weak as this is, however, perturbation theoretic calculation may not always be allowed. In electron plasma, Coulomb interaction is weak but because of divergence in 0-momentum transfer caused by long-range part, particle-hole pair, screening the above divergence, plays a dominant role in energy. On the other hand, in the case of short-range nuclear interaction, it seems that there is no positive reason to apply the particle-hole pair method. Rather, it suggests to apply the perturbation theory. We expect, however, that in terms of physical consideration of collective motion the

* The above situation is illustrated in Fig. 5.

residual interaction may be considered to be long range. So, some mechanism may be requested which makes the original short-range force become long range. One suggestion for the above may be particle-hole pair.

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Appendix

Explicit form of dispersion relation

From Eq. (5·14),

$$D(k, \omega) = \text{real part of } W(k, \omega). \tag{A·1}$$

$W(k, \omega)$ may be given from Eq. (5·9) as follows,

$$W(k, \omega) = -i\mathcal{S}(k) L(k, \omega),$$

where,

$$L(k, \omega) = \int d(\boldsymbol{X}-\boldsymbol{X}') d(t-t')\, e^{i\boldsymbol{k}(\boldsymbol{X}-\boldsymbol{X}')-i\omega(t-t')} C_1(x\,;\, x') C_1(x'\,;\, x), \tag{A·2}$$

because of the assumption of translation invariant. Since C_1 is one-particle propagator, we can express it in terms of one-particle wave function $\varphi_\alpha(x)$ defined by Eq. (3·10) as

$$C_1(x\,;\, x') = i[\theta(t-t') \sum_k^{\text{unocc.}} - \theta(t'-t) \sum_k^{\text{occ.}}] \varphi_k(x) \varphi_k^*(x'), \tag{A·3}$$

where the function θ is defined as follows,

$$\begin{aligned} \theta(x) &= 1 \quad \text{for} \quad x > 0, \\ &= 0 \quad \text{for} \quad x < 0. \end{aligned} \tag{A·4}$$

From Eq. (A·2) taking account of Eqs. (A·3) and (2·12), we obtain for $W(k, \omega)$ as follows,

$$W(k, \omega) = \mathcal{S}(k)[A(k, \omega) + iB(k, \omega)],$$

where A and B are the real functions respectively and expressed as

$$A(k, \omega) = \frac{1}{\Omega} \sum_{k,l} (1-n_k) n_l \cdot \frac{-2\Delta_{kl}}{\Delta_{kl}^2 - \omega^2(k)}, \tag{A·5}$$

$$B(k, \omega) = \frac{-\pi}{\Omega} \sum_{k,l} (1-n_k) n_l [\delta(\Delta_{kl}+\omega) + \delta(\Delta_{kl}-\omega)], \tag{A·6}$$

where,

$$\Delta_{kl} = E_k - E_l$$

and n_i represents the occupation number in state i. Here, for simplicity, we adopt the plane wave in a large box as $\varphi(x)$, that is,

$$\varphi_i(x)=\Omega^{-1/2}\exp[i\boldsymbol{k}_i\boldsymbol{X}-iE_it].$$

Thus we may express the dispersion relation explicitly as

$$\frac{1}{\Omega}\sum_{k,l}(1-n_k)n_l\cdot\frac{-2\mathfrak{V}(k)\Delta_{kl}}{\Delta_{kl}^2-\omega^2(k)}=1. \qquad (A\cdot 7)$$

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Pion Theory of Nuclear Forces and Low Energy *P*-Wave Phase Shifts

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It is shown that low energy behaviors of the triplet *P*-wave phase shifts in proton-proton scattering below 20 Mev, after being corrected for vacuum polarization effect and relativistic effects and for effect of the coupling with *F*-wave, confirm the existence of the repulsive central tail of the one-pion-exchange potential.

§ 1. Introduction

In the course to establish pion theory of nuclear forces according to a plan shown in the Taketani theory,[1] information afforded by behaviors of *P*-wave phase shifts at low energies may be said to secure the most important position. According to this plan, we should analyze phenomena of nuclear forces in the following way: (i) fully relying upon the one-pion-exchange potential (OPEP) that has the classical correspondence and dominates at distances $x>1.5$,* (ii) taking features of the two-pion-exchange potential (TPEP) into account only qualitatively in the region $1.5>x>0.7$, (iii) and treating interactions in the inner region completely phenomenologically. Then the importance of the low energy behaviors of the *P*-wave phase shifts comes from the facts that they are scarcely spoiled by nuclear interactions at small inter-nucleon distances, which are hard to be treated by the theory, and moreover, they are relatively insensitive to detailed properties of the two-pion-exchange potential.**

In 1954, two of the autors (S.O. and R.T.) showed that the behavior of the weighted average of three triplet *P*-wave phase shifts at low energies, which was observed to be negative and not to vary rapidly in the energy range $2\sim4$ Mev in p-p scattering, was a consequence of a hump of the pion-theoretical potential outside the pion range which is due to the one-pion-exchange process.[2] And this result together with determination of the pion-nucleon coupling constant from properties of deuteron[3] was regarded to bring the existence of the tail part of

* x is the inter-nucleon distance in unit of the pion Compton wave length.

** Nuclear interactions inside one-half the impact parameter scarcely affect the phase shifts. See ref. 14).

OPEP to light, and to confirm the validity of the Taketani theory. Later in 1958, Ohmura reported* that when new experimental data[4] were used and when the vacuum polarization effect as well as relativistic effects were taken into account for the p-p scattering analysis, the resultant P-wave phase shift became much smaller[5] and had not the magnitude negative enough to establish the hump of OPEP.

In the present paper, however, it is shown that P-wave phase shifts corrected for the vacuum polarization effect and the relativistic effects confirm the existence of the repulsive central part of OPEP in the energy region below 20 Mev.

§ 2. Weighted average of *P*-wave phase shift

The most important and interesting parameter in analyzing low energy p-p angular distribution is a weighted average of three triplet P-wave phase shifts, ${}^3\delta^1_{av}$.

Qualitatively speaking, the situation is as follows. The triplet P-wave nuclear scattering amplitude interferes with the Coulomb scattering amplitude mainly through

$$\sin {}^3\delta^1_{av} = (1/18)\,(\sin(2\,{}^3\delta_0{}^1) + 3\sin(2\,{}^3\delta_1{}^1) + 5\sin(2\,{}^3\delta_2{}^1)),^{**}$$

when each ${}^3\delta_J{}^1$ is small. Then p-p angular distribution data around the interference minimum angle determine ${}^3\delta^1_{av}$ with a very good accuracy. Uncertainty of ${}^3\delta^1_{av}$ arising from the interference term of singlet nuclear scattering amplitude with Coulomb scattering amplitude is rather small since the singlet nuclear scat-

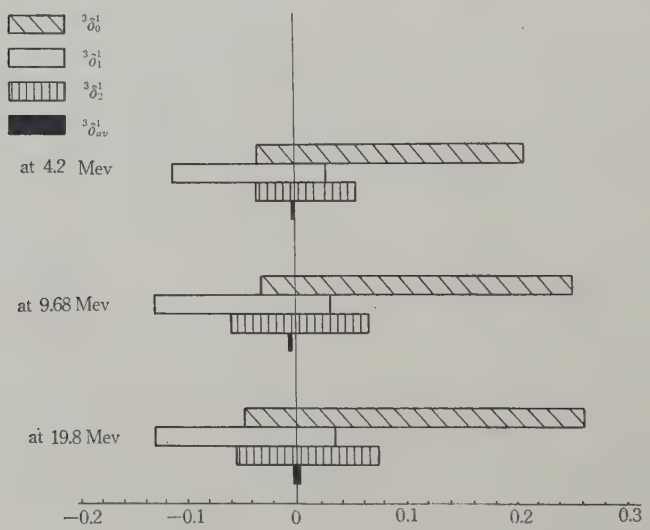


Fig. 1. Distribution of MacGregor's P-wave phase shifts and that of ${}^3\delta^1_{av}$.

* In a meeting at Institute for Nuclear Research, June, 1958.

** ${}^3\delta_J{}^L$ is the triplet nuclear bar phase shift.

tering amplitude, which is the main part of the nuclear scattering, is almost fixed by p-p angular distribution at large angles $60° < \theta \leq 90°$.[6]

This situation is well illustrated in Fig. 1, which shows that the width of distribution of MacGregor's[7] ${}^3\delta_0{}^1$ is larger than 0.3, that of ${}^3\delta_1{}^1$ about 0.2, that of ${}^3\delta_2{}^1$ is larger than 0.1, while that of ${}^3\delta_{av}^1$ is only about 0.005. Noting that Fig. 1 includes all phase shifts of MacGregor (from solution I to IV, and changing singlet S- and D-wave phase shifts ${}^1\delta_0$ and ${}^1\delta_2$*) we can observe that ${}^3\delta_{av}^1$ is a very good parameter for describing angular distribution of p-p scattering.[7']

Thus to reproduce the energy dependence of ${}^3\delta_{av}^1$ (see Fig. 3) is a necessary condition for any p-p potential. Once the energy-dependence of ${}^3\delta_{av}^1$ is reproduced, then there are a number of ways of combinations of ${}^3\delta_0{}^1$, ${}^3\delta_1{}^1$ and ${}^3\delta_2{}^1$ to reproduce the data.

In 1954, we attempted to interpret the negative ${}^3\delta_{av}^1$ below 4.2 Mev known at that time by the repulsive central part of OPEP. In the present paper, the following points are taken into account that were not taken into account before:

(i) Relativistic effects and vacuum polarization effect. They are considered in MacGregor's phase shifts,[5),7] from which ${}^3\delta_{av}^1$ in Fig. 1 is calculated.

(ii) The coupling of ${}^3P_2$-wave with the ${}^3F_2$-wave by the tensor force. In 1958, we noticed the importance of the coupling even at low energies.[8] This point is improved in the present paper, and we use the nuclear bar phase shift.**

(iii) Since few precise data were available above 5 Mev in 1954, it was not possible to get any information on energy-dependence of ${}^3\delta_{av}^1$.***

These faults are all improved in the present paper and the presence of the central part of OPEP is confirmed.

§ 3. Evidence for tail of the one-pion-exchange potential

We have calculated the proton-proton triplet P-wave phase shifts by the following potentials:

$$V(x) = (1/3)\,(g_e^2/4\pi)\,\mu\left[a\frac{e^{-x}}{x} + b\left(\frac{e^{-x}}{x}\right)^2 + S_{12}\left\{\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\frac{e^{-x}}{x} - 4\left(\frac{e^{-x}}{x}\right)^2\right\}\right] \quad \text{for} \quad x \geq 0.7, \qquad (1)$$

---

* ${}^1\delta_L$ is the singlet nuclear bar phase shift.

** Numerical integration of Schrödinger equations was made by IBM-704 computor of Japan Meteorological Agency. Numerical solutions of the equations with the potential (1) together with the Coulomb potential are continued smoothly to the Coulomb wave functions and then nuclear bar phase shifts are obtained.

*** ${}^3\delta^1{}_{av}$ at 18.2 Mev lies fairly apart from the smoothly energy-dependent curve in Fig. 3. Perhaps the data should be corrected by some systematic errors. Hara and Miyazawa, who applied a dispersion relation to p-p scattering found that the data are consistent with the dispersion relation only when $g^2{}_e/4\pi < 0.04$.[15] In 1956, using the data, we concluded the presence of an attractive central potential inside $x<1$ with depth 0~70 Mev,[3] but it is now clear that this conclusion should be replaced by a new one as stated in the present paper.

where parameters a and b are chosen as in Table I. The effective coupling constant $g_e^2/4\pi$ is varied from 0.06 to 0.08. Cases of $a=0$ correspond to absence of the repulsive central part of OPEP, while those of $b<0$ correspond to presence of an attractive central part of TPEP. These potentials are shown in Fig. 2. The tensor part of the potential (1), which consists of the OPEP-part and an attractive TPEP-part is fixed since its effects on ${}^3\delta^1_{av}$ are small as will be discussed later.

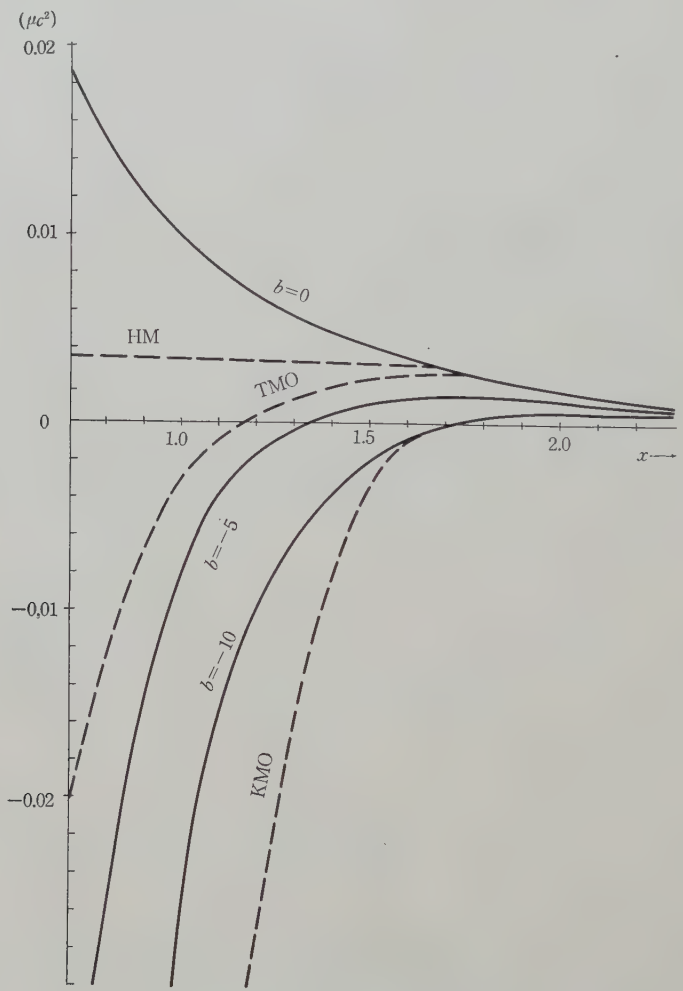


Fig. 2. Shapes of central potential of (1), which are compared with TMO,[9] KMO[10] and HM[11] potential.

Resultant phase shifts when we assume the zero cut-off for the potential inside $x=0.7$ and a hard core of the radius $x=0.3$ are shown in Figs. 3 and 4.*

* Copies of the numerical results may be obtained from one of the authors (W. W.).

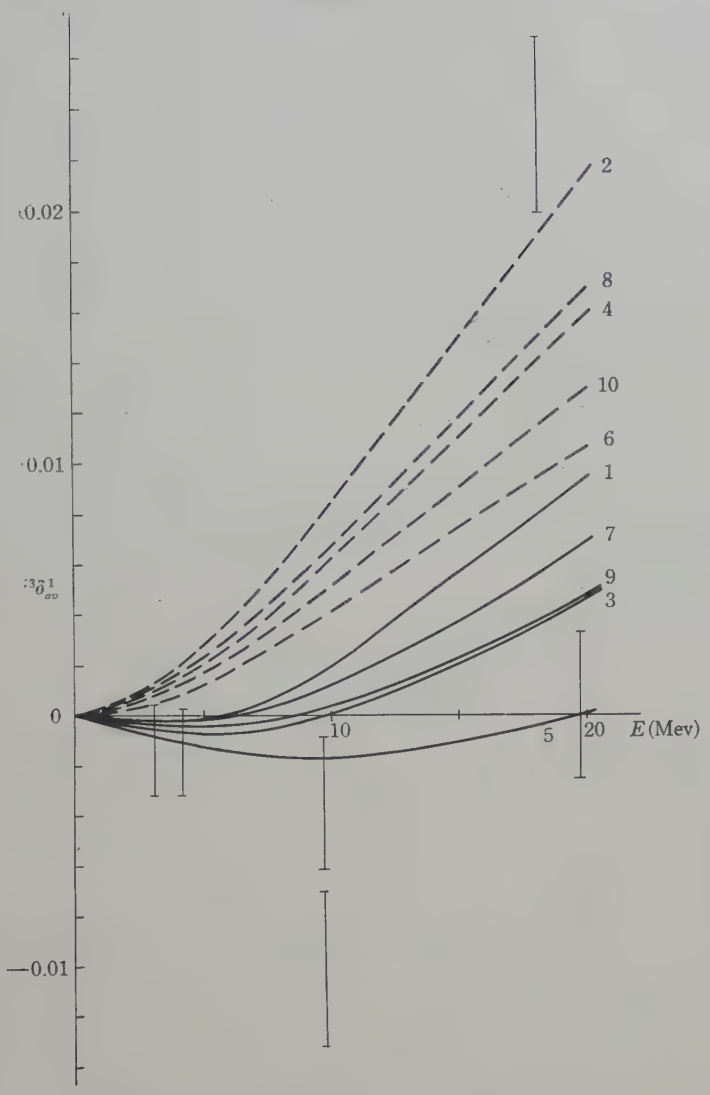


Fig. 3. Energy dependence of $^3\delta^1_{av}$. Figures on the curves specify No. of cases in Table I.
——— $a=1$, corresponding to the presence of OPEP-tail
------ $a=0$, corresponding to the absence of OPEP-tail

Table I. Parameters of potential (1)

Case	$g^2_e/4\pi$	a	b
1	0.08	1	−10
2	〃	0	〃
3	〃	1	− 5
4	〃	0	〃
5	〃	1	0
6	〃	0	〃
7	0.07	1	−10
8	〃	0	〃
9	0.06	1	〃
10	〃	0	〃

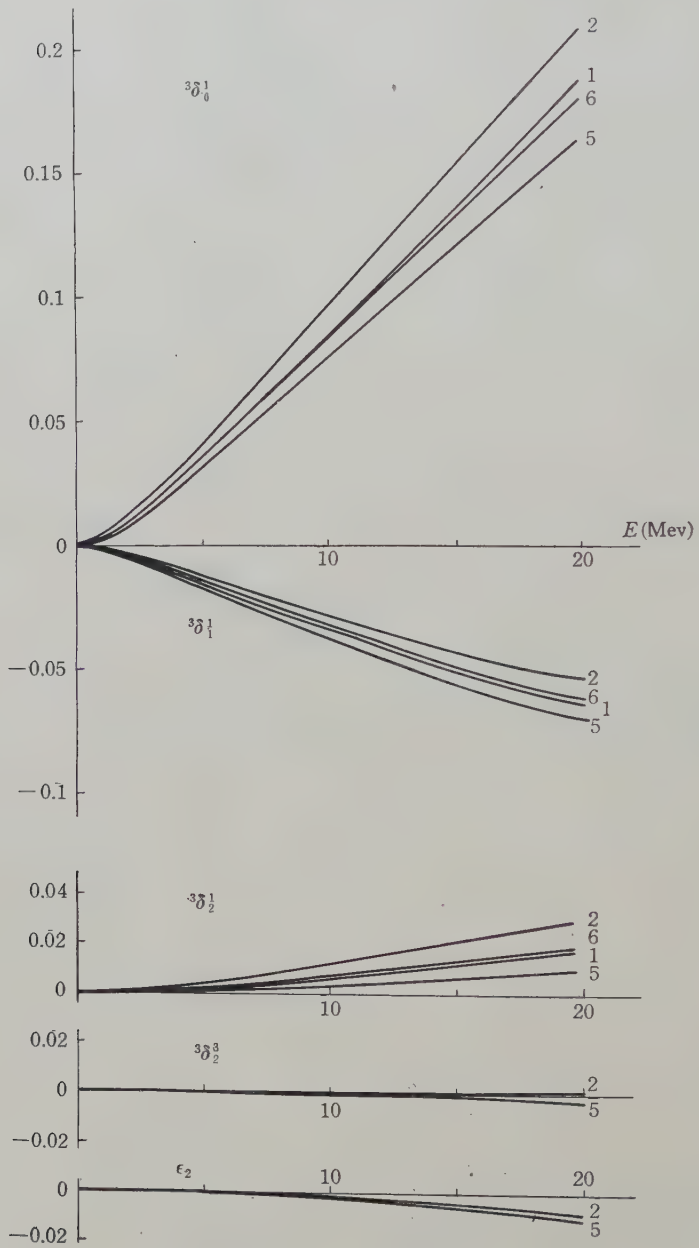


Fig. 4. Each nuclear bar phase shift are plotted for the cases 1, 2, 5 and 6.

Effects of uncertainty of the inner potential for $x<0.7$ are small compared with those due to OPEP-tail. For instance, when we assume an attractive central potential of depth -100 Mev in the region $0.7>x>0.3$, then all phase shifts increase and in consequence the weighted average ${}^3\delta_{av}^1$ increases at most as the

column (A) below, and when the radius of the hard core is changed by ± 0.1, ${}^3\delta^1_{av}$ changes as the column (B) below:

			(A)	(B)
at	1.855 Mev	by about	0.0002	∓ 0.00009,
at	9.68 Mev	by about	0.004	∓ 0.001,
at	19.8 Mev	by about	0.01	∓ 0.003.

From Fig. 3, we can conclude that:

(i) When $a=0$, ${}^3\delta^1_{av}>0$ for all energies, while when $a=1$, ${}^3\delta^1_{av}<0$ at least for $E<6$ Mev. Thus, in order to reproduce ${}^3\delta^1_{av}<0$ below 10 Mev, $a>0$ is required.

(ii) Thus, the presence of a repulsive central part outside the pion range has been confirmed.

§ 4. Some remarks

We give a few supplementary remarks:

(i) The assumption of the tensor part of Eq. (1) in the present analysis is justified in various ways:

(a) For ${}^3\delta^1_{av}$, some non-drastic changes of the shape of the tensor potential may be of minor importance, since effects of the tensor potential itself cancell one another in the first Born approximation.

(b) In studying properties of deuteron, the presence of the negative tensor OPEP was established in the triplet even state and its coupling constant was determined as $0.06<g_e^2/4\pi<0.09$, although nothing could be said definitely of the central potential.[3] Then once one admits the symmetrical theory, the positive tensor OPEP in the triplet odd state is naturally deduced.

(c) The positive sign of the tensor potential is also consistent with positive values of the polarization.[8]

(d) The tensor part of TPEP is relatively small in the region $x>0.7$ and is attractive.[1]

(ii) In Fig. 5, p-p angular distribution is calculated from some typical potentials (cases 2, 5 and 6) at 9.7 Mev as an example. The singlet D-wave phase shift is numerically calculated with OPEP tail, while singlet S-wave phase shift is chosen phenomenologically. Fit with the latest experimental data[12] for the case 5 is very satisfactory, which reflects the fact that ${}^3\delta^1_{av}$ is well reproduced in this case. Fit of the case 6 is not so good in the interference region $\theta<30°$, because ${}^3\delta^1_{av}$ lies outside the allowable region in Fig. 1. Thus we see that the pion-theoretical potential satisfies the necessary condition (the energy-dependence of ${}^3\delta^1_{av}$) as well as the sufficient condition to fit the data.

(iii) In the nuclear scattering region, the theoretical curve of the case 5 lies on the lower limit of the experimental data. To improve this fit (about 1% increase) without destroying the good fit in the interference region, we need slight

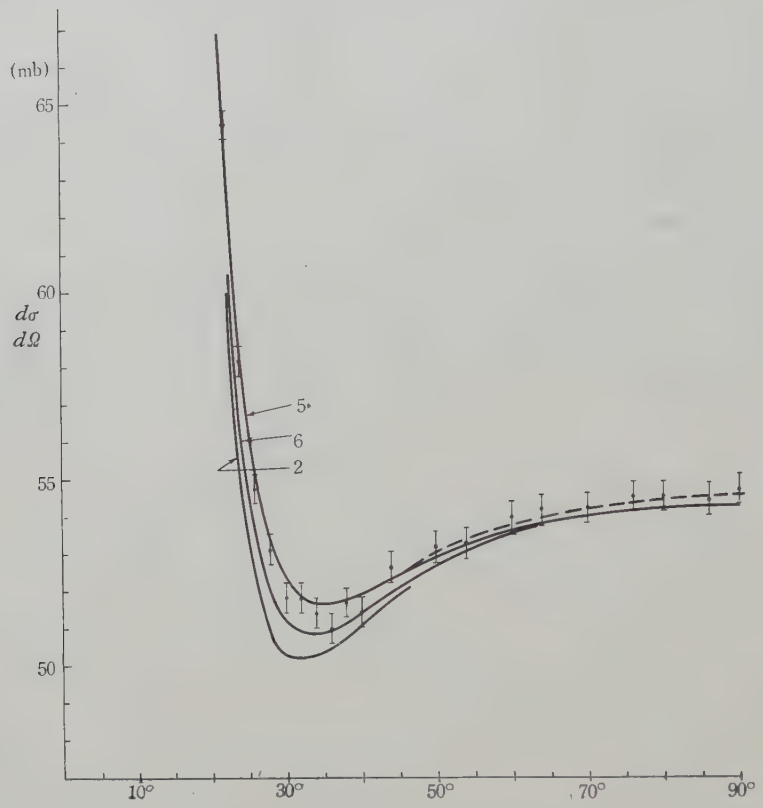


Fig. 5. Angular distribution at 9.7 Mev using the potentials for cases 2, 5 and 6. ${}^1\delta_2=0.00286$, which is obtained with the OPEP-tail, while ${}^1\delta_0$ is assumed as 0.973. The dashed curve is a modification of the case 5 adding a spin-orbit coupling potential (${}^3\delta^1{}_0=0.0326$, ${}^3\delta^1{}_1=-0.0544$, ${}^3\delta^1{}_2=0.0230$, ${}^1\delta_2=0.00286$, ${}^1\delta_0=0.969$).

modification of the P-wave phase shifts without changing ${}^3\delta^1_{av}$. This is possible by adding small tensor potential with the range one-half the pion Compton wave length (within the uncertainty of the two-pion-exchange potential) and/or a spin-orbit coupling potential.*

(iv) Fig. 3 might indicate that either the attractive central part of TPEP is weak ($b\sim 0$), or the coupling constant $g_e{}^2/4\pi$ is somewhat smaller than 0.08. (One would simply expect that when $g_e{}^2/4\pi$ increases, the repulsive central part of OPEP is strengthened and the resultant ${}^3\delta^1_{av}$ decreases. But this is not necessarily the case. —Compare cases 9, 7 and 1.— This is perhaps due to effects of the tensor potential, which vanish in the first Born approximation but not in the second one acting attractively.[13])

* An example is given by a dashed curve in Fig.5.

(v) It is to be noted that we need a long-range repulsive central potential to reproduce the energy-dependence of ${}^3\delta^1_{av}$, as can be seen from Fig. 3, when the tail of the tensor part of OPEP is present.

(vi) Thus the presence of the tail of the one-pion-exchange potential is confirmed in the low energy phenomena, namely, the tensor part is from properties of the deuteron and the central part from behaviors of ${}^3\delta^1_{av}$. We would like to require more accurate experimental data at such low energies and expect that they would enable us to make further refinement of the theory.

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The Theory of Superconductivity in the Tamm-Dancoff Approximation

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The Tamm-Dancoff approximation is applied to the interaction Hamiltonian used by Bogoliubov in his theory of superconductivity. It is shown that (i) the requirement that there is no virtual production of a single pair of electron and hole without phonons, or (ii) the electron-phonon coupling parameter tends to zero, leads to essentially the same compensating equation as obtained by Bogoliubov.

§ 1. Introduction

The theory of superconductivity has been developed by Bardeen-Cooper-Schrieffer[1] (BCS) and independently by Bogoliubov, Tolmachev and Shirkov.[2] The BCS theory is based on an interaction Hamiltonian between electrons which is attractive for electrons with opposite momenta and spins and close to the Fermi surface. The interaction Hamiltonian used by Bogoliubov et al.[2] explicitly makes use of the phonons as the agents for the interaction between electrons. The work of Bogoliubov et al.[2] besides justifying the BCS approach is also able to take into account the effect of coulomb interaction and predict the collective modes of excitations. Perturbation theory could not be applied directly to both the theories, since the energy gap between the normal state and the superconducting state is $\sim\exp(-1/g^2)$ where g is the coupling constant. The procedure of Bogoliubov[2] is to forbid the occurrence of the "dangerous" diagrams in which an electron-hole pair without phonons can be produced in the virtual state so that the matrix elements for such processes contain vanishing energy denominators. This leads to a compensating integral equation whose solutions predict the superconducting state besides the normal state.

In this note we have applied the Tamm-Dancoff[3] approximation to the interaction Hamiltonian used by Bogoliubov and investigated the conditions leading to the compensating equation.

§ 2. The electron-phonon interaction Hamiltonian in Bogoliubov's Theory of Superconductivity

The starting point of Bogoliubov's[2] theory of superconductivity is the Hamiltonian

$$H_F=\sum_{k,s} E(k)\, a_{ks}^{+} a_{ks}+\sum_{q} \omega(q)\, b_q^{+} b_q+H_{int},$$

$$H_{int}=g \sum_{\substack{k,k',s \\ q=k'-k}} \sqrt{\frac{\omega(q)}{2V}}\, a_{ks}^{+} a_{k's}\, b_q^{+}+\text{adj.}, \tag{1}$$

where $E(k)$, $\omega(q)$ are the electron and phonon energies, g is the coupling constant, V is the volume of the system, a_{ks}^{+} (a_{ks}) is the electron creation (annihilation) operator of momentum $\boldsymbol{k}$ and spin s ($s=+$ or $-$ corresponding to the values $s=+1/2$ or $-1/2$), b_q^{+} (b_q) is the phonon creation (annihilation) operator of momentum $\boldsymbol{q}$ and q_M is the cutoff momentum. The phenomenon of superconductivity is brought about by the superposition (mixing) of electron and hole states. This is accomplished by means of a canonical transformation and thereby introducing the new Fermi operators (the quasi-particle operators) α as follows

$$\begin{aligned} \alpha_{k0}&=u_k a_{k,+}-v_k a_{-k,-}^{+}, \\ \alpha_{k1}&=u_k a_{-k,-}+v_k a_{k,+}^{+}, \end{aligned} \tag{2a}$$

or

$$\begin{aligned} a_{k,+}&=u_k \alpha_{k0}+v_k \alpha_{k1}^{+}, \\ a_{-k,-}&=u_k \alpha_{k1}-v_k \alpha_{k0}^{+}, \end{aligned} \tag{2b}$$

where u_k and v_k are real and satisfy the relations

$$\begin{aligned} &u_k^2+v_k^2=1, \\ &u_{-k}=u_k, \quad v_{-k}=v_k, \end{aligned} \tag{3}$$

in order that the quasi-particle operators α conserve the commutation properties of the Fermi operators. It is to be noted from Eq. (2a) that the α's are operators of definite helicity. Further the Hamiltonian H_F is replaced by

$$H=H_F-\lambda N, \tag{4}$$

where $N=\sum_{k,s} a_{ks}^{+} a_{ks}$ and λ is a parameter determined by the condition that $\bar{N}=N_0$, N_0 being the total number of electrons. This avoids the introduction of the particle conserving condition. The transformed Hamiltonian is then given by

$$\begin{aligned} H&=U+H_0+H_{int}, \\ H_{int}&=H_1+H_2+H_3, \end{aligned} \tag{5}$$

where

$$U=\text{constant}=2\sum_{k} (E(k)-\lambda) v_k^2,$$

$$H_1 = \sum_{\substack{k,k' \\ q=k'-k}} g\sqrt{\frac{\omega(q)}{2V}} \left\{ u_k v_{k'} (\alpha_{k0}^{+} \alpha_{k'1}^{+} + \alpha_{-k'0}^{+} \alpha_{-k1}^{+}) \right.$$

$$\left. + u_{k'} v_k (\alpha_{k1} \alpha_{k'0} + \alpha_{-k'1} \alpha_{-k0}) \right\} b_q^{+} + adj.,$$

$$H_2 = \sum_{k,k'} g\sqrt{\frac{\omega(q)}{2V}} \left\{ u_k u_{k'} (\alpha_{k0}^{+} \alpha_{k'0} + \alpha_{-k1}^{+} \alpha_{-k'1}) \right.$$

$$\left. - v_k v_{k'} (\alpha_{k'1}^{+} \alpha_{k1} + \alpha_{-k'0}^{+} \alpha_{-k0}) \right\} b_q^{+} + adj.,$$

$$H_3 = 2 \sum_k (E(k) - \lambda) u_k v_k (\alpha_{k0}^{+} \alpha_{k1}^{+} + \alpha_{k1} \alpha_{k0}),$$

$$H_0 = \sum_k (E(k) - \lambda) (u_k^2 - v_k^2) (\alpha_{k0}^{+} \alpha_{k0} + \alpha_{k1}^{+} \alpha_{k1}) + \sum_q \omega(q) b_q^{+} b_q. \qquad (6)$$

The procedure of Bogoliubov et al.[2] consists in observing that the interaction H_3 can create in the virtual state two quasi-particles (without phonons) of equal and opposite momentum from vacuum (all states up to the Fermi surface filled up). Similarly, H_1 and H_2 together can produce two quasi-particles. Such processes (diagrams) are called dangerous, since in the perturbation theory their matrix elements will contain the energy denominator $2\epsilon(k) = 2|E(k) - E_F|$, E_F being the energy of the electron at the Fermi surface, which diverges on integration as $k \to k_F$. The technique is then to equate the sum of the contributions of these diagrams to zero and obtain the compensating equation which is effectively an integral equation in $u_k v_k$ (obtained only to g^2 approximation). The trivial solution of this integral equation corresponds to the normal state, and the non-trivial solution which has an energy lower than the energy of the normal state is identified as the superconducting state; the energy difference (gap) being $\sim \exp(-2/\rho)$ where $\rho \sim g^2$.

§ 3. Tamm-Dancoff approximation

In the Tamm-Dancoff[3] method the wave function $\Psi(t)$ for the actual system under consideration, with the actual energy W, is expanded in occupation number space i.e. $\psi^{(e,\nu)}$-the eigenstate of H_0 corresponding to e electrons and holes and ν phonons. $\Psi(t)$ satisfies the Schrödinger equation

$$-i\frac{\delta \Psi(t)}{\delta t} = H\Psi(t) = W\Psi(t) \qquad (7)$$

where

$$H = H_0 + H_{int},$$

$$H_0 \psi_\lambda^{(e,\nu)} = E_\lambda^{(e,\nu)} \psi_\lambda^{(e,\nu)}, \qquad (8)$$

where the index λ denotes momentum, spin etc. Also

$$\Psi(t) = \sum_{\lambda, e, \nu} a_\lambda^{(e,\nu)} \psi_\lambda^{(e,\nu)} e^{iWt}. \qquad (9)$$

Substituting Eqs. (9) and (8) in Eq. (7), we obtain

$$(W-E_\lambda^{(e;\nu)})\,a_\lambda^{(e;\nu)}=\sum_{\lambda,e',\nu'}\langle e;\nu,\lambda|H_{int}|e';\nu',\lambda\rangle\,a_\lambda^{(e';\nu')}. \tag{10}$$

The amplitudes on the R.H.S. of Eq. (10) can be eliminated by writing Eq. (10) for these amplitudes. We now consider the vacuum state (all states up to $k=k_F$ filled up) with amplitude $a^{(0;0)}$ and bare particle energy $E^{(0;0)}$. We can choose $E^{(0;0)}=0$ so that the bare energy of an electron or hole of momentum $\boldsymbol{k}$ is $\epsilon(k)=|E(k)-E_F|$. Working with the transformed Hamiltonian, Eqs. (5) and (6), we observe that the only non zero contributions to the sum $\sum_{\lambda,e,\nu}\langle 0|H_{int}|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}$ are

$$\begin{aligned}
&\sum_{\lambda,e,\nu}\langle 0|H_3|e;\nu,\lambda\rangle\,a_\lambda^{(e,\nu)}=2\sum_k(E(k)-\lambda)\,u_k v_k\,a^{(k0,k1;0)},\\
&\sum\langle 0|H_1|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}=\sum_{\substack{k,k'\\ q=k'-k}}g\sqrt{\frac{\omega(q)}{2V}}\,(u_k v_{k'}+u_{k'}v_k)\,a^{(k0,k'1;q)}.
\end{aligned}\tag{11}$$

Making the approximation that $e\leq 2$ and $\nu\leq 1$, we write below the other relevant non zero matrix elements:

$$\begin{aligned}
&\sum\langle 0|\alpha_{k1}\alpha_{k0}H_3|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}=2(E(k)-\lambda)\,u_k v_k\,a^{(0;0)},\\
&\sum\langle 0|\alpha_{k1}\alpha_{k0}H_2|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}\\
&\quad=\sum_{k',q=k'-k}g\sqrt{\frac{\omega(q)}{2V}}\,(u_{k'}u_k-v_k v_{k'})\,[a^{(k0,k'1;q)}+a^{(k'0,k1;-q)}],\\
&\sum\langle 0|\alpha_{k'1}\alpha_{k0}b_q H_1|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}=g\sqrt{\frac{\omega(q)}{2V}}\,(u_k v_{k'}+u_{k'}v_k)\,a^{(0;0)},\\
&\sum\langle 0|\alpha_{k'1}\alpha_{k0}b_q H_2|e;\nu,\lambda\rangle\,a_\lambda^{(e;\nu)}\\
&\quad=g\sqrt{\frac{\omega(q)}{2V}}\,(u_k u_{k'}-v_k v_{k'})\,[a^{(k0,k1;0)}+a^{(k'0,k'1;0)}].
\end{aligned}\tag{12}$$

Using Eqs. (11) and (12), we obtain from Eq. (10),

$$\begin{aligned}
w(0;0)\,a^{(0;0)}&=2\sum_k(E(k)-\lambda)\,u_k v_k\,a^{(k0,k1;0)}\\
&\quad+\sum_{k,k'}g\sqrt{\frac{\omega(q)}{2V}}\,(u_k v_{k'}+u_{k'}v_k)\,a^{(k0,k'1;q)},
\end{aligned}\tag{13}$$

$$\begin{aligned}
w(k,k';0)\,a^{(k0,k1;0)}&\simeq 2\sum_k(E(k)-\lambda)\,u_k v_k\,a^{(0;0)}\\
&\quad+\sum_{k'}g\sqrt{\frac{\omega(q)}{2V}}\,(u_{k'}u_k-v_k v_{k'})\,[a^{(k0,k'1;q)}+a^{(k'0,k1;-q)}],
\end{aligned}\tag{14}$$

$$\begin{aligned}
w(k,k';q)\,a^{(k0,k'1;q)}&\simeq g\sqrt{\frac{\omega(q)}{2V}}\Big\{(u_k v_{k'}+u_{k'}v_k)\,a^{(0;0)}\\
&\quad+(u_k u_{k'}-v_k v_{k'})\,[a^{(k0,k1;0)}+a^{(k'0,k'1;0)}]\Big\},
\end{aligned}\tag{15}$$

where

$$w(k_1, k_2, \cdots, k_i; q_1, q_2, \cdots q_j) = W - \sum_i \epsilon(k_i) - \sum_j \omega(q_j),$$

$$\epsilon(k_i) = |E(k) - E_F|. \tag{16}$$

From Eqs. (13), (15) and (14), (15) we obtain

$$w(0;0)\, a^{(0;0)} = 2\sum_k f(k)\, a^{(k0,k1;0)} + 2\left(\frac{g^2}{2V}\right)\sum_{k,k'} \frac{\omega(q)}{w(k, k'; q)} (u_k v_{k'} + u_{k'} v_k)^2 a^{(0;0)}, \tag{17}$$

$$w(k, k; 0)\, a^{(k0,k1;0)} = 2f(k)\, a^{(0;0)} + g^2 \sum_{k'} f_1(k, k')[a^{(k0,k1;0)} + a^{(k'0,k'1;0)}], \tag{18}$$

where

$$f(k) = (\widetilde{E}(k) - \lambda) u_k v_k + (u_k^2 - v_k^2)\left(\frac{g^2}{2V}\right)\sum_{k'} \frac{\omega(q)}{w(k, k'; q)} u_{k'} v_{k'}, \tag{19}$$

$$\widetilde{E}(k) = E(k) + \left(\frac{g^2}{2V}\right)\sum_{k'} \frac{\omega(q)}{w(k, k'; q)} (u_{k'}^2 - v_{k'}^2),$$

$$f_1(k, k') = \frac{1}{2V} \frac{2\omega(k'-k)}{w(k, k'; q)} (u_k u_{k'} - v_k v_{k'})^2.$$

We can now use Eq. (18) to eliminate the amplitude $a^{(k0,k1;0)}$ from Eq. (17). In the g^2 approximation for the equation for $a^{(0,0)}$, strictly speaking, only the first term on the right of Eq. (18) should be retained. However since the R.H.S. of Eq. (17) involves a summation over k, there is a value of $k = k_0$, such that

$$w(k_0, k_0; 0) = W - 2\epsilon(k_0) = 0, \quad \text{for } W \geq 0, \tag{20}$$

and substitution of $a^{(k0,k1;0)}$ in Eq. (17) gives an infinite contribution unless $f(k) = 0$ in Eq. (18). One may argue that around $k \sim k_0$, one should put the L.H.S. of Eq. (18) to zero and then $a^{(k0,k1;0)}$ is given by

$$[\sum_{k'} f_1(k, k')]\, a^{(k0,k1;0)} = -\frac{2}{g^2} f(k)\, a^{(0;0)} - \sum_{k'} f_1(k, k')\, a^{(k'0,k'1;0)}. \tag{21}$$

However, from Eq. (21) it is apparent that if $g^2 \to 0$, there is a class of solutions for which $f(k) \to 0$ as $g^2 \to 0$. Alternatively by following Bogoliubov and others[2] if we right from the start impose the condition that the coefficient of the amplitude $a^{(k0,k1;0)}$ in the equation for $a^{(0;0)}$, Eq. (17), vanish, we again obtain $f(k) = 0$, which is essentially the compensating equation obtained by Bogoliubov[2]; the trivial solution $(u_k, v_k) = (0, 1)$ corresponds to the normal state while the non-trivial solution gives rise to the superconducting state provided $W \geq 0$.

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Pion-Pion Interaction and Nuclear Forces

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Nuclear potentials arising from the pion-pion resonance in the state $I=J=1$ are calculated. Only the part proportional to $\tau\tau$ is considered. The resonant state is described both in the ρ-meson formalism and in the chain approximation method. The potentials are calculated explicitly in configuration space, at first in the ρ-meson formalism with its mass 600 Mev and the strength of coupling which corresponds to the width $\sim$60 Mev. The potentials are found to be of the nature of the two-pion exchange potentials. In particular there appear a strong attractive L-S potential and a repulsive central potential in the ^3O-state, a strong repulsive tensor potential in the ^3E-state, and a repulsive L-dependent potential in the ^1E-state. Among these the tensor potential in the ^3E-state and the central potential in the ^3O-state can be considered to be too strong and violate some of the facts which have already been established. Therefore a careful examination of the results will offer a test for or against the assumption of the pion-pion resonance. The chain approximation method gives similar results. The electromagnetic form factors of the nucleon are reexamined and a remark is made concerning future investigation.

§ 1. Introduction

Since the work by Frazer and Fulco[1] who calculated the electromagnetic form factors of the nucleon, the resonant state of pion-pion scattering in the state $I=J=1$ has been recognized to be important. G. Takeda and his collaborators proposed an approximate method in which the resonant state is replaced by an actual particle with $I=J=1$, which they called a ρ-meson.[2] This replacement was achieved by assuming the width Γ of the resonance to be infinitesimal, or more precisely, retaining only the first order term (the narrow width approximation). They applied this method to pion-nucleon scattering, and found that the second and the third resonances observed around 700 Mev[3] can be reproduced as a result of the pion-pion resonance. The mass of ρ-meson was set equal to the resonance energy, which they assumed to be 600 Mev according to Frazer and Fulco:

$$m=600 \text{ Mev}=4.29.*$$

The width of the resonance is, on the other hand, related to the strength of the coupling between ρ-meson and pions. For later reference we shall quote their result on the interaction Hamiltonian with somewhat different notation:

* In the following we shall use units in which the pion mass is unity.

$$H=h_r\phi_i T^k_{ij}\partial_\mu\phi_j\varphi_\mu{}^k,\quad T^k_{ij}=\varepsilon_{ijk},$$
$$h^2=h_r{}^2/4\pi\approx 5. \qquad (1\cdot1)^*$$

While Takeda et al. started from the assumption of the resonance in pion-pion scattering, Y. Miyamoto made an attempt to reproduce that resonance from more fundamental interaction between pions.[4] He examined the calculation of pion-pion scattering by Chew and Mandelstam[5] and found that their formalism is equivalent to the so-called "chain approximation" method in perturbation theory, as far as the contribution from the "right-hand cut" is concerned. He introduced the interaction Hamiltonian between pions given by

$$H=-4\pi a(\phi_i T^k_{ij}\partial_\mu\phi_j)^2. \qquad (1\cdot2)$$

This Hamiltonian is responsible for pion-pion scattering for p-waves as well as s-waves. The coupling constant a can be identified as the p-wave scattering length. The interaction Hamiltonian (1·2) need not be considered as the truely fundamental interaction. Rather it will be more reasonable to consider it as an effective interaction resulting from, for example, the virtual formation and reannihilation of a nucleon pair. Therefore we should reasonably cut off the integration of the virtual momentum by an appropriate value, which we shall assume to be the order of a nucleon mass. He found that the resonance determined by Frazer and Fulco, and used by Takeda et al., can be reproduced with

$$a\simeq 0.2, \text{ for the cutoff } 2M.^{**}$$

The above authors showed that the pion-pion resonance in the state $I=J=1$, if it exists, has an appreciable effect on the electromagnetic structure of the nucleon and pion-nucleon scattering. If this is true, we should expect that the resonance will affect also nucleon-nucleon scattering. In particular this is the case with the nuclear forces in the region II ($0.7<r<1.5$). This is because the two-pion part (the pion-current part), which is considered to dominate the isovector part of the electromagnetic form factors of the nucleon, has a close correspondence to the two-pion exchange potential (TPEP) in the nuclear force. We can hardly expect that the resonance affects the electromagnetic structure appreciably while affecting the nuclear potential in the region II little.

Furthermore the two-pion exchange potentials calculated thus far do not give satisfactory agreement with observation. For example, these potentials give too weak an L-S force to account for the experiment on the depolarization in the proton-proton scattering at the energy around 150 Mev.[6]

Miyamoto made a tentative estimation and found that the resonant state gives a very strong L-S force in the desired direction.[7] But some care will be needed in drawing conclusions, since the resonance is expected to affect the other types

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* ϕ_i and $\varphi_\mu{}^k$ describe a pion of isospin index i and a ρ-meson of isospin index k, respectively.
** M is the nucleon mass.

of potentials, e. g. central, tensor, and so on, which also should be compared with experiment. From the analysis which will be presented in the following, we find that the above expectation is true and some of potentials may be affected too much and violate results which have already been established. In this situation a careful examination of the results will offer a test for or against the assumption of a resonance in pion-pion scattering.

There are, of course, some uncertainties in the theoretical treatment of the resonance, which will be considered later. The results we have obtained, however, show some qualitative features which are characteristic of the assumption of the pion-pion resonance. Therefore, our results are expected to offer a basis for a further investigation of the problems both of the pion-pion interaction and of nuclear forces.

In this paper we shall confine ourselves to the calculation of that part of the potential, which is proportional to $\tau\tau$, leaving the consideration of the part, which is proportional to 1 in isospace, to a forthcoming paper. Also we shall not make a detailed comparison with the experiments, leaving that to another article.

In § 2, the outline of the calculation is described, in § 3 the explicit form of the potentials are given, in § 4 the results are discussed briefly. In § 5 the electromagnetic form factors are reexamined by the same method applied to the calculation of the potentials. In § 6 the results are summarized and a remark is made concerning future investigation.

§ 2. Outline of the calculation

The kinematics of the non-static potential in momentum representation were worked out by J. Goto and S. Machida,[8] which we reproduce here for the sake of completeness. The calculation is made in the center of mass system of the two nucleons, and the notation is indicated in Fig. 1. The matrix element is decomposed as follows :

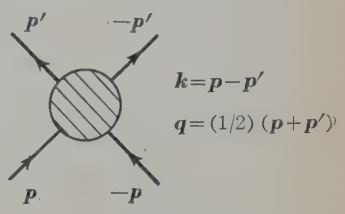
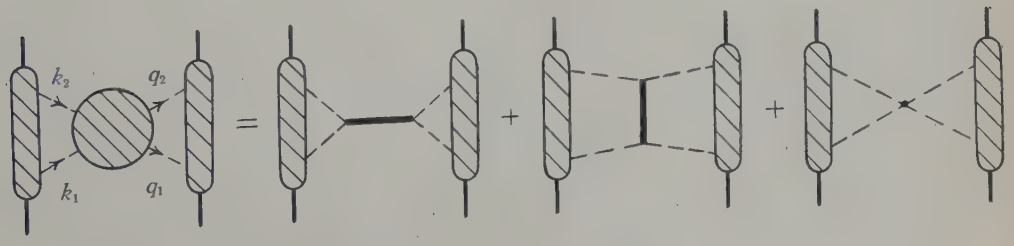


Fig. 1.



(A) (B) (C)

Fig. 2.

$$V(\boldsymbol{k}^2, \boldsymbol{q}^2, \boldsymbol{kq}) = V^1 + \boldsymbol{\tau}\boldsymbol{\tau} V^\tau,$$
$$V^\lambda = V_0^\lambda + i(\boldsymbol{S}\cdot\boldsymbol{k}\times\boldsymbol{q}) V_1^\lambda + (\boldsymbol{k}\boldsymbol{\sigma})(\boldsymbol{k}\boldsymbol{\sigma}) V_2^\lambda + (\boldsymbol{q}\boldsymbol{\sigma})(\boldsymbol{q}\boldsymbol{\sigma}) V_3^\lambda$$
$$+ (\boldsymbol{k}\times\boldsymbol{q}\cdot\boldsymbol{\sigma})(\boldsymbol{k}\times\boldsymbol{q}\cdot\boldsymbol{\sigma}) V_4^\lambda + (\boldsymbol{\sigma}\boldsymbol{\sigma})[V_5^\lambda + (\boldsymbol{k}\times\boldsymbol{q})^2 V_6^\lambda],$$
$$(\lambda = 1, \tau). \tag{2·1}$$

We shall consider the graphs in Fig. 2. Here the graphs (A) and (B) represent the contributions arising from the resonance in the state $I=J=1$, while the graph (C) comes from an interaction of the form $\lambda_\tau \phi^4$.

We shall assign the factor

$$T_{\mu\nu}(k^2) = \delta_{\mu\nu} T(k^2),$$
$$T(k^2) = \begin{cases} \dfrac{12\pi a}{D(k^2)}, & \text{in the chain approximation method,} \\ \dfrac{h_\tau^2}{k^2+m^2}, & \text{in the } \rho\text{-meson formalism,} \end{cases} \tag{2·2}$$

to a heavy line in Fig. 2, which stands for the resonant state. Here $D(k^2)$ is given by[4]

$$D(k^2) = 1 + \frac{2a}{\pi}\left(1+\frac{k^2}{4}\right)\left[\log 2\Lambda - \sqrt{1+\frac{4}{k^2}}\,\log\left(\sqrt{1+\frac{k^2}{4}}+\sqrt{\frac{k^2}{4}}\right)\right], \tag{2·3}$$

where Λ is the cutoff momentum.

It should be remarked that the method, here designated as the ρ-meson formalism, is not the one based on the vector meson theory in the rigorous meaning of a field theory, because the propagator of a vector meson is, strictly speaking, of the form

$$\frac{\delta_{\mu\nu} + k_\mu k_\nu / m^2}{k^2+m^2}. \tag{2·4}$$

We are considering the ρ-meson as a phenomenological substitute rather than as a true elementary particle. It should be remembered that we are able to determine only the first term ($\delta_{\mu\nu}$) of the numerator in (2·4), leaving the second term undetermined, as long as we introduce the ρ-meson as the pole term in the scattering amplitude of the real pions. This is analogous to the situation that we encounter in the introduction of the isobaric particle for the 3-3 resonance.[9] We might expect that the second term in the numerator of (2·4) will be cancelled by some other terms, at least asymptotically, if the resonance can be considered to result from interactions of the first kind, or renormalizable. From these considerations we have chosen, for convenience, the form in (2·2).

We can now construct the effective Hamiltonian for pion-pion scattering. Corresponding to the pion-pion scattering part of the graphs (A), (B), and (C) in Fig. 2, we have

$$\langle q_1 q_2|H_A|k_1 k_2\rangle = -2P_1(k_1-k_2)\cdot(q_1-q_2)T(k^2),$$

$$\langle q_1 q_2|H_B|k_1 k_2\rangle = P_1\{(k_1+q_1)\cdot(k_2+q_2)T[(k_1-q_1)^2]-(k_1+q_2)\cdot(k_2+q_1)T[(k_1-q_2)^2]\}$$
$$+(2P_0-P_2)\{(k_1+q_1)\cdot(k_2+q_2)T[(k_1-q_1)^2]+(k_1+q_2)\cdot(k_2+q_1)T[(k_1-q_2)^2]\},$$

$$\langle q_1 q_2|H_C|k_1 k_2\rangle = 20(2P_0-P_2)\lambda_r, \qquad (2\cdot5)$$

where P_I is the projection operator for the state of the two pions with isospin I.

The potential function is given by the two-nucleon expectation value of these effective Hamiltonians:

$$V = V^A + V^B + V^C,$$
$$V^A = \langle \boldsymbol{p}', -\boldsymbol{p}'|H_A|\boldsymbol{p}, -\boldsymbol{p}\rangle,$$
$$V^B = \langle \boldsymbol{p}', -\boldsymbol{p}'|H_B|\boldsymbol{p}, -\boldsymbol{p}\rangle,$$
$$V^C = \langle \boldsymbol{p}', -\boldsymbol{p}'|H_C|\boldsymbol{p}, -\boldsymbol{p}\rangle. \qquad (2\cdot6)$$

V^A, V^B and V^C correspond to the graphs (A), (B) and (C) in Fig. 2.

It is easily seen that V^A can be written as

$$V^A = -\langle \boldsymbol{p}'|\phi_i T^k_{ij}\partial_\mu\phi_j|\boldsymbol{p}\rangle T(k^2)\langle -\boldsymbol{p}'|\phi_m T^k_{mn}\partial_\mu\phi_n|-\boldsymbol{p}\rangle, \qquad (2\cdot7)$$

where the pion fields are evaluated at the origin of the coordinate system. Now we observe that the matrix elements $\langle \boldsymbol{p}'|\phi_i T^k_{ij}\partial_\mu\phi_j|\boldsymbol{p}\rangle$ and $\langle -\boldsymbol{p}'|\phi_m T^k_{mn}\partial_\mu\phi_n|-\boldsymbol{p}\rangle$ in (2·7) are of a form similar to the nucleon expectation value of the pion electric current. Indeed the electric current of a pion is given by

$$j_\mu = -e\phi_i T^3_{ij}\partial_\mu\phi_j, \qquad (2\cdot8)$$

the nucleon expectation value of which is given by

$$\langle \boldsymbol{p}'|j_\mu|\boldsymbol{p}\rangle = \bar{u}_{p'}\tau_3[i\gamma_\mu G_1(k^2)+\sigma_{\mu\nu}k_\nu G_2(k^2)]u_p, \qquad (2\cdot9)$$

where $G_1(k^2)$ and $G_2(k^2)$ are the two-pion parts (the pion-current parts) of the nucleon electromagnetic form factor. These G's, of course, contribute to the observed *isovector* part of the nucleon electromagnetic form factors. By comparing (2·7), (2·8) and (2·9), we have obviously

$$\langle \boldsymbol{p}'|\phi_i T^k_{ij}\partial_\mu\phi_j|\boldsymbol{p}\rangle = -(1/e)\bar{u}_{p'}\tau_k[i\gamma_\mu G_1+\sigma_{\mu\nu}k_\nu G_2]u_p, \qquad (2\cdot10)$$

from charge independence considerations. Thus, we have the simple expression for V^A:

$$V^{1A} = 0,$$
$$V^{\tau A} = -(1/e^2)\bar{u}_{p'}(i\gamma_\mu G_1+\sigma_{\mu\nu}k_\nu G_2)u_p T(k^2)\bar{u}_{-p'}(i\gamma_\mu G_1-\sigma_{\mu\lambda}k_\lambda G_2)u_{-p}. \qquad (2\cdot11)$$

Here it should be noted that $G_1(k^2)$ and $G_2(k^2)$ are the form factors with the pion-pion resonance *neglected*; otherwise the resonant state would be taken into account twice.

We cannot apply a similar method to V^B, because the momentum appearing

in the denominator is not a constant (k^2) but must be integrated over. A rigorous calculation of the graph (B) in Fig. 2 is somewhat cumbersome due to the so-called overlapping integral. In view of the approximate character of the whole problem we shall apply a simplifying method and quickly consider the qualitative or semi-quantitative features of the results. We thus make the point approximation in which the heavy line in Fig. 2 (B) is shrunk into a point, or $T(k^2)$ is replaced by $T(0)$. Then, we have

$$\langle q_1 q_2|H_B|k_1 k_2\rangle = -P_1(k_1-k_2)\cdot(q_1-q_2)T(0) + (2P_0-P_2)(3k^2-k_1^2-k_2^2-q_1^2-q_2^2)T(0), \tag{2·12}$$

and

$$V^{1B}=(3/8)T(0)\bar{u}_{p'}u_p\cdot\bar{u}_{-p'}u_{-p}[(3k^2+4)G_3(k^2)-4C]G_3(k^2),$$
$$V^{\tau B}=-(1/2)(1/e^2)\bar{u}_{p'}(i\gamma_\mu G_1+\sigma_{\mu\nu}k_\nu G_2)u_p T(0)\bar{u}_{-p'}(i\gamma_\mu G_1-\sigma_{\mu\lambda}k_\lambda G_2)u_{-p}, \tag{2·13}$$

where $G_3(k^2)$ is a form factor for the scalar interaction analogous to the vector and tensor form factors $G_1(k^2)$ and $G_2(k^2)$ and permits a spectral representation of the form (2·16), while C is a constant independent of k^2. We shall not discuss V^{1B} in this paper. We write down, however, the formula for V^C for the sake of completeness:

$$V^{1C}=(15/2)\lambda_\tau\bar{u}_{p'}u_p\cdot\bar{u}_{-p'}u_{-p}[G_3(k^2)]^2,$$
$$V^{\tau C}=0. \tag{2·14}$$

We see that $V^{\tau B}$ is very similar to $V^{\tau A}$. Indeed we would have

$$V^{\tau A}=2V^{\tau B}, \tag{2·15}$$

if we applied the point approximation also to $V^{\tau A}$.

We have treated V^A and V^B in different ways. There is, however, some justification for this method. Consider the matrix elements corresponding to the graphs (A) and (B) in Fig. 2, as the functions of the square of the momentum transfer k^2. We shall carry on the discussion, for the moment, in the ρ-meson formalism. V^A has a pole at $k^2=-m^2$ in addition to a cut along $k^2\leqq -4$. This means that $V^A(r)$, the fourier transform of $V^A(k^2)$, has a term like e^{-mr}/r, in addition to the contribution from the cut. As we shall see later, the cut contribution can be approximated roughly e^{-br}/r with $b\sim 4$. If we choose $m\sim 4$, we can expect the pole contribution to make potential appreciably wider.

V^B has, on the contrary, no such pole; the contribution from the heavy line lies along the cut $k^2\leqq -4$. Therefore the deviation from the above point approximation can be considered to appear only in the relatively inner region of the

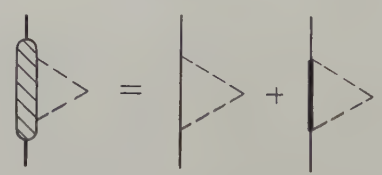


Fig. 3.

configuration space.

Now we shall calculate $G_1(k^2)$ and $G_2(k^2)$ from the graphs in Fig. 3, where the heavy line represents the isobaric particle with $I=J=3/2$. These G's were partially calculated by Chew et al.[10] and by Federbush et al.[11] We can show that their calculation in which the rescattering correction comes only from the 3-3 resonance with the narrow width approximation is equivalent to the perturbation calculation of the graphs in Fig. 3, in which the 3-3 isobar is replaced by a Rarita-Schwinger particle with the appropriate interaction.[9] We assume the spectral representations for the G's :

$$G_\alpha(k^2)=\frac{1}{\pi}\int_4^{4M^2}\frac{g_\alpha(\kappa^2)}{k^2+\kappa^2}d\kappa^2, \quad (\alpha=1, 2). \tag{2·16}$$

In accordance with the discussion of Federbush et al. we have cut off the integration with respect to the square of the energy of the two-pion state by $(2M)^2$.

We have assumed a subtraction neither for G_1 nor for G_2, though Chew et al. and Federbush et al. assumed a G_1 with one subtraction. The reason is that we are calculating the contribution only from the graphs in Fig. 3 ; the subtraction terms correspond to the other kinds of graphs which represent the wave function renormalization of the nucleon, the nucleon current contribution, and so on. In the case of the electromagnetic form factor it is necessary to include all of these contributions for gauge invariance, while this is not the case in the present calculation. The introduction of a cutoff in (2·16) was also necessary to make G_1 finite.

The calculated net values and the mean square radius are shown in Table I.

Table I. Form factors calculated from the graphs in Fig. 3.

	$G_1(0)/e$	$2MG_2(0)/e$	r_1^2	r_2^2
Pion-current contribution				
Born approximation	0.66	0.87	0.19	0.22
3-3 isobar included	0.26	1.63	0.12	0.17
Observed isovector part	(0.50)	1.85	0.33	0.33

§ 3. Potentials

Substituting the normalized Dirac spinors for $\bar{u}$ and u in (2·11) and (2·13), and comparing with (2·1), we obtain the results :

$$\begin{aligned} V_i^{\tau A}(k^2)&=T(k^2)\,\mathcal{V}_i(k^2), \\ V_i^{\tau B}(k^2)&=(1/2)T(0)\,\mathcal{V}_i(k^2), \quad (i=0, \cdots, 6) \end{aligned} \tag{3·1}$$

where

$$e^2\,\mathcal{V}_0(k^2)=G_1^2(k^2),$$

$$
\begin{aligned}
e^2\,{}^t\mathcal{V}_1(k^2) &= -\frac{2}{M}G_1\left(\frac{3}{4}\frac{1}{M}G_1+2G_2\right),\\
e^2\,{}^t\mathcal{V}_2(k^2) &= \frac{1}{4M^2}G_1^2+\frac{1}{M}G_1G_2+G_2^2,\\
e^2\,{}^t\mathcal{V}_3(k^2) &= \frac{1}{2M^2}e^2\,{}^t\mathcal{V}_5(k^2),\\
e^2\,{}^t\mathcal{V}_4(k^2) &= -\frac{1}{M^2}\left(\frac{3}{16}\frac{1}{M^2}G_1^2+\frac{3}{2}\frac{1}{M}G_1G_2+\frac{5}{2}G_2^2\right),\\
e^2\,{}^t\mathcal{V}_5(k^2) &= -k^2e^2\,{}^t\mathcal{V}_2(k^2),\\
e^2\,{}^t\mathcal{V}_6(k^2) &= \frac{1}{M^2}\left(\frac{1}{8}\frac{1}{M^2}G_1^2+\frac{1}{M}G_1G_2+\frac{3}{2}G_2^2\right).
\end{aligned}
\tag{3·2}
$$

In each of the ${}^t\mathcal{V}_i$ we have neglected higher order terms with respect to k^2/M^2 and q^2/M^2.

The ${}^t\mathcal{V}_i(k^2)$ in (3·2) turn out to be well approximated by the appropriate Yukawa potentials in configuration space. For example, $1/{}^t\mathcal{V}_4(k^2)$ is plotted as a function of k^2 in Fig. 4. From the gradient of the approximated straight line we can determine the intrinsic range of the equivalent Yukawa potential. Thus the $V_i{}^{rB}(k^2)$ can be easily transformed into potentials in configuration space. In order to transform $V_i{}^{rA}(k^2)$ in the similar manner, we shall do the calculation at first in the ρ-meson formalism. In this case we can use the formula

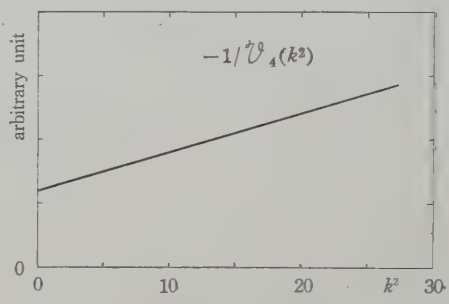


Fig. 4. $1/{}^t\mathcal{V}_4(k^2)$ is nearly linear with respect to k^2. Other ${}^t\mathcal{V}_i(k^2)$ are also of similar forms.

$$
\begin{aligned}
&(2\pi)^{-3}\int d\boldsymbol{k}\, e^{-i\boldsymbol{k}\boldsymbol{r}}\frac{1}{\boldsymbol{k}^2+m^2}\frac{1}{\boldsymbol{k}^2+b^2}\\
&\quad=\begin{cases}\dfrac{1}{b^2-m^2}\left(\dfrac{e^{-mr}}{4\pi r}-\dfrac{e^{-br}}{4\pi r}\right), & \text{for } b\neq m,\\[2ex] \dfrac{1}{2m}\dfrac{e^{-mr}}{4\pi}, & \text{for } b=m.\end{cases}
\end{aligned}
$$

We shall consider the case of the chain approximation in the later sections.

Defining the potentials in configuration space by

$$
\begin{aligned}
V^\lambda=&V_0^\lambda(r)+(\boldsymbol{\sigma}\boldsymbol{\sigma})V_S^\lambda(r)+(\boldsymbol{LS})V_{LS}^\lambda(r)+S_{12}V_T^\lambda(r)\\
&+(\boldsymbol{q}\boldsymbol{\sigma})(\boldsymbol{q}\boldsymbol{\sigma})V_q^\lambda(r)+(\boldsymbol{L}\boldsymbol{\sigma})(\boldsymbol{L}\boldsymbol{\sigma})V_{L\sigma}^\lambda(r)+(\boldsymbol{\sigma}\boldsymbol{\sigma})\boldsymbol{L}^2V_{LL}^\lambda(r),
\end{aligned}
\tag{3·3}
$$

we have the $V^r(r)$ as follows:

$$V_0^{TA}(r)=1.091\frac{1}{r}(e^{-z}-e^{-y}),$$

$$V_0^{TB}(r)=0.1318\frac{1}{r}e^{-z},\quad y=4.286\,r,\quad z=3.733\,r,$$

$$V_S^{TA}(r)=2.611\left(1-\frac{2}{y}\right)e^{-y},$$

$$V_S^{TB}(r)=2.611\ \frac{1}{y}e^{-y},$$

$$V_{LS}^{TA}(r)=-1.629\frac{1}{r^3}[(z+1)e^{-z}-(y+1)e^{-y}],$$

$$V_{LS}^{TB}(r)=-0.0945\frac{1}{r^3}(z+1)e^{-z},\quad z=4.030\,r,$$

$$V_T^{TA}(r)=-1.306\left(1+\frac{1}{y}\right)e^{-y},$$

$$V_T^{TB}(r)=-1.306\left(1+\frac{3}{y}+\frac{3}{y^2}\right)\frac{e^{-y}}{y},$$

$$V_{L\sigma}^{TA}(r)=0.1057\frac{1}{r^2}\left[78.73\left(1+\frac{3}{y}+\frac{3}{y^2}\right)\frac{e^{-y}}{y}\right.$$
$$\left.-84.58\left(1+\frac{3}{z}+\frac{3}{z^2}\right)\frac{e^{-z}}{z}\right],$$

$$V_{L\sigma}^{TB}(r)=0.04985\frac{1}{r^3}\left(1+\frac{3}{z}+\frac{3}{z^2}\right)e^{-z},$$
$$z=4.390\,r,$$

$$V_{LL}^{TA}(r)=-0.06852\frac{1}{r^2}\left[78.73\left(1+\frac{3}{y}+\frac{3}{y^2}\right)\frac{e^{-y}}{y}\right.$$
$$\left.-84.23\left(1+\frac{3}{z}+\frac{3}{z^2}\right)\frac{e^{-z}}{z}\right],$$

$$V_{LL}^{TB}(r)=-0.03030\frac{1}{r^3}\left(1+\frac{3}{z}+\frac{3}{z^2}\right)e^{-z},$$
$$z=4.384\,r. \qquad (3\cdot4)$$

It should be noted that these functions are valid only for $r\gtrsim 0.2$, since we have justified the "straight line approximation" for $1/\mathcal{V}_i(k^2)$ only up to $k\lesssim 5$. It is meaningless to compare the singularities with respect to r in the potentials in (3·4) with those appearing in the static potentials, for example, the TMO potentials.[12]

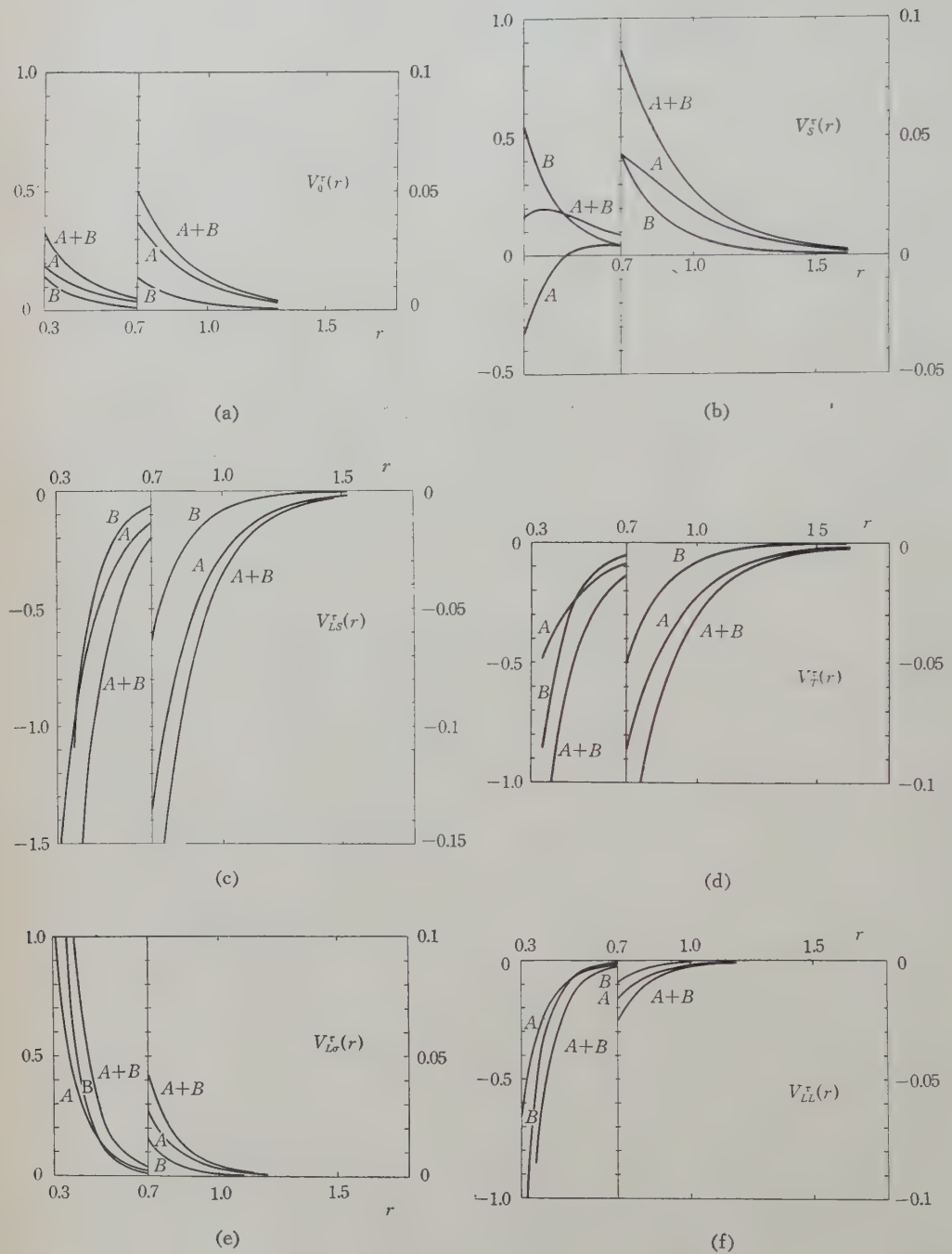


Fig. 5. Potentials calculated from (3·4). Both r and $V(r)$ are measured in units in which the pion mass is unity. Curves denoted by A, B and $A+B$ represent $V^{\tau A}(r)$, $V^{\tau B}(r)$ and $V^{\tau}(r) = V^{\tau A}(r) + V^{\tau B}(r)$, respectively.

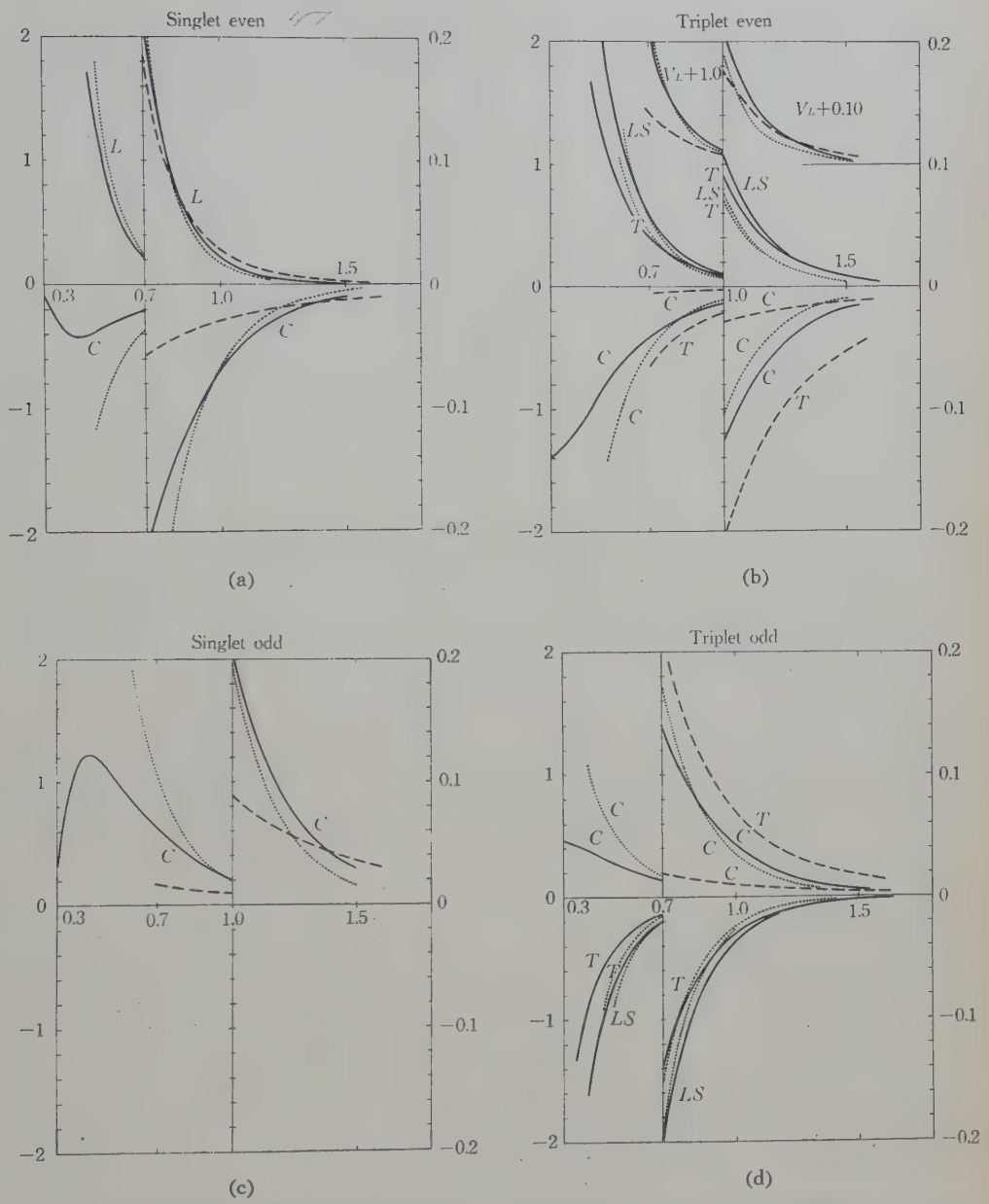


Fig. 6. Potentials are plotted for each state of orbital and spin angular momentum. Units are the same with those of Fig. 5. Solid curves: Calculated from (3·4). Dotted curves: $m\to\infty$ is assumed with $T(0)/4\pi$ remaining unchanged (0.272). Broken curves: The one-pion exchange potentials.

$V_i^{\tau A}(r)$, $V_i^{\tau B}(r)$ and $V_i^{\tau}(r)=V_i^{\tau A}(r)+V_i^{\tau B}(r)$ are plotted in Fig. 5. By comparing these plots with the TMO,[12] BW[13] or FST potentials,[14] we can con-

firm that the potentials here considered are of the nature of two-pion exchange potentials. Furthermore we can easily see the difference in the qualitative or semi-quantitative features between $V^{\tau A}$ and $V^{\tau B}$. The former is less singular due to the contribution from the pion-pion resonance.

In Fig. 6, various potentials are plotted for each state of orbital and spin angular momentum. The one-pion exchange potential (OPEP) is also plotted for the sake of comparison. We have used the notation :

$$V_C^{\tau}(r) = V_0^{\tau}(r) + \langle \boldsymbol{\sigma}\boldsymbol{\sigma} \rangle V_S^{\tau}(r),$$

$$V_L^{\tau}(r) = \langle (\boldsymbol{L}\boldsymbol{\sigma})(\boldsymbol{L}\boldsymbol{\sigma}) \rangle V_{L\sigma}^{\tau}(r) + \langle \boldsymbol{\sigma}\boldsymbol{\sigma} \rangle \boldsymbol{L}^2 V_{LL}^{\tau}(r),$$

where $\langle\ \rangle$ denotes the expectation value of the operators involved. $V_L(r)$ in the ${}^1E$-state is specifically plotted for the ${}^1D_2$-state, and that in the ${}^3E$-state for the ${}^3D_2$-state. The $V_L(r)$ denoted by OPEP are the ones calculated by N. Hoshizaki and S. Machida, for the case of pv-coupling.[15]

§ 4. Discussion of the obtained potentials

In this section we shall discuss in a rather crude manner the features of the potentials obtained. A detailed analysis will be made in a separate paper. Also we should keep in mind that we have calculated only the V^{τ}-part, leaving the V^1-part uncalculated.

(i) *${}^1E$-state* :

We have a strong attractive central potential. It may be noted that we have a repulsive V_L. At $r=1$, this is nearly equal to V_L from the OPEP calculated by Hoshizaki and Machida in the case of pv-coupling, though our V_L is more singular than HM.

(ii) *${}^3E$-state* :

We have a strong repulsive tensor potential. This reduces the attractive OPEP tensor potential to about half at $r=1$. This may destroy the success of the OPEP in explaining the quadrupole moment of the deuteron. This tensor potential will affect also the binding of the deuteron, though we must, in this case, take into account the strong attractive central potential obtained.

(iii) *${}^3O$-state* :

We have a strong repulsive central potential, which may be objectionable from the viewpoint that the OPEP together with the weak attractive TPEP gives a good fit to the observed phase shift at low energy (~ 4 Mev). We have a strong attractive L-S potential. This amounts to about a half of that proposed by Gammel and Thaler,[16] and to about twice that of HM.[17] We have an attractive tensor potential, which is about three times deeper than TMO.

Our potentials should be added to the TPEP without regard to the pion-pion interaction. Though there are, so far, no complete results which fully take into account the effect of recoil and the 3-3 resonance at the same time, we

can consider that the static TPEP or the non-static TPEP calculated by Hoshizaki and Machida[17] give some measure of the complete TPEP without regard to the pion-pion resonance. Combining these potentials with our results we may conclude that the pion-pion resonance here considered affects the nuclear potentials in the region II too strongly and violates some of the facts already established.[18] In particular, V_T in the ^3E-state, and V_C in the ^3O-state will serve as a test for or against the assumption of the pion-pion resonance, as it stands here.

On the other hand there are some uncertainties in the treatment of $G_\alpha(k^2)$ and the pion-pion resonance, which will be discussed in the next section. Therefore it seems worthwhile to see what modifications in $G_\alpha(k^2)$ and the pion-pion resonance will be required in order to make the potentials reasonably weaker.

In Fig. 6, we have also plotted the results when m is assumed infinitely heavy with $h_r^2/m^2=T(0)$ remaining unchanged. In this case Eq. (2·15) is applied. We find that the potentials with $m\to\infty$ are very close to those with $m=600$ Mev at $r\approx 1$, while the values at $r\approx 1.5$ are substantially reduced (to about a half in many cases). This shows that the values of the potentials at $r\approx 1$ are governed for the most by $T(0)=h_r^2/m^2$ alone. In other words the nature of the spatial extension of the potentials is governed mainly by the $G_\alpha(k^2)$, and we cannot make the potentials sufficiently narrow by assuming a higher resonance energy.

As will be seen later, $T(k^2)$ in the chain approximation is of a form somewhat more singular than that in the ρ-meson formalism when they are transformed into configuration space (see Fig. 12). Therefore we can consider that the potentials in the chain approximation lie between the curves of $m=600$ Mev and $m\to\infty$ in the ρ-meson formalism, if the same $T(0)$ is assumed.

The depths of the potentials are governed by the $\mathcal{V}_t(0)$, which are the products of two $G_\alpha(0)$ and $T(0)$. Here we only show in Table II what changes the $\mathcal{V}_t(0)$ undergo when the $G_\alpha(0)$ are changed. We find that $\mathcal{V}_0(0)$ and $\mathcal{V}_1(0)$ change appreciably with a change in $G_1(0)$, while $\mathcal{V}_2(0)$, $\mathcal{V}_4(0)$ and $\mathcal{V}_6(0)$ change little. $\mathcal{V}_2(0)$, $\mathcal{V}_4(0)$ and $\mathcal{V}_6(0)$ are affected greatly by a change in $G_2(0)$.

Table II. Changes of the $\mathcal{V}_t(0)$ are represented as the ratio $\mathcal{V}_t(0)\to\eta_t\mathcal{V}_t(0)$.

	η_0	η_1	η_2	η_4	η_6
$G_1(0)\to(1/2)G_1(0)$, $G_2(0)\to G_2(0)$	0.25	0.47	0.87	0.91	0.91
$G_1(0)\to 2G_1(0)$, $G_2(0)\to G_2(0)$	4.00	2.22	1.30	1.18	1.20
$G_1(0)\to G_1(0)$, $G_2(0)\to(1/2)G_2(0)$	1.00	0.54	0.33	0.30	0.30
$G_1(0)\to G_1(0)$, $G_2(0)\to 2G_2(0)$	1.00	1.89	3.46	3.65	3.62

§ 5. Electromagnetic form factors in perturbation theory

In § 3 we have calculated the potentials in the ρ-meson formalism with the

values $m=600$ Mev and $h^2=5$, which were taken from the dispersion theoretical calculation of Frazer and Fulco. But we are now calculating in perturbation theory and it is desirable to reexamine the electromagnetic form factors also in this context. In the chain approximation method Miyamoto determined the scattering length a which reproduces the rms radius r_2^V. It is also desirable to calculate the electromagnetic form factors in more detail, since the magnetic form factor has been measured only for relatively large $k^2(k^2 \gtrsim 9)$. For these reasons we shall calculate the electromagnetic form factors, especially the magnetic form factor, in some detail in perturbation theory. This will permit us to see what modifications can be applied to the results in § 3 and § 4.

Before entering into the consideration of the pion-pion resonance we shall reexamine the calculation of the $G_\alpha(k^2)$.

(i) *The electromagnetic form factors with the pion-pion interaction neglected*

The spectral functions $g_\alpha(\kappa^2)$ are plotted in Fig. 7. We find that the contributions from the 3-3 isobar are very large, especially for large κ^2. This exhibits the insufficiency of the Legendre polynomial expansion as has already been pointed out.[1),11)] But it is remarkable that the calculated $G_2(0)$ with the 3-3 isobar included gives close agreement with the observed value. If we consider the graph in Fig. 8 (a), as in Miyamoto's paper, then the value of $G_2(0)$ remains unchanged with the introduction of the pion-pion resonance. In this situation the $G_2(k^2)$ given above, which includes the 3-3 isobar, will have some justification even without the rigorous validity of the method of calculation. But this very point seems to be important and will be discussed later.

We can give an intuitive explanation for the negative contribution to $G_1(k^2)$

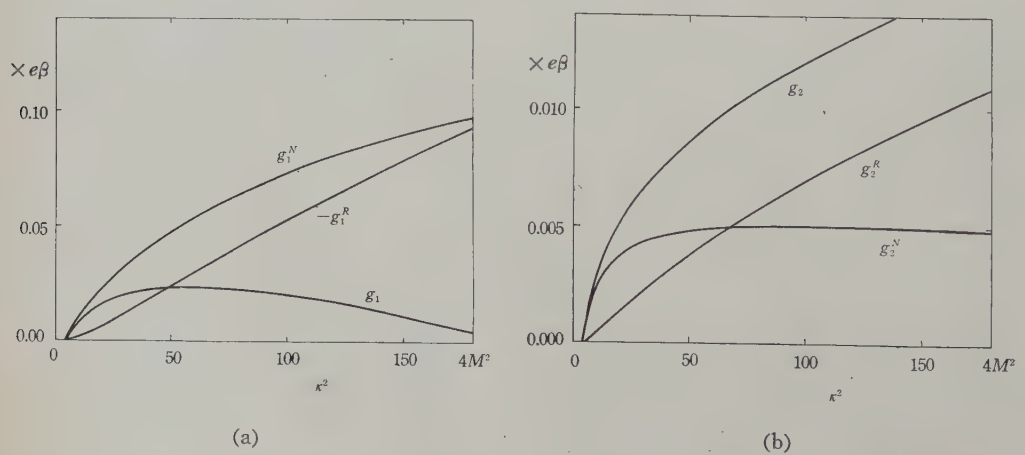


Fig. 7. Spectral functions. $g_\alpha{}^N(\kappa^2)$ and $g_\alpha{}^R(\kappa^2)$ $(\alpha=1, 2)$ are the contributions from the Born approximation term and the 3-3 isobar term respectively. $g_\alpha(\kappa^2)$ is given by $g_\alpha(\kappa^2)=g_\alpha{}^N(\kappa^2)+g_\alpha{}^R(\kappa^2)$. $\beta\sim 15$ is the square of the pseudoscalar pion-nucleon coupling constant.

from the 3-3 isobar. In the state in which a proton is dissociated into a "core" isobar and an outer pion, charge independence consideration shows that a negative pion is emitted most frequently. This is shown schematically in Fig. 9. In any case, $G_1(0)$ cannot be compared with experiment in contrast to $G_2(0)$. Therefore we should consider mainly the quantities which are stable with the variation of $G_1(0)$. These are V_2, V_4 and V_6 as seen in Table II. From this point of view $V_T(r)$ and $V_L(r)$ are most suitable for a comparison between theory and experiment.

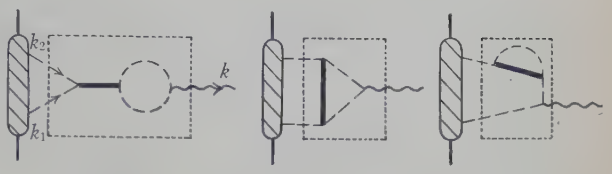


Fig. 8.

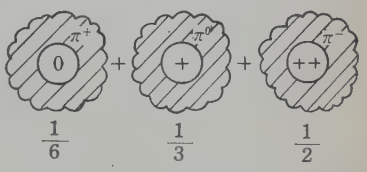


Fig. 9. Proton structure considered as a "core" 3-3 isobar surrounded by an outer pion. The probability for each component state can be calculated from the Clebsch-Gordan coefficients and is shown in the Figure.

(ii) *The pion-pion resonance*

Let us first consider the graph Fig. 8 (a) in accordance with Miyamoto. The part surrounded by the dotted line is a part of the electromagnetic form factor of a pion and is denoted by $F_\pi(k^2)$. Corresponding to the graph Fig. 8(a),* we have

$$(k_1+k_2)_\mu[F_\pi(k^2)-1]=(2\pi)^{-4}T(k^2)(k_1+k_2)_\nu \mathcal{J}_{\mu\nu}(k), \qquad (5\cdot1)$$

where $\mathcal{J}_{\mu\nu}(k)$ is a "2-vertex" connecting a photon and the resonant state. From gauge invariance $\mathcal{J}_{\mu\nu}(k)$ should be of the form

$$\mathcal{J}_{\mu\nu}(k)=(k^2\delta_{\mu\nu}-k_\mu k_\nu)\mathcal{J}(k^2). \qquad (5\cdot2)$$

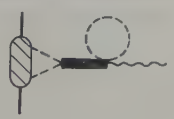
After separating the term directly proportional to $\delta_{\mu\nu}$, which is of the same form as the self-energy of a photon, we are left with $\mathcal{J}(k^2)$ which diverges logarithmically with respect to the virtual momentum. $\mathcal{J}(k^2)$ can be written more conveniently in the form of a spectral representation:

$$\mathcal{J}(k^2)=-\frac{\pi^2}{3}\int_4^{(2\Lambda)^2} d\xi \frac{1}{k^2+\xi}\left(1-\frac{4}{\xi}\right)^{3/2}, \qquad (5\cdot3)$$

where the integration can be carried out to give the form

---

* The interaction Hamiltonian contains derivatives of the fields. Therefore, various "catastrophic terms" should be added from gauge invariance. An example is illustrated below.



$$\mathcal{J}(k^2)=-\frac{2\pi^2}{3}\left[\log 2\Lambda+\frac{4}{k^2}-\left(1+\frac{4}{k^2}\right)^{3/2}\log\left(\sqrt{1+\frac{k^2}{4}}+\sqrt{\frac{k^2}{4}}\right)\right]. \quad (5\cdot4)$$

This is very similar to $D(k^2)$ defined in (2·3). Upon substituting (5·4) and (5·2) in (5·1), we obtain

$$F_\pi(k^2)=\begin{cases}\dfrac{D(0)}{D(k^2)},\\ 1-\dfrac{h^2}{6\pi}\dfrac{k^2}{k^2+m^2}\left[\log 2\Lambda+\dfrac{4}{k^2}-\left(1+\dfrac{4}{k^2}\right)^{3/2}\log\left(\sqrt{1+\dfrac{k^2}{4}}+\sqrt{\dfrac{k^2}{4}}\right)\right].\end{cases} \quad (5\cdot5)$$

For the graph in Fig. 8 (a) we can write

$$\boldsymbol{G}_\alpha(k^2)=G_\alpha(k^2)F_\pi(k^2), \quad (\alpha=1, 2) \quad (5\cdot6)$$

where $\boldsymbol{G}_\alpha(k^2)$ is a form factor with the pion-pion interaction included. It is obvious that

$$\boldsymbol{G}_\alpha(0)=G_\alpha(0), \quad (5\cdot7)$$

from $F_\pi(0)=1$.

In Fig. 11 we have plotted $\boldsymbol{G}_2(k^2)$ for the various parameters chosen for $F_\pi(k^2)$, both in the ρ-meson formalism and in the chain approximation. In Fig. 11 we have also indicated the value of $T(0)$ which is considered to govern the depth of the potentials. The curve with $m=600$ Mev, $h^2=5$ in the ρ-meson formalism is found not to reproduce the observed form factor very well. If we choose, in the ρ-meson formalism, other curves which give a better fit, $T(0)/4\pi$ turns out to be larger ($\gtrsim 0.54$) than that used in § 3 (0.27). Therefore the potentials calculated in § 3 should be considered to have been underestimated by about a factor of 2, if calculated in the ρ-meson formalism.

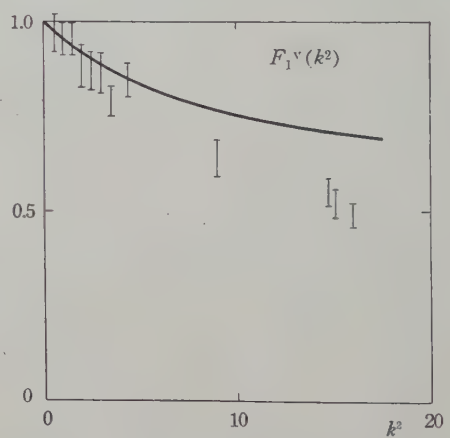


Fig. 10. $F_1^V(k^2)=\boldsymbol{G}_1(k^2)/\boldsymbol{G}_1(0)$. $F_\pi(k^2)$ is the one in the ρ-meson formalism with $h^2=10$, $m=600$ Mev and $\Lambda=2M$, which gives a good fit for $F_2^V(k^2)$.

In the case of the chain approximation, we find that the choice with $a=0.2$ and $\Lambda=2M$, used by Miyamoto gives indeed a good fit. This choice gives $T(0)/4\pi=0.47$, which is 1.70 times larger than the value used in § 3. In any case we cannot expect to make $T(0)$ appreciably smaller without giving up trying to explain the electromagnetic form factors.

In order to see the nature of the singularity of the Fourier transform of $T(k^2)$

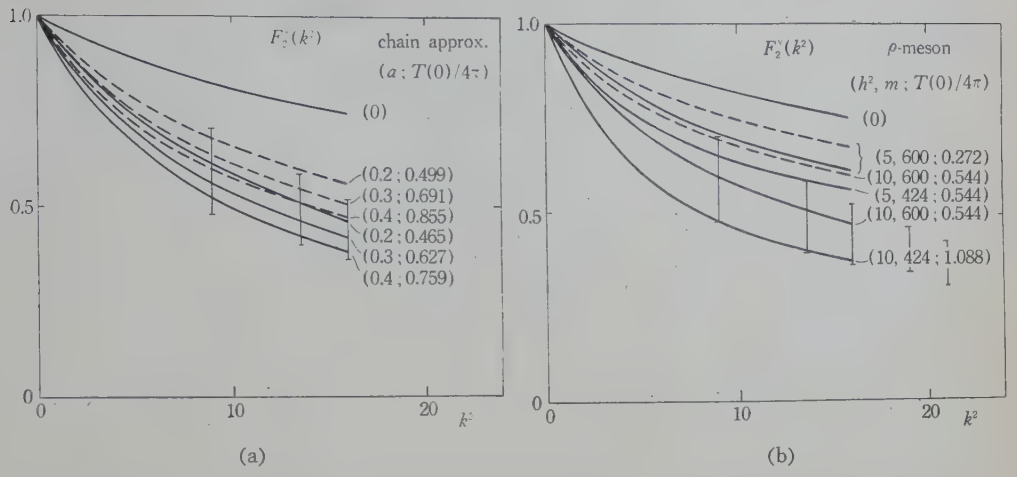


Fig. 11. $F_2^V(k^2)=G_2(k^2)/G_2(0)$. Broken curves and solid curves correspond to $\Lambda=M$ and $\Lambda=2M$ respectively, where Λ is the cutoff. In the case of the ρ-meson formalism, values of h^2, m (in Mev) and $T(0)/4\pi$ are indicated in parentheses, while in the case of the chain approximation method, values of a and $T(0)/4\pi$ are indicated. Curves denoted by (0) are the form factors with the pion-pion resonance neglected.

in the chain approximation method, we have plotted in Fig. 12 the quantity $T(k^2)/T(0)$ for several cases in which the observed electromagnetic form factor can be well reproduced. From these plots we can infer that the potentials calculated in the chain approximation method are of forms somewhat more singular than those represented in Fig. 6 by solid curves but less singular than those represented by dotted curves.

In Fig. 10 we have plotted the $G_1(k^2)=G_1(k^2)F_\pi(k^2)$ calculated by a method similar to that used with $G_2(k^2)$. We cannot obtain satisfactory agreement between theory and experiment, if we maintain the good agreement in $G_2(k^2)$. From this reason also it is desirable to consider the quantities which do not depend on $G_1(k^2)$ appreciably.

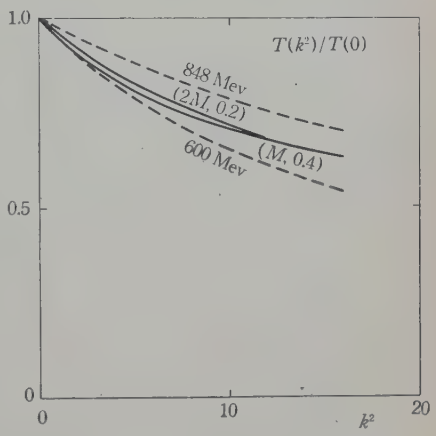


Fig. 12. Broken curves represent the results of the ρ-meson formalism with indicated masses. Solid curves represent the results of the chain approximation method: Λ and a are indicated in parentheses.

§ 6. Summary and concluding remarks

We have calculated the potentials from the graphs in Fig. 2; only the V^τ-part are considered in this paper. We have given the explicit forms in con-

figuration space in the ρ-meson formalism with $m=600$ Mev and $h^2=5$, which were taken from the results of Frazer and Fulco, and adopted by Takeda et al. Eqs. (3·4) and the curves in Fig. 5 and Fig. 6 exhibit the qualitative or semi-quantitative features of the potentials which result from the pion-pion resonance in the state $I=J=1$. Possible changes can be inferred when the parameters are changed or the chain approximation method is adopted. Reexamining the electromagnetic structure of the nucleon both in the ρ-meson formalism and in the chain approximation method, we have found that the potentials given by (3·4) cannot be expected to be an overestimate. Rather the values at $r\approx 1$ might have been somewhat underestimated.

As the main results there appear a strong attractive L-S potential and a repulsive central potential in the ${}^3O$-state, a strong repulsive tensor potential and an attractive central potential in the ${}^3E$-state, and a repulsive L-dependent potential in the ${}^1E$-state. These are all of the nature of a two-pion exchange potential and affect the nuclear forces in the region II. Among the potentials quoted above the repulsive tensor potential in the ${}^3E$-state may be objectionable if we remember that the attractive one-pion exchange tensor potential succeeded in accounting for the quadrupole moment of the deuteron, though more careful analysis will be needed to draw any conclusions.

Since $G_1(k^2)$ seems to be subject to more uncertainties than $G_2(k^2)$ in the theoretical treatment, it is desirable to consider the potentials which are insensitive to a change in $G_1(k^2)$. For this reason also, the tensor potential will be most suitable for the test for or against the assumption of a pion-pion resonance in the state $I=J=1$.

Finally we shall discuss the possibility of making the potentials weaker. As stated already, we are forced to make $G_2(k^2)$ smaller in order to make the tensor potential weaker. In this connection we should remember that we have assumed that only the graph (a) in Fig. 8 contributes to the electromagnetic form factors of the nucleon. On account of this we arrived at Eq. (5·7) and consequently had to adopt a "large" $G_2(k^2)$ which gives $2MG_2(0)/e=1.63$.

On the other hand, gauge invariance considerations are particularly important when we discuss the electromagnetic form factors at $k^2=0$. Namely, other graphs in Fig. 8 should be taken into account. In this case the electromagnetic form factor of the pion can be written as follows:

$$F_\pi(k_1^2, k_2^2, k^2)=1+k^2 f_1(k^2)+(k_1^2+1)f_2(k_1^2, k_2^2, k^2)+(k_2^2+1)f_3(k_1^2, k_2^2, k^2), \tag{6·1}$$

which satisfies the condition

$$F_\pi(-1,-1, 0)=1. \tag{6·2}$$

---

* $f_2=f_3\equiv 0$, for the graph (a) in Fig. 8.

It should be noted, however, that we have, in general,

$$F_\pi(k_1^2, k_2^2, 0) \neq 1, \tag{6·3}$$

for $k_1^2 \neq -1$, or $k_2^2 \neq -1$. In the calculation of the electromagnetic form factors of the nucleon, pions are not in a free state, and we have in general

$$G_\alpha(0) \neq G_\alpha(0), \tag{6·4}$$

due to (6·3). Frazer and Fulco obtained, in fact, $G_2(0)$ which is not equal to $G_2(0)$, although the reason is not very obvious.

There is a chance that $G_2(k^2)$ is nearly equal to that calculated in the Born approximation ($2MG_2(0)/e=0.87$) and the difference from the observed value comes from the pion-pion resonance in the manner stated above. Then we are permitted to use a "small" $G_2(k^2)$ and would obtain the weak potentials.

This really seems to be the case, since the 3-3 isobar contribution to $G_2(k^2)$, calculated in the previous sections, seems unjustifiably large. The situation is, however, not very simple. For some reference we have calculated the potentials with the $G_\alpha(k^2)$ in the Born approximation; the value at $r=1$ are shown in Table III. From this Table we find that the tensor potential has been reduced only a little. This is because $G_1(k^2)$ is very large in the Born approximation ($G_1(0)/e=0.66$). Thus we must face the difficult problem of treating the $G_1(k^2)$.

Table III. Values of the potentials at $r=1$.

$G_\alpha(k^2)$	$V_0^\tau(1)$	$V_S^\tau(1)$	$V_{LS}^\tau(1)$	$V_T^\tau(1)$	$V_{L\sigma}^\tau(1)$	$V_{LL}^\tau(1)$
Born approximation	0.0708	0.0179	−0.0640	−0.0196	0.00161	−0.00140
3-3 isobar included	0.0143	0.0277	−0.0356	−0.0300	0.00446	−0.00272

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Note added in proof:

(i) $\sqrt{16k_{10}k_{20}q_{10}q_{20}}$ should be multiplied to the left hand sides of Eqs. (2·5) and (2·12).
(ii) In $V_S^{\tau A}$ and $V_T^{\tau A}$ in Eqs. (3·4), we have set $z=y$, since they are nearly equal to each other.

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Transition Amplitudes in Perturbation Theory and Dyson's Integral Representation

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It is shown that Dyson's integral representation for the matrix element of a causal commutator between the vacuum and an arbitrary state is closely related to the transition amplitude in the lowest order of perturbation. This relation is used for clarifying physical meanings of the parameters of integration. The integral representation for the absorptive parts of vertex functions previously introduced by one of the present authors (K. Y.) comes out as a simple example. It can be applied also for the absorptive parts of scattering amplitudes, in which case, however, such a simple connection with perturbation theory is not obtained as is the case with vertex functions.

§ 1. Introduction

Recently, one of us (K. Y.) showed[1] that the absorptive part $A(z_1 z_2 \sigma^2)$ of vertex function has the following integral representation:

$$A(z_1 z_2 \sigma^2) = \int dm_1 dm_2 dm_3 \varphi(\sigma^2 : m_1 m_2 m_3) A^P(z_1 z_2 \sigma^2 : m_1 m_2 m_3), \qquad (1)$$

provided z_1 and z_2 are real-negative. Here A^P denotes the absorptive part of vertex function in the lowest order of perturbation theory, corresponding to the diagram in Fig. 1. (The letters attached to the lines denote the "masses" of corresponding particles.)

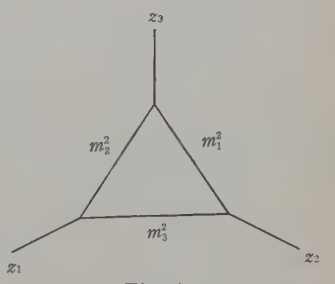


Fig. 1.

The support (the region outside of which the concerned function vanishes) of the weight-function φ is determined from the spectral condition. This result lets us conjecture that there exists an integral representation similar to (1) also for more general case: e. g. for scattering amplitudes, etc.

Perturbation theory yields a good source for causal functions, and these functions in perturbation theory belong, of course, to the set of functions allowed in general theory. What we want to know is to what extent the set allowed in general theory is larger compared with the functions in perturbation theory. The expression (1) shows that, as far as vertex functions concern, the essential difference between general and perturbation theory consists only in the σ-dependences of the weight-function φ. Then it will be worth-while to investigate the expression corresponding to (1)

for scattering amplitudes.

As is seen in the previous paper,[1] the expression (1) was obtained after transforming the parameters in Dyson's integral representation[2] to m_1, m_2 and m_3. And we got a graphical interpretation by regarding these new parameters as the masses of the propagators corresponding to the internal lines. In this paper, for more general purposes we shall investigate Dyson's integral representation when one of the "external" states is the vacuum, and it will be found that it has a close relation to scattering amplitudes in the lowest order of perturbation theory.

The essence of our method consists in rewriting of Dyson's integral representation by new parameters which admit some physical interpretation. The expression (1) is then obtained as a special example.

Some kind of a graphical interpretation is possible also for scattering amplitudes, but in this case the absorptive part does not have such a close relation to that in perturbation theory as is the case with vertex function. It will be easily understood from the above-mentioned feature of Dyson's integral representation that it is a particular character of vertex functions to have such a representation as (1).

§ 2. Transition amplitude and Dyson's representation

We begin with calculating the scattering amplitude T^P in the lowest order of perturbation theory, for the process illustrated in Fig. 2, in which particles with masses $\sqrt{z_1}$ and $\sqrt{z_2}$ undergo an inelastic scattering through an exchange of a virtual particle (mass M) yielding final particles with masses m_1 and m_2. Apart from irrelevant factors T^P is a simple propagator:

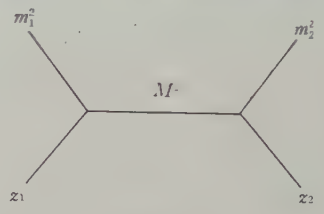


Fig. 2.

$$T^P=(M^2-k^2)^{-1} \tag{2}$$

with k, the transfer energy-momentum. If we rewrite k^2 in terms of the total energy σ and the scattering angle θ in the center of mass system, we get after elementary calculations

$$\begin{aligned} T^P(z_1 z_2 \sigma^2 :\ & m_1 m_2 M : \cos\theta) \\ =&\Big[M^2+\frac{1}{2}\sigma^2-\frac{1}{2}(z_1+z_2+m_1^2+m_2^2) \\ &+\frac{1}{2\sigma^2}(z_1-z_2)(m_1^2-m_2^2) \\ &-\frac{1}{2\sigma^2}\sqrt{\lambda(z_1 z_2 \sigma^2)\,\lambda(m_1^2 m_2^2 \sigma^2)}\cos\theta\Big]^{-1} \end{aligned} \tag{3}$$

with

$$\lambda(x_1 x_2 x_3) \equiv x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 - 2x_3x_1 - 2x_1x_2. \tag{4}$$

Next we consider the following quantity:

$$f(q) = \int dx \exp(iq\cdot x) \langle 0 | [A(x/2), B(-x/2)] | n \rangle \theta(x), \tag{5}$$

where A and B are current operators. $|0\rangle$ and $|n\rangle$ stand respectively for the vacuum state and any state with total energy-momentum p_n. We impose spectral conditions:

$$\begin{aligned} &\langle 0|A|m\rangle = 0 \text{ unless } p_m \geq a^2, \\ &\langle 0|B|m\rangle = 0 \text{ unless } p_m \geq b^2. \end{aligned} \tag{6}$$

If we employ a special Lorentz-frame with $p_n = (\sigma, 0)$, $\sigma > 0$, Dyson's integral representation for $f(q)$:

$$f(q) = \int d^4u \int d\kappa \varphi(\kappa, u, p_n) [\kappa^2 - (q-u)^2]^{-1} \tag{7}$$

holds with $\varphi(\kappa, u, p_n)$ vanishing unless

$$\begin{aligned} &|u_0| + |\boldsymbol{u}| \leq \sigma/2, \qquad \kappa \geq \kappa_0 \\ &\kappa_0 \equiv \text{Max}\,[0,\ a - \sqrt{(\sigma/2+u_0)^2 - \boldsymbol{u}^2},\ b - \sqrt{(\sigma/2-u_0)^2 - \boldsymbol{u}^2}]. \end{aligned} \tag{8}$$

The weight-function φ is in general a function of κ, u_0, $|\boldsymbol{u}|$, $p_n^2 = \sigma^2$, $\boldsymbol{e} = \boldsymbol{u}/|\boldsymbol{u}|$ and other variables to indicate the state $|n\rangle$. Here we adopt a new set of variables (M, m_1, m_2) instead of $(\kappa, u_0, |\boldsymbol{u}|)$:

$$\begin{aligned} &M = \kappa, \\ &m_1^2 = (p_n/2+u)^2 = (\sigma/2+u_0)^2 - \boldsymbol{u}^2, \\ &m_2^2 = (p_n/2-u)^2 = (\sigma/2-u_0)^2 - \boldsymbol{u}^2. \end{aligned} \tag{9}$$

We introduce further:

$$\begin{aligned} &z_1 = (p_n/2+q)^2 = (\sigma/2+q_0)^2 - \boldsymbol{q}^2, \\ &z_2 = (p_n/2-q)^2 = (\sigma/2-q_0)^2 - \boldsymbol{q}. \end{aligned} \tag{10}$$

Fig. 3 illustrates the meanings of these new quantities: the initial state is composed of two particles m_1 and m_2 which exchange a virtual particle M to become final particles z_1 and z_2. The expression (8) for the support of φ is now replaced by

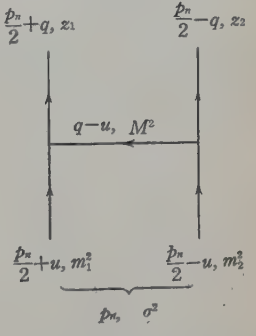


Fig. 3.
Illustration for the transformation (9) and (10). To each line is attached the corresponding momentum and mass.

$$\begin{aligned} &M \geq 0, \quad m_1 \geq 0, \quad m_2 \geq 0, \\ &M + m_1 \geq a, \quad M + m_2 \geq b, \quad m_1 + m_2 \leq \sigma. \end{aligned} \tag{11}$$

With these new variables (7) takes the following form:

$$f(q) = \int dM dm_1 dm_2 d^2\boldsymbol{e}\varphi(\sigma : m_1 m_2 M : \boldsymbol{e})$$
$$\times (m_1 m_2/2\sigma^2)\sqrt{\lambda(m_1^2 m_2^2 \sigma^2)}\; T^P(z_1 z_2 \sigma^2 : m_1 m_2 M : \boldsymbol{e}\cdot\boldsymbol{e}_0), \quad (12)$$

where $\boldsymbol{e}_0$ is the unit vector in the direction of $\boldsymbol{q}$. It is worth noting that the factor $(1/2\sigma^2)\sqrt{\lambda(m_1^2 m_2^2 \sigma^2)} = |\boldsymbol{u}|/\sigma$, which comes from the Jacobian of the transformation, is just the statistical weight for the "final" state.

Consequently, Dyson's integral representation, transformed as above, can be interpreted as the scattering amplitude corresponding the diagram in Fig. 2, integrated over a region determined from the spectral conditions with a suitable weight function besides the statistical weight, provided q is expressed (this is so in many cases) as $(p_1 - p_2)/2$ where p_1 and p_2 are energy-momentum four-vectors of real particles.

§ 3. Applications

We shall apply our result in the preceding section to definite problems.

As a first example we take the absorptive part of vertex function, which can be written down as[3]

$$A(k_1^2 k_2^2 (k_1+k_2)^2) = \sum_n \int dx \exp(iq\cdot x)\theta(x)$$
$$\times \langle 0|[A(x/2), B(-x/2)]|n\rangle\langle n|C(0)|0\rangle \delta(k_1+k_2-p_n), \quad (13)$$
$$q = (k_1-k_2)/2.$$

Adopting a special Lorentz-frame with $k_1+k_2=(\sigma, 0)$, we see that the representation (12) can be used for the right-hand side of (13). Thus we obtain

$$A(z_1 z_2 \sigma^2) = \int d\kappa dm_1 dm_2 d^2\boldsymbol{e} \sum_{p_n=(\sigma,0)} \varphi(\sigma : m_1 m_2 \kappa \boldsymbol{e}, n)$$
$$\times \langle n|C(0)|0\rangle (m_1 m_2/2\sigma^2)\sqrt{\lambda(m_1^2 m_2^2 \sigma^2)}$$
$$\times T^P(z_1 z_2 \sigma^2 : m_1 m_2 \kappa : \boldsymbol{e}\cdot\boldsymbol{e}_0). \quad (14)$$

The $\boldsymbol{e}$-dependence of the weight-function disappears after the summation with respect to n is performed, so that the integration with respect to the angular variables can be carried out. It is easy to see, by explicitly calculating the matrix elements in (13) in perturbation theory, that the following relation holds:

$$\int d^2\boldsymbol{e}(\sqrt{\lambda(m_1^2 m_2^2 \sigma^2)}/2\sigma^2) T^P(z_1 z_2 \sigma^2 : m_1 m_2 \kappa : \boldsymbol{e}\cdot\boldsymbol{e}_0)$$
$$= 2\pi A^P(z_1 z_2 \sigma^2 : m_1 m_2 \kappa). \quad (15)$$

The factor 2π in the right-hand side is attached in order that A^P may coincide with that used in the previous paper.[1] (Note that the roles of m_1 and m_2 are interchanged in this paper.) Using (15) we obtain (1) immediately with a new

weight-function. It is an essential point for the proof of (1) that the weight-function in (14) becomes e-independent after the summation with respect to n, so that we could carry out the e-integration.

As a second application we take the absorptive part of scattering amplitudes. The concerned quantity is closely related to the following function :[4),5)]

$$\begin{aligned} M=\sum_n \int dx\, dy \exp\,(iq'\cdot x - iq\cdot y)\,\langle 0|[A(x/2),\, B(-x/2)]|n\rangle \\ \times\langle n|[C(y/2),\, D(-y/2)]|0\rangle\theta(x)\theta(y)\delta(k_1+k_2-p_n), \\ q'=(k_3-k_4)/2,\ \ q=(k_1-k_2)/2. \end{aligned} \tag{18}$$

where k_1 and k_2 (k_3 and k_4) are the momenta of the particles in the initial (final) state.

Applying (12) to (18) and carrying out the summation with respect to n we get :

$$\begin{aligned} M=\int dm_1\, dm_2\, dm_3\, dm_4\, d\kappa_1\, d\kappa_2\, d^2\boldsymbol{e}_1\, d^2\boldsymbol{e}_2 \\ \times\varphi(\sigma : m_1 m_2 m_3 m_4 \kappa_1 \kappa_2 : \boldsymbol{e}_1\cdot\boldsymbol{e}_2)\ (m_1 m_2 m_3 m_4/4\sigma^4) \\ \times\sqrt{\lambda(m_1^2 m_2^2\sigma^2)\lambda(m_3^2 m_4^2\sigma^2)}\ T^P(z_1 z_2\sigma^2 : m_1 m_2\kappa_1 : \boldsymbol{e}\cdot\boldsymbol{e}_1) \\ \times T^P(z_3 z_4\sigma^2 : m_3 m_4\kappa_2 : \boldsymbol{e}'\cdot\boldsymbol{e}_2), \end{aligned} \tag{19}$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ are respectively the unit vectors in the directions of the initial and final relative momenta : $\boldsymbol{q}$ and $\boldsymbol{q}'$. In this case the $\boldsymbol{e}_1$-, $\boldsymbol{e}_2$-dependences of the weight-function are not wholly removed so that we cannot carry out the integration with respect to all four angles. The remaining angle-variable $\boldsymbol{e}_1\cdot\boldsymbol{e}_2$ cannot be interpreted as a mass of internal line, and in case such an angle-integration remains, the concerned quantity cannot have such a close relation to its perturbative counterpart as the absorptive part of vertex functions formerly treated.

The scattering amplitude can be equally well expressed as

$$T=\int dx \exp\,(iq'\cdot x)\,\langle 0|[A(x/2),\, B(-x/2)]\,|i\rangle^{in}\theta(x) \tag{20}$$

$$=\int dx \exp\,(-iq\cdot x)^{out}\langle f|[C(x/2),\, D(-x/2)]|0\rangle\theta(x)\,, \tag{21}$$

where $|i\rangle^{in}$ and $|f\rangle^{out}$ denote the initial and final states respectively with incoming and outgoing boundary conditions. These expressions are used by Lehmann[5)] in his work on the analyticity with respect to the momentum transfer. The integral representation (12) may be applied also to them, but in these cases too there remains an angle-integration, thus forcing us to give up a complete graphical interpretation.

Though except the absorptive part of vertex functions the examples treated above did not admit a complete graphical interpretation, yet the remaining para-

meter of integration is a quantity with a clear physical meaning, i. e. the scattering angle in the center-of-mass system. Therefore it may be convenient in many respects to use the parameters m_t, κ, e even in the cases when we do not have a complete graphical interpretation.

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Variational Calculation of the Scattering Lengths in Electron-Hydrogen Scattering

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The non-relativistic Schrödinger equation of electron plus hydrogen-atom system is solved by the Hulthén type variational method at the limit of zero incident electron energy. 8 and 5 parameter trial functions are used for the singlet and triplet states respectively, and the following upper bounds on the scattering length a are obtained,**

$$a_s \leqq 6.217 a_0 \text{ (singlet)}, \quad a_t \leqq 2.272 a_0 \text{ (triplet)}.$$

Some discussions are given on the accuracy of the result. It is shown that the relation $a_{0H} > a_{sH} > a_K > a(\text{true})$ holds for trial functions of the same form if $a(\text{true})$ is positive.***

§ 1. Introduction

Scattering of electron by hydrogen atom is the simplest of the problems of atomic collision, and many papers have been published on this subject. The results of these calculations based on different methods are useful for testing the validity of various approximations to be applied for more complex atoms. In this connection, it is remarkable that the Hartree-Fock approximation furnishes rather good result.[1] However, to take the correlation between the electrons correctly, it is necessary to have some other method, and for that purpose the variational method is expected to be the most effective.[2]

The energy of bound state for two electrons in the field of an infinitely heavy nucleus (He atom and H^- ion) has been calculated by this method with utmost accuracy. For example, in the recent work by Pekeris,[3] the number of adjustable parameters amounts from two hundreds to more than a thousand. In the collision problem, on the contrary, the number of parameters has been very small, and did not exceed more than three except for a few recent works.

Meanwhile, the effective range theory applied to e-H scattering[4] showed to

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** a_0 is the Bohr radius. In this paper we use the atomic units.

*** a_{0H}, a_{sH} and a_K are the scattering lengths calculated according to the variational methods of original Hulthén, second Hulthén and Kohn.

reproduce very well the phase shifts calculated variationally by Massey-Moiseiwitch.[2] In this theory, the result is expressed by two constants, the scattering length a and the effective range r_0. In this paper, an attempt is made to find these quantities more accurately by calculating the wave function for the zero energy of incident electron with increased number of parameters. In § 2, the method and the result of calculation and in § 3, discussions on their precision with a few related remarks are given. In the Appendix inequalities among the scattering lengths calculated according to the variational methods of Kohn, original Hulthén and second Hulthén are derived.

§ 2. Calculation

So far, the trial function in the collision problem seems to have been chosen quite ingeneously, but little attention has been paid in systematizing the choice. For the bound state of two-electron system, the trial function of Hylleraas type has been known to be excellent in saving the labor of calculation and giving good results. For the scattering problem, however, straightforward application of Hylleraas method is not practicable since the wave function does not damp exponentially with the increasing distance from the atom. The following is its modification for the electron with zero incident energy:

$$\begin{aligned}\Psi_{\pm} &= (e^{-r_1}/\sqrt{\pi})\{(1-e^{-r_2})/r_2+A_{\pm}\} \\ &\quad \pm (e^{-r_2}/\sqrt{\pi})\{(1-e^{-r_1})/r_1+A_{\pm}\}+f_{\pm}, \qquad (0)\end{aligned}$$

$$f_+ = \frac{e^{-k+s}}{\sqrt{\pi}}(a_1+a_2 s+a_3 u+a_4 s^2+a_5 t^2+a_6 u^2+a_7 us+\cdots), \qquad (1)$$

$$f_- = \frac{e^{-k-s}}{\sqrt{\pi}}(b_1 t+b_2 st+b_3 ut+b_4 u^2 t+\cdots), \qquad (2)$$

$$s=r_1+r_2, \quad t=r_2-r_1, \quad u=r_{12}.$$

The atomic units are used throughout this paper. $A_{\pm}$, $k_{\pm}$, a_1, $\cdots$, b_1, $\cdots$ are the variational parameters, $+$ for the singlet and $-$ for the triplet state, and $-1/A_{\pm}$ are the approximate values for the scattering length for each state.

For the variational calculation of Hulthén type,[5] the integrals

$$L_{\pm} = \int \Psi_{\pm}(H-E_0)\Psi_{\pm} dV_1 dV_2 \qquad (3)$$

are to be evaluated. According to Kohn method (abbreviated as K), a_i, A_+ are determined from

$$\frac{\partial L_+}{\partial a_i}=0, \quad \frac{\partial (L_+/A_+^2)}{\partial (1/A_+)}=4\pi, \quad \text{(singlet)} \qquad (4)$$

and similarly b_i, A_- for the triplet. Then by calculating

$$a_K = -\frac{1}{A_\pm} + \frac{L_\pm}{4\pi A_\pm^2}, \tag{5}$$

the improved scattering lengths are obtained. (4) is equivalent to the statement that a_K is stationary. According to the original Hulthén (OH) and the second Hulthén (SH) methods, the approximate values of the scattering length a_{OH} and a_{SH} are found, in the singlet case for example, respectively by

$$\frac{\partial L_+}{\partial a_l} = 0, \quad L_+ = 0, \quad a_{OH} = -\frac{1}{A_+}, \tag{6}$$

and

$$\frac{\partial L_+}{\partial a_l} = 0, \quad \frac{\partial L_+}{\partial A_+} = -4\pi, \quad \frac{1}{a_{SH}} = -A_+ - \frac{L_+}{4\pi}. \tag{7}$$

The effective range r_0 is then calculated[4] as

$$r_{0\pm} = \frac{1}{4\pi}\int (\varphi_\pm^2 - \Psi_\pm^2)\, dV_1 dV_2 \mp 8(1+2A_\pm)^2, \tag{8}$$

where the function $\Psi_\pm$ is given by the formula (0) with the parameter values obtained by the above procedures, and

$$\varphi_\pm = (e^{-r_1}/\sqrt{\pi})\left\{\frac{1}{r_2} + A_\pm\right\} \pm (e^{-r_2}/\sqrt{\pi})\left\{\frac{1}{r_1} + A_\pm\right\}.$$

In our calculation, the value of k in (1) is fixed and is taken as 1.

The results* by the Kohn method are shown in Tables I and II.

The improvement of approximation in introducing more terms in the wave function is apparent. Some of the features of these results are described below:
1) The monotonic decrease of the values of scattering lengths. This fact is in conformity with the theorem of Spruch et al.[6] that the value of scattering length

Table I. Kohn method (singlet)

a_+	6.21703	6.265325	6.29313	6.29313	6.30145	8.14507
r_{0+}	2.72	2.74	2.74	2.75	2.67	3.29
A_+	−0.161496	−0.159462	−0.158145	−0.158108	−0.159718	−0.119446
a_1	−0.970459	−0.985632	−0.994617	−0.995021	−0.904527	−0.847368
a_2	−0.240422	−0.295492	−0.359955	−0.359787	−0.431943	−0.247686
a_3	0.204058	0.307846	0.428238	0.428278	0.437357	
a_4	−0.062945	−0.027134	−0.014726	−0.014819		
a_5	0.004684	−0.000494	0.000242			
a_6	−0.025485	0.031395				
a_7	0.100793					

* The computation was carried out by the Parametric Computor I of the University of Tokyo.

Table II. Kohn method (triple)

a_-	2.27151	2.31477	2.34942	2.34944	2.35416
r_{0-}	−1.33	0.774	1.19	1.15	1.38
A_-	−0.409116	−0.425679	−0.425115	−0.423991	−0.440943
b_1	−0.363019	−0.013998	−0.095265	−0.089002	
b_2	0.048852	0.090787	0.003685		
b_3	0.184777	−0.148346			
b_4	−0.059373				

calculated by the Kohn method gives the upper bound of the true value, even though the bound state exists, if the trial function is subject to some criterion.*

2) The approximate value $a_{+K}(a_{-K})$ obtained with a_1 and a_2 (b_1 and b_2) terms is fairly close to the Hartree-Fock values,[7]

$$a_+ = 8.058, \quad r_{0+} = 3.02,$$

$$a_- = 2.347, \quad r_{0-} = 1.22.$$

Inclusion of $a_3 u$ causes rather large decrease in the value of a_+—from 8.15 to 6.30, as is seen from Table I,—and subsequent improvement in adding more terms is small. On the other hand, the decrease of a_- by adding the terms $b_3 ut$, $b_4 u^2 t$ is small compared with the case of the singlet. Somewhat irregular variation of b_i in increasing the number of terms makes us suspicious about the rapidness of convergence.

The result of similar calculation based on the SH-method is shown in Tables III and IV. It is remarkable that both the K method and the SH-method give almost the same results, though $a_K < a_{SH}$. In the case of SH, $a_\pm$ are monotonically

Table III. Second Hulthén method (singlet)

a_+	6.21711	6.265334	6.29325	6.29327	6.30171	8.15360
r_{0+}	2.73	2.74	2.73	2.73	2.69	3.23
A_+	−0.161344	−0.159495	−0.158303	−0.158149	−0.159744	−0.118032
a_1	−0.974867	−0.984635	−0.989495	−0.990328	−0.906524	−0.836797
a_2	−0.239675	−0.295746	−0.361777	−0.361355	−0.433412	−0.235311
a_3	0.203866	0.308037	−0.430191	0.430911	0.434112	
a_4	−0.063270	−0.027010	−0.013977	−0.014283		
a_5	0.003964	−0.000336	0.001076			
a_6	−0.025357	0.031442				
a_7	0.100191					

* The trial function f_+ for H^- ion gives a lower energy than 1/2 atomic units.

Table IV. Second Hulthén method (triplet)

a_-	2.27496	2.31491	2.34942	2.34945	2.35596
r_{0-}	−1.19	0.843	1.20	1.15	1.38
A_-	−0.430623	−0.430137	−0.425484	−0.424991	−0.433333
b_1	−0.369523	−0.015012	−0.095422	−0.086631	
b_2	0.078116	0.096871	0.004111		
b_3	0.178494	−0.149958			
b_4	−0.059644				

decreasing for increasing number of parameters and this fact is proved in the Appendix.

Table V shows the results for 8 parameters (singlet), and 5 parameters (triplet) by the OH-method.

Table V. Original Hulthén method

a_+	6.21740
r_{0+}	2.77
A_+	−0.160839
a_1	−0.989451
a_2	−0.237187
a_3	0.203223
a_4	−0.064346
a_5	0.001586
a_6	−0.024931
a_7	0.098198

a_-	2.27630
r_{0-}	−1.10
A_-	−0.439309
b_1	−0.372149
b_2	0.089934
b_3	0.175959
b_4	−0.059754

On comparing the results of the same order trial function in three methods, we have the relation

$$a_{\mathrm{OH}} > a_{\mathrm{SH}} > a_{\mathrm{K}} > a(\mathrm{true}).$$

The proof that $a_{\mathrm{OH}} > a_{\mathrm{SH}} > a(\mathrm{true})$ when $a(\mathrm{true}) > 0$, and a plausible explanation of the relation $a_{\mathrm{SH}} > a_{\mathrm{K}}$ are given in the Appendix.

§ 3. Discussion

Since $a_\pm$ given in Tables I and II are known to give the upper bounds of scattering lengths, the results can be compared with those of other workers. Rosenberg, Spruch and O'Malley (RSO)[8] use seven adjustable parameters, and obtain $a_+ = 6.23$ and $a_- = 1.93$ by the Kohn method. Our result is a little better for a_+, so the function (1) is not very bad for this case. a_+ also agrees quite well with $a_+ = 6.17$, obtained on the assumption of shape independent approximation using H^- function.[4]

For the triplet state, however, RSO obtain a considerably smaller value of a_- than ours. Fairly large number of terms would be needed to attain the value $a_-=1.93$ in our calculation. In the triplet, the wave function of RSO has much smaller damping factor for the terms responsible for the correlation, so that a better convergence can be expected by taking smaller value of k_- in (2). (Needless to say, it is better to vary $k_\pm$, or to carry out calculation for several values of $k_\pm$.)

Another consideration to be taken in is the effect of the long-range correlation. According to the adiabatic approximation, an electron very far from the atom is subject to the potential $-4.5/r^4$, due to the long range nature of Coulomb force.[9] This potential is better to be taken into account in determining the asymptotic form of radial part of the wave function which is $r(1-4.5/2r^2+\cdots)-a(1-4.5/6r^2\cdots)$ instead of $r-a$.[10] However, r_0 diverges in (8) for the solution of this asymptotic form, suggesting that[10] $k\cot\delta$ is not a function of k^2 but

$$k\cot\delta=-\frac{1}{a}+c_1k+c_2k^2\ln(c_3k)+c_4k^2+\cdots$$

with a linear term in k. (The irregular behavior of r_0 in the Table II might be understood by the supposition that the effect of this long range correlation is effective in the triplet.)

Martin, Seaton and Wallace (MSW)[11] obtained by Kohn method as the upper bound $a_+=6.37$, $a_-=1.99$ using a rather simple trial function, but including the long range correlation—a satisfactory results considering the crudeness of their function, and seems to indicate the importance of the long-range effect. Recently, Temkin[12] et al. obtained, in finer approximation, $a_+=5.8$ and $a_-=1.9$. Their method is not variational so that the results are not certain to be the upper bounds. Nevertheless, these values will be a good estimate. Unfortunately, the experiments so far carried out are not so exact as to allow the comparison.

It is, however, doubtful that the long-range correlation is effective only for the triplet state, so that the spin direction of electron at a great distance from the atom has much influence on the scattering amplitude. It would be more natural to suppose that the Pauli principle keeps the electrons in the vicinity of the atomic nucleus more apart from each other in the triplet than in the singlet state to diminish the value of k for the triplet state. It may also be that the inclusion of long-range correlation serves to improve the results for both states, especially for $k^2\approx0$, allowing us to obtain better figures with smaller number of parameters in the trial functions.

To sum up, with $k=1$, the trial function (1) is a good approximation but not (2), for which a smaller value of k is expected to improve the convergence. Whether adding other terms accounting for the long-range correlation, or increasing the number of parameters in (1), (2), (3) with suitable values for k gives better result is not clear.

Appendix

Proof of $a_{OH} > a_{SH} > a(true)$, *and explanation of* $a_{SH} > a_K$ *when* $a(true) > 0$

From the theorem of Spruch et al. and Eq. (5), we have

$$\frac{L_-}{4\pi A_-^2} - \frac{1}{A_-} > a(\text{true}). \tag{A1}$$

Since the value of $a_-(\text{true})$ is positive, the quadratic form with respect to A_-, and b_i, $(L_-/4\pi) - A_-$, is positive definite, and has a minimum. (L_- is defined by (3), (0) and (2)). As the coefficients of the quadratic terms do not change by the addition of the linear term $2A_-$, the second property is also valid for $(L_-/4\pi) + A_-$; that is, $(L_-/4\pi) + A_-$ has a minimum. Therefore, $-1/a_{SH} = (L_-/4\pi) + A_-$ is minimum for any variation of the wave function, and if f_- is "exact", this minimum value is equal to $a_-(\text{true})$. If (A1) is valid (as is the case with 8 parameter trial function) in the singlet state, the same conclusion can be drawn.

In the SH method all the (adjustable) parameters are adjusted in such a way that $-1/a_{SH}$ reaches a minimum value as small as possible. In the OH method parameters are adjusted to give the condition $L_- = 0$ rather than to make $-1/a_{SH}$ a minimum. In other words, the minimization of $-1/a_{SH} \equiv (L_-/4\pi) + A_-$ in the OH method is incomplete. Therefore, we have generally

$$-\frac{1}{a_{SH}} < -\frac{1}{a_{OH}}. \tag{A2}$$

We shall now consider the relation between a_K and a_{SH}. In this respect it would be worthwhile to calculate the second order variations of a_K and a_{SH} around the exact solution. From the theorem of Spruch et al., we know that

$$\frac{\partial^2 a_K}{\partial A^2} > 0.$$

Calculating the second derivative of a_K step by step, we get

$$\begin{aligned}\frac{\partial^2 (4\pi a_K)}{\partial A^2} &= \frac{\partial^2}{\partial A^2}\left(\frac{L}{A^2} - \frac{4\pi}{A}\right) = \frac{1}{A^2}\frac{\partial^2 L}{\partial A^2} + \frac{8\pi}{A^3} \\ &= \frac{1}{A^2}\frac{\partial^2}{\partial A^2}(L + 4\pi A) + \frac{8\pi}{A^3} = -\frac{1}{A^2}\frac{\partial^2}{\partial A^2}\left(\frac{4\pi}{a_{SH}}\right) + \frac{8\pi}{A^3},\end{aligned} \tag{A3}$$

where the use has been made of $L = 0$ and $\partial L/\partial A = -4\pi$ of (7). Since the first derivative of a_{SH} vanishes, (A3) is equivalent to

$$\frac{\partial^2 a_K}{\partial A^2} = \frac{\partial^2 a_{SH}}{\partial A^2} - 2a^3. \tag{A4}$$

Then we expect from (A4) that

$$a_K < a_{SH}, \quad \text{for } a > 0. \tag{A5}$$

From (A2), (A5) and $a_K > a(\text{true})$, we finally get

$$a_{OH} > a_{SH} > a_K > a(\text{true}).$$

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On Energy Matrices for the Independent Particle Model

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A method which makes use of Fourier transforms of two-body interactions for the calculations of energy matrices in the independent particle model is proposed. The non-central interactions as well as central ones can be easily expanded into series of tensor products of spherical harmonics by this procedure. Furthermore, the radial integrals can be reduced to simple integrals which involve the Fourier transforms of the radial dependence of the interactions. For the harmonic oscillator wave functions, the procedure can be easily carried out and explicit formulas for the integrals are obtained. Useful tables for the calculations of the integrals for the central, tensor and spin-orbit interactions are given.

§ 1. Introduction

The matrix elements of two-body interactions are necessary in the investigation of nuclear properties by means of the independent particle model. The calculations of these matrix elements are usually carried out by expanding the interactions into series of Legendre polynomials.[1] For the central interaction, it is well known that this procedure is quite easy if one applies the methods of tensor operator proposed by Racah.[2] For the non-central interactions, the situations are somewhat complicated and the interactions have been discussed by Talmi,[3] Hope and others.[4]

In the next two sections of this paper, it is shown that the two-body interactions can be easily expanded into series of the products of spherical harmonics by considering Fourier transforms of the interactions, and the matrix elements can be obtained by straightforward application of the method of tensor operators. This procedure is much simpler and more systematic than those proposed so far for the treatment of the non-central interactions.

The radial integrals which appear in this expansion involve the variables r_1 and r_2, the distances of the two particles from the origin, in separated forms in the integrands, because the Fourier transform is considered. Therefore, the

integrations over r_1 and r_2 are manageable and the integrals can be expressed by means of simple ones. However, it is difficult to carry out the integrations over r_1 and r_2 analytically for the wave functions with arbitrary radial dependence.

In order to avoid this difficulty, we assume three-dimensional isotopic harmonic oscillator wave functions for one-particle wave functions and consider the radial integrals in the last few sections of this paper. Then, the radial integrals can be obtained as linear combinations of simple integrals. The coefficients are expressed in terms of the expansion coefficients of the product of two associated Laguerre polynomials into power series. Simple recurrence formulas between the latter coefficients are obtained and they are useful for the evaluation of the radial integrals. The simple integrals into which the radial integrals are reduced have linear relations with the Talmi integrals which have been introduced as the integrals over relative coordinates after the separation of the center-of-mass motion of two particles in the harmonic oscillator potential.[5] By making use of the relations, the expansion coefficients of the radial integrals into the Talmi integrals, which have been given by several authors in some cases for the central interactions,[6] are easily obtained. Although the method which expands the two particle wave function into products of the wave functions of the relative and the center-of-mass motions and calculates the matrix elements of the interaction directly has been proposed,[7] our procedure seems to be simpler than that for the calculations of the energy matrices of individual configurations.

§ 2. The central and tensor interactions

Two-particle matrix elements of the interactions are exclusively considered in this paper, since it is well known that matrix elements of the two-body interactions for many-particle configurations can be reduced to the two-particle ones.[1),8)] Let us first decompose the tensor interaction into spin and ordinary space parts, as an example. This interaction is defined as $S_{12}V(r)$, where $S_{12}=(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})/r^2 -1/3\cdot(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ and $\boldsymbol{r}=\boldsymbol{r}_2-\boldsymbol{r}_1$ $(r=|\boldsymbol{r}|)$. It is easy to write the S_{12} as a scalar product of tensor operators of second rank which operate spin and ordinary spaces, respectively :[3]

$$S_{12}V(r)=(2/3)^{1/2}([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]^{(2)}\cdot V(r)\boldsymbol{C}^{(2)}(\Omega)), \tag{1}$$

where Ω denotes the direction of the vector $\boldsymbol{r}$ and $C_q^k(\Omega)=[4\pi/(2k+1)]^{1/2}Y_{kq}(\Omega)$ are the unnormalized spherical harmonics.* It is seen from (1) that the spatial part of the tensor interaction is taken to be $V(r)\boldsymbol{C}^{(2)}(\Omega)$. In this section, we discuss the interactions the spatial parts of which has the form of $V(r)\boldsymbol{C}^{(\kappa)}(\Omega)$, κ being zero or even positive integers. If κ equals zero, the interaction is central, since

* For the notation, see reference 9). The formulas used in this paper are also contained in it, unless otherwise stated.

$\boldsymbol{C}^{(0)}=1$, and if κ equals two, the interaction is tensor. The spin parts are simple to be handled and only the spatial parts have to be treated in detail.

In order to separate the variables $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the spatial part of the interaction, we introduce its Fourier transform $\boldsymbol{v}^{(\kappa)}(\boldsymbol{p})$:

$$\boldsymbol{v}^{(\kappa)}(\boldsymbol{p})=(2\pi)^{-3}\int V(r)\,\boldsymbol{C}^{(\kappa)}(\Omega)\exp(i\boldsymbol{p}\cdot\boldsymbol{r})\,d\boldsymbol{r}. \tag{2}$$

By making use of the well-known formula

$$\exp(i\boldsymbol{p}\cdot\boldsymbol{r})=\sum_k i^k(2k+1)j_k(pr)\,(\boldsymbol{C}^{(k)}(\Omega)\cdot\boldsymbol{C}^{(k)}(\Omega_p)), \tag{3}$$

where $j_k(pr)$ are the spherical Bessel functions and Ω_p denotes the direction of vector $\boldsymbol{p}$, and the orthonormality relation of the spherical harmonics, $\boldsymbol{v}^{(\kappa)}(\boldsymbol{p})$ becomes

$$\boldsymbol{v}^{(\kappa)}(\boldsymbol{p})=(4\pi)^{-1}i^{\kappa}v_{\kappa}(p)\,\boldsymbol{C}^{(\kappa)}(\Omega_p), \tag{4}$$

where

$$v_{\kappa}(p)=\frac{2}{\pi}\int_0^{\infty}V(r)\,j_{\kappa}(pr)\,r^2\,dr. \tag{5}$$

Then, the spatial part of the interaction is expressed by the Fourier inverse transform as

$$V(r)\,\boldsymbol{C}^{(\kappa)}(\Omega)=(4\pi)^{-1}i^{\kappa}\int v_{\kappa}(p)\,\boldsymbol{C}^{(\kappa)}(\Omega_p)\exp(-i\boldsymbol{p}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_1))\,d\boldsymbol{p}.$$

In this form of $V(r)\boldsymbol{C}^{(\kappa)}(\Omega)$, the variables $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are separated and the integration over Ω_p can be carried out by applying (3) for $\exp(i\boldsymbol{p}\cdot\boldsymbol{r}_1)$ and $\exp(-i\boldsymbol{p}\cdot\boldsymbol{r}_2)$, respectively. Taking into account the symmetry relation of the Clebsch-Gordan coefficients, we have the result:

$$\begin{aligned}V(r)\,\boldsymbol{C}^{(\kappa)}(\Omega)=&\sum_{k_1,k_2}i^{k_1-k_2+\kappa}(2k_1+1)\,(2k_2+1)/(2\kappa+1)\\&\times(k_1\,0\,k_2\,0|\kappa\,0)\,v^{(k_1,k_2;\,\kappa)}(r_1,\,r_2)[\boldsymbol{C}_1^{(k_1)}\times\boldsymbol{C}_2^{(k_2)}]^{(\kappa)},\end{aligned} \tag{6}$$

where $\boldsymbol{C}_i^{(k)}$ means $\boldsymbol{C}^{(k)}(\Omega_i)$ and

$$v^{(k_1,k_2;\,\kappa)}(r_1,\,r_2)=\int_0^{\infty}j_{k_1}(pr_1)j_{k_2}(pr_2)\,v_{\kappa}(p)\,p^2dp. \tag{7}$$

In (6), the summations over k_1 and k_2 are restricted by $|k_1-k_2|\leqq\kappa\leqq k_1+k_2$ and $k_1+k_2+\kappa=$even from the properties of the Clebsch-Gordan coefficients. The formula (6) is useful for the calculation of the reduced matrix elements. They are easily obtained by means of the method of tensor operators[2),9)] as

$$\begin{aligned}(l_1l_2L\|\,V(r)\,C^{(\kappa)}(\Omega)\,\|l_1'l_2'L')=&\sum_{k_1,k_2}i^{k_1-k_2+\kappa}(2k_1+1)\,(2k_2+1)/(2\kappa+1)\cdot(k_1\,0\,k_2\,0|\kappa\,0)\\&\times[(2L+1)\,(2L'+1)\,(2\kappa+1)]^{1/2}(l_1\|C^{(k_1)}\|l_1')\,(l_2\|C^{(k_2)}\|l_2')\\&\times X\begin{pmatrix}l_1&l_2&L\\l_1'&l_2'&L'\\k_1&k_2&\kappa\end{pmatrix}R^{(k_1,k_2;\,\kappa)}(l_1l_2\,;\,l_1'l_2'),\end{aligned} \tag{8}$$

where $R^{(k_1,k_2;\kappa)}(l_1l_2;l_1'l_2')$ are radial integrals:

$$R^{(k_1,k_2;\kappa)}(l_1l_2;l_1'l_2')=\int_0^\infty\int_0^\infty R_{l_1}(r_1)R_{l_2}(r_2)v^{(k_1,k_2;\kappa)}(r_1,r_2)R_{l_1'}(r_1)R_{l_2'}(r_2)r_1^2\,dr_1r_2^2dr_2, \tag{9}$$

$R_l(r)$ being the radial functions of the one-particle wave functions. (In this section, the principal quantum numbers n are omitted for the sake of simplicity.)

For the central interaction $\kappa=0$, (6) gives, by putting $k_1=k_2=k$,

$$V(r)=\sum_k(2k+1)v^{(k)}(r_1,r_2)P_k(\cos\omega), \tag{10}$$

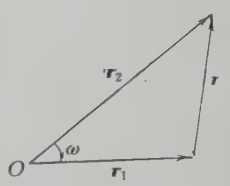


Fig. 1

where ω is the angle between the two radii vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ (Fig. 1). $P_k(\cos\omega)$ are the Legendre polynomials, and $v^{(k)}(r_1,r_2)$ means $v^{(k,k;0)}(r_1,r_2)$. By inserting the $v_0(p)$ of (5) into (7) and making use of a formula of the Bessel functions:[10]

$$\frac{2}{\pi}\int_0^\infty j_k(pr_1)j_k(pr_2)j_0(pr)\,p^2\,dp=P_k(\cos\omega)/(2r_1r_2r), \tag{11}$$

it is given by

$$v^{(k)}(r_1,r_2)=\frac{1}{2}\int_{-1}^{1}V(r)P_k(\cos\omega)\,d(\cos\omega). \tag{12}$$

The matrix element is reduced from (8) to

$$\begin{aligned}&(l_1l_2LM|V(r)|l_1'l_2'LM)\\&=\sum_k(-)^{l_1+l_2'-L}(l_1\|C^{(k)}\|l_1')(l_2\|C^{(k)}\|l_2')W(l_1l_2l_1'l_2';Lk)(2k+1)R^{(k)}(l_1l_2;l_1'l_2'),\end{aligned} \tag{13}$$

which agrees with the well-known formula given by Racah.[2]

For the tensor interaction given by (1), the matrix elements are easily obtained in the LS-coupling as

$$\begin{aligned}&(l_1l_2SLJM|S_{12}V(r)|l_1'l_2'S'L'JM)=(2/3)^{1/2}(S\|[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]^{(2)}\|S')\\&\times(l_1l_2L\|V(r)C^{(2)}(\Omega)\|l_1'l_2'L')(-)^{S+L'-J}W(SLS'L';J2),\end{aligned} \tag{14}$$

where $(S\|[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]^{(2)}\|S')=2\cdot5^{1/2}\delta_{S,1}\delta_{S',1}$ and the reduced matrix elements for the spatial part are given by putting $\kappa=2$ in (8). The summations over k_1 and k_2 are restricted by $|k_1-k_2|\leqq2\leqq k_1+k_2$ and $k_1+k_2=$even. If the matrix elements of the tensor interaction in jj-coupling is to be calculated, the interaction may be transformed from (1) and (6) by changing the coupling shemes of the tensor operators as

$$\begin{aligned}&S_{12}V(r)=(2/3)^{1/2}\sum_{k_1,k_2,K}i^{k_1+k_2}(2k_1+1)(2k_2+1)(k_10k_20|20)\\&\times W(k_11k_21;K2)v^{(k_1,k_2;2)}(r_1,r_2)([\boldsymbol{\sigma}_1\times\boldsymbol{C}_1^{(k_1)}]^{(K)}\cdot[\boldsymbol{\sigma}_2\times\boldsymbol{C}_2^{(k_2)}]^{(K)}).\end{aligned} \tag{15}$$

For $\kappa=2$, the radial dependences $v^{(k_1,k_2;2)}(r_1, r_2)$ of the interaction can be written in somewhat different forms. Carrying out partial integrations in (7) after putting $j_2(\rho)=\rho^2(\rho^{-1}d/d\rho)^2 j_0(\rho)$, and applying the recurrence formulas for the Bessel functions, they are given by

$$v^{(k,k;2)}(r_1, r_2)=r_1 r_2\{(2k+3)w^{(k-1)}(r_1, r_2)+(2k-1)w^{(k+1)}(r_1, r_2)\}/(2k+1) - (r_1^2+r_2^2)w^{(k)}(r_1, r_2), \tag{16a}$$

and

$$v^{(k,k+2;2)}(r_1, r_2)=r_2^2 w^{(k)}(r_1, r_2)-2r_1 r_2 w^{(k+1)}(r_1, r_2)+r_1^2 w^{(k+2)}(r_1, r_2). \tag{16b}$$

Naturally, $v^{(k_1,k_2;\kappa)}(r_1, r_2)=v^{(k_2,k_1;\kappa)}(r_2, r_1)$ from the definition (7). Here, the $w^{(k)}(r_1, r_2)$ is defined by

$$w^{(k)}(r_1, r_2)=\frac{1}{2}\int_{-1}^{1} V(r)/r^2\cdot P_k(\cos\omega)\,d(\cos\omega),$$

corresponding to (12) for the central interaction.

§ 3. The spin-orbit interactions

Since the two-body spin-orbit interactions explicitly contain the momentum operators $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of the interacting particles, the treatment in the preceding section must be modified. While there are some types of the spin-orbit interactions, we confine ourselves to the Case-Pais type[11] in the following consideration and the procedure given here can be applied also to the other types with slight changes. The interaction is given by

$$V_{LS}=V(r)\,((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot[(\boldsymbol{r}_2-\boldsymbol{r}_1)\times(\boldsymbol{p}_2-\boldsymbol{p}_1)]), \tag{17}$$

where the square bracket without superfix denotes a vector product.

Let us separate the variables $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in the spatial part of (17) as in the preceding section. Since $\boldsymbol{r}_2-\boldsymbol{r}_1=\boldsymbol{r}=rC^{(1)}(\Omega)$, by making use of (6) for $rV(r)C^{(1)}(\Omega)$, and changing the coupling shemes of the tensor operators, we have

$$V(r)[(\boldsymbol{r}_2-\boldsymbol{r}_1)\times(\boldsymbol{p}_2-\boldsymbol{p}_1)]=\sum_{k_1,k_2,K} i^{k_1-k_2}(2k_1+1)(2k_2+1)[2(2K+1)/3]^{1/2} \times(k_1 0 k_2 0|10)\,W(k_1 1\, k_2 1;\, K1)\,u^{(k_1,k_2;1)}(r_1, r_2) \times\{(-)^{1+k_1-K}[[\boldsymbol{C}_1^{(k_1)}\times\boldsymbol{p}_1]^{(K)}\times\boldsymbol{C}_2^{(k_2)}]^{(1)}+[\boldsymbol{C}_1^{(k_1)}\times[\boldsymbol{C}_2^{(k_2)}\times\boldsymbol{p}_2]^{(K)}]^{(1)}\}, \tag{18}$$

where

$$u^{(k_1,k_2;1)}(r_1, r_2)=\frac{2}{\pi}\int_0^\infty j_{k_1}(pr_1)\,j_{k_2}(pr_2)\,p^2\,dp\int_0^\infty rV(r)\,j_1(pr)\,r^2\,dr. \tag{19}$$

The summations over k_1, k_2 and K in (18) are restricted by the conditions: $k_1+k_2=$odd, $|k_1-k_2|\leqq 1\leqq k_1+k_2$, $|k_1-1|\leqq K\leqq k_1+1$ and $|k_2-1|\leqq K\leqq k_2+1$. Therefore, K equals either k_1 or k_2, and (18) can be rewritten as

$$V(r)[(\boldsymbol{r}_2-\boldsymbol{r}_1)\times(\boldsymbol{p}_2-\boldsymbol{p}_1)]=\sum_{k,k'} i^{k+k'}(2k+1)(2k'+1)2^{1/2}(k'010|k0)\,W(k1k1;k'1)$$
$$\times(\{u^{(k,k';1)}(r_1,r_2)[[\boldsymbol{C}_1^{(k)}\times\boldsymbol{p}_1]^{(k)}\times\boldsymbol{C}_2^{(k')}]^{(1)}+u^{(k',k;1)}(r_1,r_2)[\boldsymbol{C}_1^{(k')}\times[\boldsymbol{C}_2^{(k)}\times\boldsymbol{p}_2]^{(k)}]^{(1)}\}$$
$$+\{u^{(k',k;1)}(r_1,r_2)[[\boldsymbol{C}_1^{(k')}\times\boldsymbol{p}_1]^{(k)}\times\boldsymbol{C}_2^{(k)}]^{(1)}-u^{(k,k';1)}(r_1,r_2)[\boldsymbol{C}_1^{(k)}\times[\boldsymbol{C}_2^{(k')}\times\boldsymbol{p}_2]^{(k)}]^{(1)}\}),\tag{18}'$$

where $k'=k\pm1$. The radial parts of the interaction $u^{(k,k';1)}(r_1,r_2)$ can be expressed in terms of those of the central interaction $v^{(k)}(r_1,r_2)$. They are obtained from (19) by making use of $j_1(\rho)=-dj_0(\rho)/d\rho$ in a similar way to the derivation of (16).

$$u^{(k',k;1)}(r_1,r_2)=i^{k-k'+1}\{r_1v^{(k)}(r_1,r_2)-r_2v^{(k')}(r_1,r_2)\},\quad(k'=k\pm1)\tag{20}$$

and $u^{(k,k';1)}(r_1,r_2)=u^{(k',k;1)}(r_2,r_1)$, where $v^{(k)}(r_1,r_2)$ is defined by (12).

Since the operator $[\boldsymbol{C}_i^{(k)}\times\boldsymbol{p}_i]^{(K)}$ cannot be easily handled for the calculations of matrix elements, we rewrite it so as to separate $(\boldsymbol{r}_i\cdot\boldsymbol{p}_i)=i^{-1}r_id/dr_i$. On observing that $\boldsymbol{p}_i=r_i^{-2}\{\boldsymbol{r}_i(\boldsymbol{r}_i\cdot\boldsymbol{p}_i)-[\boldsymbol{r}_i\times\boldsymbol{l}_i]\}$ ($\boldsymbol{l}_i$ are the orbital angular momenta), it can be expressed as

$$ir_i[\boldsymbol{C}_i^{(k)}\times\boldsymbol{p}_i]^{(K)}=(k010|K0)\,\boldsymbol{C}_i^{(K)}r_i\partial/\partial r_i-\boldsymbol{D}_i^{(k,K)},\tag{21}$$

where $\boldsymbol{D}_i^{(k,K)}$ do not involve differential operator and it is defined as

$$\boldsymbol{D}_i^{(k,K)}=2^{1/2}[\boldsymbol{C}_i^{(k)}\times[\boldsymbol{C}_i^{(1)}\times\boldsymbol{l}_i]^{(1)}]^{(K)}.$$

Simpler forms of $\boldsymbol{D}_i^{(k,K)}$ can be obtained by changing the coupling shemes oft he tensor operators:

$$\boldsymbol{D}_i^{(k,k)}=[(2k-1)/(k+1)]^{1/2}[\boldsymbol{C}_i^{(k-1)}\times\boldsymbol{l}_i]^{(k)}=[(2k+3)/k]^{1/2}[\boldsymbol{C}_i^{(k+1)}\times\boldsymbol{l}_i]^{(k)},\tag{22a}$$

and

$$[(2k-1)/(k+1)]^{1/2}\boldsymbol{D}_i^{(k-1,k)}=[(2k+3)/k]^{1/2}\boldsymbol{D}_i^{(k+1,k)}=[\boldsymbol{C}_i^{(k)}\times\boldsymbol{l}_i]^{(k)}.\tag{22b}$$

The spatial part of the interaction is thus given by introducing (20), (21) and (22) into (18)′ and by manipulation with respect to the Clebsch-Gordan and the Racah coefficients involved as

$$V(r)[(\boldsymbol{r}_2-\boldsymbol{r}_1)\times(\boldsymbol{p}_2-\boldsymbol{p}_1)]$$
$$=\sum_k(-)^k(2k+1)\sum_{k'=k\pm1}[(2k'+1)/3]^{1/2}\{(v^{(k)}-v^{(k')}r_2/r_1)[[\boldsymbol{C}_1^{(k)}\times\boldsymbol{l}_1]^{(k')}\times\boldsymbol{C}_2^{(k)}]^{(1)}$$
$$+(v^{(k)}-v^{(k')}r_1/r_2)[\boldsymbol{C}_1^{(k)}\times[\boldsymbol{C}_2^{(k)}\times\boldsymbol{l}_2]^{(k')}]^{(1)}\}$$
$$+\sum_k(-)^k[(2k+1)/3]^{1/2}(\{-(k+1)v^{(k-1)}r_2/r_1+(2k+1)v^{(k)}-kv^{(k+1)}r_2/r_1\}$$
$$\times[[\boldsymbol{C}_1^{(k)}\times\boldsymbol{l}_1]^{(k)}\times\boldsymbol{C}_2^{(k)}]^{(1)}\tag{23}$$
$$-\{-(k+1)v^{(k-1)}r_1/r_2+(2k+1)v^{(k)}-kv^{(k+1)}r_1/r_2\}[\boldsymbol{C}_1^{(k)}\times[\boldsymbol{C}_2^{(k)}\times\boldsymbol{l}_2]^{(k)}]^{(1)})$$
$$+\sum_k(-)^k[k(k+1)(2k+1)/3]^{1/2}\{v^{(k-1)}-v^{(k+1)}\}(r_2\partial/\partial r_1-r_1\partial/\partial r_2)[\boldsymbol{C}_1^{(k)}\times\boldsymbol{C}_2^{(k)}]^{(1)},$$

where $v^{(k)}$ means $v^{(k)}(r_1,r_2)$. This expression is rather complicated at first sight,

but the reduced matrix elements $(l\|[\boldsymbol{C}^{(k')}\times\boldsymbol{l}]^{(k)}\|l')$ is very simple and the angular parts of the matrix elements are easily calculated. Corresponding to $(l\|C^{(k)}\|l')=(2l'+1)^{1/2}(l'0k0|l0)$, the reduced matrix elements of $[\boldsymbol{C}^{(k')}\times\boldsymbol{l}]^{(k)}$ is obtained as

$$(l\|[\boldsymbol{C}^{(k')}\times\boldsymbol{l}]^{(k)}\|l')=[2l'(l'+1)(2l'+1)]^{1/2}(k'011|k1)(l'1k-1|l0).$$
$$(k'=k,\ k\pm1)$$

Especially for l^2 configurations, the contributions from the second and third sums over k in (23) vanish as can readily be seen, and the calculations of the matrix elements become very simple.

§ 4. Radial integrals in terms of the harmonic oscillator wave functions

The radial integrals which are necessary in the evaluation of the interaction matrix elements can be obtained from (7) and (9). Inserting (7) into (9), they are

$$R^{(k_1,k_2;\pi)}(n_1l_1,\ n_2l_2;\ n_1'l_1',\ n_2'l_2')=\int_0^\infty v_\pi(p)\,p^2\,dp$$
$$\times\int_0^\infty R_{n_1l_1}(r_1)R_{n_1'l_1'}(r_1)j_{k_1}(pr_1)r_1^2\,dr_1\int_0^\infty R_{n_2l_2}(r_2)R_{n_2'l_2'}(r_2)j_{k_2}(pr_2)r_2^2\,dr_2,$$

where principal quantum numbers n are explicitly introduced and the expression for $v_\pi(p)$ is given by (5). If the integrations over r_1 and r_2 can be carried out, the radial integrals will be obtained as simple integrals over p. However, it seems difficult to carry out the integrations analytically over r_1 and r_2 for single particle wave functions with arbitrary radial dependence, and we confine ourselves to the harmonic oscillator wave functions which are used quite often for the analysis of the nuclear spectroscopy.

The wave functions of three-dimensional isotropic harmonic oscillator are given as[5]

$$R_{nl}(r)=N_{nl}\exp(-\nu r^2/2)r^l v_{n,l}(\nu r^2), \tag{24}$$

where $\nu=m\omega/\hbar$, $\hbar\omega$ is energy quantum, and m is mass. N_{nl} is the normalization constant:

$$N_{nl}^2=\frac{(2l+2n+1)!!}{2^n n![(2l+1)!!]^2}(2\nu)^{l+3/2}\left(\frac{2}{\pi}\right)^{1/2}. \tag{25}$$

And $v_{n,l}(\rho)$ is the associated Laguerre polynomial:

$$v_{n,l}(\rho)=\sum_\mu\binom{n}{\mu}\frac{(2l+1)!!}{(2l+2\mu+1)!!}(-2\rho)^\mu. \tag{26}$$

In order to carry out the integration $\int_0^\infty R_{nl}(r)R_{n'l'}(r)j_k(pr)r^2dr$ with the har-

monic oscillator wave functions (24), we expand the product of the two associated Laguerre polynomials into power series and define the expansion coefficients $a_m(nl, n'l')$ by

$$v_{n,l}(\rho)v_{n',l'}(\rho)=\frac{(2l+1)!!\,(2l'+1)!!}{(2l+2n+1)!!\,(2l'+2n'+1)!!}\sum_{s=0}^{n+n'}a_{l+l'+2s}(nl,\,n'l')(2\rho)^s. \quad (27)$$

Then, the product of two harmonic oscillator wave function is written down as

$$R_{nl}(r)R_{n'l'}(r)=M(nl,\,n'l')^{-1/2}(2/\pi)^{1/2}\exp(-\nu r^2)\textstyle\sum_m a_m(nl,\,n'l')(2\nu)^{(m+3)/2}r^m, \quad (28)$$

where $m=l+l'+2s$ $(s=0,\,1,\,\cdots,\,n+n')$ and

$$M(nl,\,n'l')=2^{n+n'}n!\,n'!\,(2l+2n+1)!!\,(2l'+2n'+1)!! \quad (29)$$

Thus, by making use of a formula of the integral involving the Bessel function[12]

$$\int_0^\infty \exp(-\nu r^2)j_k(pr)\,r^{m+2}\,dr$$
$$=\frac{(m+k+1)!!}{(2k+1)!!}\left(\frac{\pi}{2}\right)^{1/2}(2\nu)^{-(m+3)/2}\exp\left(-\frac{p^2}{4\nu}\right)\left(\frac{p^2}{2\nu}\right)^{k/2}v_{(m-k)/2,k}\left(\frac{p^2}{4\nu}\right), \quad (30)$$

where $m-k=\text{even}\geqq 0$ and $v_{n,l}(\rho)$ is defined by (26), the integration can be easily carried out.

$$\int_0^\infty R_{nl}(r)R_{n'l'}(r)j_k(pr)r^2\,dr=M(nl,\,n'l')^{-1/2}\textstyle\sum_m a_m(nl,\,n'l')$$
$$\times(m+k+1)!!/(2k+1)!!\cdot\exp(-p^2/4\nu)(p^2/2\nu)^{k/2}v_{(m-k)/2,k}(p^2/4\nu). \quad (31)^*$$

Actually, $(m-k)/2=(l+l'-k)/2+s$ are non-negative integers, since there are factors of the type $(l\|\boldsymbol{C}^{(k)}\|l')$ in (8) which give the conditions $|l-l'|\leqq k\leqq l+l'$ and $l+l'-k$ is even. Then, the radial integral is obtained by the application of (31) for the integrations over r_1 and r_2:

$$R^{(k_1,k_2;\kappa)}(n_1l_1,\,n_2l_2;\,n_1'l_1',\,n_2'l_2')$$
$$=(M_1M_2)^{-1/2}\textstyle\sum_{m_1,m_2}a_{m_1}(n_1l_1,\,n_1'l_1')a_{m_2}(n_2l_2,\,n_2'l_2')f^{(k_1,k_2;\kappa)}(m_1,\,m_2), \quad (32)$$

where the abbreviations $M_i=M(n_il_i,\,n_i'l_i')$ $(i=1,\,2)$ are used, and by taking (27) into account again, $f^{(k_1,k_2;\kappa)}(m_1,\,m_2)$ are given as

---

* In (31), if one takes the limit $p\to 0$, the following expression is obtained.

$$\int_0^\infty R_{nl}(r)\,r^k R_{n'l'}(r)\,r^2\,dr=(2\nu)^{-k/2}M(nl,\,n'l')^{-1/2}\textstyle\sum_m(m+k+1)!!\,a_m(nl,\,n'l').$$

This expression is useful for the calculations of the matrix elements for the nuclear moments, β and γ transitions when the harmonic oscillator wave functions are used as one particle wave functions.

$$f^{(k_1,k_2;\kappa)}(m_1, m_2) \equiv (2\nu)^{\bar{m}+3}\frac{2}{\pi}\int_0^\infty\int_0^\infty v^{(k_1,k_2;\kappa)}(r_1, r_2)$$

$$\times\exp\{-\nu(r_1^2+r_2^2)\}\, r_1^{m_1+2} r_2^{m_2+2}\, dr_1\, dr_2 = \sum_m a_{2m}\left(\frac{m_1-k_1}{2}k_1,\ \frac{m_2-k_2}{2}k_2\right)J_m^{(\kappa)}. \quad (33)$$

The summation over m is restricted by $m=\bar{k}, \bar{k}+1, \cdots \bar{m}$, where $\bar{k}=(k_1+k_1)/2$ and $\bar{m}=(m_1+m_2)/2$. $J_m^{(\kappa)}$ are simple integrals which involve the Fourier transform $v_\kappa(p)$ of the radial dependence $V(r)$ of the interaction

$$J_m^{(\kappa)} = (2\nu)^{-m}\int_0^\infty \exp(-p^2/2\nu)v_\kappa(p)p^{2m+2}\,dp. \quad (34)$$

By (32) and (33), the radial integrals which involve any quartets of quantum numbers nl (naturally $l_1+l_2+l_1'+l_2'=$even from the conservation of parity), can be expressed as linear combinations of simple integrals $J_m^{(\kappa)}$ given by (34). The coefficients which appear in both (32) and (33) are defined by (27). The $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ has a close connection with $R^{(k_1,k_2;\kappa)}(0l_1, 0l_2; 0l_1', 0l_2')$ which is given by (32) when

$$R^{(k_1,k_2;\kappa)}(0l_1, 0l_2; 0l_1', 0l_2')$$
$$=[(2l_1+1)!!\,(2l_2+1)!!\,(2l_1'+1)!!(2l_2'+1)!!]^{-1/2} f^{(k_1,k_2;\kappa)}(l_1+l_1', l_2+l_2'). \quad (35)$$

The above expressions for the radial integrals hold for the central and tensor interactions which were discussed in § 2. For the spin-orbit interactions, one needs further considerations, since there appear two types of integrals, one of which contains the derivatives with respect to r_1 or r_2 and the other r_2/r_1 or r_1/r_2 in the integrand. The calculation of these integrals is given in the Appendix.

§ 5. The coefficients $a_m(nl, n'l')$

The coefficients $a_m(nl, n'l')$ defined by (27) proved very useful in the calculation of the radial integrals in terms of the harmonic oscillator wave functions. In this section, the convenient formulas for the evaluation of the coefficients are derived and those for $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ are obtained.

It is evident from the definitions (26) and (27) that

$$a_{l+l'+2s}(nl, n'l') = (-)^s \sum_{(\mu+\mu'=s)} \binom{n}{\mu}\binom{n'}{\mu'} \begin{matrix}(2l+2n+1)!! & (2l'+2n'+1)!! \\ (2l+2\mu+1)!! & (2l'+2\mu'+1)!!\end{matrix}, \quad (36)^*$$

In the case of $l=l'$, it may be written in another form which is useful in the application to the central interaction in the next section,

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* The coefficients $a_m(nl, n'l')$ have the following relation to $B(nl, n'l', p)$ defined by Brody, Jacob and Moshinski. (Nuclear Phys. **17** (1960), 16)

$$B(nl, n'l', p) = M(nl, n'l')^{-1/2}(2p+1)!!\, a_{2p}(nl, n'l').$$

$$a_{2l+2s}(nl, n'l) = (-)^s \frac{(2l+2n+1)!!\,(2l+2n'+1)!!}{(2l+2s+1)!!}$$
$$\times \sum_{\mu} \frac{2^{\mu} n!\, n'!}{\mu!\,(n-\mu)!\,(n'-\mu)!\,(2l+2\mu+1)!!} \binom{n+n'-2\mu}{s-2\mu}. \tag{37}$$

The coefficients are naturally symmetric with respect to the exchange of their arguments:

$$a_m(nl, n'l') = a_m(n'l', nl). \tag{38}$$

For some special values of n and n', the simple expressions are obtained:

$$a_{l+l'+2s}(0l, 0l') = \delta_{s,0}, \tag{39}$$

and

$$a_{l+l'+2s}(0l, n'l') = (-)^s \binom{n'}{s} \frac{(2l'+2n'+1)!!}{(2l'+2s+1)!!}. \tag{40}$$

Table I. $a_m(nl, n'l')$ $l+l'$=even

nl $n'l'$	$m=0$	2	4	6	8	10	12
00 00	1						
01 01	·	1					
02 00	·	1					
02	·	·	1				
10 00	3	−1					
02	·	3	−1				
10	9	−6	1				
03 01	·	·	1				
03	·	·	·	1			
11 01	·	5	−1				
03	·	·	5	−1			
11	·	25	−10	1			
04 00	·	·	1				
02	·	·	·	1			
10	·	·	3	−1			
04	·	·	·	·	1		
12 00	·	7	−1				
02	·	·	7	−1			
10	·	21	−10	1			
04	·	·	·	7	−1		
12	·	·	49	−14	1		
20 00	15	−10	1				
02	·	15	−10	1			
10	45	−45	13	−1			
04	·	·	15	−10	1		
12	·	105	−85	17	−1		
20	225	−300	130	−20	1		
05 01	·	·	·	1			
03	·	·	·	·	1		
11	·	·	·	5	−1		
05	·	·	·	·	·	1	

Table I. (Continued) $a_m(nl, n'l')$ $l+l'$=even

$nl\ n'l'$	$m=0$	2	4	6	8	10	12
13 01	·	·	9	−1			
03	·	·	·	9	−1		
11	·	·	45	−14	1		
05	·	·	·	·	9	−1	
13	·	·	·	81	−18	1	
21 01	·	35	−14	1			
03	·	·	35	−14	1		
11	·	175	−105	19	−1		
05	·	·	·	35	−14	1	
13	·	·	315	−161	23	−1	
21	·	1225	−980	266	−28	1	
06 00	·	·	·	1			
02	·	·	·	·	1		
10	·	·	·	3	−1		
04	·	·	·	·	·	1	
12	·	·	·	·	7	−1	
20	·	·	·	15	−10	1	
06	·	·	·	·	·	·	1
14 00	·	·	11	−1			
02	·	·	·	11	−1		
10	·	·	33	−14	1		
04	·	·	·	·	11	−1	
12	·	·	·	77	−18	1	
20	·	·	165	−125	21	−1	
06	·	·	·	·	·	11	−1
14	·	·	·	·	121	−22	1
22 00	·	63	−18	1			
02	·	·	63	−18	1		
10	·	189	−117	21	−1		
04	·	·	·	63	−18	1	
12	·	·	441	−189	25	−1	
20	·	945	−900	258	−28	1	
06	·	·	·	·	63	−18	1
14	·	·	·	693	−261	29	−1
22	·	·	3969	−2268	450	−36	1
30 00	105	−105	21	−1			
02	·	105	−105	21	−1		
10	315	−420	168	−24	1		
04	·	·	105	−105	21	−1	
12	·	735	−840	252	−28	1	
20	1575	−2625	1470	−330	31	−1	
06	·	·	·	105	−105	21	−1
14	·	·	1155	−1260	336	−32	1
22	·	6615	−8505	3318	−546	39	−1
30	11025	−22050	15435	−4620	651	−42	1

Actually, we used (39) in obtaining (35). Furthermore, the following recurrence formulas hold between the coefficients*

* The recurrence formulas between the coefficients $a_m(nl, n'l')$ are easily derived from those between the associated Laguerre polynomials:

$$(2n+2l+1)v_{n,l}(\rho) = (2l+1)v_{n,l-1}(\rho) + 2n v_{n-1,l}(\rho),$$
$$2\rho v_{n,l}(\rho) = (2l+1)\{v_{n,l-1}(\rho) - v_{n+1,l-1}(\rho)\},$$

and the definition of the coefficients (27).

Table II. $a_m(nl, n'l')$ $l+l'=$odd

nl $n'l'$	$m=1$	3	5	7	9	11
01 00	1					
02 01	·	1				
10 01	3	−1				
03 00	·	1				
02	·	·	1			
10	·	3	−1			
11 00	5	−1				
02	·	5	−1			
10	15	−8	1			
04 01	·	·	1			
03	·	·	·	1		
11	·	·	5	−1		
12 01	·	7	−1			
03	·	·	7	−1		
11	·	35	−12	1		
20 01	15	−10	1			
03	·	15	−10	1		
11	76	−65	15	−1		
05 00	·	·	1			
02	·	·	·	1		
10	·	·	3	−1		
04	·	·	·	·	1	
12	·	·	·	7	−1	
20	·	·	15	−10	1	
13 00	·	9	−1			
02	·	·	9	−1		
10	·	27	−12	1		
04	·	·	·	9	−1	
12	·	·	63	−16	1	
20	·	135	−105	19	−1	
21 00	35	−14	1			
02	·	35	−14	1		
10	105	−77	17	−1		
04	·	·	35	−14	1	
12	·	245	−133	21	−1	
20	525	−560	190	−24	1	
06 01	·	·	·	1		
03	·	·	·	·	1	
11	·	·	·	5	−1	
05	·	·	·	·	·	1
13	·	·	·	·	9	−1
21	·	·	·	35	−14	1
14 01	·	·	11	−1		
03	·	·	·	11	−1	
11	·	·	55	−16	1	
05	·	·	·	·	11	−1
13	·	·	·	99	−20	1
21	·	·	385	−189	25	−1
22 01	·	63	−18	1		
03	·	·	63	−18	1	
11	·	315	−153	23	−1	
05	·	·	·	63	−18	1
13	·	·	567	−225	27	−1
21	·	2205	−1512	350	−32	1

Table II. (Continued) $a_m(nl, n'l')$ $l+l'$=odd

$nl\,n'l'$	m=1	3	5	7	9	11
30 01	105	−105	21	−1		
03	·	105	−105	21	−1	
11	525	−630	210	−26	1	
05	·	·	105	−105	21	−1
13	·	945	−1050	294	−30	1
21	3675	−5145	2310	−434	35	−1

$$a_m(nl, n'l') = 2n a_m(n-1\,l, n'l') + a_{m-1}(nl-1, n'l'), \tag{41a}$$

and

$$a_m(nl, n'l') = (2l+3)\,a_{m+1}(n-1\,l+1, n'l') - a_m(n-1\,l+2, n'l'). \tag{41b}$$

These recurrence formulas are quite useful in obtaining the values of the $a_m(nl, n'l')$ starting from (39) or (40), although it is not difficult to calculate them by the direct application of (36) or (37).

In Tables I and II, the values of the coefficients $a_m(nl, n'l')$, which are necessary for the evaluation in (32), are listed for $l+l'$ even and odd, respectively.

The properties of the $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ can also be obtained from those of the $a_m(nl, n'l')$ by (33). First of all, from (38), the symmetry property of the $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ with respect to the exchange of their arguments is

$$f^{(k_1,k_2;\kappa)}(m_1, m_2) = f^{(k_2,k_1;\kappa)}(m_2, m_1). \tag{38}'$$

For the special cases of $m_1=k_1$ and $m_2=k_2$, we have, corresponding to (39),

$$f^{(m_1,m_2;\kappa)}(m_1, m_2) = J_{\bar{m}}^{(\kappa)}, \tag{39}'$$

and, for $m_1=k_1$,

$$f^{(m_1,k_2;\kappa)}(m_1, m_2) = \sum_{\mu} (-)^{\mu} \binom{(m_2-k_2)/2}{\mu} \frac{(m_2+k_2+1)!!}{(2k_2+2\mu+1)!!} J_{\bar{m}+\mu}^{(\kappa)}. \tag{42}$$

Four recurrence formulas for the $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ are obtained from those between the coefficients $a_m(nl, n'l')$ with common suffices m, which are derived from (41a) and (41b). Those formulas are, for example,

$$\begin{aligned} f^{(k_1,k_2;\kappa)}(m_1, m_2) &= (m_1+k_1+1) f^{(k_1,k_2;\kappa)}(m_1-2, m_2) - f^{(k_1+1,k_2+1;\kappa)}(m_1-1, m_2+1) \\ &\quad + (m_2-k_2) f^{(k_1+1,k_2+1;\kappa)}(m_1-1, m_2-1), \\ f^{(k_1,k_2;\kappa)}(m_1, m_2) &= (m_1+k_1+1) f^{(k_1,k_2;\kappa)}(m_1-2, m_2) \\ &\quad + (m_1-k_1-2) f^{(k_1+2,k_2;\kappa)}(m_1-2, m_2) - f^{(k_1+2,k_2;\kappa)}(m_1, m_2), \end{aligned} \tag{43}$$

which hold irrespective of the values of κ.

§ 6. The integrals $J_m^{(\kappa)}$

As was shown in § 4, any radial integral can be expanded into a linear combination of the simple integrals $J_m^{(\kappa)}$, if we assume the harmonic oscillator

wave function for the radial wave function. $J_m^{(\kappa)}$ was defined by (34) and it is an integral containing the Fourier transform $v_\kappa(p)$ of the interaction.

Without the knowledge of the radial dependence of interaction $V(r)$, we can obtain the relation between the $J_m^{(\kappa)}$ and the Talmi integral I_l, where I_l is defined as[5]

$$I_l=\int_0^\infty V(\sqrt{2}\,r)R_{0l}^2(r)r^2dr. \tag{44}$$

I_l may be written more explicitly by employing (24) and changing the integration variable

$$I_l=\left(\frac{2}{\pi}\right)^{1/2}\frac{\nu^{l+3/2}}{(2l+1)!!}\int_0^\infty V(r)\exp\left(-\frac{\nu r^2}{2}\right)r^{2l+2}dr. \tag{44}'$$

On the other hand, the $J_m^{(\kappa)}$ may be expressed in terms of $V(r)$, by the introduction of the Fourier inverse transform (5) and carrying out the integration over p by (30)

$$J_m^{(\kappa)}=\frac{\nu^{(\kappa+3)/2}}{2^m}\frac{(2m+\kappa+1)!!}{(2\kappa+1)!!}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty V(r)\exp\left(-\frac{\nu r^2}{2}\right)v_{m-\kappa/2,\kappa}\left(\frac{\nu r^2}{2}\right)r^{\kappa+2}dr.$$

Then, expanding $v_{m-\kappa/2,\kappa}(\nu r^2/2)$ into a series of r^2 by (26), it is easy to obtain the relation between $J_m^{(\kappa)}$ and I_l:

$$J_m^{(\kappa)}=\frac{(2m+\kappa+1)!!}{2^m}\sum_l(-)^l\binom{m-\kappa/2}{l}\frac{(2l+\kappa+1)!!}{(2l+2\kappa+1)!!}I_{l+\kappa/2}. \tag{45}$$

For the central interaction ($\kappa=0$), the above relation is simplified:

$$J_m^{(0)}=\frac{(2m+1)!!}{2^m}\sum_l(-)^l\binom{m}{l}I_l. \tag{46}$$

In this case, I_l may also be expressed in terms of $J_m^{(0)}$ from (46)

$$I_l=\sum_m(-)^m\binom{l}{m}\frac{2^m}{(2m+1)!!}J_m^{(0)}.$$

For the tensor interaction ($\kappa=2$), the relation is given by

$$J_m^{(2)}=\frac{(2m+3)!!}{2^m}\sum_l(-)^l\binom{m-1}{l}\frac{1}{2l+5}I_{l+1}. \tag{47}$$

It is apparent that the radial integral of the tensor interaction does not involve I_0. From (46) and (47), the relation between the $J_m^{(2)}$ and $J_m^{(0)}$ may be obtained:

$$J_m^{(2)}=\frac{3}{2}\sum_{m'=0}^{m-1}\frac{(m-1)!}{m'!}J_{m'}^{(0)}-J_m^{(0)}. \tag{48}$$

The radial integral $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ may be expanded in terms of I_l instead of $J_m^{(\kappa)}$ in (33). By the relation (45) between $J_m^{(\kappa)}$ and I_l it becomes

$$f^{(k_1,k_2;\kappa)}(m_1, m_2) = \sum_l (-)^l \frac{(2l+\kappa+1)!!}{(2l+2\kappa+1)!!} \sum_m \frac{(2m+\kappa+1)!!}{2^m} \binom{m-\kappa/2}{l} \times a_{2m}\left(\frac{m_1-k_1}{2}k_1, \frac{m_2-k_2}{2}k_2\right) I_{l+\kappa/2}. \tag{49}$$

For the central interaction, by putting $k_1=k_2=k$ and inserting (37) into (49), we have

$$f^{(k)}(m_1, m_2) \equiv f^{(k,k;0)}(m_1, m_2) = (m_1+k+1)!!\,(m_2+k+1)!! \sum_l (-)^l \sum_m (-)^{m-k} \times \frac{1}{2^m}\binom{m}{l} \sum_\mu \frac{2^\mu[(m_1-k)/2]!\,[(m_2-k)/2]!}{\mu![(m_1-k)/2-\mu]!\,[(m_2-k)/2-\mu]!\,(2k+2\mu+1)!!} \binom{\bar{m}-k-2\mu}{m-k-2\mu} I_l. \tag{50}$$

This form of the expansion has been obtained by Ford and Konopinski[6] from the consideration of Gaussian type interaction. If I_l is proportional to z^l (z is a variable), the summations over l and m in (50) can be carried out by putting $I_l \sim z^l$:

$$f^{(k)}(m_1, m_2|z) \sim \frac{(m_1+k+1)!!\,(m_2+k+1)!!}{(2k+1)!!}\left(\frac{1+z}{2}\right)^{\bar{m}}\left(\frac{1-z}{1+z}\right)^k \times {}_2F_1\left((k-m_1)/2, (k-m_2)/2\,;\, k+\frac{3}{2}\,;\, \left(\frac{1-z}{1+z}\right)^2\right), \tag{51}$$

where ${}_2F_1(\alpha, \beta\,;\gamma\,;\,x)$ is a hypergeometric function. Some other forms of $f^{(k)}(m_1, m_2|z)$ can be obtained by the transformation of the hypergeometric function and those of $f^{(k)}(m_1, m_2)$ can also be derived by expanding them into power series of z.

Now we calculate the explicit formulas for the integral $J_m^{(0)}$ and I_l from (5), (34) and (45) for some radial dependences which are used for the phenomenological analysis in the nuclear spectroscopy. The simplest cases are the short- and long-range limits of the radial dependences. In these cases, the values of $J_m^{(0)}$, $J_m^{(2)}$, I_l and $f^{(k)}(m_1, m_2)$ are listed in Table III.

Table III. The short- and long-range limits

	short-range limit	long-range limit
$V(r)$	$\delta(r)/r^2$	1
$J_m^{(0)}$	$(2m+1)!!/2^m\cdot(2\nu^3/\pi)^{1/2}$	$\delta_{m,0}$
$J_m^{(2)}$	0	$3/2\cdot(m-1)!$
I_l	$\delta_{l,0}(2\nu^3/\pi)^{1/2}$	1
$f^{(k)}(m_1, m_2)$	$(2\bar{m}+1)!!/2^{\bar{m}}\cdot(2\nu^3/\pi)^{1/2}$	$(m_1+1)!!\,(m_2+1)!!\,\delta_{k,0}$

The values of $f^{(k)}(m_1, m_2)$ may also be obtainable from (51) by putting $z=0$ and 1 for the short- and long-range limits, respectively. For the interactions with finite range, we consider Gaussian and Yukawa potentials.

i) Gaussian potential: $V(r)=\exp(-r^2/r_0^2)$ (r_0=force range)

$$J_m^{(0)}=(2m+1)!!/2^m\cdot\lambda^3(1+\lambda^2)^{-(m+3/2)}, \tag{52}$$

and

$$I_l=[\lambda^2/(1+\lambda^2)]^{l+3/2}, \tag{53}$$

where $\lambda=r_0(\nu/2)^{1/2}$ is the non-dimensional range parameter. Hence, by assuming $z=\lambda^2/(1+\lambda^2)$, (51) can be used for the radial integrals.[6]

ii) Yukawa potential: $V(r)=\exp(-r/r_0)/(r/r_0)$

We introduce the parameters $\mu=1/(2\lambda)$ and $\beta=p(2\nu)^{-1/2}$. Then, from (5) and (34), we obtain

$$J_m^{(0)}=\frac{2}{\pi\mu}\int_0^\infty \exp(-\beta^2)\frac{\beta^{2m+2}}{\beta^2+\mu^2}\,d\beta.$$

For these integrals, we get a recurrence formula

$$J_m^{(0)}=(2m-1)!!/2^m\cdot\mu^{-1}\pi^{-1/2}-\mu^2 J_{m-1}^{(0)}, \quad (m\geqq 0)$$

and for $m=-1$,

$$J_{-1}^{(0)}=\mu^{-2}\{1-\Phi(\mu)\}\exp(\mu^2),$$

where $\Phi(\mu)=2/\pi^{1/2}\cdot\int_0^\mu \exp(-t^2)\,dt$ is an error function. From these relations, the expression for $J_m^{(0)}$ may be obtained:

$$J_m^{(0)}=\frac{1}{\mu\sqrt{\pi}}\sum_{n=0}^{m}\frac{(2m-2n-1)!!}{2^{m-n}}(-\mu^2)^n-(-\mu^2)^m\{1-\Phi(\mu)\}\exp(\mu^2), \tag{54}$$

although the direct application of the above recurrence formula and the initial value seems more practical in the numerical calculation of $J_m^{(0)}$. The expression for I_l can be written down by making use of (46) and $J_m^{(2)}$ are obtainable from (48).

§ 7. An example: The $(0d)(1s)$ configuration

In order to illustrate the systematic procedure to be followed when we want to calculate the matrix elements, we take the simple configuration $(0d)(1s)$. The radial integrals, which are necessary for both the central and the tensor interactions, are reduced to the $f^{(k_1,k_2;\kappa)}(m_1, m_2)$ by (32).

$$R^{(k_1,k_2;\kappa)}(0d1s;\,0d1s)=\{9f^{(k_1,k_2;\kappa)}(4,0)-6f^{(k_1,k_2;\kappa)}(4,2)+f^{(k_1,k_2;\kappa)}(4,4)\}/90,$$

and

$$R^{(k_1,k_2;\kappa)}(0d1s;\,1s0d)$$
$$=\{9f^{(k_1,k_2;\kappa)}(2,2)-3f^{(k_1,k_2;\kappa)}(4,2)-3f^{(k_1,k_2;\kappa)}(2,4)+f^{(k_1,k_2;\kappa)}(4,4)\}/90.$$

Therefore, for the central interaction, the direct integral $F^0(0d, 1s)$ and the exchange integral $G^2(0d, 1s)$ are given by (33) and Table I as follows:

$$F^0(0d, 1s) = R^{(0,0;0)}(0d1s\,;\,0d1s) = \{9(15J_0 - 10J_1 + J_2) - 6(45J_0 - 45J_1 + 13J_2 - J_3) + (225J_0 - 300J_1 + 130J_2 - 20J_3 + J_4)\}/90,$$

and

$$G^2(0d, 1s) = 5R^{(2,2;0)}(0d1s\,;\,1s0d) = 5\{9J_2 - 6(7J_2 - J_3) + (49J_2 - 14J_3 + J_4)\}/90,$$

where we write J_m instead of $J_m{}^{(0)}$.

For the tensor interaction, the direct integral $R^{(2,0;2)}(0d1s\,;\,0d1s)$ and the exchange integral $R^{(2,2;2)}(0d1s, 1s0d)$ are given in the same way as follows:

$$R^{(2,0;2)}(0d1s\,;\,0d1s) = \{9(7J_1{}^{(2)} - J_2{}^{(2)}) - 6(21J_1{}^{(2)} - 10J_2{}^{(2)} + J_3{}^{(2)}) + (105J_1{}^{(2)} - 85J_2{}^{(2)} + 17J_3{}^{(2)} - J_4{}^{(2)})\}/90,$$

and

$$R^{(2,2;2)}(0d1s\,;\,1s0d) = \{9J_2{}^{(2)} - 6(7J_2{}^{(2)} - J_3{}^{(2)}) + (49J_2{}^{(2)} - 14J_3{}^{(2)} + J_4{}^{(2)})\}/90.$$

The relations between $J_m{}^{(2)}$ and $J_m{}^{(0)}$ (which are denoted as J_m in this section) are given by (48):

$$J_1{}^{(2)} = 3/2 \cdot J_0 - J_1,$$
$$J_2{}^{(2)} = 3/2(J_0 + J_1) - J_2,$$
$$J_3{}^{(2)} = 3/2(2J_0 + 2J_1 + J_2) - J_3,$$
$$J_4{}^{(2)} = 3/2(6J_0 + 6J_1 + 3J_2 + J_3) - J_4.$$

In the above expression, J_m may be rewritten in terms of the Talmi integral I_l by the use of (46).

Appendix

The radial integrals for the spin-orbit interaction

For the calculation of the matrix elements of the spin-orbit interaction, it is seen from (23) that the radial integrals which involve $v^{(k)}(r_1, r_2)$, $v^{(k')}(r_1, r_2)r_2/r_1$ and $\{v^{(k-1)}(r_1, r_2) - v^{(k+1)}(r_1, r_2)\}(r_2\partial/\partial r_1 - r_1\partial/\partial r_2)$ are necessary. The first integral is given in (32) as

$$\int_0^\infty\int_0^\infty R_{n_1 l_1}(r_1)R_{n_2 l_2}(r_2)v^{(k)}(r_1, r_2)R_{n_1' l_1'}(r_1)R_{n_2' l_2'}(r_2)r_1^2\,dr_1 r_2^2\,dr_2 = (M_1 M_2)^{-1/2}\sum_{m_1, m_2} a_{m_1}(n_1 l_1, n_1' l_1')a_{m_2}(n_2 l_2, n_2' l_2')f^{(k)}(m_1, m_2),$$

where M_i, $a_{m_i}(n_i l_i, n_i' l_i')$ $(i=1, 2)$ and $f^{(k)}(m_1, m_2)$ are defined by (29), (27) and (33), respectively. The second integral is also simple and it can be easily calculated in quite a similar way to the first one.

$$\int_0^\infty \int_0^\infty R_{n_1 l_1}(r_1) R_{n_2 l_2}(r_2) v^{(k')}(r_1, r_2) r_2/r_1 R_{n_1' l_1'}(r_1) R_{n_2' l_2'}(r_2) r_1^2 dr_1 r_2^2 dr_2$$
$$= (M_1 M_2)^{-1/2} \sum_{m_1, m_2} a_{m_1}(n_1 l_1, n_1' l_1') a_{m_2}(n_2 l_2, n_2' l_2') f^{(k')}(m_1-1, m_2+1).$$

The calculation of the third integral which involve differentiations with respect to r_1 and r_2 is a littel more tedious. By taking into account

$$(d/dr - l/r) R_{nl}(r) = -(\nu/2)^{1/2} \{(2l+2n+3)^{1/2} R_{n,l+1}(r) + (2n)^{1/2} R_{n-1,l+1}(r)\},$$

and calculating by the similar procedure as above, we have

$$\int_0^\infty \int_0^\infty R_{n_1 l_1}(r_1) R_{n_2 l_2}(r_2) v^{(k')}(r_1, r_2) \{(r_2 \partial/\partial r_1 - r_1 \partial/\partial r_2)$$
$$\times R_{n_1' l_1'}(r_1) R_{n_2' l_2'}(r_2)\} r_1^2 dr_1 r_2^2 dr_2$$
$$= (M_1 M_2)^{-1/2} \sum_{m_1, m_2} [\{l_1' a_{m_1}(n_1 l_1, n_1' l_1') - a_{m_1-1}(n_1 l_1, n_1' l_1'+1)\}$$
$$\times a_{m_2}(n_2 l_2, n_2' l_2') f^{(k')}(m_1-1, m_2+1)$$
$$- a_{m_1}(n_1 l_1, n_1' l_1') \{l_2' a_{m_2}(n_2 l_2, n_2' l_2')$$
$$- a_{m_2-1}(n_2 l_2, n_2' l_2'+1)\} f^{(k')}(m_1+1, m_2-1)].$$

Note added in proof: Tables I and II are sufficient in the evaluation of (32) up to the $N=6$ shell, but, in the evaluation of (33), they are not enough for the shells higher than $N=3$. Tables which can be applied up to the $N=6$ shell for the evaluation of (33) will be published in the Bulletin of the Tokyo Institute of Technology.

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Photopion Production and $(\gamma, 3\pi)$ Interaction

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The photoproduction of a neutral pion is investigated to obtain information on the $(\gamma, 3\pi)$ interaction. An amplitude of photoproduction is obtained in terms of the isovector form factors of nucleon and a coupling constant representing the above interaction. This interaction is found to have a fairly large effect on the angular distribution of the neutral pion production, especially on the coefficient of $\cos\theta$. Comparisons with experiments impose a crude limitation on the magnitude of the coupling constant.

§ 1. Introduction

The purpose of this paper is to investigate the angular distribution of the photopion production from the proton, in order to get some information on the $(\gamma, 3\pi)$ interaction, or, on the reaction

$$\gamma+\pi\rightarrow\pi+\pi. \qquad \text{(A)}$$

Recently many theoretical attempts have been made on the problem of the electromagnetic structure of the nucleon. As for its isovector part, the dispersion theoretical approaches[1),2),3)] have been quite successful. Indeed, if the assumed pion-pion resonance in $I=J=1$ state is found in experiments, we may say to have understood the essential features of the isovector structure, at least as far as the experimental data are now available.[4)] On the other hand, on the isoscalar part of the structure no satisfactory attempts have been made yet. In the dispersion theory of the electromagnetic structure, the most dominant contribution to the spectral function of the isoscalar form factors will be that of the three-pion intermediate state with $I=0$, $J=1$ and odd parity, which is the least massive intermediate state that connects the nucleon-antinucleon pair state and the virtual photon state. But we have neither experimental nor theoretical knowledge at all on the matrix element $\langle 0|\, j_\mu|3\pi\rangle$ which we need in calculating the above spectral functions. This is one of the chief reasons why we cannot perform any meaningful calculations.

Consequently, it will be desirable to obtain some information on the matrix element from some other phenomena. In general the matrix element $\langle 0|\, j_\mu|3\pi\rangle$ is expressed in terms of an invariant function of k^2 and two other variables, where k is the four-momentum of the virtual photon. When $k^2=0$, i.e. the photon is real, the function is the amplitude of the reaction (A). It will be adequate to

study this function first, because the knowledge about it will be obtained to some extent from the analysis of the single or double photopion production experiments. Investigation of the function with non-vanishing k^2 remains as a future problem.

Here we take up the photopion production of a neutral pion from the proton. The experimental data on the photopion production in the energy region up to the first (3, 3) resonance have been explained fairly well by the static theory[5] and the relativistic dispersion theory[6] which do not take into account the effect of the $(\gamma, 3\pi)$ interaction. Therefore the inclusion of the effect of this interaction must not change the status largely. Especially, in case of the neutral pion production, the amplitude usually considered is very simple, so that we shall probably be able to obtain the upper limit of the strength of the $(\gamma, 3\pi)$ interaction from the study of the angular distribution in this energy region.

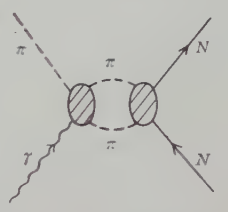


Fig. 1. The simplest Feynman diagram of photopion production that involves the $(\gamma, 3\pi)$ interaction.

In Fig. 1 we show the simplest photoproduction diagram that involves the $(\gamma, 3\pi)$ interaction. Crudely speaking, we express this interaction by an effective Hamiltonian[7]

$$H_{\gamma 3\pi}=i4\pi e\lambda\varepsilon_{\mu\nu\sigma\lambda}\frac{1}{\mu^3}A_\mu\frac{\partial\varphi^1}{\partial x_\nu}\frac{\partial\varphi^2}{\partial x_\sigma}\frac{\partial\varphi^3}{\partial x_\lambda}\equiv -j_\mu^{(3\pi)}A_\mu, \qquad (1)^*$$

and try to obtain the upper limit of the coupling constant λ.

It is also interesting to see if this interaction plays an important role in the photopion production in the higher energy region (above the first resonance). Indeed the position of the singularity of the amplitude in the $\cos\theta$ plane[8] due to the $(\gamma, 3\pi)$ interaction approaches the physical range nearer for the higher incident energies. Furthermore, there has been a hope that the electric dipole gamma ray, which is responsible for the second $d_{3/2}$ $(I=1/2)$ resonance in the photopion production, may be absorbed with this interaction.[9] We shall briefly discuss the photoproduction in the high energy region.

In § 2 we shall state about the general features of the amplitude of the reaction (A), and also about our method of the approximation. In § 3 the photopion production amplitude will be shown to be expressed in terms of λ and the (isovector) electromagnetic form factors of the nucleon. In § 4 the angular distribution of the photoproduction will be calculated and compared with experiment. In the Appendix we shall discuss the application of the one-level formula to the second resonance of pion-nucleon scattering, in order to guess about the d-wave phase shift.

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* We use the unrationalized unit for the coupling constants e, f and λ throughout this paper, i.e. $e^2=1/137$ and $f^2\simeq 0.08$.

§ 2. Amplitude of the reaction $\gamma+\pi\to\pi+\pi$

We first consider the amplitude of the reaction (A). The assignments of four-momenta, charges, and polarization to each particle are shown in Fig. 2. Here all the four-momenta correspond formally to the incoming particles.[8] The amplitude is written as follows:[2]

$$S_{fi}=\delta_{fi}-2\pi i\delta(p_1+p_2+p_3+k)(16\,w_1w_2w_3k)^{-1/2}R,$$

$$R=-(8w_1w_2w_3)^{1/2}\varepsilon_\mu\langle -p_2\beta\,;\,-p_3\gamma|j_\mu^{(S)}|p_1\alpha\rangle$$

$$=\frac{4\pi e}{\sqrt{2}\,\mu^3}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_\mu p_\nu^{\,1}p_\sigma^{\,2}p_\lambda^{\,3}A(s_1,s_2,s_3). \qquad (2)$$

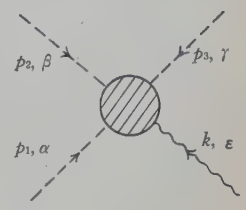


Fig. 2. Feynman diagram of the reaction (A).

The cross section in the center of mass system is

$$\frac{d\sigma}{d\Omega}=\frac{p}{k}\left|\frac{R}{16\pi w_p}\right|^2.$$

As is the case with the pion-pion scattering,[10] the amplitude A is an invariant function of the following three invariant variables:

$$s_1=-(k+p_1)^2=4w_p^{\,2},$$

$$s_2=-(k+p_2)^2=\mu^2-2kw_p+2kp\cos\theta',$$

$$s_3=-(k+p_3)^2=\mu^2-2kw_p-2kp\cos\theta', \qquad (3)$$

where $\boldsymbol{p}=-\boldsymbol{p}_2=\boldsymbol{p}_3$ $(w_p^{\,2}=\boldsymbol{p}^2+\mu^2)$ and θ' is the angle between $\boldsymbol{k}$ and $\boldsymbol{p}$, both in the center of mass system. These three variables have a relation

$$s_1+s_2+s_3=3\mu^2,$$

so that only two of them are independent.

We assume here that the amplitude $A(s_1,s_2,s_3)$ satisfies the Mandelstam representation.[11] From the viewpoint of the perturbation theory no subtractions seem to be necessary. We, however, use the representation with one subtraction for the practical convenience; that is, since there appears at least one nucleon loop in any Feynman diagrams representing the reaction (A) (we neglect the current of strange particles here), it is useful to represent the effect of higher mass singularities $(s\geq(2m)^2)$ by a subtraction constant. We introduce, therefore, as in the pion-pion scattering a coupling constant λ by the value of the amplitude at the symmetrical point $s_1=s_2=s_3=\mu^2$:

$$\lambda=A(\mu^2,\mu^2,\mu^2).$$

Considering the fact that $A(s_1,s_2,s_3)$ is a totally symmetric function of s_1, s_2 and s_3 owing to the crossing relations, we have the following representation:

$$A(s_1,s_2,s_3)=\lambda+\sum_i\frac{s_i-\mu^2}{\pi}\int ds'\frac{f(s')}{(s'-\mu^2)(s'-s_i)}$$

$$+\sum_{(i,j)}\frac{(s_i-\mu^2)(s_j-\mu^2)}{\pi^2}\int ds'\,dt'\,\frac{\rho(s',t')}{(s'-\mu^2)(s'-s_i)(t'-\mu^2)(t'-s_j)}. \tag{4}$$

In the reaction (A), the final pions are produced only in odd l states, with the absorption of magnetic l-th pole gamma rays. If only the singularities due to the intermediate two pion states with $l=1$ are kept in Eq. (4), we have a plausible form of solution of Eq. (4) :

$$A(s_1,s_2,s_3)=\lambda\exp\{u(s_1)+u(s_2)+u(s_3)\}, \tag{5}$$

where

$$u(s)=\frac{s-\mu^2}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\delta(s')\,ds'}{(s'-\mu^2)(s'-s-i\varepsilon)}. \tag{6}$$

$\delta(s)$ is the phase shift of the p-wave pion-pion scattering with the total energy $\sqrt{s}$. This form is analogous to the pion form factor introduced by Federbush, Goldberger and Treiman[2] and by Frazer and Fulco,[3] and has all the properties desired, i.e. the position of singularities, the phase, and the symmetry property.

In the next section, we need the projection of $A(s_1,s_2,s_3)$ into the $l=1$ state,

$$A^{(1)}(s_1)=\frac{3}{4}\int_{-1}^{1}d\cos\theta'\,\sin^2\theta'\,A(s_1,s_2,s_3). \tag{7}$$

Remembering that the pion form factor $F_\pi(s)$ is expressed[2),3)] as

$$F_\pi(s)=\exp\left\{\frac{s}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\delta(s')\,ds'}{s'(s'-s-i\varepsilon)}\right\}, \tag{8}$$

we can write

$$A^{(1)}(s_1)=\frac{\lambda F_\pi(s_1)}{F_\pi(\mu^2)}C(s_1), \tag{9}$$

where

$$C(s_1)=\frac{3}{4}\int_{-1}^{1}d\cos\theta'\,\sin^2\theta'\,e^{u(s_2)+u(s_3)}. \tag{10}$$

The explicit form of $C(s_1)$ cannot be obtained until the pion-pion phase shift $\delta(s)$ is known. As an example, we show in Fig. 3 $C(s_1)$ which has been calculated from the approximate pion-pion amplitude of Frazer and Fulco,[3] .

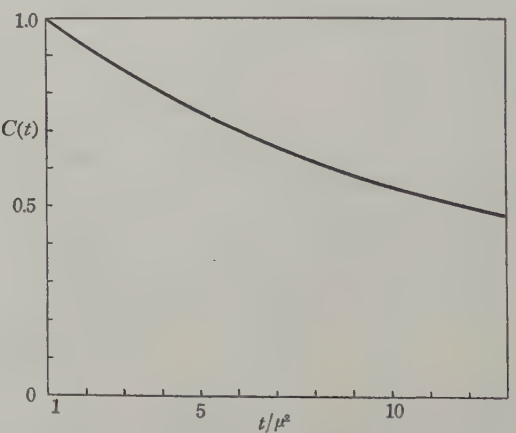


Fig. 3. The function $C(t)$ which shows the effect of the so-called "left-hand singularities" on the amplitude of the reaction (A).

$$f_{\pi\pi}(s)=\left(\frac{\nu+1}{\nu^3}\right)^{1/2}e^{i\delta}\sin\delta=\frac{\Gamma}{\nu_r-\nu[1-\Gamma\alpha(\nu)]-i\Gamma[\nu^3/\nu+1]^{1/2}},$$

where $\nu=(s/4\mu^2)-1$. For other notations see reference 3). We have used the relation[3]

$$\exp u(s)=\frac{s+s_0}{\mu^2+s_0}\frac{f_{\pi\pi}(s)}{f_{\pi\pi}(\mu^2)},$$

where $-s_0$ is the position of the pole of $f_{\pi\pi}(s)$. The values of parameters are $\nu_r=1.5$, $\Gamma=0.4$ and $s_0=2612\mu^2$.

In the next section we approximate $C(s_1)$ by unity. This corresponds to neglecting all the singularities except that of s_1 in Eq. (4), and is analogous to the ladder approximation adopted in the pion-pion scattering problem by Chew and Mandelstam,[10] by Okubo[12] and by Miyamoto.[13]

Finally we notice that, if we calculate the "coupling constant" by the lowest order perturbation theory (Fig. 4) and take the limit of infinitely heavy nucleon mass, we obtain the value

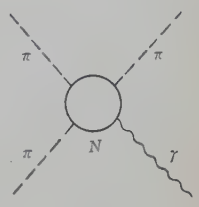


Fig. 4. The lowest order Feynman diagram for the reaction (A).

$$\lambda_{pert}=\frac{8}{\pi}f^3\approx 0.057, \qquad (11)$$

where the sign is same as that of f.

§ 3. Amplitude of photopion production

Let us investigate now the amplitude of photopion production via the above discussed $(\gamma, 3\pi)$ interaction. In general, the photoproduction amplitudes are divided[6] into three parts, $T^{(+)}$, $T^{(-)}$ and $T^{(0)}$, which are proportional to $\delta_{\alpha 3}$, $1/2\cdot[\tau_\alpha, \tau_3]$ and τ_α, respectively. Here α is the charge variable of the outgoing pion. We are interested in the part due to the isoscalar current, i.e. $T^{(0)}$:

$$T^{(0)}=-\left(\frac{2\omega_q E_p E_{p'}}{16\pi^2 m^2}\right)^{1/2}\varepsilon_\mu\langle p'; q\alpha|j_\mu^{(S)}|p\rangle, \qquad (12)$$

where $(\boldsymbol{p}, E_p)$, $(\boldsymbol{p}', E_p')$, $(\boldsymbol{q}, \omega_q)$ and $(\boldsymbol{k}, k)$ are the momenta and energies of the initial and final nucleons, outgoing pion and incident gamma ray, respectively. ε_μ is the polarization of the gamma ray. The cross section is

$$\frac{d\sigma}{d\Omega}=\frac{q}{k}|T|^2.$$

We have neglected the relativistic factor $m/(E_{p'}+\omega_q)$ in Eq. (12). As is shown in reference 6), $T^{(0)}$ is expressed in terms of four invariant amplitudes, each of which is again the function of the three variables, $s=-(k+p)^2$, $t=-(k-p)^2$ and $\bar{t}=-(k-p')^2$, where

$$s+t+\bar{t}=2m^2+\mu^2.$$

First we explain the essential points of our argument in a crude but intuitive way. As stated in § 1, we consider the Feynman diagram shown in Fig. 1, and replace the $(\gamma, 3\pi)$ interaction in it by the effective Hamiltonian Eq. (1). Then the photoproduction amplitude corresponding to the diagram, which we call $T^{(2\pi)}$, becomes

$$T^{(2\pi)}=-\left(\frac{2w_qE_pE_{p'}}{16\pi^2m^2}\right)^{1/2}\varepsilon_\mu\langle p';\, q\alpha|j_\mu^{(2\pi)}|p\rangle. \tag{13}$$

Further we take into account only the pion-pion interaction between the two "internal" pion lines in the diagram. This corresponds to putting $C(s)=1$ (see Eq. (9)). Then

$$\begin{aligned}T^{(2\pi)}&=\left(\frac{E_pE_{p'}}{m^2}\right)^{1/2}\frac{e\lambda}{\mu^3}\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_\mu q_\nu\left\langle p'\left|\frac{\partial\varphi^\beta}{\partial x_\sigma}\frac{\partial\varphi^\gamma}{\partial x_\lambda}\right|p\right\rangle\\&=\left(\frac{E_pE_{p'}}{4\pi m^2}\right)^{1/2}\frac{i\lambda}{2\mu^3}\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_\mu q_\nu k_\sigma\langle p'|j_\lambda^{(\pi)\alpha}|p\rangle,\end{aligned} \tag{14}$$

where

$$j_\lambda^{(\pi)\alpha}=\frac{\sqrt{4\pi}e}{2}\varepsilon_{\alpha\beta\gamma}\left(\frac{\partial\varphi^\beta}{\partial x_\lambda}\varphi^\gamma-\varphi^\beta\frac{\partial\varphi^\gamma}{\partial x_\lambda}\right). \tag{15}$$

$j_\lambda^{(\pi)3}$ is nothing but the pion current, so that $\langle p'|j_\mu^{(\pi)\alpha}|p\rangle$ is expressed in terms of $F_i^{(2\pi)}$ $(i=1, 2)$, which are the pion current contributions to the electromagnetic form factors of the nucleon. Then we have

$$\begin{aligned}T^{(2\pi)}=&\frac{i\lambda}{2\mu^3F_\pi(\mu^2)}\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_\mu q_\nu k_\sigma\bar{u}(p')\tau_\alpha\left\{\frac{e}{2}F_1^{(2\pi)}(t)i\gamma_\lambda\right.\\&\left.+\frac{e}{2m}\frac{g_p'-g_n}{2}F_2^{(2\pi)}(t)i\sigma_{\lambda\rho}(p-p')_\rho\right\}u(p).\end{aligned} \tag{16}$$

The appearance of $F_\pi(\mu^2)$ in the above equation is justified below.

More rigorously, we consider the dispersion relation of the photoproduction amplitude $T^{(0)}$ with respect to the momentum transfer t, with the energy s fixed. We take up only the singularity of the two pion intermediate states $(t\geq(2\mu)^2)$. The effect of the nucleon pole $(\bar{t}=m^2)$ has been considered by Chew, Goldberger, Low and Nambu,[6] and the singularity of a nucleon and a pion $(\bar{t}\geq(m+\mu)^2)$ may be neglected unless we discuss the second resonance. Then the absorptive part of $T^{(2\pi)}$ is

$$\begin{aligned}A^{(2\pi)}=&-\left(\frac{2w_qE_p}{16\pi^2m}\right)^{1/2}\varepsilon_\mu\pi\sum_{\substack{p_1p_2\\\beta\gamma}}\bar{u}(p')\langle q\alpha|j_\mu^{(S)}|p_1\beta;\, p_2\gamma\rangle\langle p_1\beta;\, p_2\gamma|f|p\rangle\\&\times\delta(p_1+p_2+p'-p).\end{aligned} \tag{17}$$

In this expression we take only the intermediate state with $l=1$ and neglect the

states with $l \geq 3$. With this approximation we have from Eqs. (2) and (9)

$$\langle q\alpha | j_\mu^{(S)} | p_1\beta\,;\, p_2\tau\rangle = \frac{-i\sqrt{4\pi}\,\lambda C(t)}{2t^3(2\omega_q)^{1/2}F_\pi(t^2)}\,\varepsilon_{\mu\nu\sigma\lambda}q_\nu k_\sigma\langle 0|j_\lambda^{(V)}|p_1\beta\,;\,p_2\tau\rangle,$$

and consequently

$$A^{(2\pi)}(t) = \frac{1}{\sqrt{4\pi}}\frac{i\lambda}{2t^3F_\pi(t^2)}\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_\mu q_\nu k_\sigma A_\lambda^{(2\pi)}(t)C(t). \tag{18}$$

Here $A_\lambda^{(2\pi)}(t)$ is the part of the absorptive part of $(E_pE_{p'}/m^2)^{1/2}\langle p'|j_\lambda^{(V)}|p\rangle$ that comes from the two pion intermediate states. Since $C(t)$ is not known without the knowledge of the pion-pion scattering phase shift, we put $C(t)=1$ here. If Frazer and Fulco's amplitude is correct, then we see, from Fig. 3, that the error introduced by this approximation will be 50% at most, assuming that the important contribution to the dispersion relation integral comes from the region $t\sim 10t^2$.[3] From the absorptive part Eq. (18) with $C(t)=1$, we obtain the amplitude Eq. (16). Here $F_i^{(2\pi)}(t)$ is the two-pion intermediate state contribution to the isovector form factors $F_i^{(V)}(t)$.

Seeing the successful theory of Frazer and Fulco,[3] we may say that the (isovector) magnetic form factor of the nucleon $F_2^{(V)}(t)$ satisfies the dispersion relation without the subtraction and that its absorptive part is well represented by the two-pion state contribution. Hence we may safely replace $F_2^{(2\pi)}(t)$ in Eq. (16) by $F_2^{(V)}(t)$. On the other hand, the things are not so clear with the electric form factor $F_1^{(V)}(t)$. Indeed there may probably be some core of isovector charge at the center of the pion cloud. Here we assume the existence of an unextended core of charge $ez/2$, and set

$$F_1^{(V)}(t) = z + F_1^{(2\pi)}(t).$$

This procedure reflects to some extent the fact in the lowest order perturbation theory that on the one hand we have a finite result for $F_2^{(V)}$ but a divergent one for $F_1^{(V)}$, and on the other hand we should have a convergent amplitude $T^{(2\pi)}$ if we used the conventional interaction Hamiltonian instead of the effective Hamiltonian $H_{\gamma 3\pi}$ (Fig. 5).

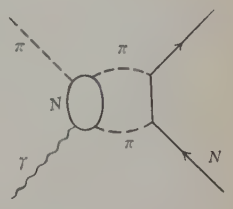


Fig. 5. The lowest order Feynman diagram for the photopion production that involves the $(\gamma, 3\pi)$ interaction.

As for the value of the parameter z, we can say nothing with any confidence. In the static theory (one pion approximation of Sachs[14]) the following relation holds :

$$1 \geq z \geq \frac{1}{3}\,;$$

in the relativistic dispersion theory Chew[15] has estimated

$$1 - z \approx 1.28\,;$$

and the experiments[4] seem to indicate $z \lesssim 0.4$. But fortunately our main result is independent of the value of z, as is shown in the next section.

Finally we take the static limit for the sake of simplicity. The result is

$$T^{(2\pi)} = -\frac{2ef^3}{\pi\mu^3}\frac{\lambda}{\lambda_{pert}}\tau_\alpha\Big\{(F_1^{(V)}(t)-z)\boldsymbol{k}\times\boldsymbol{\varepsilon}\cdot\boldsymbol{q} + F_2^{(V)}(t)\frac{g_p'-g_n}{2}\frac{w_q}{m}i\boldsymbol{\sigma}\cdot\boldsymbol{Q}\times(\boldsymbol{\varepsilon}\times\boldsymbol{Q})\Big\}, \tag{19}$$

where

$$\boldsymbol{Q}=\boldsymbol{q}-\boldsymbol{k}.$$

In the above equation we have put $F_\pi(\mu^2)\approx 1$ and used Eq. (11).

§ 4. Cross section and comparison with experiment

The cross section of photopion production with the amplitude $T^{(2\pi)}$ alone is of the order of $\mu^{-2}e^2f^6(\lambda/\lambda_{pert})^2\approx 0.07(\lambda/\lambda_{pert})^2\times 10^{-30}\,\mathrm{cm}^2$ when k and q are of the order of μ. This may be neglected unless λ is much larger than λ_{pert}. In the following we shall investigate the interference between the amplitude $T^{(2\pi)}$ and the usually considered amplitude. As stated in § 1 we consider only the case of the neutral pion production, because in this case the amplitude is very simple on account of the lack of the large s-wave and retardation terms.

In the low energy region (up to about $E_\gamma\approx 400$ Mev), the main amplitude is the magnetic dipole one leading to the final p-state with $I=J=3/2$. According to Chew and Low,[5] it has the form

$$T(M1)=\frac{2}{3}efA\frac{e^{i\alpha_{33}}\sin\alpha_{33}}{q^3}[2\boldsymbol{k}\times\boldsymbol{\varepsilon}\cdot\boldsymbol{q}+i\boldsymbol{\sigma}\cdot(\boldsymbol{k}\times\boldsymbol{\varepsilon})\times\boldsymbol{q}], \tag{20}$$

where $A=(g_p-g_n)/4\cdot\mu/m\cdot f^{-2}\approx 2.2$. We consider the interference of $T^{(2\pi)}$ with this amplitude alone:

$$\frac{d\sigma^{(2\pi,p)}}{d\Omega}=\frac{4e^2f^4A}{3\pi\mu^2}\frac{\lambda}{\lambda_{pert}}\frac{w_q}{\mu}\sin 2\alpha_{33}\Big[-(F_1^{(V)}(t)-z)(1-\cos^2\theta) + \frac{g_p'-g_n}{2}\frac{w_q}{2m}F_2^{(V)}(t)\left(1-2\left(v_\pi+\frac{1}{v_\pi}\right)\cos\theta+3\cos^2\theta\right)\Big]. \tag{21}$$

Besides, there are small amplitudes such as the magnetic dipole ones leading to other p-states and the electric dipole one leading to s-state. We neglect the former as they are numerically very small, and as the latter we adopt the expression of Chew, Goldberger, Low and Nambu:[6]

$$T(E1,s)=\frac{ef}{\mu}\left[\frac{2}{3}i(\alpha_1-\alpha_3)F_s-\frac{g_p+g_n}{2m}w_q\right]i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}, \tag{22}$$

where

$$F_s = 1 - \frac{1}{2}\left(1 + \frac{1-v_\pi^2}{2v_\pi} \ln \frac{1-v_\pi}{1+v_\pi}\right).$$

As an example we show in Fig. 6 the angular distribution of the photopion production when $E_\gamma = 300$ Mev. The cross sections with and without the amplitude $T^{(2\pi)}$ are shown. We have set $\lambda = \lambda_{pert}$ and $z = 0.3$ in Eq. (21). The experimental data of the phase shifts (Fig. 7) have been used. We see that the effect of $T^{(2\pi)}$ is within the experimental error when $\lambda \approx \lambda_{pert}$.

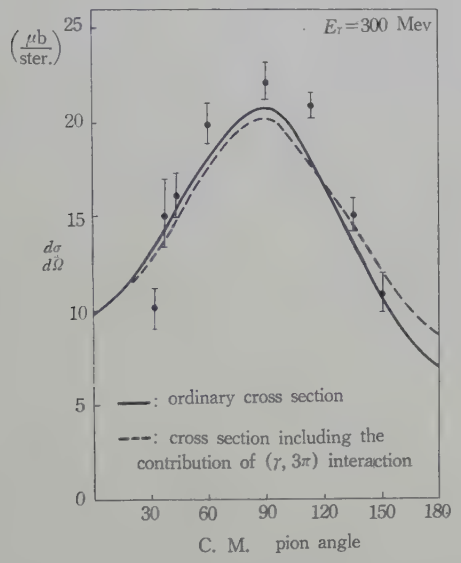


Fig. 6. The theoretical angular distribution for neutral photopion production, compared with experiment. The energy of incident gamma ray is 300 Mev in the laboratory system. The solid curve is the ordinary cross section which does not involve the contribution of the $(\gamma, 3\pi)$ interaction and the dotted curve involves it. The experimental data are those of references 16)～18) which are all quoted in reference 18).

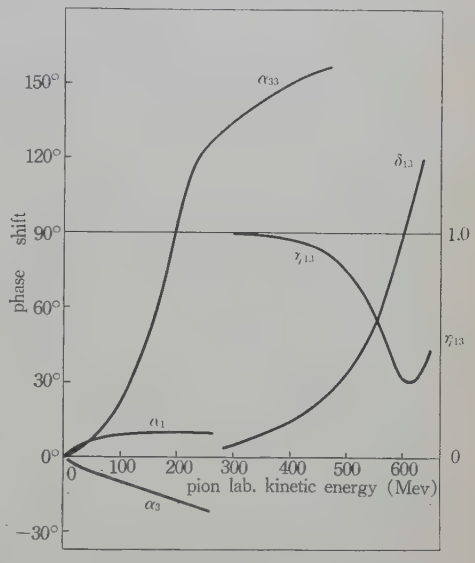


Fig. 7. The phase shifts of the pion-nucleon scattering which have been used in the calculation of the angular distributions of photopion production. α_{2I}, α_{33} and δ_{13} are the phase shifts of s-wave ($I=1/2$ or $3/2$), p-wave ($I=J=3/2$) and d-wave ($I=1/2$, $J=3/2$) scatterings, respectively. α_{2I} and α_{33} have been taken from the experimental data, and δ_{13} has been obtained from the one-level formula. η_{13}, the magnitude of the elastic amplitude in the $d_{3/2}$-state, is also shown.

To make things clearer we express the cross section in the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta$$

and take up the coefficient B. Without $T^{(2\pi)}$, B comes from the interference of p- and s-waves :[6]

$$B^{(s,p)} = \frac{e^2 f^2 \Lambda}{\mu^2}\left[\frac{8}{9}\frac{\mu}{q}(\alpha_1 - \alpha_3) F_s \sin^2\alpha_{33} - \frac{2}{3}\frac{g_p + g_n}{2}\frac{\mu w_q}{mq}\sin 2\alpha_{33}\right]. \quad (23)$$

When $T^{(2\pi)}$ is taken into account, from Eq. (20),

$$B^{(2\pi)}=\frac{8e^2f^4\Lambda}{3\pi\mu^2}\frac{\lambda}{\lambda_{pert}}\frac{w_q}{\mu}\sin 2\alpha_{33}\left\{-\frac{g_p'-g_n}{2}\frac{w_q}{m}\frac{1}{2}\left(v_\pi+\frac{1}{v_\pi}\right)\right.$$
$$\left.+\frac{\langle r^2\rangle^{(V)}}{6}\left[-\frac{2}{5}w_q^2v_\pi+\frac{g_p'-g_n}{2}\frac{w_q}{m}\left(w_q^2\left(\frac{12}{5}v_\pi+\frac{1}{v_\pi}\right)-\frac{\mu^2}{2}\left(v_\pi+\frac{1}{v_\pi}\right)\right)\right]\right\}. \tag{24}$$

Here we have approximated the form factors $F_i^{(V)}$ as

$$F_i^{(V)}(t)\approx 1-\frac{1}{6}\langle r_i^2\rangle^{(V)}(-t)\approx 1-\frac{1}{6}\langle r_i^2\rangle^{(V)}[2w_q^2(1-v_\pi\cos\theta)-\mu^2],$$

and also replaced the expression of the form $B\cos\theta+D\cos^3\theta$ by $(B+3/5\cdot D)\cos\theta$, according to the least mean square principle. It is to be noticed that $B^{(2\pi)}$ is independent of the parameter z. Using the experimental values for mean square radii $\langle r_i^2\rangle^{(V)}(\approx 0.32\times\mu^{-2})$ and the pion-nucleon scattering phase shifts, we have calculated $B^{(sp)}$ and $B^{(sp)}+B^{(2\pi)}$. The results are shown in Fig. 8, together with the experimental data. We see that the inclusion of the amplitude $T^{(2\pi)}$, i.e. the effect of $(\gamma, 3\pi)$ interaction, is very important for the coefficient B, and that λ cannot take a much larger value than λ_{pert}. More precise experiments of angular distribution and the inclusion of the small amplitudes neglected here will make the limitation more severe.

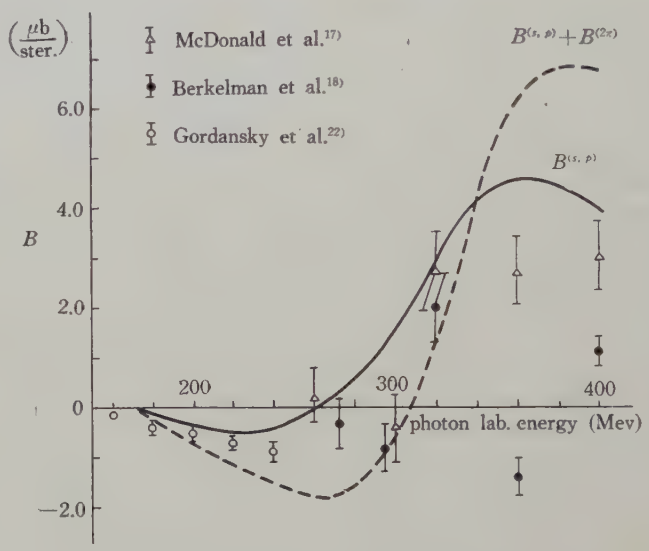


Fig. 8. The theoretical prediction for the coefficient B of the angular distribution of neutral photopion production, compared with several experiments. The dotted curve involves the effect of the $(\gamma, 3\pi)$ interaction. The experimental data of Berkelman et al. are the summary of references 16)~19).

Now we consider the photoproduction in higher energy region (between the first and the second resonances) and here again we want to see the effect of the $(\gamma, 3\pi)$ interaction on the angular distribution. As the main amplitudes we take up only the magnetic dipole one, Eq. (20), and the electric dipole one leading to the final d-state with $I=1/2$ and $J=3/2$. As the latter we assume a phenomenological form

$$T(E1, d)=iefCk\mu\frac{\eta_{13}e^{2i\delta_{13}}-1}{q^5}[3i\boldsymbol{\sigma}\cdot\boldsymbol{q}\,\boldsymbol{q}\cdot\boldsymbol{\varepsilon}-q^2i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}], \tag{25}$$

where δ_{13} is the phase shift and η_{13} the magnitude of the elastic channel amplitude in the corresponding pion-nucleon scattering. This expression is obtained from the assumptions that the partial photoproduction cross section in this state satisfies the one-level formula,[23),24)] that the amplitude is proportional to the wave function of the final pion-nucleon state at the surface of the nucleon, and that the elastic and γ widths Γ_π and Γ_γ of the second resonance are proportional to q^5 and k^3 respectively. It is a generalization of what Bernardini has quoted in the Kiev Conference report.[9)] If we take[9),25),26)] the resonant cross sections $\sigma_{\gamma\pi^0}$ and $\sigma_{N\pi}$ at the second resonance to be 30 μb and 32 mb, respectively, which indicates $\eta=0.34$ at this energy, we obtain $C^2=5.9$. (Here we put $C=-\sqrt{5.9}$.) We neglect other small amplitudes completely, simply because we have little knowledge on the corresponding phase shifts.

The differential cross section due to the interference of $T^{(2\pi)}$ with $T(M1)$ has been shown in Eq. (21), and that with $T(E1, d)$ is

$$\frac{d\sigma^{(2\pi,d)}}{d\Omega}=\frac{2e^2f^4C}{\pi\mu^2}\frac{\lambda}{\lambda_{pert}}\frac{w_q^3}{qm\mu}\eta_{13}\sin 2\delta_{13}\frac{g_p'-g_n}{2}F_2^{(V)}(t)$$
$$\times\left[\left(\frac{1}{v_\pi}-v_\pi\right)+\cos\theta-\left(\frac{3}{v_\pi}+v_\pi\right)\cos^2\theta+3\cos^3\theta\right]. \tag{26}$$

In principle we should subtract a part corresponding to d-wave from $T^{(2\pi)}$, so that the amplitude $T(E1, d)$ may have the phase indicated in Eq. (25). But this will make no important change numerically.

In Figs. 9 and 10 we show the calculated angular distribution for $E_\gamma=450$ Mev and 600 Mev, together with the experimental data. The phase shifts that we have used in the calculation are shown in Fig. 7. α_{33}'s are the experimental value, while δ_{13} and η_{13} are obtained from the one-level formula applied to the second resonance of the pion nucleon scattering. The details are shown in the Appendix.

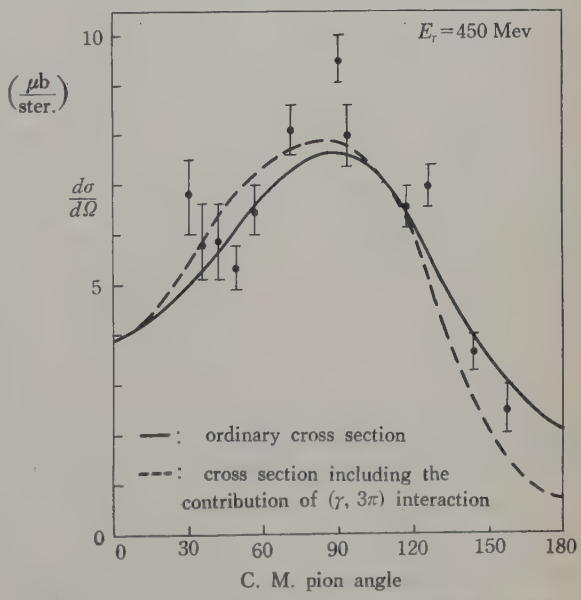


Fig. 9. The theoretical and experimental angular distribution for gamma ray energy (lab. system) of 450 Mev. The notation is the same with that of Fig. 6. The experimental data are those of references 16)~21).

We see from Figs. 9 and 10 that the $(\gamma, 3\pi)$ interaction has rather large effect on the photopion production at high energy. Although the effect is within the experimental errors at present, one should take it into account correctly when one makes a quantitative theory in future.

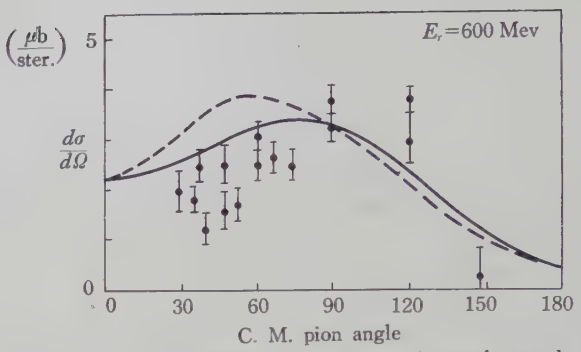


Fig. 10. The theoretical and experimental angular distribution for gamma ray energy of 600 Mev. The notation is the same with that of Fig. 6.

§ 5. Discussion

Several authors have calculated the isoscalar form factors of the nucleon, or, more precisely, the three-pion intermediate state contribution to them, using the effective Hamiltonian Eq. (1) and the static[7] or relativistic[27),28)] perturbation theory. The relativistic calculations give the mean square radius of charge distribution $\langle r_1^2 \rangle$[(8)] which is smaller than the experimental value ($\approx 0.32\mu^{-2}$) by factor five or more if the value λ_{pert} is used as λ. Of course these results should not be taken too seriously because the effective Hamiltonian (1) is rather a crude approximation and the results depend on the cutoff considerably. But combining the above results and the conclusion of this paper that λ cannot be much larger than λ_{pert}, we feel that the perturbation calculation of isoscalar form factors is insufficient and that it will be necessary to take the pion-pion interaction[29)] into account as is the case with the isovector form factors. To perform a quantitatively reliable calculation, it is necessary to know the matrix element $\langle 0| j_\mu |3\pi \rangle$ with time-like k^2, and this remains as a very difficult problem. Anyway the precise value of λ is needed first of all, and, for that purpose, it is desirable to obtain more detailed data on both the angular distribution of the γ-π^0 production and the small phase shifts of the pion-nucleon scattering.

Acknowledgements

The author would like to express his sincere thanks to Professor Gyo Takeda for his guidance and valuable discussions throughout this work, and to Professor Ken-ichi Ono for his continual encouragement. The author is also indebted to Professor Satio Hayakawa, who stressed first the importance of the $(\gamma, 3\pi)$ interaction in the problem of photomeson production.

It has been called to our attention that J. S. Ball [Phys. Rev. Letters **5**(1960), 73] has studied the $(\gamma, 3\pi)$ interaction using the essentially same method with ours. He has investigated the positive-negative ratio of the photopion production and obtained the limitation $|\lambda| < 3.5\lambda_{pert}$. He has approximated the function

$C(t)$ as $6.7\mu^2/(t+5.7\,\mu^2)$, based on the argument of H. S. Wong [Phys. Rev. Letters **5**(1960), 70].

Appendix

We apply the one level resonance formula to the second resonance of the pion nucleon scattering:

$$\sigma(E)=2\pi\lambda^2\frac{\Gamma_{el}\Gamma}{(E-E_0)^2+\Gamma^2/4}. \tag{A1}$$

The total width Γ is the sum of the elastic and inelastic widths, Γ_{el} and Γ_{inel}, which are approximately represented by the one-pion and two-pion widths, respectively:

$$\Gamma=\Gamma_{el}+\Gamma_{inel},$$
$$\Gamma_{el}=\Gamma_{\pi},$$
$$\Gamma_{inel}\approx\Gamma_{2\pi}.$$

We extract from the widths the penetration factor and the "invariant phase space volume", i.e. the phase space volume divided by the energy, assuming the residual parts to be constant. First, the elastic width Γ_{π} is proportional to an invariant quantity $\gamma_2(E)$

$$\gamma_2(E)=\int\frac{d^3p\,d^3q}{\varepsilon_p w_q}\delta^3(\boldsymbol{P}-\boldsymbol{p}-\boldsymbol{q})\delta(E-\varepsilon_p-w_q)v_2. \tag{A2}$$

$\boldsymbol{q}$ and $\boldsymbol{p}$ are the momenta of the final pion and nucleon, and w_q and ε_p are their energies, respectively. v_l is the penetration factor corresponding to an l state. In the center of mass system, $P=0$ and

$$v_2=v_2(ka)=\frac{(ka)^4}{9+3(ka)^2+(ka)^4}.$$

k is the magnitude of the pion momentum in the center of mass system. Here we have assumed the nonrelativistic form of the penetration factor. Of course we do not know an unambiguous way of defining the interaction radius a. Performing the integral in Eq. (A2), we have the familiar form

$$\gamma_2(E)=\frac{4\pi k}{M}v_2(ka), \tag{A3}$$

where $M^2=E^2-\boldsymbol{P}^2$.

Next we consider the inelastic width $\Gamma_{2\pi}$. We assume that the "compound state" emits an s-wave pion first, leaving the residual part in a state (N_1) with $J=3/2$ and odd parity, then N_1 emits a p-wave pion. $\Gamma_{2\pi}$ is proportional to

$$\int \frac{d^3q'}{w_{q'}} \frac{d^3q}{w_q} \frac{d^3p}{\varepsilon_p} \delta^3(\boldsymbol{q}+\boldsymbol{q}'+\boldsymbol{p})\delta(w_q+w_{q'}+\varepsilon_p-E)$$

$$= \int \frac{d^3q'}{w_{q'}} \gamma_1(E-w_{q'}), \tag{A4}$$

where $\boldsymbol{q}'$, $w_{q'}$, $\boldsymbol{q}$ and w_q are the momenta and energies of the s- and p-wave pions, respectively. $\gamma_1(E)$ is defined by Eq. (A2), v_2 being replaced by v_1. Inserting Eq. (A3) into (A4) and changing the integration variable to M, the "mass" of N_1, we have

$$\Gamma_{2\pi} \propto 16\pi^2 \frac{1}{E} \int_{m+\mu}^{E-\mu} dMqkv_1(ka). \tag{A5}$$

Here

$$k=\frac{1}{2M}[(M^2-(m+\mu)^2)(M^2-(m-\mu)^2)]^{1/2},$$

$$q=\frac{1}{2E}[((E+\mu)^2-M^2)((E-\mu)^2-M^2)]^{1/2},$$

and

$$v_1(ka)=\frac{(ka)^2}{1+(ka)^2}.$$

In Fig. 11 we show the widths Γ_π and $\Gamma_{2\pi}$, both normalized to unity at the second resonance energy E_0 ($T_\pi \approx 610$ Mev). We have chosen[23] $a=0.88\ \mu^{-1}$.

The phase shift δ_{13} and the magnitude of the elastic channel amplitude η_{13} of the pion nucleon scattering in the $d_{3/2}$ ($I=1/2$) state are derived from the relation

$$\eta e^{2i\delta}=1-\frac{i\Gamma_{el}}{E-E_0+i\Gamma/2}.$$

We have

$$\eta^2=1-\frac{\Gamma_{el}\Gamma_{inel}}{(E-E_0)^2+\Gamma^2/4} \tag{A6}$$

and

$$\tan\delta=\frac{\Gamma}{2(E_0-E)}\left(1-\frac{2}{1+\eta}\frac{\Gamma_{inel}}{\Gamma}\right). \tag{A7}$$

Using the values of parameters

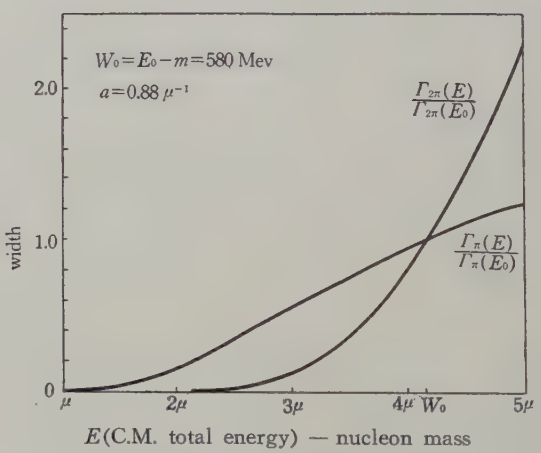


Fig. 11. The elastic and inelastic widths of the second excited state of the nucleon, calculated from Eqs. (A3) and (A5) and normalized to unity at the resonance energy $W_0=580$ Mev.

$$W_0 = E_0 - m = 580 \text{ Mev} \quad (T_{\pi(Lab)} = 610 \text{ Mev}),$$
$$\sigma(E_0) = 32 \text{ mb}$$

and

$$\Gamma(E_0) = 102 \text{ Mev},$$

with the relation

$$\sigma(E_0) = 8\pi \lambda\!\!\!^{-}_0{}^2 \frac{\Gamma_{el}(E_0)}{\Gamma(E_0)}$$

we have calculated δ_{13} and η_{13}, and have shown the results in Fig. 7.

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Letters to the Editor

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A New Solution of the Clock Paradox

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October 17, 1960

It is shown that the problem of "clock paradox" can be solved within the framework of special relativity, and the Einstein's statement is justified by a very simple consideration on the ground of directly observable data.

When a light source and an observer are in relative motion with velocity v along the straight line joining these two, the observed frequency is given by (ν_0 is the proper frequency):

$$\nu_1=\nu_0(1-\beta)/\sqrt{1-\beta^2},\quad \beta=v/c, \qquad (1)$$

when they are receding from each other, and

$$\nu_2=\nu_0(1+\beta)/\sqrt{1-\beta^2}, \qquad (2)$$

when they are approaching to each other.

Now, let a space traveller O' leave the point where another O stays in an inertial system traverse a distance L along a straight line, and return to the original point, with uniform speed v throughout the whole course. What is the relation between the readings of the clock carried by O and O'? In this paper we assume that O and O' measure their proper times by means of their own standard atom-clocks of identical construction with proper frequency ν_0, and that these atoms emits continually the light of the same frequency ν_0. Further, we neglect the time required for accerelating or decelerating O'.

The lapse of time during the whole journey of O', properly measured by O, is $\Delta t=2L/v$. During this time, the number of oscillation of the atom-clock of O is $N=\nu_0(2L/v)$, while that of vibrations of the atom-clock carried by O' during his whole journey, observed by O is given by

$$N'=\nu_1(L/v+L/c)+\nu_2(L/v-L/c). \qquad (3)$$

The term $\nu_1(L/c)$ of (3) gives the number of light waves which were emitted from O' in his going way and are still in the course of propagation toward O at the moment of the turn of O'. All these waves reach O within the time L/c after the turn. Thus, the light of frequency ν_1 is observed by O during the time $(L/v+L/c)$, and the light of frequency ν_2 is observed during the time $(L/v-L/c)$. Substituting (1) and (2) in (3), we get $N'=N\sqrt{1-\beta^2}$.

From the above argument, the lapse

of proper time of O' during the whole journey, $\Delta t' = N'/\nu_0$, is judged to be $\sqrt{1-\beta^2}$ times the lapse of proper time of O. And the proper time proves to elapse for O' only by $(L/v)\sqrt{1-\beta^2}$ both in his going and in his returning journey.

Next, the number of light waves emitted from O and observed by O' in the going way amounts to $N_1 = \nu_1(L/v)\sqrt{1-\beta^2}$. Again in the returning way, $N_2 = \nu_2(L/v)\sqrt{1-\beta^2}$. The Doppler effect of light is symmetric for both O and O', and the uniform rectilinear motion is perfectly relative in this point. The total number of vibrations of the atom-clock at O, during the whole journey of O', is reckoned by O' also to be N:

$$N_1 + N_2 = (\nu_1 + \nu_2)(L/v)\sqrt{1-\beta^2} = \nu_0(2L/v) = N. \qquad (4)$$

Thus, the relation between the total numbers of vibration of the atom-clocks of O and O', is one and the same viewed from either O or O'. And the space traveller is concluded to age only $\sqrt{1-\beta^2}$ times, compared with his fellows staying at home in an inertial system.

The significant point in this paper is the existence of the terms, $\nu_1(L/c) - \nu_2(L/c)$ in (3). This is due to the following circumstances. The change of the frequency of light from O' begins to be observed by O at the time L/c after the turn of O', because the light from O' propagates with constant velocity c towards the observer O in an inertial system, independently of the motion of the light source O', and it takes also the time L/c for the effect of the turn of O' to reach O; while the apparent change of the frequency of light from O begins to be observed by O' at the very moment of the turn, because the propagation of light from O is stationary in any inertial system (in this case, it is stationary, as long as O' is moving uniformly relative to O), but the change of frequency appears at once when O' begins an accerelated motion.

The above judgment with regard to the asymmetrical aging of O and O' is justified also by geometrical consideration in a Minkowski diagram. In the figure, OT is the world line of O, OT' and $T'T$ are the world lines of O' in his going and returning journey; $\overline{OT} = 2L/v$, $\overline{OT'} = \overline{T'T} = (L/v)\cdot\sqrt{1-\beta^2}$, $\theta = \tan^{-1}\beta$, $T_1T' = L/c$.

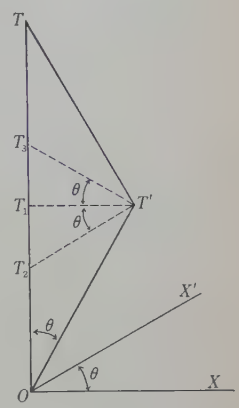


Fig. 1.

Now, O regards T' as simultaneous with T_1, which is the middle point of OT. But O' regards, T_2 and T_3, as simultaneous with T' respectively, before and after the turn. Again,

$$\overline{OT_2} = \overline{OT'}\sqrt{1-\beta^2} = (L/v)(1-\beta^2) = L/v - vL/c^2 = \overline{T_3T}.$$

At the moment of the turn of O', the point on the world line of O which is regarded by O' as simultaneous with T', jumps by the amount

$$\overline{T_2T_3} = 2\overline{T_1T'}\tan\theta = 2(L/c)\cdot\beta = 2vL/c^2.$$

This is due to the relativity of judgment with regard to the simultaneity of separated events in different inertial frames, and it is clear that there is no paradox in the Einstein's statement of asymmetrical ageing between a space traveller and his stay-at-home fellows.

Rigorous consideration including accerelated motions and gravitational fields will be given in another paper, in connection with the criticism of various solutions of this problem of "clock paradox" preceding the present paper.

The Effect of Short-Range Order on Paramagnetic Resonance

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December 19, 1960

The aim of this note is to investigate the effect of the short-range ordering of spins on paramagnetic resonances in ferro- and antiferro-magnets at high temperatures. The calculations is based on the general theory of paramagnetic resonance of Kubo and Tomita[1] (hereafter called I).

Let us consider a spin system, whose Hamiltonian H consists of the exchange interaction H_{ex} and the dipole-dipole interaction H_{d-d} and the case in which the arrangement of spins is of a simple cubic type and the magnitude of spins is one-half. The same notation as in I will be used throughout the present note. Then, the Hamiltonians are expressed by

$$H_{ex}=-\frac{1}{2}\sum_{jk}J_{jk}(\sigma_j\cdot\sigma_k), \qquad (1)$$

and

$$H_{d-d}\equiv\sum_{\gamma=-2}^{+2}H_\gamma=\sum_\gamma\frac{1}{4}\sum_{jk}\Phi_{jk}{}^\gamma\{jk\}^\gamma. \qquad (2)$$

Here σ is the pauli spin matrix.

We shall make the following assumptions and approximations.

i) The exchange interaction is so much stronger than dipole-dipole interaction, that the line shape of the paramagnetic resonance is Lorentzian in the higher temperature limit. This allows us to assume that the density matrix of the system is determined approximately by the exchange interaction.

ii) Correlation function determining the width of paramagnetic resonance is approximated by the Gaussian form, which is obtained by the expansion of time from the exact form.

iii) We take into account only the exchange and dipole-dipole interactions between the nearest neighbours.

iv) We use the expansion method of Opechowski[2] to evaluate the average value of some function of spin operator and neglect quantities higher than fourth order in J/kT, where J is the nearest neighbour exchange integral.

According to the assumption ii), correlation function $f_\gamma(\tau)$ is expressed by

$$f_\gamma(\tau)=\exp(i\bar{\omega}_\gamma\tau)\exp\left(-\frac{1}{2}\omega^2{}_{e\gamma}\tau^2\right), \qquad (3)$$

where

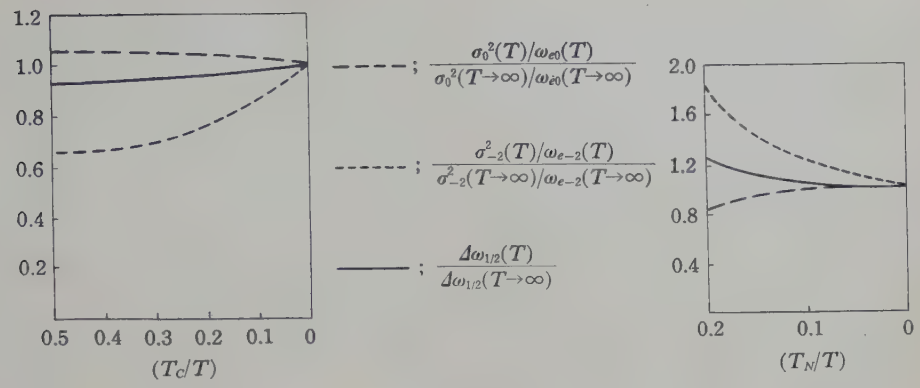


Fig. 1.

$$\bar{\omega}_\gamma=\frac{1}{\hbar}\frac{\langle[[H_{ex}[M_-, H_{-\gamma}]]H_\gamma]M_+\rangle}{\langle[[M_-, H_{-\gamma}]H_\gamma]M_+\rangle} \quad (4)$$

and

$$\omega_{e\gamma}{}^2=\frac{1}{\hbar^2}\times\frac{\langle[[H_{ex}[H_{ex}[M_-, H_{-\gamma}]]]H_\gamma]M_+\rangle}{\langle[[M_-, H_{-\gamma}]H_\gamma]M_+\rangle} \quad (5)$$

and the notation $\langle A\rangle$ denotes $\mathrm{Tr}\,\rho A$ with

$$\rho=e^{-\beta H_{ex}}/\mathrm{Tr}\,e^{-\beta H_{ex}}.$$

By the use of Eq. (3), we obtain the following expression of total half width $(\Delta\omega)_{1/2}$.

$$(\Delta\omega)_{1/2}=\sqrt{\frac{\pi}{2}}\times\left[\frac{\sigma_0^2}{\omega_{e0}}+\frac{\sigma_{-2}^2}{\omega_{e-2}}\exp(-2\omega_Z^2/\omega_{e-2}^2)\right], \quad (6)$$

where

$$\sigma_\gamma{}^2=\hbar^{-2}\frac{\langle[[M_-, H_{-\gamma}]H_\gamma]M_+\rangle}{\langle M_-M_+\rangle}, \quad (7)$$

and $\hbar\omega_Z$ is the Zeeman energy of spin. In the derivation of Eq. (6), we have used the fact that quantity $\bar{\omega}_\gamma$ of Eq. (4) contributes only the term of the forth and higher order in J/kT and can be omitted, and the term including $\sigma_\gamma=\pm 1$ are not present, because of the approximation iii).

The results in the case $\omega_Z^2\ll\omega_\gamma^2$ are expressed in Fig. 1, where the Curie temperature T_C and the Neél temperature T_N is used as the temperature scale and, as the numerical values of T_C and T_N, we used those theoretically calculated by Kastellijin and Kranendonk.[3] We can see in Fig. 1 that the temperature variation of the calculated half-width is almost determined by the non-secular part ($\gamma=-2$) of the dipole-dipole interaction. It is interesting to note that the anomalous broadening[4] observed in the paramagnetic resonance of $Cu(NH_3)_4SO_4H_2O$ is qualitatively similar to our results in antiferromagnets.

In order to compare our results with the experimental temperature dependence[5] of the half width in antiferromagnets, which is almost flat except in the vicinity of the Neél temperature, where the half width increases very steeply when temperature decreases, it is necessary to carry out the calculation, which is free from the assumptions iii) and iv) and the calculation in the case of other spin value in ad-

dition to one-half spin.

The detailed result of our calculations will be given in the series of "Science Report of Tôhoku University".

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A Note on the Treatment of Bound State Problem

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November 4, 1960

Usually the bound state problem is treated by the Bethe-Salpeter equation. However, if a consistent relativistic Hamiltonian formalism existed and a bound state should behave like a relativistic particle, the bound state problem could be treated in principle more simply. Namely it is enough to consider the eigenvalue problem of mass operator,

$$-M^2\Phi\equiv(P^2-H^2)\Phi=-m^2\Phi. \quad (1)$$

To the author's opinion this method is preferable to usual Bethe-Salpeter's one as the former is the more faithful method to the direct meaning of the mass of the composite system. Unfortunately we have no completely consistent relativistic Hamiltonian formalism as yet. But, putting the complete invariance aside, we can investigate the properties of the aforementioned eigenvalue problem. It is the purpose of this note to show a method to handle Eq. (1) and its relation to the ordinary Schrödinger equation.

Putting $H=H_0+H_I$ Eq. (1) becomes

$$(P^2-H_0^2-H_0H_I-H_IH_0 \\ -H_I^2)\Phi=-m^2\Phi. \quad (1')$$

Let us consider the interaction between two sorts of particles. We denote their creation-annihilation operators and masses by $A^{\pm}$, $B^{\pm}$ and m_a, m_b respectively.* For simplicity here we assume that both particles are spin zero, and adopt the following form for H_I,

$$H_I=\sum a_{\alpha\beta\gamma\delta}A^+{}_\alpha B^+{}_\beta A^-{}_\gamma B^-{}_\delta ,$$

$$a_{\alpha\beta\gamma\delta}=\int f(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r}) \\ \bar{\varphi}_\alpha(\boldsymbol{p})\bar{\phi}_\beta(\boldsymbol{q})\varphi_\gamma(\boldsymbol{r})\times \\ \times\phi_\delta(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r})\,d\boldsymbol{p}d\boldsymbol{q}d\boldsymbol{r} ,$$

but these assumptions are not any essential restrictions. To guarantee the hermiticity of H_I, f must satisfy the relation

$$f(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s})=\bar{f}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{p},\boldsymbol{q}).$$

In order to investigate the bound state of two particles we put

$$\Phi=\sum a_{\alpha\beta}A^+{}_\alpha B^+{}_\beta\Phi_0,$$

$$a_{\alpha\beta}=\int\bar{\varphi}_\alpha(\boldsymbol{p})\bar{\phi}_\beta(\boldsymbol{q})\psi(\boldsymbol{p},\boldsymbol{q})\,d\boldsymbol{p}d\boldsymbol{q} ,$$

where Φ_0 is the vacuum state. Putting the above H_I and Φ into (1′) we

* In the following the same notations and method as developed in the author's previous paper[2] will be used.

obtain the following equation for $\psi(\boldsymbol{p},\boldsymbol{q})$ in the lowest order approximation,

$$\{-m_a^2-m_b^2+2(\boldsymbol{p}\cdot\boldsymbol{q}-\omega_p W_q)\}\times\psi(\boldsymbol{p},\boldsymbol{q})-\int f(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r})\times(\omega_p+W_q+\omega_r+W_{p+q-r})\times\psi(\boldsymbol{r},\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{r})d\boldsymbol{r}=-m^2\psi(\boldsymbol{p},\boldsymbol{q}),$$

where $\omega_p=\sqrt{m_a^2+\boldsymbol{p}^2}$, $W_q=\sqrt{m_b^2+\boldsymbol{q}^2}$. (The situation is not changed even if the higher order term is taken into account.) It is convenient to use the variables $\boldsymbol{s}=(\boldsymbol{p}+\boldsymbol{q})/2$, $\boldsymbol{t}=(\boldsymbol{p}-\boldsymbol{q})/2$, and we use $\psi(\boldsymbol{s},\boldsymbol{t})$ instead of $\psi(\boldsymbol{p},\boldsymbol{q})$, then the above equation becomes

$$\{-m_a^2-m_b^2+2(\boldsymbol{s}^2-\boldsymbol{t}^2-\omega_{s+t}W_{s-t})\}\times\psi(\boldsymbol{s},\boldsymbol{t})-\int f(\boldsymbol{s}+\boldsymbol{t},\boldsymbol{s}-\boldsymbol{t},\boldsymbol{s}+\boldsymbol{r},\boldsymbol{s}-\boldsymbol{r})(\omega_{s-t}+W_{s-t}+\omega_{s+r}+W_{s-r})\times\psi(\boldsymbol{s},\boldsymbol{r})d\boldsymbol{r}=-m^2\psi(\boldsymbol{s},\boldsymbol{t}). \quad (2)$$

This is the fundamental equation which corresponds to the conventional Bethe-Salpeter equation, and the hermiticity of H_I guarantees that of the operator which appears in the lefthand side of this equation.

Let $\psi(\boldsymbol{s},\boldsymbol{t})$ be an eigenfunction of Eq. (2), then multiplying an arbitrary function of $\boldsymbol{s}$ to it we get another eigenfunctions belonging to the same eigenvalue, so the eigenvalue of Eq. (2) is infinitely degenerated. This degeneracy shows the arbitrariness with respect to total momentum and this is a desirable feature for composite particles.

Now, put $\boldsymbol{s}=0$ in Eq. (2) and denote $\psi(0,\boldsymbol{t})$ by $\psi(\boldsymbol{t})$, then we get

$$\{-m_a^2-m_b^2-2(\boldsymbol{t}^2+\omega_t W_t)\}\psi(\boldsymbol{t})-\int f(\boldsymbol{t},-\boldsymbol{t},\boldsymbol{r},-\boldsymbol{r})\times(\omega_t+W_t+\omega_r+W_r)\psi(\boldsymbol{r})d\boldsymbol{r}=-m^2\psi(\boldsymbol{t}). \quad (3)$$

Assuming that this procedure has a meaning mathematically, each eigenvalue of Eq. (2) is also an eigenvalue of Eq. (3), but in general inverse is not true. Next we perform the non-relativistic approximation in Eq. (3). We put $m=(m_a+m_b-E)$ and neglect the second order term of E, and use the approximation

$$m_a^2+m_b^2+2(\boldsymbol{t}^2+w_t W_t)\approx(m_a+m_b)^2+2\boldsymbol{t}^2\frac{(m_a+m_b)^2}{2m_a m_b}.$$

Further we assume that $f(\boldsymbol{t},-\boldsymbol{t},\boldsymbol{r},-\boldsymbol{r})\approx V(\boldsymbol{t}-\boldsymbol{r})$ in the non-relativistic approximation then we obtain

$$\frac{m_a+m_b}{2m_a m_b}\boldsymbol{t}^2\psi(\boldsymbol{t})+\int V(\boldsymbol{t}-\boldsymbol{r})\psi(\boldsymbol{r})d\boldsymbol{r}=-E\psi(\boldsymbol{t}).$$

This is exactly the Schrödinger equation for potential V.

Thus it turns out that our eigenvalue problem Eq. (1) can serve as a relativistic generalization of the Schrödinger equation for the bound state problem. Of course, the above arguments are rough and it is necessary to investigate further the nature of Eq. (2) and its relation to Eq. (3).

1) E. E. Salpeter and H. A. Bethe, Phys. Rev. **84** (1951), 1232.
2) S. Sato, Prog. Theor. Phys. **23** (1960), 717.

Electric Polarizability of the Neutron

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Interest has been growing in the electric polarizability of the neutron recently.[1] This is concerned with the nonrelativistic limit of the two-photon interaction of nucleon. On the other hand, the calculations of electric and magnetic form factors are concerned with one-photon interaction. The term photon should be understood in a loose sense; it need not be a free photon when it represents an external electromagnetic field. The purpose of this note is to summarize the kinematical aspects of the two-photon interaction as a preliminary survey of further covariant investigations of this problem.

Let us denote the four-momenta of incoming (outgoing) nucleon and photon by p and q (p' and q'), respectively. Photon's polarization vectors are denoted by a and a'. The problem is to list the complete set of gauge-invariant and Lorentz invariant matrix elements. The answer is not unique, but there are many equivalent sets. The straightforward application of the method described below, [A]–[E], leads to the results listed in (8). The Dirac spinor in the initial (final) state is denoted by $u(p)$ ($u'(p')$). The kinematical constraints which reduce the number of independent members of the set are as follows.

Conservation of four-momenta;

$$p+q=p'+q'. \tag{1}$$

Dirac equations for the initial and final states for the nucleon:

$$(i\boldsymbol{p}+m)u(p)=u'(p')(i\boldsymbol{p}'+m)=0, \tag{2}$$

where $\boldsymbol{A}$ means $A_\mu\gamma_\mu$. We also have

$$p^2=p'^2=-m^2. \tag{3}$$

[A] The constraint, Eq. (1), reduces the number of independent vectors to three. We choose the set

$$q,\ q',\ \text{and}\ P=(1/2)(p+p'). \tag{4}$$

[B] The constraint, Eq. (3), reduces the number of independent scalars to four. We choose the set

$$q^2,\ q'^2,\ q\cdot q',\ \text{and}\ P\cdot q=(P\cdot q'). \tag{5}$$

[C] The constraint, Eq. (2), reduces Dirac matrices in the members of the final answer, (8), to one of the following three

$$1,\ \sigma_{\mu\nu},\ \text{and}\ \gamma_5. \tag{6}$$

The consistency of this choice can be checked after the step [D], where we use Eq. (2) in a modified form

$$u(p)=(1/2i\,m)[2\boldsymbol{P}+\boldsymbol{q}'-\boldsymbol{q}]u(p) \tag{2'}$$

and the corresponding one for $u'(p')$.

---

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[D] We are interested in bilinear forms in a and a'; first of all, they should be Lorentz invariant. a and a' may or may not appear in the form of a scalar product

$$(a\cdot a'),\ (a\cdot x) \text{ or } (a'\cdot x), \quad (7)$$

where x is any one of three vectors in (4). The cofactor to make up a Lorentz invariant bilinear form, together with a member of (7), is an invariant which is formed with use of Dirac matrices taken from (6). The factor to be multiplied by a Dirac matrix is a tensor which is formed by using some of a, a', q and q' not more than once.

[E] The final step is to take a proper linear combination of Lorentz invariants found in [D] in order to get a gauge-invariant expression. The criterion for the gauge-invariance is that the replacement of $a(a')$ by $q(q')$ gives rise to an expression which vanishes identically in $a'(a)$. In this way we have twenty expressions.

The following is one complete set.

$$\begin{aligned}
&M_1=[f\cdot f'], \quad M_2=(P\cdot f\cdot f'\cdot P),\\
&M_3=(P\cdot f\cdot f'\cdot q), \quad M_4=(q\cdot f\cdot f'\cdot P),\\
&M_5=(q\cdot f\cdot f'\cdot q'), \quad M_6=[f\cdot\sigma\cdot f'],\\
&M_{7\sim15}=(x\cdot f\cdot\sigma\cdot f'\cdot y),\\
&M_{16}=[\sigma\cdot f](P\cdot f'\cdot q'),\\
&M_{17}=[\sigma\cdot f](q\cdot f'\cdot q'),\\
&M_{18}=[\sigma\cdot f'](P\cdot f\cdot q),\\
&M_{19}=[\sigma\cdot f'](q'\cdot f\cdot q),\\
&M_{20}=[f'\cdot\bar{f}]\gamma_5,
\end{aligned} \quad (8)$$

where x and y can be any one taken from (4). The abbreviations for contraction of tensors are as follows

$$\begin{aligned}
&[A\cdot B]=A_{\mu\nu}B_{\nu\mu}, \quad (x\cdot A\cdot y)=x_\mu A_{\mu\nu}y_\nu,\\
&(A\cdot B)_{\mu\nu}=A_{\mu\lambda}B_{\lambda\nu}.
\end{aligned} \quad (9)$$

f and f' mean the electromagnetic field component for the photon q and q', respectively; $\bar{f}$ is the dual of f.

Finally a remark is due about the derivation of the electric polarizability. The most general form of the S matrix element should be put as

$$\begin{aligned}
S=i\int dq\,dq'\delta(p'+q'-p-q)\\
\times\sum_{i=1}^{20}A_i(\bar{u}'(p')M_iu(p)),
\end{aligned} \quad (10)$$

where A_i is a function of scalars listed in (5). In the limit of constant electrostatic field all components vanish except for

$$f_{i4}=E_i\delta(q),\ f'_{i4}=-E_i\delta(q'),\ i=1,\ 2,\ 3. \quad (11)$$

It turns out that only M_1 and M_2 make contribution. On the other hand, the definition of the electric polarizability α appears in the S matrix element in the form

$$\begin{aligned}
S&=-i\int dx H_{eff}(x)\\
&=-i\int dx\bar{\psi}(x)\frac{\alpha}{2}\boldsymbol{E}^2\psi(x),
\end{aligned} \quad (12)$$

where H_{eff} is the effective Hamiltonian in the nonrelativistic limit. By comparing (12) with (10), we have

$$\alpha=\frac{2}{(2\pi)^4}(2A_1-m^2A_2). \quad (13)$$

1) Further literatures will be found in: L. L. Foldy, Phys. Rev. Letters **3** (1959), 105; V. I. Goldansky, O. A. Karpukhin, A. V. Kutsenko and V. V. Pavlovskaya, Nuclear Phys. **18** (1960), 473.
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Theory of Polaron Mobility

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The polaron mobility is calculated by making use of the general theory of electrical conductivity. We take the states determined by Feynman's trial action as the unperturbed states and treat the difference between the true action and the trial action as a perturbation. Numerical values of the polaron mobility at very low temperatures are given and are discussed in comparison with the results obtained by Shultz and by Morita.

§ 1. Introduction

We have investigated the static properties of polaron at finite temperatures in the previous paper[1] (hereafter referred to as I). We shall investigate the polaron mobility in the present paper. In the same way as in I, we calculate the polaron mobility by using Feynman's path-integral method. In the polaron problem, the interaction between an electron and lattice vibrations is so large that the usual perturbation theoretic treatment fails. Therefore the calculation of the mobility in the present paper will be based on the general theory of electrical conductivity recently developed by many authors,[2] which is applicable to a system which does not allow us to set up the Boltzmann equation. We take the states determined by Feynman's trial action as unperturbed states and treat the difference between the trial action and the true action as a perturbation which yields a decay of electronic current correlation.

The following assumptions will be made in the present paper. i) The electrical conductivity is determined by the asymptotic form of a correlation function of electronic current. ii) This asymptotic form has the property of exponential decay in time. These assumptions are valid in the case of weak interaction. It is not sure whether or not they are applicable to the present problem which includes the case of strong interaction. In the present treatment, however, the states determined by Feynman's trial action instead of the states of a free electron are chosen as the unperturbed states. It is expected that the perturbation in the former case may be made weaker than in the latter case and that the above mentioned assumptions may be valid. In fact the self-energy of polaron at 0°K calculated by using these assumptions agrees with that obtained by Feynman[3] as will be shown in § 3. Therefore, the approxi-

mation used in the present calculation for the mobility may not be so bad even for strong interactions.

In § 2, the calculation of electrical conductivity, using the path-integral method, is discussed. Then, it is applied to the calculation of the polaron mobility in the case of a weak interaction. In § 3, we shall calculate the polaron mobility in the case of strong interaction based on the evaluation in § 2. In § 4, the result will be discussed in comparison with the results by Shultz and by Morita.

§ 2. Calculation of the polaron mobility

(in the case of weak interaction)

In this paper, we take the units so that $\hbar=1$, the electron mass $m=1$, and the optical phonon frequency $=1$.

The conductivity σ is generally expressed by a current correlation function $\chi(t)$ in the following way:

$$\sigma=\beta\int_0^\infty \chi(t)\,dt, \tag{1}$$

$$\chi(t)=\langle j_x(t)j_x+j_xj_x(t)\rangle/2, \tag{2}$$

where $\beta=1/kT$, j_x is the x component of electronic current operator and

$$j_x(t)=e^{-tHt}j_xe^{tHt},$$

H being the Hamiltonian of the system. In this equation and in what follows $\langle A\rangle$ denotes $\mathrm{Tr}\,\rho A/\mathrm{Tr}\,\rho$ where ρ is the density matrix defined by $e^{-\beta H}$. The following equation is easily derived:

$$\begin{aligned}&\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_xe^{iHt}j_x)\\&=-e^2\iint d\boldsymbol{X}_1\,d\boldsymbol{X}_2\,(\partial/\partial X_{2x}^{\beta-\lambda})\,(\partial/\partial X_{1x}^{\lambda})\left\{\iint d\boldsymbol{Q}_1\,d\boldsymbol{Q}_2\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\,\boldsymbol{X}_2\boldsymbol{Q}_2\,;\,\beta-\lambda)\right.\\&\left.\times\rho(\boldsymbol{X}_2\boldsymbol{Q}_2,\,\boldsymbol{X}_1\boldsymbol{Q}_1\,;\,\lambda)\right\},\end{aligned} \tag{3}$$

where e is the electronic charge, $\lambda=-it$, and $\boldsymbol{X}$ is the electron coordinate, $\boldsymbol{Q}$ denotes the many-dimensional coordinates of phonons, and $\partial/\partial X_{2x}^{\beta-\lambda}$ operates only on $\boldsymbol{X}_2$ in $\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\,\boldsymbol{X}_2\boldsymbol{Q}_2\,;\,\beta-\lambda)$ and not on $\boldsymbol{X}_2$ in $\rho(\boldsymbol{X}_2\boldsymbol{Q}_2,\,\boldsymbol{X}_1\boldsymbol{Q}_1\,;\,\lambda)$. In what follows, we try to rewrite the density matrices by the use of the path-integral method.

Let us take the Hamiltonian as follows:

$$H=(p^2/2)+\sum_{\boldsymbol{k}}M\omega_{\boldsymbol{k}}q_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}+\sum_{\boldsymbol{k}}M\omega_{\boldsymbol{k}}{}^2q_{\boldsymbol{k}}{}^2/2+\sum_{\boldsymbol{k}}p_{\boldsymbol{k}}{}^2/2M, \tag{4}$$

where $\gamma_{\boldsymbol{k}}(\boldsymbol{X})$ is some function of the electronic coordinate $\boldsymbol{X}$, and $p_{\boldsymbol{k}}$ are the coordinate and its conjugate momentum of the $\boldsymbol{k}$-th normal mode of lattice

vibrations, respectively, and M is the reduced mass of the lattice ions. In the polaron problem, $\gamma_k=(2^{3/2}\alpha\pi/V)^{1/2}\cdot 1/k\cdot e^{ik\cdot X}$ and $\omega_k=1$ for all k, where V is the volume of the crystal and α is the coupling constant used in I. The density matrix ρ in the path-integral representation is given by

$$\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\ \boldsymbol{X}_2\boldsymbol{Q}_2;\ u)\{\prod_k (M\omega_k/2\pi\sinh\omega_k u)\}^{-1/2}$$

$$=\int\exp\Big[-\Big\{\int_0^u \frac{1}{2}\dot{\boldsymbol{X}}^2(t)\,dt+\sum_k P(q_{1k},\ q_{2k},\ \boldsymbol{X}(t))\Big\}\Big]\mathfrak{D}\begin{pmatrix}\boldsymbol{X}(t):\boldsymbol{X}_2(u)\\ \boldsymbol{X}_1(0)\end{pmatrix}, \quad (5)$$

where

$$P(q_{1k},\ q_{2k},\ \boldsymbol{X}(t))=[(q_{1k}^2+q_{2k}^2)\cosh\omega_k u-2q_{1k}\cdot q_{2k}$$
$$+2A_k\cdot q_{1k}+2B_k\cdot q_{2k}+2C_k]/2\sinh\omega_k u, \quad (6a)$$

and

$$A_k=\int_0^u\gamma_k(\boldsymbol{X}(t))\sinh\omega_k t\,dt,\quad B_k=\int_0^u\gamma_k(\boldsymbol{X}(t))\sinh\omega_k(u-t)\,dt,$$

$$C_k=-\int_0^u dt\int_0^t ds\,\gamma_k(\boldsymbol{X}(t))\gamma_k(\boldsymbol{X}(s))\sinh\omega_k(u-t)\sinh\omega_k s. \quad (6b)$$

$\mathfrak{D}\begin{pmatrix}\boldsymbol{X}(t):\boldsymbol{X}_2(u)\\ \boldsymbol{X}_1(0)\end{pmatrix}$ means that the path-integration must be carried out under the conditions $\boldsymbol{X}(0)=\boldsymbol{X}_1$ and $\boldsymbol{X}(u)=\boldsymbol{X}_2$.

Using (5) and (6), we obtain, after some calculations, the following equation for the polaron problem:

$$(e^{\beta/2}/e^{\beta}-1)^{-3N}\iint d\boldsymbol{Q}_1\,d\boldsymbol{Q}_2\,\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\ \boldsymbol{X}_2\boldsymbol{Q}_2;\ \beta-\lambda)\rho(\boldsymbol{X}_2\boldsymbol{Q}_2,\ \boldsymbol{X}_1\boldsymbol{Q}_1:\lambda)$$

$$=\iint\exp\Big[-\int_0^{\beta-\lambda}\frac{1}{2}\dot{\boldsymbol{X}}'^2(t)\,dt-\int_0^{\lambda}\frac{1}{2}\dot{\boldsymbol{X}}''^2(t)\,dt+Q(\boldsymbol{X}'\boldsymbol{X}'')\Big]\mathfrak{D}\begin{pmatrix}\boldsymbol{X}'(t):\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{pmatrix}$$

$$\times\mathfrak{D}\begin{pmatrix}\boldsymbol{X}''(t):\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{pmatrix}, \quad (7a)$$

where N is the total number of unit cells in the crystal and

$$Q(\boldsymbol{X}',\ \boldsymbol{X}'')=Q_1(\boldsymbol{X}'\boldsymbol{X}'';\ \beta\lambda)+S_{cross}(\boldsymbol{X}'\boldsymbol{X}'';\ \beta\lambda).$$

Q_1 and S_{cross} are given by

$$Q_1(\boldsymbol{X}'\boldsymbol{X}'';\ \beta,\ \lambda)V(e^{\beta}-1)/2^{3/2}\alpha\pi$$

$$=\sum_w\frac{1}{w^2}\Big[\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma\,I_w(\boldsymbol{X}''(\tau),\ \boldsymbol{X}''(\sigma)\ ;\ 1)+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma\,I_w(\boldsymbol{X}''(\tau),\ \boldsymbol{X}'(\sigma)\ ;\ 1)\Big], \quad (7b)$$

$$S_{cross}(\boldsymbol{X}'\boldsymbol{X}''\,;\,\beta\lambda)\,V(e^{\beta}-1)/2^{3/2}\alpha\pi=\Big[\sum_{w}\frac{1}{w^2}\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma)\,;\,1)\Big], \tag{7c}$$

where

$$I_w(\boldsymbol{x}(\tau),\boldsymbol{y}(\sigma)\,;\,\chi)=e^{\chi(\beta-\tau+\sigma)-iw(\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma))}+e^{\chi(\tau-\sigma)+iw(\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma))}. \tag{7d}$$

In this section, we hereafter confine ourselves to the case in which the interaction of the electron with phonons can be treated as a small perturbation. In order to calculate the mobility in such a case, we shall make the approximations that (1) the correlation function of electronic current is assumed to have the property of simple exponential decay in time and that (2) all averages over the canonical distribution are replaced by those of the unperturbed system. Substituting (7a) in (3) and omitting the unimportant numerical factor $(e^{\beta/2}/e^{\beta}-1)^{-3N}$ which is cancelled by the normalization factor for phonon, we get

$$\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_x e^{iHt}j_x)=-e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})\,(\partial/\partial X_{1x}^{\lambda})G(\boldsymbol{X}_1,\,\boldsymbol{X}_2), \tag{8a}$$

where

$$\begin{aligned}G(\boldsymbol{X}_1,\,\boldsymbol{X}_2)=&\iint\mathfrak{D}\begin{pmatrix}\boldsymbol{X}':\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{pmatrix}\mathfrak{D}\begin{pmatrix}\boldsymbol{X}'':\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{pmatrix}\\ &\times\exp\Big[-\int_0^{\beta-\lambda}\frac{1}{2}\,\dot{\boldsymbol{X}}'^2(t)dt-\int_0^{\lambda}\frac{1}{2}\,\dot{\boldsymbol{X}}''^2(t)dt\Big]\times\Big[1+(2^{3/2}\pi\alpha/(e^{\beta}-1)V)\\ &\times\Big\{\sum_{w}\frac{1}{w^2}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma\, I_w(\boldsymbol{X}''(\tau),\,\boldsymbol{X}''(\sigma):1)\\ &+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}'(\sigma):1)+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma):1)\Big\}\Big\}\Big].\end{aligned} \tag{8b}$$

It should be noted here that the approximation

$$\exp[Q(\boldsymbol{X}',\,\boldsymbol{X}'')]\simeq 1+Q(\boldsymbol{X}',\,\boldsymbol{X}'') \tag{8c}$$

was used to obtain (8b). As is seen from (7b), $Q(\boldsymbol{X}'\boldsymbol{X}'')$ is proportional to the coupling constant α which is small in the present case.

The perturbational treatment in the present path-integral method consists of replacing $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma)))$, $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma)))$, and $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma)))$ by their average values over the path with a free action $-\int_0^{\lambda}1/2\cdot\dot{\boldsymbol{X}}''^2(t)\,dt$ and $-\int_0^{\beta-\lambda}1/2\cdot\dot{\boldsymbol{X}}'^2(t)\,dt$. In the same way as in I, these average values are given by

$$\langle e^{\pm(i\boldsymbol{w}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma))}\rangle_{\boldsymbol{X}_2\boldsymbol{X}_1;\lambda}$$
$$=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)(\tau-\sigma)/\lambda)\exp\left(-\frac{w^2}{2}(\tau-\sigma)\left(1-\frac{(\tau-\sigma)}{\lambda}\right)\right),$$
$$\langle e^{\pm(i\boldsymbol{w}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma))}\rangle_{\boldsymbol{X}_1\boldsymbol{X}_2;\beta-\lambda}$$
$$=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)(\tau-\sigma)/\beta-\lambda)\exp\left(-\frac{w^2}{2}(\tau-\sigma)\left(1-\frac{(\tau-\sigma)}{\beta-\lambda}\right)\right),$$
$$\langle e^{\pm i\boldsymbol{w}\cdot\boldsymbol{X}'(\tau)}\rangle_{\boldsymbol{X}_1\boldsymbol{X}_2;\beta-\lambda}=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)\tau/\beta-\lambda)\exp\left(-\frac{w^2}{2}\tau\left(1-\frac{\tau}{\beta-\lambda}\right)\right),$$
$$\langle e^{\pm i\boldsymbol{w}\cdot\boldsymbol{X}''(\sigma)}\rangle_{\boldsymbol{X}_2\boldsymbol{X}_1;\lambda}=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_1-\boldsymbol{X}_2)\sigma/\lambda)\exp\left(\frac{w^2}{2}\sigma\left(1+\frac{\sigma}{\lambda}\right)\right). \tag{9}$$

We shall give another method to calculate the average values of Eq. (9) in Appendix I, which will be used in the calculation in § 3. Noting that the identity

$$-\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{1x}^{\beta-\lambda})f_{\beta-\lambda}(\boldsymbol{X}_1,\boldsymbol{X}_2)(\partial/\partial X_{2x}^{\lambda})f_\lambda(\boldsymbol{X}_2,\boldsymbol{X}_1)$$
$$=\iint d\boldsymbol{K}_1 d\boldsymbol{K}_2 K_{1x}K_{2x}g_{\beta-\lambda}(\boldsymbol{K}_1,\boldsymbol{K}_2)g_\lambda(\boldsymbol{K}_2,\boldsymbol{K}_1) \tag{10a}$$

holds with

$$f_\lambda(\boldsymbol{X}_1\boldsymbol{X}_2)=(2\pi)^{-3}\int g_\lambda(\boldsymbol{K}_1,\boldsymbol{K}_2)\,e^{i(\boldsymbol{K}_1\boldsymbol{X}_1-\boldsymbol{K}_2\boldsymbol{X}_2)}d\boldsymbol{K}_1 d\boldsymbol{K}_2$$

and that the density matrix $\rho_0(\boldsymbol{X}_1\boldsymbol{X}_2;u)$ for a free electron is $(V/2\pi/u)^3$ $\times\exp(-(\boldsymbol{X}_1-\boldsymbol{X}_2)^2/2u)$, and furthermore, using (8) and (9) one can obtain

$$\mathrm{Tr}(e^{-\beta H}e^{-tHt}j_x e^{iHt}j_x)=\sum_{\boldsymbol{k}}e^2k_x^2e^{-k^2\beta/2}\Big[1+\sum_{w}\{2^{3/2}\alpha/Vw^2(e^\beta-1)\}$$
$$\times\Big\{\int_0^\lambda d\tau(\lambda-\tau)(e^{\beta-\tau(1-k\cdot w+w^2/2)}+e^{-\tau(-1+k\cdot w+w^2/2)})$$
$$+\int_0^{\beta-\lambda}d\tau(\beta-\lambda-\tau)(e^{\beta-\tau(1-k\cdot w+w^2/2)}+e^{-\tau(-1+k\cdot w+w^2/2)})$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{\beta-(\tau-\sigma)(1-k\cdot w+w^2/2)}+e^{(\tau-\sigma)(-1+k\cdot w+w^2/2)})\Big\}\Big]. \tag{11}$$

The first four terms appearing in brackets in the right-hand side of Eq. (11) are simplified by using the formulae for a large time t

$$\sum_{\boldsymbol{w}}F(\boldsymbol{w})\Big[\int_0^{\beta+it}ds(\beta+it-s)e^{-f(\boldsymbol{w})s}+\int_0^{-it}ds(-it-s)\,e^{-f(\boldsymbol{w})s}\Big]\simeq 2\pi t\sum_{\boldsymbol{w}}F(\boldsymbol{w})\delta(f(\boldsymbol{w})), \tag{A·2·1}$$

which will be proved in Appendix II, Here $F(\omega)$ and $f(\omega)$ are some functions of ω. The remaining terms which come from the terms S_{cross} of Eq. (7c) become quantities being independent of t for a large time t as will be shown in Appendix II. Then, one obtains

$$\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_x e^{iHt}j_x) \simeq \sum_k e^2 k_x^2 e^{-k^2\beta/2}(1-t/\tau_k) \tag{12a}$$

for a large time t, where

$$1/\tau_k = \{2\pi\cdot 2^{3/2}\alpha\pi/V(e^\beta-1)\}\sum_w \frac{1}{w^2}\left\{e^\beta\,\delta\left(1-\boldsymbol{k}\cdot\boldsymbol{w}+\frac{w^2}{2}\right)+\delta\left(-1+\boldsymbol{k}\cdot\boldsymbol{w}+\frac{w^2}{2}\right)\right\}. \tag{12b}$$

This expression of τ_k agrees with the collision time obtained by the ordinary perturbation theory with the electron-phonon interaction as a perturbing Hamiltonian. In the perturbation theory, the energy shift ΔE_n and the collision time τ_n of an unperturbed state are given, respectively, by

$$\Delta E_n = P\sum_m |\langle m|H'|n\rangle|^2/(E_n^0-E_m^0),$$

$$1/2\pi\tau_n = \sum_m \delta(E_n^0-E_m^0)|\langle m|H'|n\rangle|^2,$$

where H' is the perturbing Hamiltonian and E_m^0 is the energy of the unperturbed state m. Comparing these two equations with one another, it is expected that the self energy ΔE_k corresponding to τ_k given by Eq. (12) becomes

$$\Delta E_k = \{2^{3/2}\alpha\pi/(e^\beta-1)V\}\sum_w \{e^\beta P/(1-\boldsymbol{k}\cdot\boldsymbol{w}+w^2/2)+P/(-1+\boldsymbol{k}\cdot\boldsymbol{w}+w^2/2)\}. \tag{13}$$

This result also agrees with that obtained by the ordinary perturbation treatment.

Adopting the assumption that the asymptotic form of a correlation function of electronic current has the property of simple exponential decay in time and using of Eqs. (1), (2) and (12a), we get, as the expression of the mobility μ_0,

$$\mu_0 = e\beta\sum_k k_x^2 e^{-(k^2/2)\beta}\int_0^\infty e^{-t/\tau_k}dt/\sum_k e^{-(k^2/2)\beta}.$$

Substituting the value τ_k of Eq. (12b) in this equation, the mobility μ_0 at very low temperatures is given by

$$\mu_0 = e/2\bar{n}\alpha, \tag{14}$$

where

$$\bar{n} = 1/(e^\beta-1).$$

§ 3. Calculation of the polaron mobility

(in the case of strong interaction)

We shall calculate the polaron mobility in the case of strong interaction by using the evaluation in § 2. We take states determined by Feynman's trial action as the unperturbed states and treat the difference between it and the true action as a perturbation.

We shall start with the approximate expression

$$\mathrm{Tr}(e^{-(\beta+it)H} j_x e^{iHt} j_x) = -e^2 \iint d\boldsymbol{X}_1 d\boldsymbol{X}_2 (\partial/\partial X_{2x}^{\beta-\lambda})(\partial/\partial X_{1x}^{\lambda}) G(\boldsymbol{X}_1, \boldsymbol{X}_2), \quad (15a)$$

where

$$G(\boldsymbol{X}_1, \boldsymbol{X}_2) = \iint \mathfrak{D}\left(\boldsymbol{X}'(t) : \begin{matrix}\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{matrix}\right) \mathfrak{D}\left(\boldsymbol{X}''(t) : \begin{matrix}\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{matrix}\right)$$
$$\times \exp[S_{0,\beta-\lambda}(\boldsymbol{X}') + S_{0,\lambda}(\boldsymbol{X}'') + S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}', \boldsymbol{X}'')](1 + F(\boldsymbol{X}'\boldsymbol{X}''; \beta, \lambda)), \quad (15b)$$

and

$$F(\boldsymbol{X}'\boldsymbol{X}''; \beta, \lambda) = 2^{-3/2}\alpha(e^{\beta}-1)^{-1}\Big\{\int_0^{\lambda} d\tau \int_0^{\tau} d\sigma F_1(\boldsymbol{X}''(\tau), \boldsymbol{X}''(\sigma))$$
$$+ \int_0^{\beta-\lambda} d\tau \int_0^{\tau} d\sigma F_1(\boldsymbol{X}'(\tau), \boldsymbol{X}'(\sigma)) + \int_0^{\beta-\lambda} d\tau \int_0^{\lambda} d\sigma F_1(\boldsymbol{X}'(\tau), \boldsymbol{X}''(\sigma))\Big\}$$
$$+ C(e^{\beta\omega}-1)^{-1}/2\Big\{\int_0^{\lambda} d\tau \int_0^{\tau} d\sigma F_2(\boldsymbol{X}''(\tau), \boldsymbol{X}''(\sigma))$$
$$+ \int_0^{\beta-\lambda} d\tau \int_0^{\tau} d\sigma F_2(\boldsymbol{X}'(\tau), \boldsymbol{X}'(\sigma)) + \int_0^{\beta-\lambda} d\tau \int_0^{\lambda} d\sigma F_2(\boldsymbol{X}'(\tau), \boldsymbol{X}''(\sigma)\Big\}, \quad (15c)$$

where

$$S_{0u}(\boldsymbol{X}) = -\frac{1}{2}\int_0^u \dot{\boldsymbol{X}}^2(t)\,dt - C(e^{\beta\omega}-1)^{-1}/2 \int_0^u d\tau \int_0^{\tau} d\sigma F_2(\boldsymbol{X}(\tau), \boldsymbol{X}(\sigma)), \quad (15d)$$

$$S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}'\boldsymbol{X}'') = C(e^{\beta\omega}-1)^{-1}/2 \int_0^{\beta-\lambda} d\tau \int_0^{\lambda} d\sigma F_2(\boldsymbol{X}(\tau), \boldsymbol{X}(\sigma)), \quad (15e)$$

$$F_1(\boldsymbol{x}(\tau), \boldsymbol{y}(\sigma)) = |\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma)|^{-1}(e^{(\beta-\tau+\sigma)} + e^{(\tau-\sigma)}), \quad (15f)$$

$$F_2(\boldsymbol{x}(\tau), \boldsymbol{y}(\sigma)) = |\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma)|^{2}(e^{\omega(\beta-\tau+\sigma)} + e^{\omega(\tau-\sigma)}). \quad (15g)$$

Here S_{0u} denotes Feynman's trial action. To derive (15) from (3), we considered $F(\boldsymbol{X}'\boldsymbol{X}''; \beta\lambda)$ small, since it corresponds to the difference between the true action and the trial action, and we used an approximation similar to (8c)

in the case of weak interaction. C and ω are variational parameters which should be determined so as to minimize the free energy of the system.

Let us first consider the term which does not include the terms $F(\boldsymbol{X}'\boldsymbol{X}'';\beta\lambda)$ in Eq. (15b), that is,

$$-e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})(\partial/\partial X_{1x}^{\lambda})\iint \mathfrak{D}\left((\boldsymbol{X}'(t):\begin{matrix}\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{matrix}\right)$$
$$\times\mathfrak{D}\left(\boldsymbol{X}''(t):\begin{matrix}\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{matrix}\right)e^{\bar{S}_0(\boldsymbol{X}'\boldsymbol{X}'')}, \tag{16}$$

where

$$\bar{S}_0(\boldsymbol{X}'\boldsymbol{X}'')=S_{0,\beta-\lambda}(\boldsymbol{X}')+S_{0,\lambda}(\boldsymbol{X}'')+S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}'\boldsymbol{X}'').$$

As was shown in I, the states determined by the trial action S_0 are equivalent to the states of a system with the Lagrangian

$$L_0=(\dot{\boldsymbol{X}}^2+M\dot{\boldsymbol{q}}^2-\kappa(\boldsymbol{q}-\boldsymbol{X})^2)/2, \tag{17}$$

where

$$\kappa=M\omega^2 \quad\text{and}\quad M=4C/\omega^2.$$

The Hamiltonian H_0 corresponding to the Lagrangian L_0 is expressed by

$$H_0=-\frac{1}{2m_0}\partial^2/\partial\boldsymbol{x}_0^2+\left(-\frac{1}{2\eta}\partial^2/\partial\boldsymbol{r}^2+\frac{\kappa}{2}\boldsymbol{r}^2\right), \tag{18}$$

where the new variables $\boldsymbol{x}_0$ and $\boldsymbol{r}$ defined by

$$\boldsymbol{x}_0=(M\boldsymbol{q}+\boldsymbol{X})/(M+1),\quad \boldsymbol{r}=\boldsymbol{X}-\boldsymbol{q} \tag{19}$$

are used instead of X and q, and also

$$m_0=\nu^2/\omega^2,\quad \eta=4C/\nu^2.$$

The energy eigenvalues of this Hamiltonian are given by

$$E_0(\boldsymbol{k},\, n_1n_2n_3)=\boldsymbol{k}^2/2m_0+(n_1+n_2+n_3+3/2)\nu, \tag{20}$$

where $\boldsymbol{k}$ is the wave number specifying a free motion for $\boldsymbol{x}_0$ and n_1, n_2 and n_3 are the quantum numbers of the harmonic oscillator for $\boldsymbol{r}$.

It is not so difficult to show, by performing calculations similar to that used in § 2, that

$$\text{the quantity (16)}=\sum_{mm'}e^{-(\beta-\lambda)E_m^0}e^{-\lambda E_{m'}^0}(m|j_x|m')(m'|j_x|m)$$
$$=\mathrm{Tr}(e^{-\beta H_0}e^{-iH_0t}j_xe^{iH_0t}j_x), \tag{21}$$

where we simply express the eigenstates $|\boldsymbol{k}\,n_1n_2n_3)$ and their eigenvalues $E_0(\boldsymbol{k}n_1n_2n_3)$ by $|m)$ and E_m^0, respectively. Using Eq. (21) and noting that

$$\partial/\partial\boldsymbol{X}=(m_0)^{-1}\partial/\partial\boldsymbol{x}_0+\partial/\partial\boldsymbol{r}, \tag{22}$$

we obtain that

$$\text{the quantity (16)}=\sum_{\boldsymbol{k}n_1n_2n_3}e^2\exp\{-\beta E_0(\boldsymbol{k}n_1n_2n_3)\}[kx^2/m_0^2$$
$$+(\kappa\eta)^{1/2}(n_1+1)e^{-\lambda\nu}/2+(\kappa\eta)^{1/2}n_1e^{\lambda\nu}/2]. \tag{23}$$

Next, we consider the term including $F(\boldsymbol{X}'\boldsymbol{X}''\,;\beta\lambda)$ in Eq. (15). This quantity is transformed into

$$-e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})\,(\partial/\partial \boldsymbol{X}_{1x}^{\lambda})\Big[\iint \mathfrak{D}\Big(\boldsymbol{X}'(t):\frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\Big)\mathfrak{D}\Big(\boldsymbol{X}''(t):\frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\Big)\Big]e^{\bar{S}_0(\boldsymbol{X}'\boldsymbol{X}'')}$$

$$\times\Big[\sum_{\boldsymbol{u}} C_{\boldsymbol{u}}{}^2(e^{\beta}-1)^{-1}\Big\{\int_0^{\lambda} d\tau\int_0^{\tau} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}''(\tau),\,\boldsymbol{X}''(\sigma):1)+\int_0^{\beta-\lambda} d\tau\int_0^{\lambda} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}'(\tau),\,\boldsymbol{X}'(\sigma):1)$$

$$+\int_0^{\beta-\lambda} d\tau\int_0^{\lambda} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma)\,;1)\Big\}+\sum_{\boldsymbol{u}} D_{\boldsymbol{u}}{}^2(e^{\beta\omega}-1)^{-1}\Big\{\int_0^{\lambda} d\tau\int_0^{\tau} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}''(\tau),\,\boldsymbol{X}''(\sigma)\,;\omega)$$

$$+\int_0^{\beta-\lambda} d\tau\int_0^{\tau} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}'(\tau),\,\boldsymbol{X}'(\sigma)\,;\omega)+\int_0^{\beta-\lambda} d\tau\int_0^{\lambda} d\sigma\, I_{\boldsymbol{u}}(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma)\,;\omega)\Big\}\Big], \qquad (24)$$

where we used the notation of Eq. (7d) and the identities

$$|\boldsymbol{X}_1-\boldsymbol{X}_2|^{-1}=4\pi\sum_{\boldsymbol{u}} e^{i\boldsymbol{u}\cdot(\boldsymbol{X}_1-\boldsymbol{X}_2)}/V\boldsymbol{u}^2,$$

$$|\boldsymbol{X}_1-\boldsymbol{X}_2|^{2}=-8\pi^3\sum_{\boldsymbol{u}}\delta(\boldsymbol{u})\boldsymbol{\Gamma}_{\boldsymbol{u}}{}^2 e^{i\boldsymbol{u}\cdot(\boldsymbol{X}_1-\boldsymbol{X}_2)}/V$$

and the definitions

$$C_{\boldsymbol{u}}=\sqrt{2^{3/2}\pi\alpha/V\boldsymbol{u}},\quad D_{\boldsymbol{u}}=-i\sqrt{2^2\pi^3 C\delta(\boldsymbol{u})\boldsymbol{\Gamma}_{\boldsymbol{u}}{}^2/V}. \qquad (25)$$

Now we replace $e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma))}$, $e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma))}$ and $e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma))}$ by their average values $\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma))}}$, $\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma))}}$ and $\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma))}}$ taken along the path with the action $S_{0,\beta-\lambda}(\boldsymbol{X}')+S_{0,\lambda}(\boldsymbol{X}'')+S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}'\boldsymbol{X}'')$ as was done in § 2. These average values are calculated as follows. Noting that

$$\iint\mathfrak{D}\Big(\boldsymbol{X}'(t):\frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\Big)\mathfrak{D}\Big(\boldsymbol{X}''(t):\frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\Big)e^{\bar{S}_0(\boldsymbol{X}',\boldsymbol{X}'')}$$

$$=\sum_{mm'}\iint d\boldsymbol{q}_1 d\boldsymbol{q}_2 e^{-(\beta-\lambda)E_m^0}e^{\lambda E_{m'}^0}\psi_m^{0*}(\boldsymbol{X}_1\boldsymbol{q}_1)\psi_m{}^0(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^{0*}(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^0(\boldsymbol{X}_1\boldsymbol{q}_1),$$

where $\psi_m{}^0(\boldsymbol{X}_1\boldsymbol{q}_1)$ is the eigen-function of H_0, and making use of Eqs. (A·1·1) and (A·1·2), it is shown that

$$\Big[\iint\mathfrak{D}\Big(\boldsymbol{X}'(t):\frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\Big)\mathfrak{D}\Big(\boldsymbol{X}''(t):\frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\Big)e^{\bar{S}_0(\boldsymbol{X}',\boldsymbol{X}'')}\Big]\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma))}}$$

$$=\sum_{mm'}\iint d\boldsymbol{q}_1 d\boldsymbol{q}_2 e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(\boldsymbol{X}_1\boldsymbol{q}_1)\psi_m{}^0(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^{0*}(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^0(\boldsymbol{X}_1\boldsymbol{q}_1)$$

$$\times\langle m'|e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\tau)}e^{\mp i\boldsymbol{u}\cdot\boldsymbol{X}(\sigma)}|m'\rangle, \qquad (26a)$$

$$\Big[\iint\mathfrak{D}\Big(\boldsymbol{X}'(t):\frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\Big)\mathfrak{D}\Big(\boldsymbol{X}''(t):\frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\Big)e^{\bar{S}_0(\boldsymbol{X}',\boldsymbol{X}'')}\Big]\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma))}}$$

$$=\sum_{mm'}e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(\boldsymbol{X}_1\boldsymbol{q}_1)\psi_m{}^0(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^{0*}(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^0(\boldsymbol{X}_1\boldsymbol{q}_1)$$

$$\times\langle m|e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\tau)}e^{\mp i\boldsymbol{u}\cdot\boldsymbol{X}(\sigma)}|m\rangle, \qquad (26b)$$

and

$$\left[\iint \mathfrak{D}\left(\boldsymbol{X}'(t):\begin{matrix}\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{matrix}\right)\mathfrak{D}\left(\boldsymbol{X}''(t):\begin{matrix}\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{matrix}\right)e^{\bar{S}_0(X',X'')}\right]\overline{e^{\pm iu\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma))}}$$

$$=\lim_{\sigma'\to 0,\tau'\to 0}\sum_{mm'}\iint d\boldsymbol{q}_1 d\boldsymbol{q}_2 e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(\boldsymbol{X}_1\boldsymbol{q}_1)\psi_m{}^0(\boldsymbol{X}_2\boldsymbol{q}_2)$$

$$\times\psi_{m'}^{0*}(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^0(\boldsymbol{X}_1\boldsymbol{q}_1)\langle m|e^{\pm iu\cdot\boldsymbol{X}(\tau)}e^{\mp iu\cdot\boldsymbol{X}(\tau')}|m\rangle\langle m'|e^{\pm iu\cdot\boldsymbol{X}(\sigma')}e^{\mp iu\cdot\boldsymbol{X}(\sigma)}|m'\rangle, \tag{26c}$$

where

$$e^{\pm iu\cdot\boldsymbol{X}(\tau)}=e^{H_0\tau}e^{\pm iu\cdot\boldsymbol{X}}e^{-H_0\tau}.$$

The quantity (24) can be replaced, using of Eq. (26), by

$$\sum_u C_u{}^2(e^{\beta}-1)^{-1}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma(e^{(\beta-\tau+\sigma)}f_u^{\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}f_u^{\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma(e^{(\beta-\tau+\sigma)}f_u^{\beta-\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}f_u^{\beta-\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{(\beta-\tau+\sigma)}g_u^{\lambda,\beta-\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}g_u^{\lambda,\beta-\lambda,-}(\tau,\sigma))\Big\}$$

$$+\sum_u D_u{}^2(e^{\beta\omega}-1)^{-1}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma(e^{\omega(\beta-\tau+\sigma)}f_u^{\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}f_u^{\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma(e^{\omega(\beta-\tau+\sigma)}f_u^{\beta-\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}f_u^{\beta-\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{\omega(\beta-\tau+\sigma)}g_u^{\lambda,\beta-\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}g_u^{\lambda,\beta-\lambda,-}(\tau,\sigma))\Big\}, \tag{27a}$$

where

$$f_u^{\lambda,\pm}(\tau,\sigma)=\sum_{kn_1n_2n_3}e^{-\beta E_0(kn_1n_2n_3)}e^2\{k_x{}^2/m_0{}^2+\sqrt{\kappa\eta}(n_1+1)e^{-\lambda\nu}/2$$

$$+\sqrt{\kappa\eta}\,n_1e^{\lambda\nu}/2\}h_u^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma), \tag{27b}$$

$$f_u^{\beta-\lambda,\pm}(\tau,\sigma)=\sum_{kn_1n_2n_3}e^{-\beta E_0(kn_1n_2n_3)}e^2\{k_x{}^2h_u^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma)/m_0{}^2$$

$$+(n_1+1)\sqrt{\kappa\eta}\,e^{-\lambda\nu}h_u^{\pm}(\boldsymbol{k},n_1+1,n_2n_3;\tau\sigma)/2$$

$$+n_1\sqrt{\kappa\eta}\,e^{\lambda\nu}h_u^{\pm}(\boldsymbol{k},n_1-1,n_2n_3;\tau\sigma)/2\}, \tag{27c}$$

and

$$g_{\boldsymbol{u}}^{\lambda,\beta-\lambda,\pm}(\tau,\sigma)=\lim_{\tau'\to 0,\sigma'\to 0}\sum_{\boldsymbol{k}n_1n_2n_3}e^{-\beta E_0(\boldsymbol{k}n_1n_2n_3)}e^2$$
$$\times\{k_x{}^2h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\tau')h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\sigma'\sigma)/m_0{}^2+(n_1+1)\sqrt{\kappa\eta}\,e^{-\lambda\nu}$$
$$\times h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\tau')h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k},n_1+1,n_2n_3;\sigma'\sigma)/2+n_1\sqrt{\kappa\eta}\,e^{\lambda\nu}$$
$$\times h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\tau')h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k},n_1-1,n_2n_3;\sigma'\sigma)/2\},\tag{27d}$$

and notation

$$h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma)=(\boldsymbol{k}n_1n_2n_3|e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\tau)}e^{\mp i\boldsymbol{u}\cdot\boldsymbol{X}(\sigma)}|\boldsymbol{k}n_1n_2n_3)\tag{27e}$$

is employed, for simplicity.

For convenience of calculations in what follows, let us consider operators $X_s(\tau)$ and $e^{i\boldsymbol{X}_0(\tau)\cdot\boldsymbol{u}}$ defined below. $X_s(\tau)$ is defined by

$$X_s(\tau)=x_{0s}(\tau)+\eta r_s(\tau)=x_{0s}(\tau)+\sqrt{\eta/2\nu}(b_s{}^*e^{-\nu\tau}+b_se^{\nu\tau}),\tag{28}$$

where the suffix s stands for one of x, y and z-components, $b_s{}^*$ and b_s denote the creation and annihilation operators of the harmonic oscillator for r_s. In Eq. (28), the first equality comes from Eq. (18) and the second equality from the definition

$$r_s(\tau)=(b_s{}^*(\tau)+b_s(\tau))/\sqrt{2\nu\eta}\,.$$

$e^{ix_0(\tau)\cdot u}$ is defined by

$$e^{ix_0(\tau)\cdot u}=e^{H_0\tau}e^{ix_0\cdot u}e^{-H_0\tau}=\exp\left(\frac{p_0{}^2-(\boldsymbol{p}_0-\boldsymbol{u})^2}{2m_0}\right)e^{ix_0\cdot u},\tag{29}$$

where $\boldsymbol{p}_0$ is the momentum conjugate to the coordinate $\boldsymbol{x}_0$. It is shown, according to Eqs. (28) and (29), that

$$h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\langle n_1n_2n_3|\prod_s\exp(\mp i\sqrt{\eta}\,u_sb_s{}^*e^{\nu\tau}/\sqrt{2\nu})\exp(\pm i\sqrt{\eta}\,u_sb_se^{-\nu\tau}/\sqrt{2\nu})$$
$$\times\prod_{s'}\exp(\pm i\sqrt{\eta}\,u_{s'}b_{s'}{}^*e^{\nu\sigma}/\sqrt{2\nu})\exp(\mp i\sqrt{\eta}\,u_{s'}b_{s'}e^{-\nu\sigma}/\sqrt{2\nu})|n_1n_2n_3\rangle.\tag{30}$$

If we are concerned only with the phenomena at low temperatures, it is sufficient to carry out the evaluation of Eq. (30) in the case of $n_1=n_2=n_3=0$. In such a case, we have

$$h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma)_{n_1=n_2=n_3=0}=\exp\{(k^2-(\boldsymbol{k}\pm\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}$$
$$\times\exp(-\eta u^2/2\nu)\sum_{l_s=0}^{\infty}\prod_s(\eta u_s{}^2)^{l_s}\exp(-\nu l_s(\tau-\sigma))/l_s!\,(2\nu)^{l_s},\tag{31a}$$

$$h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k},n_1+1,n_2n_3;\tau\sigma)_{n_1=n_2=n_3=0}=\exp\{(k^2-(\boldsymbol{k}\pm\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times[\eta u_x{}^2\sum_{l_s=0}^{\infty}\prod_s(\eta u_s{}^2)^{l_s}e^{-\nu(l_s-1)(\tau-\sigma)}/2\nu l_s!$$

$$+\sum_{l_1=0,l_2=0,l_3=0}^{\infty}(l_1+1)^2(\eta)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}$$
$$\times e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)}/(l_1+1)!\,l_2!\,l_3!\,(2\nu)^{l_1+l_2+l_3}]. \qquad (31b)$$

According to Eqs. (23), (27), (28), (29), (31a) and (31b), Eq. (15a) is rewritten, in the case of $n_1=n_2=n_3=0$, as

$$\mathrm{Tr}(e^{-(\beta+it)H}j_x e^{iHt}j_x)$$
$$=\sum_k e^{-k^2\beta/2m_0}e^2\{k_x^2/m_0^2+\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2\}+\sum_k e^{-k^2\beta/2m_0}e^2k_x^2/m_0^2$$
$$\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\Big\}$$
$$+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\Big\}\Big]$$
$$+\sum_k e^{-k^2\beta/2m_0}e^2\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2$$
$$\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}C_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}C_{uk}^-(\tau-\sigma))\Big\}$$
$$+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}C_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}C_{uk}^-(\tau-\sigma))\Big\}\Big]$$
$$+\sum_k e^{-k^2\beta/2m_0}e^2k_x^2/m_0^2$$
$$\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{(\beta-\tau+\sigma)}\overline{B_{uk}^+}(\tau,\sigma)+e^{(\tau-\sigma)}\overline{B_{uk}^-}(\tau,\sigma))\Big\}$$
$$+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau,\sigma)+e^{\omega(\tau-\sigma)}\overline{B_{uk}^-}(\tau,\sigma))\Big\}\Big]$$
$$+\sum_k e^{-k^2\beta/2m_0}\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2$$

$$\times\Big[\sum_{\boldsymbol{u}} C_{\boldsymbol{u}}{}^2(e^{\beta}-1)^{-1}\Big\{\int_0^{\beta-\lambda} d\tau\int_0^{\lambda} d\sigma(e^{(\beta-\tau+\sigma)}\overline{C_{\boldsymbol{u}\boldsymbol{k}}^{+}}(\tau,\sigma)+e^{(\tau-\sigma)}\overline{C_{\boldsymbol{u}\bar{\boldsymbol{k}}}}(\tau,\sigma))\Big\}$$

$$+\sum_{\boldsymbol{u}} D_{\boldsymbol{u}}{}^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^{\beta-\lambda} d\tau\int_0^{\lambda} d\sigma(e^{\omega(\beta-\tau+\sigma)}\overline{C_{\boldsymbol{u}\boldsymbol{k}}^{+}}(\tau,\sigma)+e^{\omega(\tau-\sigma)}\overline{C_{\boldsymbol{u}\bar{\boldsymbol{k}}}}(\tau,\sigma))\Big\}\Big], \tag{32a}$$

where

$$B_{\boldsymbol{u}\boldsymbol{k}}^{\pm}(\tau-\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\sum_{l_1,l_2,l_3=0}^{\infty} f_{\boldsymbol{u}}(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)}, \tag{32b}$$

$$\overline{B_{\boldsymbol{u}\boldsymbol{k}}^{\pm}}(\tau,\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big(\sum_{l_1,l_2,l_3=0}^{\infty} f_{\boldsymbol{u}}(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)\tau}\Big)\Big(\sum_{l_1',l_2',l_3'=0}^{\infty} f_{\boldsymbol{u}}(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3')\sigma}\Big), \tag{32c}$$

$$C_{\boldsymbol{u}\boldsymbol{k}}^{\pm}(\tau-\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big\{(\eta u_x{}^2/2\nu)\sum_{l_1,l_2,l_3=0}^{\infty} f_{\boldsymbol{u}}(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3-1)(\tau-\sigma)}+\sum_{l_1,l_2,l_3=0}^{\infty} g_{\boldsymbol{u}}(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)}\Big\}, \tag{32d}$$

$$\overline{C_{\boldsymbol{u}\boldsymbol{k}}^{\pm}}(\tau,\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big(\sum_{l_1,l_2,l_3=0}^{\infty} f_{\boldsymbol{u}}(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)\tau}\Big)\Big\{(\eta u_x{}^2/2\nu)$$
$$\times\sum_{l_1',l_2',l_3'=0}^{\infty} f_{\boldsymbol{u}}(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3'-1)\sigma}+\sum_{l_1',l_2',l_3'=0}^{\infty} g_{\boldsymbol{u}}(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3')\sigma}\Big\}, \tag{32e}$$

and

$$f_{\boldsymbol{u}}(l_1l_2l_3)=(\eta/2\nu)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}/l_1!\,l_2!\,l_3!, \tag{32f}$$

$$g_{\boldsymbol{u}}(l_1l_2l_3)=(\eta/2\nu)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}(l_1+1)^2/(l_1+1)!\,l_2!\,l_3!. \tag{32g}$$

The second and third term in the right-hand of Eq. (32a) are simplified by using Eqs. (A·2·1), (A·2·2) and (A·2·3), and is proportional to t. The fourth and fifth term become a constant quantities independent of t for a large time t, as will be shown in Appendix II, and can be omitted.

According to the discussion mentioned just above, we can obtain

$$\mathrm{Tr}(e^{-(\beta+it)H}j_x e^{itH}j_x)=\sum_{\boldsymbol{k}} e^{-k^2\beta/2m_0}e^2\Big[(1-|t|/\tau(\boldsymbol{k},0))\,k_x{}^2/m_0{}^2$$
$$+\sqrt{\kappa\eta}e^{i\nu t}\{1+it(\Delta E(k,1)-\Delta E(k,0))-|t|(1/\tau(\boldsymbol{k},0)+1/\tau(\boldsymbol{k},1))/2\}\Big]$$
$$\simeq\sum_{\boldsymbol{k}} e^{-k^2\beta/2m_0}e^2\Big[e^{-(|t|/\tau(k,0))}/m_0{}^2$$
$$+\sqrt{\kappa\eta}\,e^{t(\nu+\Delta E(k,1)-\Delta E(k,0))t}e^{-(|t|/2)(1/\tau(k,0)+1/\tau(k,1))}\Big], \tag{33a}$$

in the case of $n_1=n_2=n_3=0$, where

$$1/2\pi\tau(\boldsymbol{k},0)=\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu} f_u(l_1l_2l_3)(e^\beta-1)^{-1}(\delta_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} D_u^2 e^{-u^2\eta/2\nu} f_u(l_1l_2l_3)(e^{\omega\beta}-1)^{-1}$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3)), \tag{33b}$$

where we have used the notation $\epsilon_k=\boldsymbol{k}^2/2m_0$ and the abbreviations are

$$\delta_u^\pm(x,y;l_1l_2l_3)=\delta(\epsilon_k\pm x-\epsilon_{k\pm u}-(l_1+l_2+l_3)y). \tag{33c}$$

and

$$\Delta E(\boldsymbol{k},0)=\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu} f_u(l_1l_2l_3)(e^\beta-1)^{-1}(p_u^+(1,\nu;l_1l_2l_3)+e^\beta p_u^-(1,\nu;l_1l_2l_3)), \tag{33d}$$

where the abbreviations are

$$p_u^\pm(xy;l_1l_2l_3)\equiv P/(\epsilon_k\pm x-\epsilon_{k\pm u}-(l_1+l_2+l_3)y), \tag{33e}$$
$$1/2\pi\tau(\boldsymbol{k},1)=\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu) f_u(l_1l_2l_3)(e^\beta-1)^{-1}(\delta_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} D_u^2 e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu) f_u(l_1l_2l_3)$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu} g_u(l_1l_2l_3)$$
$$\times(\delta_u^+(1,\nu;l_1l_2l_3)+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} D_u^2 e^{-u^2\eta/2\nu} g_u(l_1l_2l_3)$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3)), \tag{33f}$$
$$\Delta E(\boldsymbol{k},1)=\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu) f_u(l_1l_2l_3)(e^\beta-1)^{-1}(p_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta p_u^-(1,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} D_u^2 e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu) f_u(l_1l_2l_3)$$
$$\times(p_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}p_u^-(\omega,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} C_u^2 e^{-u^2\eta/2\nu} g_u(l_1l_2l_3)$$
$$\times(p_u^+(1,\nu;l_1l_2l_3)+e^\beta p_u^-(1,\nu;l_1l_2l_3))+\sum_{u l_1 l_2 l_3} D_u^2 e^{-u^2\eta/2\nu} g_u(l_1l_2l_3)$$
$$\times(p_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}p_u^-(\omega,\nu;l_1l_2l_3)). \tag{33g}$$

The terms including D_u of $\tau(\boldsymbol{k},0)$ and $\tau(\boldsymbol{k},1)$ are shown to be zero, by taking into account that D_u^2 contains a δ-function $\delta(\boldsymbol{u})$ (see Eq. (25)) and $\nu\neq\omega$. We retain these terms, however, in order to make it easy to get insight into a relation between the energy shift and the collision time.

It is noted that $\Delta E(\boldsymbol{k},0)$ and $\tau(\boldsymbol{k},0)$ correspond to the energy shift and the collision time of the state $|\boldsymbol{k}, n_1=n_2=n_3=0)$, respectively, and $\Delta E(\boldsymbol{k},1)$ and $\tau(\boldsymbol{k},1)$ the energy shift and the collision time of the state $|\boldsymbol{k}, n_1=1, n_2=n_3=0)$ respectively. The energy shift $\Delta E(\boldsymbol{k},0)$ at $\boldsymbol{k}=0$ near 0°K is given, after some mathematical manipulations, by

$$\Delta E(\boldsymbol{k}=0,\,0)=-\sum_{u}{}'C_u^{\,2}\int_0^\infty \exp[-u^2\{\eta(1-e^{-\nu t})/2\nu+t/2m_0\}]e^{-t}\,dt$$

$$-\sum_{u} D_u^{\,2}\int_0^\infty \exp[-u^2\{\eta(1-e^{-\nu t})/2\nu+t/2m_0\}]e^{-\omega t}\,dt,$$

$$=-\alpha\nu/\sqrt{\pi}\,\omega\left\{\int_0^\infty d\tau\, e^{-\tau}/\sqrt{\tau(1+(\nu^2-\omega^2)(1-e^{-\nu\tau})/\nu\omega^2\tau)}\right\}+(3C/\nu\omega). \tag{34}$$

The energy of the ground state determined by the trial action is $(3/2)(\nu-\omega)$, which has been calculated by Feynman.[3] Therefore, the self-energy of the polaron state at 0°K is $(3/2)(\nu-\omega)+\Delta E(\boldsymbol{k}=0,\,0)$. This result agrees with that obtained by Feynman, that is, the approximation in our calculation of mobility may be considered to correspond to that which gives Feynman's result for the self-energy at 0°K.

Further, $\Delta E(\boldsymbol{k}=0,\,1)$ near 0°K is given by

$$\Delta E(\boldsymbol{k}=0,\,1)=-\sum_{u} C_u^{\,2}\int_0^\infty (1+(\eta u_x^{\,2}/\nu))\exp[-u^2\{\eta(1-e^{-\nu t})/2\nu+t/2m_0\}]e^{-(\nu+1)t}\,dt$$

$$-\sum_{u} D_u^{\,2}\int_0^\infty (1+(\eta u_x^{\,2}/\nu))\exp[-u^2\{\eta(1-e^{-\nu t})/2\nu+t/2m_0\}]e^{-(\nu+\omega)t}\,dt. \tag{35}$$

Explicit calculation of Eq. (35) is not necessary, because the contribution to the mobility at very low temperature from the term including $\Delta E(\boldsymbol{k}=0,\,1)$ is neglected as we shall mention below.

We can calculate conductivity using Eqs. (1), (2) and Eq. (33) in the same way as the derivation of Eq. (14). Near 0°K, we are interested in the collision time for the state $k=0$. If we put $\tau(\boldsymbol{k}=0,\,0)=\tau_0$ and $\tau(\boldsymbol{k}=0,\,1)=\tau_1$, we get

$$\begin{aligned} 1/\tau_0&=2\sqrt{m_0}\alpha/(e^{\beta}-1),\\ 1/\tau_1&=(2\sqrt{m_0}\alpha/(e^{\beta}-1))(1+2(\eta m_0)^2/5\nu^2). \end{aligned} \tag{36}$$

Using Eqs. (1), (33a) and (36), we get the mobility μ near 0°K given by

$$\mu=(\mu_0\exp(m_0\eta/\nu)/m_0^{3/2})+(\beta\sqrt{\kappa\eta}\,e/2)$$
$$\times(1/\tau_1+1/\tau_0)/2\{(\nu+\Delta E(1,\,0)-\Delta E(0,\,0))^2+(1/\tau_1+1/\tau_0)^2/4\}, \tag{37}$$

where μ_0 is the mobility given by Eq. (14). The second term in the right-hand side of Eq. (37) can be neglected compared with the first term at very low temperatures. Therefore, we get

$$\lim_{\beta\to\infty}(\mu/\mu_0)=\exp(m_0\eta/\nu)/(m_0)^{3/2}. \tag{38}$$

§ 4. Discussion

We compare the polaron mobility at 0°K obtained in the proceding section with those obtained by Shultz[4] and by Morita[5] which are shown in Fig. 1. Their results are given over a wide range of the magnitude of coupling constant. The present result is numerically given in Table I.

The treatment used by Shultz is as follows. He considers the "extended Hamiltonian" which gives the same energy-shift as that obtained by Feynman at $\boldsymbol{k}=0$. He says that the use of this Hamiltonian in a time-dependent formalism with the appropriate boundary conditions gives the same results as the evaluation of the path-integration in all orders of the difference between true action and trial action. He calculates the self-energy for all $\boldsymbol{k}$ to the first approximation of the damping theory, using this "extended" Hamiltonian. Then, he calculates the polaron mobility, using this self-energy as the unperturbed energy of the polaron and treating the interaction between electron and phonons as the perturbation determining polaron mobility. It seems rather difficult, however, to give the concrete justification of Shultz' treatment using the damping theory. The equivalence of the path-integration and his "extended" Hamiltonian is not verified in all the orders of the difference between true action and trial action. Therefore, the approximation contained in his result of the polaron mobility is not evident.

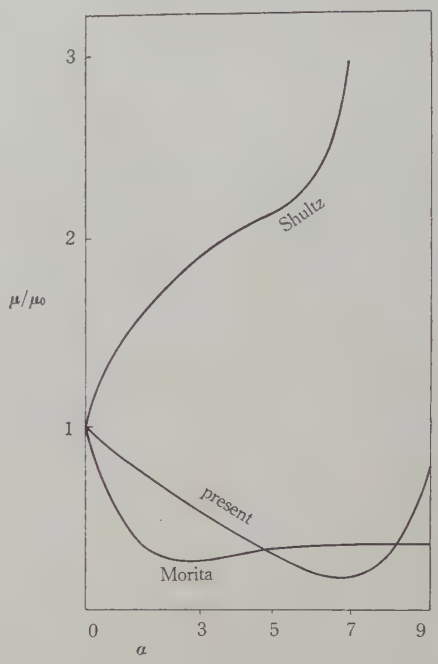


Fig. 1. Theoretical Value of Polaron mobility.

As was mentioned before, the approximation used in the present calculation for the polaron mobility at 0°K gives the same self-energy as that obtained

Table I

α	ν	ω	m_0	μ/μ_0
3	3.4	2.55	1.78	0.530
5	4.02	2.13	3.56	0.282
7	5.81	1.60	13.2	0.170
9	9.85	1.28	59.2	0.808
11	15.5	1.15	181.	45.5

Parameters at 0°K

by Feynman. Our result of the collision time (33b) includes the absorption and emission of phonons followed by the excitation of the internal state of the polaron at a finite temperature. Our result is similar to Morita's result up to $\alpha=7$, if the effective mass is put to unity in his formulation. According to the present result, the mobility of the polaron increases steeply at a large coupling constant even though the increase is slower than the result obtained by Shultz where the increase begins already at a smaller coupling constant. This increase is expected because when the coupling becomes strong the wave length of phonons, which contribute effectively to the polaron scattering, decreases more steeply than the localization length of the polaron, so that the polaron becomes insensitive to the scattering by phonons. This effective wave length is proportional to $1/\alpha^2$ and the localization length of the polaron to $1/\alpha$, as Shultz already suggested. This fact is based on the assumption that the polaron mobility is determined by the scattering of the polaron which is accompanied with phonon cloud. It has to be noted, however, that the polaron mobility related to the electronic current is determined by the scattering of the electronic part of the polaron. There will remain some problem in the behaviour of the mobility for a large coupling constant.

At finite temperatures, the deviation from $(e^{\beta}-1)$ of the temperature dependence of the mobility comes from the temperature dependence of the variational parameters ν and ω, that is, the temperature change of the polaron state. Moreover, other mechanisms for the scattering, for example, the internal excitation and the spontaneous emission of phonons that occur only at finite temperatures, change the temperature dependence of the mobility. Also the second term in the right-hand side of Eq. (37), which is neglected at very low temperature, has a contribution to the mobility at finite temperature.

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Appendix I

In this appendix, we shall prove the following two equations:

$$\int e^{[S_0(X')+i\omega\cdot X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X'(t)\,;\,\frac{X_2(u)}{X_1(0)}\right)$$
$$=\sum_m\int dq e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)(m|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|m), \qquad (A\cdot1\cdot1)$$

$$\int e^{[S_0(X')+i\omega\cdot X(\tau)]}\mathfrak{D}\left(X'(t):\frac{X_2(u)}{X_1(0)}\right)$$
$$=\lim_{\sigma\to0}\sum_m\int dq e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)(m|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|m),$$
$$=\lim_{\sigma\to0}\int e^{[S_0(X')+i\omega(X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X'(t):\frac{X_2(u)}{X_1(0)}\right), \qquad (A\cdot1\cdot2)$$

where the action S_0 corresponds to Hamiltonian H_0 and the operator $e^{i\omega\cdot X(\tau)}$ is defined by $e^{\tau H_0}e^{i\omega\cdot X}e^{-\tau H_0}$. At first, we shall derive Eq. (A·1·1).

From the meaning of path-integration, we can see that

$$\int e^{S_0(X')}(\partial^n X'(t)/\partial t^n)_{t=0}\mathfrak{D}\left(X'(t):\frac{X_2(u)}{X_1(0)}\right)=\sum_m\int dq\, e^{-uE_m^0}$$
$$\times\psi_m^{0*}(X_1 q)\psi_m{}^0(X_2 q)(m|[\underbrace{H_0[H_0\cdots[H_0}_{n},X]\cdots]]|m),$$

where $\psi_m{}^0(X, q)$ is the eigenfunction of Hamiltonian H_0. Using this equation, we can derive Eq. (A·1·1) as follows:

$$\int e^{[S_0(X')+i\omega\cdot(X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right)=\int e^{[S_0(X')+i\omega\cdot(X'(\tau-\sigma)-X'(0))}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right),$$

$$=\sum_{n=0}^{\infty}\int e^{S_0(X')}\left[\frac{\partial^n}{\partial(\tau-\sigma)^n}e^{i\omega\cdot X'(\tau-\sigma)}\right]_{\tau-\sigma=0}e^{-i\omega\cdot X'(0)}\frac{(\tau-\sigma)^n}{n!}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right),$$

$$=\sum_{n=0}^{\infty}\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1 q)\psi_m{}^0(X_2 q)$$

$$\times\left[\frac{\partial^n}{\partial(\tau-\sigma)^n}(m|e^{(\tau-\sigma)H_0}e^{i\omega X}e^{-(\tau-\sigma)H_0}e^{-i\omega X}|m)\right]_{\tau-\sigma=0}\frac{(\tau-\sigma)^n}{n!}$$

$$=\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1 q)\psi_m{}^0(X_2 q)(m|e^{(\tau-\sigma)H_0}e^{i\omega\cdot X}e^{-(\tau-\sigma)H_0}e^{-i\omega\cdot X}|m),$$

$$=\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1 q)\psi_m{}^0(X_2 q)(m|e^{\tau H_0}e^{i\omega\cdot X}e^{-\tau H_0}e^{\sigma H_0}e^{-i\omega\cdot X}e^{-\sigma H_0}|m).$$

Noting that $e^{(i\omega\cdot X(0))}=\exp\left[i\omega\int_{-\infty}^{u}\delta(t)X(t)dt\right]$ and the boundary condition $X'(0)=X_1$, we can see that

$$\int e^{[S_0(X')+i\omega\cdot X(\tau)]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right)=\lim_{\sigma\to 0}\int e^{[S_0(X')+i\omega\cdot(X(\tau)-X(\sigma))]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right).$$

(A·1·3)

This equation is also simply derived by using the superposition of action. We can derive Eq. (A·1·2) in the same way as the derivation of Eq. (A·1·1) by using Eq. (A·1·3).

Using Eqs. (A·1·1) and (A·1·2), we can easily derive the average values of Eq. (9). For example, we get

$$\langle e^{i\omega\cdot(X'(\tau)-X'(\sigma))}\rangle_{X_2X_1;\lambda}=\{\sum_k e^{-k^2\lambda/2+ik\cdot(X_2-X_1)}(k|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|k)\}$$
$$\div\{\sum_k e^{-k^2\lambda/2+ik\cdot(X_2-X_1)}\}=\exp(i\omega\cdot(X_2-X_1)(\tau-\sigma)/\lambda)$$
$$\times\exp\left(\frac{-\omega^2(\tau-\sigma)}{2}\left(1-\frac{(\tau-\sigma)}{\lambda}\right)\right).$$

In the same way as the derivation of Eqs. (A·1·1) and (A·1·2), we obtain Eq. (26). Applying Eq. (26) to the case of weak interaction, we simply get Eq. (11).

Appendix II

At first we shall derive the formulae (A·2·1) :

$$\sum_{\omega} F(\omega)\left[\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it} ds(-it-s)e^{-f(\omega)s}\right]\simeq 2\pi t\sum_{\omega} F(\omega)\delta(f(\omega)). \tag{A·2·1}$$

We obtain, after some elementary integrations,

$$\sum_{\omega} F(\omega)\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}=\sum_{\omega} F(\omega)\left[\frac{(\beta+it)}{f(\omega)}+\frac{1-e^{-f(\omega)(\beta+it)}}{f^2(\omega)}\right].$$

We are interested in the asymptotic form of the above equation for a large time t. Since $\exp(-if(\omega)t)$ oscillates rapidly in time, the term including this factor remains finite near the region in which $f(\omega)$ nearly equals zero. Therefore we can put $f(\omega)\beta\to 0$ in the numerical factor of the term including $\exp(-if(\omega)t)$ of this equation. Therefore, it holds

$$\begin{aligned}\sum_{\omega} F(\omega)\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s} &\simeq \sum_{\omega}\left[\frac{(\beta+it)}{f(\omega)}+\frac{1-e^{-if(\omega)t}}{f^2(\omega)}\right]F(\omega)\\ &=\sum_{\omega}\left[\frac{i}{f(\omega)}\left(t+\frac{\sin f(\omega)t}{f(\omega)}\right)+\frac{(1-\cos f(\omega)t)}{f^2(\omega)}\right]F(\omega)+\sum_{\omega}\frac{\beta}{f(\omega)}F(\omega).\end{aligned} \tag{A·2·2}$$

Further, we get

$$\begin{aligned}&\sum_{\omega} F(\omega)\int_0^{-it}(-it-s)e^{-f(\omega)s}\,ds\\ &\quad=\sum_{\omega}\left[-\frac{i}{f(\omega)}\left\{t+\frac{\sin f(\omega)t}{f(\omega)}\right\}+\frac{(1-\cos f(\omega)t)}{f^2(\omega)}\right]F(\omega).\end{aligned} \tag{A·2·3}$$

From the above equations, we obtain

$$\begin{aligned}&\sum_{\omega} F(\omega)\left[\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it} ds(-it-s)e^{-f(\omega)s}\right]\\ &\quad\simeq\sum_{\omega}\frac{\beta}{f(\omega)}F(\omega)+2\sum_{\omega}\frac{F(\omega)[1-\cos f(\omega)t]}{[f(\omega)]^2}.\end{aligned}$$

If we employ the procedure that is the same as that used in the theory of

ordinary time dependent perturbation, to the term including $1-\cos f(\boldsymbol{\omega})t$, this equation is transformed into

$$\sum_{\boldsymbol{\omega}} \frac{\beta F(\boldsymbol{\omega})}{f(\boldsymbol{\omega})} + 2\pi t \sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega})\, \delta(f(\boldsymbol{\omega})).$$

For a large time t, we obtain

$$\sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega}) \left[\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\boldsymbol{\omega})s} + \int_0^{-it} ds(-it-s)e^{-f(\boldsymbol{\omega})s} \right] \simeq 2\pi t \sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega})\, \delta(f(\boldsymbol{\omega})). \tag{A·2·1}$$

The last two terms in Eq. (11) appearing in § 2 have the following form:

$$\sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega}) \int_0^{\beta+it} e^{-f_1(\boldsymbol{\omega})s_1} ds_1 \int_0^{-it} e^{-f_2(\boldsymbol{\omega})s_2} ds_2.$$

After elementary integrations, we get

$$\sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega}) \int_0^{\beta+it} e^{-f_1(\boldsymbol{\omega})s_1} ds_1 \int_0^{-it} e^{-f_2(\boldsymbol{\omega})s_2} ds_2 = \sum_{\boldsymbol{\omega}} iF(\boldsymbol{\omega}) \left(\frac{1-f_1(\boldsymbol{\omega})\beta}{f_1(\boldsymbol{\omega})} \right)$$

$$\times \int_0^t e^{if_2(\boldsymbol{\omega})z} dz + \sum_{\boldsymbol{\omega}} e^{-\beta f_1(\boldsymbol{\omega})} F(\boldsymbol{\omega}) \int_0^t e^{-if_1(\boldsymbol{\omega})z_1} dz_1 \int_0^t e^{if_2(\boldsymbol{\omega})z_2} dz_2.$$

For a large time t, the right-hand side of this equation tends to

$$\sum_{\boldsymbol{\omega}} iF(\boldsymbol{\omega}) \left(\frac{1-f_1(\boldsymbol{\omega})\beta}{f_1(\boldsymbol{\omega})} \right) [\pi\delta(f_2(\boldsymbol{\omega})) + iP/f_2(\boldsymbol{\omega})]$$

$$+ \sum_{\boldsymbol{\omega}} F(\boldsymbol{\omega}) e^{-\beta f_1(\boldsymbol{\omega})} [\pi\delta(f_1(\boldsymbol{\omega})) + iP/f_1(\boldsymbol{\omega})][\pi\delta(f_2(\boldsymbol{\omega})) + iP/f_2(\boldsymbol{\omega})].$$

These terms can be neglected with comparison to the terms which are proportional to t, for a large time t.

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A New Approach to the Theory of Classical Fluids. III

—— *General Treatment of Classical Systems* ——

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A classical system enclosed in a finite volume and exerted by external forces is treated in quite a general way. The main results, which may be applicable to solids as well as to fluids, are as follows. Exact integral equations are found for the one- and two-particle distribution functions. Some thermodynamic functions are expressed in terms of these distribution functions. It is shown that there exist variational principles saying that the grand partition function is to be maximum with respect to the variations in the one- and two-particle distribution functions. Variational principles are found also for the Helmholtz free energy. It is suggested that Mayer's theory of condensation may in fact give the end point of the metastable gaseous state. It is pointed out that the hyper-netted chain approximation, proposed previously by one of the present authors, has a meaning in solids as well as in fluids.

§ 1. Introduction

In paper I[1] of this series, one-component fluid systems have been investigated by starting from the customary density expansion for the pair distribution function. An exact integral equation has been found for the pair distribution function. The Helmholtz free energy has been expressed in a form of expansion which may be expected to converge more rapidly than its customary density expansion. It has also been shown that the integral equation for the pair distribution function can be derived from the expression for the Helmholtz free energy by means of a variational principle. In paper II,[2] the results of paper I have been generalized to the case of multicomponent systems.

The chief objection to papers I and II will be that the arguments are based on the density expansion formulas so that we have no sufficient reason why the results of I and II can be applied to liquids or to classical plasmas even if the resultant expressions have definite meanings. Therefore it is desirable to develop the theory of classical fluids, not being based on the density expansion formulas. If that is carried out, it will also be possible to discuss the condensation phenomena of gases. The other objection to I and II may be that the stationary character of the variational principle which has been derived in I or II is not clear. The variational principle will be especially useful if the stationary character is found to be maximum or minimum. It is desirable to find such a

variational principle. In the present paper we shall remove these objections.

The main purpose of the present paper is to treat a classical system as generally as possible. For this purpose we shall investigate a classical system, which is enclosed in a finite voulme and is exerted by external forces, by means of a grand canonical ensemble. The system can be either a fluid or a solid. As the consequence it will become possible to obtain the expressions which are applicable to a solid as well as to a fluid. It will also become possible to discuss the phase change of the system. A part of the results of the present paper has been shown in the preliminary report.[3)]

In the present paper we shall use diagrams as an auxiliary means of expressing integrals. We shall here make a brief mention of diagrams. In the customary density expansion formulas for the thermodynamic functions or the distribution functions, the coefficients are expressed in the form of sums of integrals which can be made to correspond to diagrams. Such a correspondence of the integrals and the diagrams is usually adopted only to understand intuitively the structures of the coefficients in the expansion formulas. Recently Morita[4)] has shown that, if a diagram is made to correspond to an integral multiplied by a suitable numerical factor, analysis of expansion formulas can be made by dealing with the diagrams alone. The diagrams defined in such a way has been used in the Appendix of paper I[1)] as well as in the derivation of formulas in the hyper-netted chain approximation.[4)] These diagrams may be considered to be the same as the corresponding integrals. In the present paper we shall adopt the diagrams in this sense. It is, however, necessary to generalize the definition of the diagrams to some extent, since, as will be seen later, we shall be concerned with the series expansion formulas in powers of a quantity depending on spatial coordinates. The definition will be given in § 2 of the present paper. On the other hand it may be said that the arguments in reference 4) are somewhat lacking in the logical or mathematical rigor. The same objection may be raised to Salpeter's treatment of cluster expansion[5)] and also to the arguments in papers I and II. It is also one of the purposes of the present paper to arrange the arguments in the logically rigorous form. Lemmas which will be stated in § 2 will serve to this purpose.

The contents of the present paper are as follows. Throughout the paper, except for § 6, we consider one-component systems. In § 2 we present the definition of diagrams and some lemmas on the diagrams. These lemmas, the proofs of which are given in the Appendices, are useful for the arguments in § 4. In § 3 we express the thermodynamic functions and distribution functions, of classical systems enclosed in a finite volume, in the forms useful for the arguments in the following sections. In § 4 we express the grand partition function in the form of sum of diagrams and reduce the expression by introducing the one- and two-particle distribution functions. It is shown that the integral equations for these distribution functions are derived by means of variational principles

that the expressions, obtained in this section, for the grand partition function are to be stationary. Some thermodynamic functions are also expressed in terms of the one- and two-particle distribution functions. In § 5 we prove that the variational principles obtained in § 4 give in fact the maximum of the grand partition function. The variational principles for the Helmholtz free energy are also proved. The results of this section are arranged in the form of theorems 1～4. These theorems may be considered a summary of the results in § 4. In § 6 we consider the multicomponent systems. It is shown that the multicomponent systems can be formally reduced to the one-component systems. In § 7 we discuss the behavior of the system in the limit of an infinite volume. A special attention is paid to the condensation of gas. We make a conjecture that Mayer's theory of condensation does not give the condensation point but gives the end point of the metastable gaseous state. It is also pointed out in this section that the hyper-netted chain approximation has a meaning in solids and liquids as well as in gases. In Appendices 1～3 we give the proofs of the lemmas presented in § 2. In Appendix 4 we give the proof of theorem 2 in § 5.

§ 2. Definition of diagrams and some lemmas on the diagrams

In this section we shall first give the definition of diagrams. Then the terminology about them will be defined. Lastly we shall present six lemmas which will be useful for the arguments in § 4. These lemmas will be proved in the Appendices.

Let $f_1(\boldsymbol{r}), f_2(\boldsymbol{r}), \cdots$ be the functions of spatial coordinate $\boldsymbol{r}$ of one particle and $F_\alpha(\boldsymbol{r}, \boldsymbol{r}')$, $F_\beta(\boldsymbol{r}, \boldsymbol{r}'), \cdots$ the functions of spatial coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$ of two particles. We shall assume that the particles are confined in volume Ω and that the functions $F_\alpha(\boldsymbol{r}, \boldsymbol{r}')$'s are symmetrical in $\boldsymbol{r}$ and $\boldsymbol{r}'$. In the following $\int d\boldsymbol{r}$ will mean an integration over $\boldsymbol{r}$ within Ω. Let us first define a diagram corresponding to a product of several f_i's and F_α's.

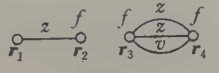
Definition 1.

A diagram which is composed of white circles ${}^{f_1}\!\circ_{\boldsymbol{r}_1}$, ${}^{f_2}\!\circ_{\boldsymbol{r}_2}$, $\cdots$, ${}^{f_n}\!\circ_{\boldsymbol{r}_n}$ and bonds $\underline{\quad F_\alpha \quad}$, $\underline{\quad F_\beta \quad}$, $\cdots$ connecting white circles

$$= \prod_{i=1}^{n} f_i(\boldsymbol{r}_i) \prod_{i>j} \frac{1}{n(\alpha, ij)!} F_\alpha(\boldsymbol{r}_i, \boldsymbol{r}_j)^{n(\alpha, ij)} \times \frac{1}{n(\beta, ij)!} F_\beta(\boldsymbol{r}_i, \boldsymbol{r}_j)^{n(\beta, ij)} \cdots,$$

where $n(\nu, ij)$ $(\nu = \alpha, \beta, \cdots)$ is the number of bonds $\underline{\quad F_\nu \quad}$ connecting ${}^{f_i}\!\circ_{\boldsymbol{r}_i}$ and ${}^{f_j}\!\circ_{\boldsymbol{r}_j}$.

When $f_i(\boldsymbol{r}_i)$ is identically unity, the corresponding white circle will be written simply as $\circ_{\boldsymbol{r}_i}$. An example of definition 1 is



$$= \frac{1}{2!} f(\boldsymbol{r}_2) f(\boldsymbol{r}_3) f(\boldsymbol{r}_4)\, z(\boldsymbol{r}_1, \boldsymbol{r}_2)\, z(\boldsymbol{r}_3, \boldsymbol{r}_4)^2 v(\boldsymbol{r}_3, \boldsymbol{r}_4).$$

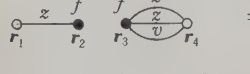
In the next place we shall consider a product, of f_i's and F_α's, in which the integrations over several coordinates have been performed. The coordinates over which the integration is performed will be represented by a black circle. Namely we have

Definition 2.

A diagram which is composed of white circles $\overset{f_1}{\circ}_{\boldsymbol{r}_1}, \overset{f_2}{\circ}_{\boldsymbol{r}_2}, \cdots, \overset{f_n}{\circ}_{\boldsymbol{r}_n}$, black circles $\overset{f_{n+1}}{\bullet}_{\boldsymbol{r}_{n+1}}, \overset{f_{n+2}}{\bullet}_{\boldsymbol{r}_{n+2}}, \cdots, \overset{f_{n+m}}{\bullet}_{\boldsymbol{r}_{n+m}}$ and bonds $\underline{\quad F_\alpha \quad}$, $\underline{\quad F_\beta \quad}$, $\cdots$ connecting the circles

$= \int d\boldsymbol{r}_{n+1} \int d\boldsymbol{r}_{n+2} \cdots \int d\boldsymbol{r}_{n+m}$ (The diagram which is obtained from the diagram of the left-hand side by changing the black circles into white ones.).

For example,



$$= \frac{1}{2!} \iint d\boldsymbol{r}_2 d\boldsymbol{r}_3 f(\boldsymbol{r}_2) f(\boldsymbol{r}_3) z(\boldsymbol{r}_1, \boldsymbol{r}_2) z(\boldsymbol{r}_3, \boldsymbol{r}_4)^2 v(\boldsymbol{r}_3, \boldsymbol{r}_4).$$

For actual use the way of numbering the black circles will be of no importance. We shall accordingly define a diagram in which no coordinates are attached to black circles as follows:

Definition 3.

A diagram which is composed of white circles $\overset{f_1}{\circ}_{\boldsymbol{r}_1}, \overset{f_2}{\circ}_{\boldsymbol{r}_2} \cdots, \overset{f_n}{\circ}_{\boldsymbol{r}_n}$, black circles $\overset{f_{n+1}}{\bullet}, \overset{f_{n+2}}{\bullet}, \cdots, \overset{f_{n+m}}{\bullet}$ and bonds $\underline{\quad F_\alpha \quad}$, $\underline{\quad F_\beta \quad}$, $\cdots$ connecting the circles

$= \dfrac{1}{m!}$ {Sum of all the topologically different diagrams which are obtained from the diagram of the left-hand side by attaching coordinates $\boldsymbol{r}_{n+1}$, $\boldsymbol{r}_{n+2}$, $\cdots$, $\boldsymbol{r}_{n+m}$ to the black circles in an arbitrary way}.

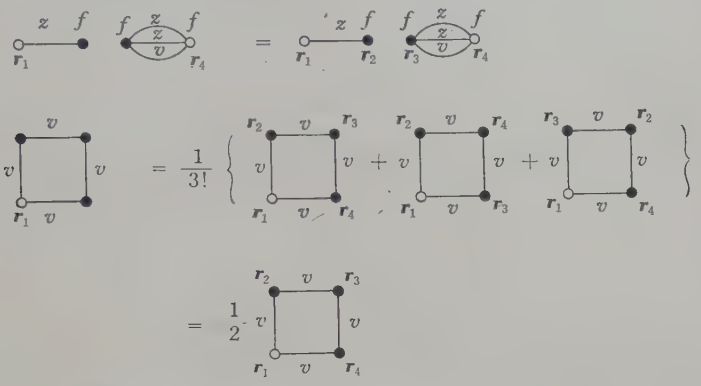
The total number of the diagrams appearing on the right-hand side of definition 3 is given by $m!/s$, where the symmetry number s denotes the number of permutations, of the coordinates of the black circles, which do not lead to a topologically different diagram. Therefore definition 3 may be rewritten as

Definition 3′.

A diagram which is composed of white circles $\overset{f_1}{\circ}_{\boldsymbol{r}_1}, \overset{f_2}{\circ}_{\boldsymbol{r}_2}, \cdots, \overset{f_n}{\circ}_{\boldsymbol{r}_n}$, black circles $\overset{f_{n+1}}{\bullet}, \overset{f_{n+2}}{\bullet}, \cdots, \overset{f_{n+m}}{\bullet}$ and bonds $\underline{\quad F_\alpha \quad}$, $\underline{\quad F_\beta \quad}$, $\cdots$ connecting the circles

$= \dfrac{1}{\text{(the symmetry number of the diagram)}}$ (Any one of the diagrams which are obtained from the diagram of the left-hand side by attaching coordinates $\boldsymbol{r}_{n+1}, \boldsymbol{r}_{n+2}, \cdots, \boldsymbol{r}_{n+m}$ to the black circles in an arbitrary way).

The diagrams appearing in the following will always mean those which are defined by definition 3 or 3′. The following two examples will serve to show the meaning of definition 3 and 3′.



It will be convenient to define here the terminology which will be useful to specify the type of diagrams.

single-bond diagram: It is the diagram in which no pairs of circles are connected by two or more bonds.

connected diagram: It is the diagram which contains at least two circles and in which there exists at least one path to reach any circle in the diagram from any other circle. The path must be understood to be a sequence of circles and bonds. A diagram will in general be composed of several separate parts, each of which is a connected diagram or one circle.

more than singly (doubly) connected diagram: It is the connected diagram in which there exist at least two (three) independent paths to reach any circle in the diagram from any other circle. Two paths are said to be independent of each other if the paths involve no common intermediate circles.

articulation circle: It is such a circle, in a connected diagram, that if one removes it, the diagram divides into two or more separate parts in such a way that at least one part contains no white circles.

articulation pair of circles: It is such a pair of circles, in a connected diagram, that if one removes the pair of circles and, if exist, the bonds connecting them, the diagram divides into two or more separate parts in such a way that at least one part contains no white circles.

s-circle: It is such a black circle, in a connected diagram containing just two white circles, that if one removes it, the diagram divides into two or more separate parts in such a way that the two white circles are contained in different parts. In other words it is such a black circle, in a connected diagram containing just two white circles, that all paths going from one white circle to the other must pass through it. It is noted that an *s*-circle

is not necessarily a black articulation circle in a connected diagram containing just two white circles and vice versa.

As will be readily seen, it follows from the above terminology that, for a connected diagram containing no white circles, "being more than singly (doubly) connected" is equivalent to "having no articulation circles (having neither articulation circles nor articulation pairs of circles)".

It will often be necessary to use the concept of the functional derivative which is a natural extension of the concept of a partial derivative. Let P be a functional of $f(\boldsymbol{r})$. When $f(\boldsymbol{r})$ is varied by any $\delta f(\boldsymbol{r})$, the corresponding first order variation in P is denoted by δP. Then the functional derivative $\delta P/\delta f(\boldsymbol{r})$ is defined by

$$\delta P=\int d\boldsymbol{r}\frac{\delta P}{\delta f(\boldsymbol{r})}\delta f(\boldsymbol{r}). \tag{2·1}$$

It is easy to see that such a definition gives a unique form of $\delta P/\delta f(\boldsymbol{r})$. When P is a functional of $F(\boldsymbol{r},\boldsymbol{r}')$, the functional derivative $\delta P/\delta F(\boldsymbol{r},\boldsymbol{r}')$ will be defined, in the similar way as above, by

$$\delta P=\iint d\boldsymbol{r}\,d\boldsymbol{r}'\frac{\delta P}{\delta F(\boldsymbol{r},\boldsymbol{r}')}\delta F(\boldsymbol{r},\boldsymbol{r}'). \tag{2·2}$$

However, if the function $F(\boldsymbol{r},\boldsymbol{r}')$ is symmetric in $\boldsymbol{r}$ and $\boldsymbol{r}'$, as is the case in the present paper, expression (2·2) fails to give a unique definition of $\delta P/\delta F(\boldsymbol{r},\boldsymbol{r}')$, because in such a case any variation must be symmetric in $\boldsymbol{r}$ and $\boldsymbol{r}'$ so that "$\iint d\boldsymbol{r}d\boldsymbol{r}'\frac{\delta P}{\delta F(\boldsymbol{r},\boldsymbol{r}')}\delta F(\boldsymbol{r},\boldsymbol{r}')=0$" means only "$\frac{\delta P}{\delta F(\boldsymbol{r},\boldsymbol{r}')}-\frac{\delta P}{\delta F(\boldsymbol{r}',\boldsymbol{r})}=0$" but does not necessarily mean "$\frac{\delta P}{\delta F(\boldsymbol{r},\boldsymbol{r}')}=0$". In order to avoid the difficulty, we shall impose a supplementary condition upon $\delta P/\delta F(\boldsymbol{r},\boldsymbol{r}')$ that it must be symmetric in $\boldsymbol{r}$ and $\boldsymbol{r}'$ if $F(\boldsymbol{r},\boldsymbol{r}')$ is symmetric. Then $\delta P/\delta F(\boldsymbol{r},\boldsymbol{r}')$ is defined uniquely by means of (2·2). It is not difficult to see that the above-defined functional derivatives have the main properties of ordinary derivatives. In the similar way, one can introduce also functional derivatives of higher order. It is noted that for practical purposes we may manipulate functional derivatives with the aid of the relations

$$\frac{\delta g(\boldsymbol{r}')}{\delta f(\boldsymbol{r})}=\begin{cases}0 & \text{if } g \text{ is not a functional of } f\\ \delta(\boldsymbol{r}-\boldsymbol{r}') & \text{if } g=f \text{ identically}\end{cases}$$

and

$$\frac{\delta G(\boldsymbol{r}_1,\boldsymbol{r}_2)}{\delta F(\boldsymbol{r},\boldsymbol{r}')}=\begin{cases}0 & \text{if } G \text{ is not a functional of } F\\ \delta(\boldsymbol{r}-\boldsymbol{r}_1)\delta(\boldsymbol{r}'-\boldsymbol{r}_2) & \text{if } G=F \text{ identically, } F \text{ not symmetric}\\ (1/2)\{\delta(\boldsymbol{r}-\boldsymbol{r}_1)\delta(\boldsymbol{r}'-\boldsymbol{r}_2)+\delta(\boldsymbol{r}-\boldsymbol{r}_2)\delta(\boldsymbol{r}'-\boldsymbol{r}_1)\} & \text{if } G=F \text{ identically, } F \text{ symmetric,}\end{cases} \tag{2·3}$$

which follow immediately from the above definitions.

Lastly we shall state some lemmas on the diagrams. It is to be remembered that we consider the diagrams and the integrals to be the same ones. The proofs of the lemmas are given in the Appendices 1∼3.

Lemma 1. Let Γ be a diagram which is composed of black circles ${}^{f_1}\bullet$, ${}^{f_2}\bullet$, $\cdots$, ${}^{f_m}\bullet$ and bonds connecting the circles. Then the following relation exists on a functional derivative of Γ:
$g(\boldsymbol{r})\cdot\delta\Gamma/\delta f(\boldsymbol{r})=$ {Sum of all the topologically different diagrams which are obtained from Γ by changing a black circle ${}^{f}\bullet$, being contained in Γ, into a white one ${}^{g}\circ_{\boldsymbol{r}}$}.

Lemma 2. Let Γ be a single-bond diagram which is composed of black circles and bonds $\overset{F_\alpha}{\text{---}}$, $\overset{F_\beta}{\text{---}}$, $\cdots$ connecting the circles. Then $\delta\Gamma/\delta F(\boldsymbol{r},\boldsymbol{r}')=(1/2)$ {Sum of all the topologically different diagrams which are obtained from Γ by removing a bond $\overset{F}{\text{---}}$, being contained in Γ, and by changing two black circles at both ends of the bond into white ones with coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$}.

Lemma 3. Let $\boldsymbol{M}$ be a set of an infinite number of topologically different diagrams and $\boldsymbol{N}$ be its subset which consists of the connected diagrams and one circle. If any diagram belonging to $\boldsymbol{M}$ is composed of separate diagrams, each belonging to $\boldsymbol{N}$, and conversely there is a unique diagram, belonging to $\boldsymbol{M}$, in which any diagram belonging to $\boldsymbol{N}$ appears arbitrary given times as its component parts, then the following relation exists on the sums of diagrams:

{Sum of all the diagrams belonging to $\boldsymbol{M}$}
$=\exp${Sum of all the diagrams belonging to $\boldsymbol{N}$}-1.

Lemma 4. Let $\boldsymbol{M}$ be a set of an infinite number of topologically different connected diagrams which have a particular group of white circles $\circ_{\boldsymbol{r}_i}$'s and $\boldsymbol{N}$ be its subset consisting only of the diagrams which are connected even if the white circles $\circ_{\boldsymbol{r}_i}$'s are removed. If any diagram belonging to $\boldsymbol{M}$ can be uniquely divided at the white circles $\circ_{\boldsymbol{r}_i}$'s into separate parts, each being a diagram belonging to $\boldsymbol{N}$ and conversely there is a unique diagram, belonging to $\boldsymbol{M}$, which is built up from the diagrams belonging to $\boldsymbol{N}$ by joining them together at the white circles $\circ_{\boldsymbol{r}_i}$'s in such a way that any diagram appears arbitrary given times, then the following relation exists:

{Sum of all the diagrams belonging to $\boldsymbol{M}$}
$=\exp${Sum of all the diagrams belonging to $\boldsymbol{N}$}-1.

Lemma 5. Let $\boldsymbol{M}$ be a set of topologically different diagrams which are composed of one white circle ${}^{f}\circ_{\boldsymbol{r}}$, black circles and bonds, and $g(\boldsymbol{r})$ be a function which is defined by a sum of all the diagrams belonging to $\boldsymbol{M}$.

Let Γ be a diagram and $\boldsymbol{N}$ be a set of all the topologically different diagrams which are built up from Γ by attaching some diagram, belonging to $\boldsymbol{M}$, to each black circle of Γ in such a way that the white circle ${}^{f}\circ_{\boldsymbol{r}}$ is superposed on the black circle of Γ. If any diagram belonging to $\boldsymbol{N}$ is reduced in a unique way to Γ by removing the diagrams belonging to $\boldsymbol{M}$ from it, then

{Sum of all the diagrams belonging to $\boldsymbol{N}$}
= {A diagram which is obtained from Γ by attaching $g(\boldsymbol{r})$ to each black circle}.

Lemma 6. Let $\boldsymbol{M}$ be a set of topologically different diagrams which are composed of two white circles ${}^{f}\circ_{\boldsymbol{r}}$, ${}^{f}\circ_{\boldsymbol{r}'}$, black circles and bonds, and $f(\boldsymbol{r})f(\boldsymbol{r}')\times G(\boldsymbol{r}, \boldsymbol{r}')$ be a function which is defined by a sum of all the diagrams belonging to $\boldsymbol{M}$. Let Γ be a diagram and $\boldsymbol{N}$ be a set of all the topologically different diagrams which are built up from Γ by inserting some diagram, belonging to $\boldsymbol{M}$, into each bond of Γ in such a way that the circles ${}^{f}\circ_{\boldsymbol{r}}$ and ${}^{f}\circ_{\boldsymbol{r}'}$ are superposed on the two circles at both ends of the bond. If any diagram belonging to $\boldsymbol{N}$ is reduced in a unique way to Γ by removing the diagrams belonging to $\boldsymbol{M}$ from it, then

{Sum of all the diagrams belonging to $\boldsymbol{N}$}
= {A diagram which is obtained from Γ by replacing each bond by a bond $\overset{G}{\frown}$}.

§ 3. Expressions in a grand canonical ensemble

In this section we shall express the thermodynamic functions and distribution functions in the forms useful for the arguments in the following sections.

Let us consider a one-component system which is enclosed in a volume Ω and which is in contact with a reservoir at a given temperature T and a given chemical potential μ. In other words we shall consider a system in a grand canonical ensemble. The potential energy of N particles will be assumed to be of the form

$$\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)+\frac{1}{2}\sum_{i\neq j}^{N}\sum_{j}^{N}\phi(\boldsymbol{r}_i, \boldsymbol{r}_j), \tag{3·1}$$

where $\psi(\boldsymbol{r})$ denotes the potential energy due to external forces and $\phi(\boldsymbol{r}, \boldsymbol{r}')$ the interaction energy of two particles. As is well known,[6] in thermal equilibrium the probability that there is exactly N particles in Ω and that these are located at the coordinates $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N(\equiv\boldsymbol{r}^N)$ is given by

$$P(N, \boldsymbol{r}^N)=\frac{1}{\Xi}\frac{z^N}{N!}\exp\left\{-\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)/kT-\frac{1}{2}\sum_{i\neq j}^{N}\sum_{j}^{N}\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}. \tag{3·2}$$

In (3·2) the quantity Ξ, being called the grand partition function, is defined by

$$\Xi=\sum_{N=0}^{\infty}\frac{z^N}{N!}\int d\boldsymbol{r}^N\exp\left\{-\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)/kT-\frac{1}{2}\sum_{i\neq j}^{N}\sum_{j}^{N}\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}. \tag{3·3}$$

The quantity z, which will be called the fugacity, is defined by

$$z=\left(\frac{2\pi mkT}{h^2}\right)^{3/2}e^{\mu/kT}=\frac{1}{\lambda^3}e^{\mu/kT}, \tag{3·4}$$

where m and h have the usual meanings. Hereafter we shall use z instead of μ.

The pressure, p, of the system is defined as

$$\frac{p\Omega}{kT}=\ln\Xi. \tag{3·5}$$

Furthermore the average number of particles, $\bar{N}$, the entropy S and the internal energy E are defined in terms of $P(N, \boldsymbol{r}^N)$ given by (3·2) as follows:

$$\bar{N}=\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P(N, \boldsymbol{r}^N)N \tag{3·6}$$

$$E=\frac{3}{2}\bar{N}kT+\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P(N, \boldsymbol{r}^N)\left\{\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)+\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)\right\} \tag{3·7}$$

$$S=\frac{3}{2}\bar{N}k-k\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P(N, \boldsymbol{r}^N)\ln\{N!\,\lambda^{3N}P(N, \boldsymbol{r}^N)\}. \tag{3·8}$$

It is easy to see that $\ln\Xi$ is related to the above-defined $\bar{N}$, E and S by

$$\ln\Xi=\bar{N}\ln(z\lambda^3)-\frac{E}{kT}+\frac{S}{k}. \tag{3·9}$$

This relation, of course, corresponds to the thermodynamical relation

$$\frac{p\Omega}{kT}=\frac{G}{kT}-\frac{E}{kT}+\frac{S}{k},$$

where G denotes the Gibbs free energy. The Helmholtz free energy A is defined as

$$\frac{A}{kT}=\bar{N}\ln(z\lambda^3)-\ln\Xi. \tag{3·10}$$

The n-particle distribution function, $\rho^{(n)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n)$,* is defined by[6]

$$\rho^{(n)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n)=\sum_{N=n}^{\infty}\int d\boldsymbol{r}_{n+1}\cdots d\boldsymbol{r}_N\frac{N!}{(N-n)!}P(N, \boldsymbol{r}^N). \tag{3·11}$$

Some thermodynamic functions are expressed in terms of the distribution functions. First it is seen from (3·6) and (3·11) that

$$\bar{N}=\int d\boldsymbol{r}\,\rho(\boldsymbol{r}). \tag{3·12}$$

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* The one-particle distribution function will be denoted simply as $\rho(\boldsymbol{r})$.

From (3·7) and (3·11) we obtain

$$E=\frac{3}{2}\bar{N}kT+\int d\boldsymbol{r}\rho(\boldsymbol{r})\phi(\boldsymbol{r})+\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\rho^{(2)}(\boldsymbol{r},\boldsymbol{r}')\phi(\boldsymbol{r},\boldsymbol{r}'). \quad (3\cdot13)$$

Lastly let us consider the isothermal compressibility κ which is defined by

$$\kappa=\frac{\Omega}{\bar{N}}\left(\frac{\partial N/\Omega}{\partial p}\right)_{\Omega,T}. \quad (3\cdot14)$$

This expression is rewritten, with the aid of (3·2), (3·3) and (3·6), as

$$\kappa=\frac{\Omega}{kT}\frac{(\partial^2\ln\Xi/\partial\ln z^2)_{\Omega,T}}{\bar{N}^2}=\frac{\Omega}{kT}\frac{\overline{(N-\bar{N})^2}}{\bar{N}^2},$$

where $(N-\bar{N})^2$ stands for the grand canonical ensemble average of $(N-\bar{N})^2$. Since (3·11) gives the relation

$$\overline{(N-\bar{N})^2}=\bar{N}+\iint d\boldsymbol{r}\,d\boldsymbol{r}'\{\rho^{(2)}(\boldsymbol{r},\boldsymbol{r}')-\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\},$$

we obtain

$$\kappa=\frac{1}{kT}\frac{\Omega}{\bar{N}}+\frac{1}{kT}\frac{\Omega}{(\bar{N}^2)}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\{\rho^{(2)}(\boldsymbol{r},\boldsymbol{r}')-\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\}. \quad (3\cdot15)$$

Eqs. (3·12), (3·13) and (3·15) are of course well-known relations.[6]

In order to express the distribution functions in the form of functional derivatives, we shall rewrite (3·3) as

$$\Xi=\sum_{N=0}^{\infty}\frac{1}{N!}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2\cdots d\boldsymbol{r}_N\prod_{i=1}^{N}z^*(\boldsymbol{r}_i)\exp\left\{-\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi(\boldsymbol{r}_i,\boldsymbol{r}_j)/kT\right\}, \quad (3\cdot16)$$

where $z^*(\boldsymbol{r})$ is defined by

$$z^*(\boldsymbol{r})=ze^{-\psi(\boldsymbol{r})/kT}. \quad (3\cdot17)$$

Then Ξ is regarded as a functional of $z^*(\boldsymbol{r})$ and $\phi(\boldsymbol{r},\boldsymbol{r}')$. It is readily verified, with the aid of (3·2) and (3·11), that $\rho^{(n)}$ is expressed in the form

$$\rho^{(n)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\cdots,\boldsymbol{r}_n)=\frac{1}{\Xi}\frac{\delta^n\Xi}{\delta\ln z^*(\boldsymbol{r}_1)\delta\ln z^*(\boldsymbol{r}_2)\cdots\delta\ln z^*(\boldsymbol{r}_n)}. \quad (3\cdot18)$$

It is also seen that the two-particle distribution function is expressed as

$$\rho^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2)=-2kT\frac{\delta\ln\Xi}{\delta\phi(\boldsymbol{r}_1,\boldsymbol{r}_2)}, \quad (3\cdot19)$$

which is a natural extension of the relation that has been derived by one of the present authors.[7]

In the following sections we shall mainly use the function $b(\boldsymbol{r},\boldsymbol{r}')$, instead of $\phi(\boldsymbol{r},\boldsymbol{r}')$, which is defined by

$$b(\boldsymbol{r},\boldsymbol{r}')=e^{-\phi(\boldsymbol{r},\boldsymbol{r}')/kT}-1. \quad (3\cdot20)$$

Then the grand partition function is written in the form

$$\Xi=\sum_{N=0}^{\infty}\frac{1}{N!}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2\cdots d\boldsymbol{r}_N \prod_{i=1}^{N} z^*(\boldsymbol{r}_i) \prod_{N\geq i>j\geq 1}\{1+b(\boldsymbol{r}_i,\boldsymbol{r}_j)\}. \tag{3·21}$$

The one- and two-particle distributions are expressed respectively as

$$\rho(\boldsymbol{r})=z^*(\boldsymbol{r})\frac{\delta\ln\Xi}{\delta z^*(\boldsymbol{r})}=\frac{\delta\ln\Xi}{\delta\ln z^*(\boldsymbol{r})} \tag{3·22}$$

$$\rho^{(2)}(\boldsymbol{r},\boldsymbol{r}')=2\{1+b(\boldsymbol{r},\boldsymbol{r}')\}\frac{\delta\ln\Xi}{\delta b(\boldsymbol{r},\boldsymbol{r}')}=2\frac{\delta\ln\Xi}{\delta\ln\{1+b(\boldsymbol{r},\boldsymbol{r}')\}}. \tag{3·23}$$

§ 4. Reduction of diagrams

In this section we shall express the grand partition function in the form of sum of diagrams and attempt to reduce such diagrams. As will be seen in the following, such a reduction will be done with the aid of the one- and two-particle distribution functions. The integral equations will be found for the distribution functions. It will be shown that these integral equations are derived also by a variational principle that the grand partition function is to be stationary. Some thermodynamic functions will also be expressed in terms of the distribution functions.

Let us start with writing (3·21) in the form

$$\Xi=\sum_{N=0}^{\infty}\frac{1}{N!}\int d\boldsymbol{r}^N \prod_{i=1}^{N} z^*(\boldsymbol{r}_i)\sum\prod_{i>j} b(\boldsymbol{r}_i,\boldsymbol{r}_j).$$

According to the definition 2 in § 2, the right-hand side of this expression is written as

$\sum_{N=0}^{\infty}\frac{1}{N!}$ {Sum of all the topologically different single-bond diagrams which are composed of black circles $^{z^*}\bullet_{\boldsymbol{r}_1}$, $^{z^*}\bullet_{\boldsymbol{r}_2}$, $\cdots$, $^{z^*}\bullet_{\boldsymbol{r}_N}$ and bonds $\overset{b}{\_\_\_\_}$}.

Some of the diagrams in the above expression will become topologically identical with each other if all the coordinates attached to black circles are removed. Partial summation over such diagrams gives a diagram defined by definition 3. It is noted that a factor $1/N!$ is essential in that case. Then the above expression is written in the form of {sum of all the topologically different single-bond diagrams which are composed of black circles $^{z^*}\bullet$ and bonds $\overset{b}{\_\_\_\_}$}. For the sake of simplicity, "all the topologically different diagrams" will be abbreviated hereafter to "all the diagrams." Furthermore we shall call the "single-bond diagram" simply the "diagram", since the diagrams other than single-bond diagrams will not appear in the following. After all the grand partition function is expressed as

$$\Xi = 1 + \{\text{Sum of all the diagrams which are composed of black circles } {}^{z^*}\bullet \text{ and bonds } \overset{b}{\frown}\}, \tag{4·1}$$

where unity on the right-hand side corresponds to the term with $N=0$ in (3·21).

The diagrams appearing in (4·1) will in general contain several separate parts, each of which is either a connected diagram or one black circle. A set of all the connected diagrams and one black circle is a subset of the set of all the diagrams in (4·1). It is easy to see that lemma 3 in § 2 is applied to the sum in (4·1). Therefore we obtain

$$\ln \Xi = {}^{z^*}\bullet + \{\text{Sum of all the connected diagrams which are composed of black circles } {}^{z^*}\bullet \text{ and bonds } \overset{b}{\frown}\}. \tag{4·2}$$

Eq. (4·2) corresponds to the usual cluster expansion in terms of the cluster integrals[8] if one assumes that $z^*(\boldsymbol{r})=z$, namely $\psi(\boldsymbol{r})=0$.

The one-particle distribution function is expressed, with the aid of lemma 1, (3·22) and (4·2), as*

$$\rho(\boldsymbol{r}) = z^*(\boldsymbol{r})[1 + \{\text{Sum of all the connected diagrams which are composed of one white circle } \circ_{\boldsymbol{r}}, \text{ black circles } {}^{z^*}\bullet \text{ and bonds } \overset{b}{\frown}\}]. \tag{4·3}$$

In the diagrams of (4·3) the white circle $\circ_{\boldsymbol{r}}$ will in general be an articulation circle. In other words the diagrams will in general be divided into several separate parts if $\circ_{\boldsymbol{r}}$ is removed. The total of the diagrams having no white articulation circle is a subset of the set of the diagrams in (4·3). Applying lemma 4 to the sum in (4·3), we obtain

$$\ln z^*(\boldsymbol{r}) = \ln \rho(\boldsymbol{r}) - \{\text{Sum of all the connected diagrams which are composed of one white circle } \circ_{\boldsymbol{r}}, \text{ black circles } {}^{z^*}\bullet \text{ and bonds } \overset{b}{\frown}, \text{ and in which the white circle } \circ_{\boldsymbol{r}} \text{ is not an articulation circle}\}. \tag{4·4}$$

Next we shall eliminate the black articulation circles from the sum in (4·4). For this purpose we shall classify the set of all the diagrams in (4·4) in the following way. For each diagram in (4·4) there is a unique diagram that has no black articulation circles. Therefore all the diagrams in (4·4) are classified by the diagrams that have no black articulation circles. We consider now all the diagrams which correspond to a particular diagram that has no black articulation circles. Each diagram in this collection of diagrams can be built up by attaching some diagram in (4·3) to each black circle in the particular diagram. Partial summation over all the diagrams in this collection gives, with the aid of lemma 5 in § 2, the diagram which is obtained from the particular

* It is noted that an isolated circle $\circ_{\boldsymbol{r}}$ is identically unity.

diagram by replacing all the black circles $\overset{z^*}{\bullet}$ by $\overset{\rho}{\bullet}$. After all (4·4) is rewritten as

$$\ln z^*(\boldsymbol{r}) = \ln \rho(\boldsymbol{r}) - \{\text{Sum of all the connected diagrams which are composed of one white circle } \circ_{\boldsymbol{r}}, \text{ black circles } \overset{\rho}{\bullet} \text{ and bonds } \overset{b}{\frown}, \text{ and which have no articulation circles}\}. \tag{4·5}$$

This equation may be regarded as an integral equation to determine the one-particle distribution function $\rho(\boldsymbol{r})$ in terms of $z^*(\boldsymbol{r})$ and $b(\boldsymbol{r}, \boldsymbol{r}')$.

In order to express $\ln \Xi$ in terms of $\rho(\boldsymbol{r})$, we shall consider the variation of $\ln \Xi$ caused by the variation in $z^*(\boldsymbol{r})$ while $b(\boldsymbol{r}, \boldsymbol{r}')$ is kept fixed. Then we obtain the following expression:

$$\begin{aligned}
\delta \ln \Xi &= \int d\boldsymbol{r}\, \rho(\boldsymbol{r})\, \delta \ln z^*(\boldsymbol{r}) \\
&= \delta \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) \ln z^*(\boldsymbol{r}) - \int d\boldsymbol{r} \ln z^*(\boldsymbol{r}) \delta\rho(\boldsymbol{r}) \\
&= \delta \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) \ln z^*(\boldsymbol{r}) \\
&\quad - \int d\boldsymbol{r} [\ln \rho(\boldsymbol{r}) - \{\text{Sum on the right-hand side of (4·5)}\}] \delta\rho(\boldsymbol{r}).
\end{aligned}$$

In this expression the first line is nothing but (3·22). In transforming the second line into the third one, use has been made of (4·5). The third line is integrated, with the aid of lemma 1, as

$$\begin{aligned}
\ln \Xi = &\int d\boldsymbol{r}\, \rho(\boldsymbol{r}) \ln z^*(\boldsymbol{r}) - \int d\boldsymbol{r}\, \rho(\boldsymbol{r}) \{\ln \rho(\boldsymbol{r}) - 1\} \\
&+ \{\text{Sum of all the more than singly connected diagrams which are composed of black circles } \overset{\rho}{\bullet} \text{ and bonds } \overset{b}{\frown}\}.
\end{aligned} \tag{4·6}$$

More than singly connected diagrams have been defined in § 2. In obtaining (4·6) we have determined the integration constant, which must be a functional independent of $z^*(\boldsymbol{r})$, by use of the fact that both $\ln \Xi$ and $\rho(\boldsymbol{r})$ vanish when $z^*(\boldsymbol{r}) = 0$. This fact follows immediately from (4·2) and (4·3). It should be noted that the integral equation, (4·5), for $\rho(\boldsymbol{r})$ is derived by the stationary condition

$$\left(\frac{\delta \ln \Xi \text{ given by (4·6)}}{\delta \rho(\boldsymbol{r})} \right)_{z^*, b} = 0, \tag{4·7}$$

where the subscripts z^* and b indicate that $z^*(\boldsymbol{r})$ and $b(\boldsymbol{r}, \boldsymbol{r}')$ are kept fixed during the variation in $\rho(\boldsymbol{r})$.* It will later be shown that this stationary condition is in fact the maximum condition.

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* Such a nomenclature will be employed also in the following.

It is easy to see that, if $z^*(\boldsymbol{r})=z$ and $\rho(\boldsymbol{r})=\rho$, (4·6) is reduced to the usual density or virial expansion and (4·5) to the usual relation connecting the fugacity z and the number density ρ.[8]

In the above, starting from the diagrams composed of separate parts (see (4·1)), we have reduced them first to the connected diagrams (see (4·2)) and then to the ones having no articulation circles (see (4·6)). The first reduction has been trivial. The second reduction has been done with the aid of the one-particle distribution function. The next natural step will be to reduce the diagrams to the ones which have no articulation pairs of circles. Such a reduction will be properly done with the aid of two-particle distribution function $\rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}')$.

Using (4·6) in (3·23) and taking notice of (4·7), we obtain, with the aid of lemma 2 in § 2,

$$\rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}')=2\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}\left(\frac{\delta \ln \Xi \text{ given by } (4\cdot6)}{\delta b(\boldsymbol{r}, \boldsymbol{r}')}\right)_{z^*,\rho}$$

$=\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}\rho(\boldsymbol{r})\rho(\boldsymbol{r}')$ [1+{Sum of all the connected diagrams which are composed of two white circles $\underset{\boldsymbol{r}}{\circ}$, $\underset{\boldsymbol{r}'}{\circ}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{b}{\text{——}}$ and which have no articulation circles and also in which the bond connecting $\underset{\boldsymbol{r}}{\circ}$ and $\underset{\boldsymbol{r}'}{\circ}$ is not allowed to appear}]

$=\rho(\boldsymbol{r})\rho(\boldsymbol{r}')$ [1+{Sum of all the connected diagrams which are composed of two white circles $\underset{\boldsymbol{r}}{\circ}$, $\underset{\boldsymbol{r}'}{\circ}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{b}{\text{——}}$ and which have no articulation circles}]. (4·8)

We shall define two functions, $g(\boldsymbol{r}, \boldsymbol{r}')$ and $v(\boldsymbol{r}, \boldsymbol{r}')$, as

$$\rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}')=\rho(\boldsymbol{r})\rho(\boldsymbol{r}')g(\boldsymbol{r}, \boldsymbol{r}')=\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\{1+v(\boldsymbol{r}, \boldsymbol{r}')\}. \quad (4\cdot9)$$

The function $g(\boldsymbol{r}, \boldsymbol{r}')$ corresponds to the radial distribution function in the case of a uniform fluid. Then (4·8) is written in the form

$v(\boldsymbol{r}, \boldsymbol{r}')=$ {Sum of all the connected diagrams which are composed of two white circles $\circ_{\boldsymbol{r}}$, $\circ_{\boldsymbol{r}'}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{b}{\text{——}}$ and which have no articulation circles} (4·10)

$=\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}$[1+{Sum of all the connected diagrams which are composed of two white circles $\circ_{\boldsymbol{r}}$, $\circ_{\boldsymbol{r}'}$, black circles $\overset{\rho}{\bullet}$ and bond $\overset{b}{\text{——}}$ and which have no articulation circles and also in which the bond connecting $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ is not allowed to appear}]−1. (4·10′)

The diagrams apearing in (4·10) or (4·10′) will in general have s-circles and articulation pairs of circles. We shall first classify all the diagrams in (4·10) into two groups. One group consists of the diagrams having no s-circle. The other consists of the diagrams which have at least one s-circle. If we de-

note the sum of all the diagrams having no s-circles by $z(\boldsymbol{r}, \boldsymbol{r}')$,* namely

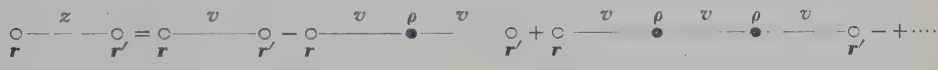
$$z(\boldsymbol{r}, \boldsymbol{r}') = \{\text{Sum of all the diagrams, among those in (4·10), which have no } s\text{-circles}\}, \tag{4·11}$$

then the sum of all the diagrams which have at least one s-circle is expressed as  . Therefore (4·10) is written in the form



(4·12)

This expression may be considered as the integral equation for $z(\boldsymbol{r}, \boldsymbol{r}')$, which is solved as



(4·12′)

Next we shall eliminate the articulation pair of $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ from (4·10′). The diagrams appearing in { } of (4·10′) will in general be divided into several separate parts if the pair of $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ is removed. Each part appearing in such a division is diagram having no articulation pair of $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$. A set of all the diagrams that have no articulation pair of $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ is a subset of the set of the diagrams appearing in { } of (4·10′). It can be seen that lemma 4 is applied to such a set and subset. Therefore, if we put

$$w(\boldsymbol{r}, \boldsymbol{r}') = \{\text{Sum of all the diagrams, among those in } \{\ \} \text{ of (4·10′), which have no articulation pair of } \circ_{\boldsymbol{r}} \text{ and } \circ_{\boldsymbol{r}'}\}, \tag{4·13}$$

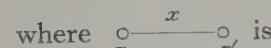
then (4·10′) is written as

$$v(\boldsymbol{r}, \boldsymbol{r}') = \{1 + b(\boldsymbol{r}, \boldsymbol{r}')\}\, e^{w(\boldsymbol{r}, \boldsymbol{r}')} - 1. \tag{4·14}$$

The diagrams in (4·13) also are classified by the presence of s-circles. Then $w(\boldsymbol{r}, \boldsymbol{r}')$ is written, correspondingly to (4·12), as



(4·15)

where  is defined by

$$x(\boldsymbol{r}, \boldsymbol{r}') = \{\text{Sum of all the diagrams, among those in (4·13), which have no } s\text{-circles}\}. \tag{4·16}$$

We shall finally eliminate all articulation pairs of circles from (4·16).** For

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* The function $z(\boldsymbol{r}, \boldsymbol{r}')$ must not be confused with z and $z^*(\boldsymbol{r})$.

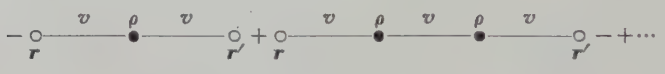
** It is to be noted that it is essential for the following argument that the diagrams in (4·16) have no s-circles. The argument, for example, cannot be applied to the diagrams in (4·10) or (4·10′).

each diagram in (4·16) there is a unique diagram that has no articulation pairs of circles. Therefore all the diagrams in (4·16) are classified by the diagrams that have no articulation pairs of circles. We consider now all the diagrams which correspond to a particular diagram that has no articulation pairs of circles. Each diagram in this collection of diagrams can be built up by inserting some diagram in (4·10) between each pair of circles in the particular diagram. According to lemma 6 in § 2, partial summation over all the diagrams in the collection under consideration gives the diagram which is obtained from the particular diagram by replacing all the bonds $\overset{b}{\frown}$ by $\overset{v}{\frown}$. Therefore (4·16) is reduced to

$x(\boldsymbol{r}, \boldsymbol{r}')=$ {Sum of all the connected diagrams which are composed of two white circles $\circ_{\boldsymbol{r}}$, $\circ_{\boldsymbol{r}'}$, black circles $^{\rho}\bullet$ and bonds $\overset{v}{\frown}$ and which have no articulation circles, no s-circles and no articulation pairs of circles, and in which the bond connecting $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ is not allowed to appear}. (4·17)

Eqs. (4·12′), (4·14) and (4·15) are brought together in the form

$$\ln\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}=\ln\{1+v(\boldsymbol{r}, \boldsymbol{r}')\}$$



$$-\{x(\boldsymbol{r}, \boldsymbol{r}') \text{ given by (4·17)}\}. \tag{4·18}$$

This equation may be regarded as the integral equation for $v(\boldsymbol{r}, \boldsymbol{r}')$ if $\rho(\boldsymbol{r})$ is known.

The expression for $\ln \Xi$ in terms of $v(\boldsymbol{r}, \boldsymbol{r}')$ will be derived in the following way. The first line of (4·8) is rewritten as

$$(\delta \ln \Xi \text{ given by (4·6)})_{z^*,\rho}$$

$$=\frac{1}{2}\iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}')\,(\delta \ln\{1+b(\boldsymbol{r}, \boldsymbol{r}')\})_{z^*,\rho}$$

$$=\frac{1}{2}\delta\Big(\iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}') \ln\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}\Big)_{z^*,\rho}$$

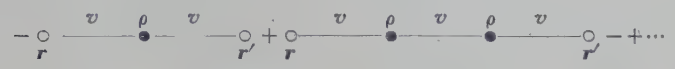
$$-\frac{1}{2}\iint d\boldsymbol{r}\, d\boldsymbol{r}' \ln\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}\,(\delta\rho^{(2)}(\boldsymbol{r}, \boldsymbol{r}'))_{z^*,\rho},$$

where the subscripts z^* and ρ indicate that $z^*(\boldsymbol{r})$ and $\rho(\boldsymbol{r})$ are kept fixed. The above expression is further rewritten, with the aid of (4·9) and (4·18), as

$$(\rho \ln \Xi \text{ given by (4·6)})_{z^*,\rho}$$

$$=\frac{1}{2}\delta\Big(\iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho(\boldsymbol{r})\rho(\boldsymbol{r}')\{1+v(\boldsymbol{r}, \boldsymbol{r}')\} \ln\{1+b(\boldsymbol{r}, \boldsymbol{r}')\}\Big)_{z^*,\rho}$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\,[\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}$$



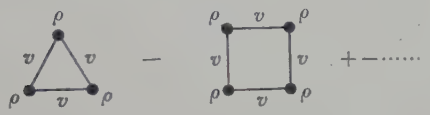
$$-\{x(\boldsymbol{r},\boldsymbol{r}')\ \text{given by (4·17)}\}]\ (\delta v(\boldsymbol{r},\boldsymbol{r}'))_{z^*,\rho}.$$

This expression is integrated, by use of lemma 2, as follows:

$$\{\ln\Xi\ \text{given by (4·6)}\}=\{\text{A functional independent of } b(\boldsymbol{r},\boldsymbol{r}')\}$$

$$+\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\,\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+b(\boldsymbol{r},\boldsymbol{r}')\}$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho(\boldsymbol{r})\rho(\boldsymbol{r}')[\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}-v(\boldsymbol{r},\boldsymbol{r}')]$$



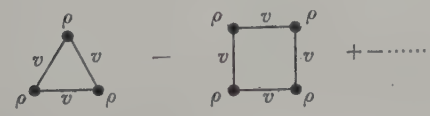
+{Sum of all the connected diagrams which are composed of black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\text{——}}$ and which have neither articulation circles nor articulation pairs of circles}.

The first term on the right-hand side is determined by using the fact that $\ln\Xi$ given by (4·6) becomes $[\int d\boldsymbol{r}\rho(\boldsymbol{r})\ln z^*(\boldsymbol{r})-\int d\boldsymbol{r}\rho(\boldsymbol{r})\{\ln\rho(\boldsymbol{r})-1\}]$ when $b(\boldsymbol{r},\boldsymbol{r}')=0$. The last term is expressed as the sum of more than doubly connected diagrams (see § 2). After all, $\ln\Xi$ is expressed in the form

$$\ln\Xi=\int d\boldsymbol{r}\rho(r)\ln z^*(\boldsymbol{r})-\int d\boldsymbol{r}\rho(\boldsymbol{r})\{\ln\rho(\boldsymbol{r})-1\}$$

$$+\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\,\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+b(\boldsymbol{r},\boldsymbol{r}')\}$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho(\boldsymbol{r})\rho(\boldsymbol{r}')[\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}-v(\boldsymbol{r},\boldsymbol{r}')]$$



+{Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\text{——}}$}. (4·19)

As is seen from the derivation of (4·19), the integral equation, (4·18), for $v(\boldsymbol{r},\boldsymbol{r}')$ is derived also by the stationary condition

$$\left(\frac{\delta \ln \Xi \text{ given by } (4\cdot19)}{\delta v(\boldsymbol{r}, \boldsymbol{r}')}\right)_{z^*, \rho, b} = 0. \tag{4·20}$$

The integral equation, (4·5), for $\rho(\boldsymbol{r})$ will be rewritten in terms of $v(\boldsymbol{r}, \boldsymbol{r}')$ in the following way. The stationary condition (4·7) can be expressed, with the aid of (4·20), as*

$$\left(\frac{\delta \ln \Xi \text{ given by } (4\cdot19)}{\delta \rho(\boldsymbol{r})}\right)_{z^*, b, v} = 0. \tag{4·21}$$

When use is made of expression (4·19), (4·21) gives us

$$\ln z^*(\boldsymbol{r}) = \ln \rho(\boldsymbol{r}) - \int d\boldsymbol{r}' \rho(\boldsymbol{r}') [\{1+v(\boldsymbol{r}, \boldsymbol{r}')\} \ln \{1+b(\boldsymbol{r}, \boldsymbol{r}')\}$$
$$- \{1+v(\boldsymbol{r}, \boldsymbol{r}')\} \ln \{1+v(\boldsymbol{r}, \boldsymbol{r}')\} + v(\boldsymbol{r}, \boldsymbol{r}')]$$

$$- \quad + \quad - + \cdots\cdots$$





$-$ {Sum of all the connected diagrams which are composed of one white circle $\circ_{\boldsymbol{r}}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\text{———}}$ and which have neither articulation circles nor articulation pairs of circles}. (4·22)

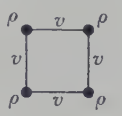
In deriving (4·22), we have used lemma 1 in § 2. A set of (4·18) and (4·22) may be regarded as the integral equation for $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ in terms of $z^*(\boldsymbol{r})$ and $b(\boldsymbol{r}, \boldsymbol{r}')$. It is to be noted that this integral equation can be derived by means of a variational principle that $\ln \Xi$ given by (4·19) is to be stationary with respect to the variations in $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ (see (4·20) and (4·21)). It will later be shown that $\ln \Xi$ given by (4·19) is in fact to be maximum.

We now express the Helmholtz free energy and the entropy in terms of $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$. Comparison of (4·19) with (3·10) gives us

$$A/kT = \int d\boldsymbol{r} \rho(\boldsymbol{r}) \frac{\psi(\boldsymbol{r})}{kT} - \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}' \rho(\boldsymbol{r}) \rho(\boldsymbol{r}') \{1+v(\boldsymbol{r}, \boldsymbol{r}')\} \ln \{1+b(\boldsymbol{r}, \boldsymbol{r}')\}$$
$$+ \int d\boldsymbol{r} \rho(\boldsymbol{r}) \{\ln \lambda^3 \rho(\boldsymbol{r}) - 1\}$$

* It should be noted that $\ln \Xi$ given by (4·19) is identical with $\ln \Xi$ given by (4·6) as a functional of $z^*(\boldsymbol{r})$, $b(\boldsymbol{r}, \boldsymbol{r}')$ and $\rho(\boldsymbol{r})$ provided that $v(\boldsymbol{r}, \boldsymbol{r}')$ in (4·19) is determined by (4·18) or (4·20). Therefore, $\ln \Xi$ in (4·7) can be replaced by $\ln \Xi$ given by (4·19).

$$+\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\rho(\boldsymbol{r})\rho(\boldsymbol{r}')[\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}-v(\boldsymbol{r},\boldsymbol{r}')]$$

$$-$$  $$+$$  $$-+\cdots\cdots$$

$-\{$Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\text{——}}\}$. (4·23)

Since the internal energy is written, in terms of the present notation, (see (3·1)) as

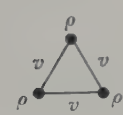
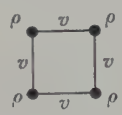
$$E/kT=\frac{3}{2}\bar{N}+\bar{N}\ln z-\int d\boldsymbol{r}\,\rho(\boldsymbol{r})\ln z^*(\boldsymbol{r})$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+b(\boldsymbol{r},\boldsymbol{r}')\}, \qquad (4\cdot24)$$

the expression for the entropy is obtained, with the aid of (3·9), as

$$S/k=\frac{5}{2}\bar{N}-\int d\boldsymbol{r}\,\rho(\boldsymbol{r})\ln\lambda^3\rho(\boldsymbol{r})$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\rho(\boldsymbol{r})\rho(\boldsymbol{r}')[\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}-v(\boldsymbol{r},\boldsymbol{r}')]$$

$$+$$  $$-$$  $$+-\cdots\cdots$$

$+\{$Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\text{——}}\}$. (4·25)

It is noted that $\psi(\boldsymbol{r})$ and $b(\boldsymbol{r},\boldsymbol{r}')$ do not appear explicitly in (4·25).

When the system under consideration is of infinite extent and also when $\rho(\boldsymbol{r})=\rho$ independent of $\boldsymbol{r}$ and $b(\boldsymbol{r},\boldsymbol{r}')=b(|\boldsymbol{r}-\boldsymbol{r}'|)$, the above introduced functions $v(\boldsymbol{r},\boldsymbol{r}')$, $z(\boldsymbol{r},\boldsymbol{r}')$, $w(\boldsymbol{r},\boldsymbol{r}')$ and $x(\boldsymbol{r},\boldsymbol{r}')$ are reduced to $v(|\boldsymbol{r}-\boldsymbol{r}'|)$, $z(|\boldsymbol{r}-\boldsymbol{r}'|)$, $w(|\boldsymbol{r}-\boldsymbol{r}'|)$ and $x(|\boldsymbol{r}-\boldsymbol{r}'|)$, respectively, which have been used in paper I. Furthermore in such a case (4·18), (4·19) and (4·23) are reduced to the expressions obtained in I (see also § 7). Therefore it may be said that the integral equation (4·8) is a generalization of the one previously derived by de Boer *et al.*,[9] Meeron[10] and also by the present authors.[1]

When one wishes to consider the average number of particles $\bar{N}$, instead of the fugacity z, to be given, then the set of (4·18) and (4·22) may be regarded as the integral equation for $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r},\boldsymbol{r}')$ in terms of N, $\psi(\boldsymbol{r})$ and $b(\boldsymbol{r},\boldsymbol{r}')$, provided that $\ln z$ appearing on the right-hand side of (4·22) (see

(3·17)) is considered as constant to be determined in such a way that $\int d\boldsymbol{r}\rho(\boldsymbol{r})=\bar{N}$. Similarly, (4·5) may be regarded as the integral equation for $\rho(\boldsymbol{r})$ in terms of $\bar{N}$, $\psi(\boldsymbol{r})$ and $b(\boldsymbol{r}, \boldsymbol{r}')$.

§ 5. Variational principle

In the preceding section we have obtained two variational principles, one of which states that $\ln \Xi$ given by (4·6) is to be stationary with respect to the variations in $\rho(\boldsymbol{r})$ (see (4·7)), and the other of which states that $\ln \Xi$ given by (4·19) is to be stationary with respect to the variations in $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ (see (4·20) and (4·21)). These variational principles will be especially useful if their stationary character is found to be maximum or minimum. We cannot say as yet that they give the maximum of $\ln \Xi$, because the arguments in the preceding section have been restricted to the case of thermal equilibrium. In this section it will be proved that these variational principles in fact give the maximum of $\ln \Xi$. The variational principles will be given also for the Helmholtz free energy. The results of this section are arranged in the form of theorems 1∼4 in the following. These theorems may be considered a summary of the results obtained in the preceding section.

Let us start from a basic variational principle for the grand canonical ensemble that $\ln \Xi$ is the maximum of trial $\ln \Xi_t$ which is defined in terms of trial probability function $P_t(N, \boldsymbol{r}^N)$ as follows:

$$\ln \Xi_t = N_t \ln(z\lambda^3) - E_t/kT + S_t/k, \tag{5·1}$$

where

$$N_t = \sum_{N=0}^{\infty} \int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N) N \tag{5·2}$$

$$E_t = \frac{3}{2} N_t kT + \sum_{N=0}^{\infty} \int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N) \{ \sum_{i=1}^{N} \psi(\boldsymbol{r}_i) + \sum_{N \geq i > j \geq 1} \sum \phi(\boldsymbol{r}_i, \boldsymbol{r}_j) \} \tag{5·3}$$

$$S_t = \frac{3}{2} N_t k - k \sum_{N=0}^{\infty} \int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N) \ln\{N! \lambda^{3N} P_t(N, \boldsymbol{r}^N)\} \tag{5·4}$$

and the trial probability function is subject to the condition

$$\sum_{N=0}^{\infty} \int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N) = 1 \text{ and } P_t(N, \boldsymbol{r}^N) \geqq 0 \text{ for all } N \text{ and } \boldsymbol{r}^N. \tag{5·5}$$

This basic variational principle will be expressed simply as

$$\ln \Xi = \underset{P_t}{\mathrm{Max}} \{\ln \Xi_t\}, \tag{5·6}$$

where the subscript P_t indicates that the variation is to be taken with respect to $P_t(N, \boldsymbol{r}^N)$. It is easy to see that the probability function maximizing $\ln \Xi_t$ is equal to $P(N, \boldsymbol{r}^N)$ given by (3·2) and maximum of $\ln \Xi_t$ is equal to $\ln \Xi$ given by (3·3), as they should be. For the sake of convenience, we shall again write the expressions of $P(N, \boldsymbol{r}^N)$ and Ξ:

$$P(N, \boldsymbol{r}^N)=\frac{1}{\Xi}\frac{z^N}{N!}\exp\left\{-\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)/kT-\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\} \quad (5\cdot 7)$$

$$\Xi=\sum_{N=0}^{\infty}\int\frac{z^N}{N!}\int d\boldsymbol{r}^N\exp\left\{-\sum_{i=1}^{N}\psi(\boldsymbol{r}_i)/kT-\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}. \quad (5\cdot 8)$$

It is to be remembered that the analysis in §§ 3 and 4 is based upon these formulas.

We now consider the trial probability function of the form

$$P_t(N, \boldsymbol{r}^N)=\frac{1}{\Xi_t^*}\frac{z^N}{N!}\exp\left\{-\sum_{i=1}^{N}\psi_t(\boldsymbol{r}_i)/kT-\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}, \quad (5\cdot 9)$$

where

$$\Xi_t^*=\sum_{N=0}^{\infty}\frac{z^N}{N!}\int d\boldsymbol{r}^N\exp\left\{-\sum_{i=1}^{N}\psi_t(\boldsymbol{r}_i)/kT-\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}. \quad (5\cdot 10)$$

The function $\psi_t(\boldsymbol{r})$ is now regarded as the trial function to be determined so as to maximize $\ln\Xi_t$. Since $P_t(N, \boldsymbol{r}^N)$ given by (5·9) satisfies the condition (5·5) and contains the rigorous one (5·7) as its special case, the variational principle (5·6) ought to give the rigorous results even if $P_t(N, \boldsymbol{r}^N)$ is of a particular form given by (5·9). Two quantities Ξ_t and Ξ_t^* must not be confused. While Ξ_t is the trial grand partition function defined by (5·1) in terms of (5·9), the quantity Ξ_t^* is rather the true grand partition function of such a model system that its potential energy is of the form

$$\sum_{i=1}^{N}\psi_t(\boldsymbol{r}_i)+\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j), \quad (5\cdot 11)$$

as is seen by comparing (5·9) and (5·10) with (5·7) and (5·8). In other words $\ln\Xi_t^*$ is written (see (3·6∼9)) as

$$\ln\Xi_t^*=N_t^*\ln(z\lambda^3)-E_t^*/kT+S_t^*/k,$$

where

$$\left.\begin{aligned} N_t^*&=\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N)N \\ E_t^*&=\frac{3}{2}N_t^*kT+\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N)\left\{\sum_{i=1}^{N}\psi_t(\boldsymbol{r}_i)+\sum_{N\geq i>j\geq 1}\sum\phi(\boldsymbol{r}_i, \boldsymbol{r}_j)\right\} \\ S_t^*&=\frac{3}{2}N_t^*k-k\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N)\ln\{N!\,\lambda^{3N}P_t(N, \boldsymbol{r}^N)\}. \end{aligned}\right\} \quad (5\cdot 12)$$

In (5·12) N_t^*, E_t^* and S_t^* stand for the average particle number, the internal energy and the entropy of the model system, respectively. Comparison of (5·12) with (5·1)—(5·4) immediately gives us

$$\ln\Xi_t=\ln\Xi_t^*+\frac{1}{kT}\sum_{N=0}^{\infty}\int d\boldsymbol{r}^N P_t(N, \boldsymbol{r}^N)\sum_{i=1}^{N}\{\psi_t(\boldsymbol{r}_i)-\psi(\boldsymbol{r}_i)\},$$

which is written, in terms of the one-particle distribution function, $\rho_t(\boldsymbol{r})$, for the model system, as

$$\ln \Xi_t = \ln \Xi_t^* + \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\psi_t(\boldsymbol{r})/kT - \psi(\boldsymbol{r})/kT\}. \tag{5·13}$$

The analysis in § 4 is applied also to the model system, as it stands. In particular we obtain, correspondingly to (4·3), (4·5) and (4·6),

$\rho_t(\boldsymbol{r}) = z_t^*(\boldsymbol{r})$ [1 + {Sum of all the connected diagrams which are composed of one white circle $\circ_{\boldsymbol{r}}$, black circles $\overset{z_t^*}{\bullet}$ and bonds $\overset{b}{\text{——}}$}] (5·14)

$\psi_t(\boldsymbol{r})/kT = \ln z - \ln \rho_t(\boldsymbol{r}) +$ {Sum of all the connected diagrams which are composed of one white circle $\circ_{\boldsymbol{r}}$, black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{b}{\text{——}}$ and which have no articulation circles} (5·15)

$$\ln \Xi_t^* = \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln z - \psi_t(\boldsymbol{r})/kT\} - \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln \rho_t(\boldsymbol{r}) - 1\}$$

+ {Sum of all the more than singly connected diagrams which are composed of black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{b}{\text{——}}$}, (5·16)

where $z_t^*(\boldsymbol{r}) = z \exp\{-\psi_t(\boldsymbol{r})/kT\}$. Using (5·16) in (5·13), we obtain

$$\ln \Xi_t = \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \ln z^*(\boldsymbol{r}) - \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln \rho_t(\boldsymbol{r}) - 1\}$$

+ {Sum of all the more than singly connected diagrams which are composed of black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{b}{\text{——}}$}. (5·17)

Furthermore (5·14) and (5·15) show that the variation with respect to $\psi_t(\boldsymbol{r})$ is equivalent to that with respect to $\rho_t(\boldsymbol{r})$. Therefore, the basic variational principle (5·6) gives us the following theorem:

Theorem 1. $\ln \Xi$ is given by the maximum of $\ln \Xi_t$ given by (5·17) when $\rho_t(\boldsymbol{r})$ is varied. The one-particle distribution function, $\rho(\boldsymbol{r})$, is equal to $\rho_t(\boldsymbol{r})$ that maximizes $\ln \Xi_t$ given by (5·17).

Since $\ln \Xi_t$ given by (5·17) is of the same form as (4·6) if $\rho_t(\boldsymbol{r})$ is replaced by $\rho(\boldsymbol{r})$, the above theorem states that the stationary character of variational principle (4·7) is to be maximum.

In quite the same way, if we consider a model system in which the potential energy is of the form

$$\sum_{i=1}^{N} \psi_t(\boldsymbol{r}_i) + \sum_{N \geq i > j \geq 1} \sum \phi_t(\boldsymbol{r}_i, \boldsymbol{r}_j), \tag{5·18}$$

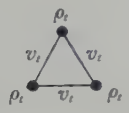
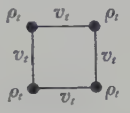
we can prove the following theorem 2. For the sake of completeness the proof is given in Appendix 4.

Theorem 2. $\ln \Xi$ is given by the maximum of $\ln \Xi_t$ given by (5·19) in the following when $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r}, \boldsymbol{r}')$ are varied independently of each other. The functions $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ are equal to $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r}, \boldsymbol{r}')$, respectively, that maximize $\ln \Xi_t$ given by (5·19).

$$\ln \Xi_t = \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \ln z^*(\boldsymbol{r}) - \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln \rho_t(\boldsymbol{r}) - 1\}$$

$$+ \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho_t(\boldsymbol{r}) \rho_t(\boldsymbol{r}') \{1 + v_t(\boldsymbol{r}, \boldsymbol{r}')\} \ln \{1 + b(\boldsymbol{r}, \boldsymbol{r}')\}$$

$$- \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho_t(\boldsymbol{r}) \rho_t(\boldsymbol{r}') [\{1 + v_t(\boldsymbol{r}, \boldsymbol{r}')\} \ln \{1 + v_t(\boldsymbol{r}, \boldsymbol{r}')\} - v_t(\boldsymbol{r}, \boldsymbol{r}')]$$

$$+$$  $-$  $+ - \cdots\cdots$

$+ \{$Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{v_t}{\text{———}}\}$. (5·19)

This theorem states that the stationary character of variational principle (4·20) and (4·21) is to be maximum.

In the above we have considered the ensemble to be characterized by z, Ω and T. It is of course appropriate to characterize the ensemble by the set of z, Ω and T. However, if one notices that the average particle number $\bar{N}$, defined by (3·6), is a monotonously increasing one-valued function of z, the same ensemble may be characterized by the set of $\bar{N}$, Ω and T. It will often be useful to consider the ensemble to be characterized by $\bar{N}$, Ω and T. For such a case the characteristic function of the ensemble will be the Helmholtz free energy, A, even though the ensemble is yet the grand canonical ensemble. In the following, we shall derive the variational principles for the Helmholtz free energy. For this purpose, we shall divide the process of the variation in (5·6) into two steps: i) a variation keeping N_t constant and ii) a variation with respect to N_t. Then the basic variational principle (5·6) is rewritten in the form

$$\ln \Xi = \underset{N_t}{\text{Max}} \{N_t \ln (z\lambda^3) - A(N_t, \Omega, T)/kT\}, \tag{5·20}$$

where

$$A(N_t, \Omega, T)/kT = \underset{P_t, \text{while } \Sigma \int P_t N = N_t}{\text{Min}} \{A_t/kT\} \tag{5·21}$$

and A_t is defined, in terms of (5·3) and (5·4), as

$$A_t/kT = E_t/kT - S_t/k. \tag{5·22}$$

By taking the variation with respect to N_t in (5·20), it is seen that $\bar{N}$, which is equal to N_t that maximizes $\ln \Xi_t$, is the solution of

$$\ln(z\lambda^3) = \frac{\partial}{\partial \bar{N}} (A(\bar{N}, \Omega, T)/kT). \tag{5·23}$$

This implies that $\ln \Xi$ is given by

$$\ln \Xi = \bar{N} \ln(z\lambda^3) - A(\bar{N}, \Omega, T)/kT \tag{5·24}$$

if $\bar{N}$ is determined by (5·23) in terms of z, Ω and T. Conversely, when $\bar{N}$ is given in addition to Ω and T, (5·23) gives such a value of z for which N_t maximizing $\ln \Xi_t$ is equal to given $\bar{N}$. $\ln \Xi$ for this value of z is determined by Eq. (5·24). A comparison of (5·24) with (3·10) shows that $A(\bar{N}, \Omega, T)$ is the Helmholtz free energy. Therefore, if $\bar{N}$, Ω and T are given, one has to calculate the Helmholtz free energy A by the variational principle

$$A/kT = \min_{P_t, \text{ while } \Sigma \int P_t N = \bar{N}} \{A_t/kT\}. \tag{5·25}$$

Then one can calculate $\ln z$ and $\ln \Xi$ by means of (5·23) and (5·24), hence other thermodynamic functions. As a result, the basic variational principle (5·6) has been reduced to the basic variational principle (5·25) for the Helmholtz free energy.

Since theorems 1 and 2 have been derived from (5·6) as its special cases, the argument reducing (5·6) to (5·25) is applied to these theorems. Theorems 1 and 2 accordingly are transformed, when the ensemble is characterized by $\bar{N}$, Ω and T, as follows:

Theorem 3. A/kT is given by the minimum of A_t/kT given by (5·26) in the following when $\rho_t(\boldsymbol{r})$ is varied while $\int d\boldsymbol{r} \rho_t(\boldsymbol{r}) = \bar{N}$ being kept constant. The one-particle distribution function, $\rho(\boldsymbol{r})$, is equal to $\rho_t(\boldsymbol{r})$ that minimizes A_t/kT given by (5·26).

$$A_t/kT = \int d\boldsymbol{r} \rho_t(\boldsymbol{r}) \psi(\boldsymbol{r})/kT - \int d\boldsymbol{r} \rho_t(\boldsymbol{r}) \{\ln \lambda^3 \rho_t(\boldsymbol{r}) - 1\}$$
$$- \{\text{Sum of all the more than singly connected diagrams which are composed of black circles } \overset{\rho_t}{\bullet} \text{ and bonds } \overset{b}{\text{——}}\}. \tag{5·26}$$

Theorem 4. A/kT is given by the minimum of A_t/kT given by (5·27) in the following when $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r}, \boldsymbol{r}')$ are varied while $\int d\boldsymbol{r} \rho_t(\boldsymbol{r}) = \bar{N}$ is kept constant. The functions $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ are equal to $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r}, \boldsymbol{r}')$, respectively, that minimize A_t/kT given by (5·27).

$$A_t/kT = \int d\boldsymbol{r} \rho_t(\boldsymbol{r}) \psi(\boldsymbol{r})/kT - \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}' \rho_t(\boldsymbol{r}) \rho_t(\boldsymbol{r}') \{1 + v_t(\boldsymbol{r}, \boldsymbol{r}')\} \ln\{1 + b(\boldsymbol{r}, \boldsymbol{r}')\}$$
$$+ \int d\boldsymbol{r} \rho_t(\boldsymbol{r}) \{\ln \lambda^3 \rho_t(\boldsymbol{r}) - 1\}$$

$$+\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho_t(\boldsymbol{r})\rho_t(\boldsymbol{r}')[\{1+v_t(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v_t(\boldsymbol{r},\boldsymbol{r}')\}-v_t(\boldsymbol{r},\boldsymbol{r}')]$$

$$-\quad \begin{array}{ccc} & \rho_t & \\ v_t & & v_t \\ \rho_t & v_t & \rho_t \end{array} \quad + \quad \begin{array}{ccc} \rho_t & v_t & \rho_t \\ v_t & & v_t \\ \rho_t & v_t & \rho_t \end{array} \quad -+\cdots\cdots$$

$-\{$Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{v_t}{\text{—}}\}$. (5·27)

It is noted that (5·27) is of the same form as (4·23) if ρ_t and v_t are replaced by ρ and v respectively. As is readily seen, theorem 4 gives an integral equation for $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r},\boldsymbol{r}')$ which is the same as the set of (4·18) and (4·22) provided that $\ln z$ in the latter is considered a constant to be determined in such a way that $\int d\boldsymbol{r}\rho(\boldsymbol{r})=\bar{N}$. Similarly theorem 3 gives an integral equation for $\rho(\boldsymbol{r})$ which is the same as (4·5).

Although the ensemble characterized by $\bar{N}$, Ω and T is not the usual canonical ensemble, the difference between these ensembles ought to be negligible for the macroscopic systems. Therefore we may say that theorems 3 and 4 are valid for the canonical ensemble if $\bar{N}$ is regarded as the number of particles.

In the variational principle that was obtained in paper I,[1] the Helmholtz free energy was made to be stationary with respect to the independent variations in three functions $v(r)$, $w(r)$ and $z(r)$ which correspond to $v(\boldsymbol{r},\boldsymbol{r}')$, $w(\boldsymbol{r},\boldsymbol{r}')$ and $z(\boldsymbol{r},\boldsymbol{r}')$ in the present paper, respectively. Although the stationary character of this variational principle is not clear, we can readily see that it does not give a minimum of the Helmholtz free energy. However, since the expression of the free energy in paper I can be reduced, by eliminating $w(r)$ and $z(r)$, to (5·27) in which it has been assumed that $\psi(\boldsymbol{r})=0$, $b(\boldsymbol{r},\boldsymbol{r}')=b(|\boldsymbol{r}-\boldsymbol{r}'|)$, $\rho_t(\boldsymbol{r})=\rho$ and $v_t(\boldsymbol{r},\boldsymbol{r}')=v(|\boldsymbol{r}-\boldsymbol{r}'|)$, the variational principle in paper I can be reduced to a special case of theorem 4.

§ 6. Multicomponent system

The results derived in the preceding sections will be generalized to the case of multicomponent systems. As will be seen in the following, the multicomponent systems can be formally reduced to the one-component systems.

Let us consider a σ-component system. The potential energy of N particles, among which there exist N_1 particles of species 1, N_2 of species 2, $\cdots$, and N_σ of species σ, will be assumed to be of the form

$$\sum_{i=1}^{N}\phi_{\nu_i}(\boldsymbol{r}_i)+\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j), \tag{6·1}$$

where $\nu_i(i=1, 2, \cdots, N)$ denotes the species of i-th particle so that

$$\sum_{i=1}^{N}\delta_{\nu_i\nu}=N_\nu \quad (\nu=1, 2, \cdots, \sigma). \tag{6·2}$$

The grand partition function is, in the present case, of the form

$$\Xi=\sum_{N_1,N_2,\cdots,N_\sigma}\frac{z_1^{N_1}z_2^{N_2}\cdots z_\sigma^{N_\sigma}}{N_1!\,N_2!\cdots N_\sigma!}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2\cdots d\boldsymbol{r}_N$$
$$\exp\left\{-\sum_{i=1}^{N}\psi_{\nu_i}(\boldsymbol{r}_i)/kT-\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j)/kT\right\},\qquad(6\cdot3)$$

where the fugacity z_ν is defined, in terms of the chemical potential μ_ν of species ν, by

$$z_\nu=(2\pi m_\nu kT/h^2)^{3/2}e^{\mu_\nu/kT}=(1/\lambda_\nu^3)e^{\mu_\nu/kT}.\qquad(6\cdot4)$$

In (6·1) and in the integrand of (6·3), the species of each particle is specified in a definite way. However, the way of specification may be arbitrary unless (6·2) is violated. The number of ways in which the species of particles are specified in such a way that (6·2) is satisfied is $N!/N_1!N_2!\cdots N_\sigma!$. Each way of specification gives the same value to the integral of (6·3). Therefore (6·3) may be written as

$$\Xi=\sum_{N_1,N_2,\cdots,N_\sigma}\frac{z_1^{N_1}z_2^{N_2}\cdots z_\sigma^{N_\sigma}}{N!}\sum_{\nu_1=1}^{\sigma}\sum_{\nu_2=1}^{\sigma}\cdots\sum_{\nu_N=1}^{\sigma}\int d\boldsymbol{r}_1 d\boldsymbol{r}_2\cdots d\boldsymbol{r}_N$$
under condition (6·2)
$$\times\exp\left\{-\sum_{i=1}^{N}\psi_{\nu_i}(\boldsymbol{r}_i)/kT-\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j)/kT\right\}$$
$$=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\nu_1=1}^{\sigma}\int d\boldsymbol{r}_1\sum_{\nu_2=1}^{\sigma}\int d\boldsymbol{r}_2\cdots\sum_{\nu_N=1}^{\sigma}\int d\boldsymbol{r}_N\prod_{i=1}^{N}z_{\nu_i}^{*}(\boldsymbol{r}_i)$$
$$\times\exp\left\{-\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi_{\nu_i\nu_j}(\boldsymbol{r}_i,\boldsymbol{r}_j)/kT\right\},\qquad(6\cdot5)$$

where

$$z_\nu^{*}(\boldsymbol{r})=z_\nu e^{-\psi_\nu(\boldsymbol{r})/kT}.\qquad(6\cdot6)$$

We shall express a set of coordinates $\boldsymbol{r}$ and species ν as $\boldsymbol{x}$ and $\sum_{\nu=1}^{\sigma}\int d\boldsymbol{r}$ as $\int d\boldsymbol{x}$. Writing $z_\nu^{*}(\boldsymbol{r})$ and $\phi_{\nu\nu'}(\boldsymbol{r},\boldsymbol{r}')$ as $z^{*}(\boldsymbol{x})$ and $\phi(\boldsymbol{x},\boldsymbol{x}')$ respectively, we can express (6·5) as the form

$$\Xi=\sum_{N=0}^{\infty}\frac{1}{N!}\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\cdots d\boldsymbol{x}_N\prod_{i=1}^{N}z^{*}(\boldsymbol{x}_i)\exp\left\{-\frac{1}{2}\sum_{i\neq j}^{N}\sum^{N}\phi(\boldsymbol{x}_i,\boldsymbol{x}_j)/kT\right\}.\qquad(6\cdot7)$$

This expresssion for the grand partition function is formally identical with (3·16) in the case of one-component system. Furthermore it is easy to see that when $z_\nu^{*}(\boldsymbol{r})$'s and $\phi_{\nu\nu'}(\boldsymbol{r},\boldsymbol{r}')$'s are varied by $\delta z_\nu^{*}(\boldsymbol{r})$'s and $\delta\phi_{\nu\nu'}(\boldsymbol{r},\boldsymbol{r}')$'s respectively, the first variation of $\ln\Xi$ is expressed in terms of one-particle distribution functions, $\rho_\nu(\boldsymbol{r})$'s, and two-particle distribution functions, $\rho_{\nu\nu'}^{(2)}(\boldsymbol{r},\boldsymbol{r}')$'s, as

$$\delta \ln \Xi = \sum_{\nu=1}^{\sigma} \int d\boldsymbol{r} \rho_\nu(\boldsymbol{r}) \delta \ln z_\nu^*(\boldsymbol{r}) - \frac{1}{2kT} \sum_{\nu=1}^{\sigma} \sum_{\nu'=1}^{\sigma} d\boldsymbol{r} d\boldsymbol{r}' \rho_{\nu\nu'}^{(2)}(\boldsymbol{r}, \boldsymbol{r}') \delta \phi_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}'), \tag{6·8}$$

which is rewritten as

$$\delta \ln \Xi = \int d\boldsymbol{x} \rho(\boldsymbol{x}) \delta \ln z^*(\boldsymbol{x}) - \frac{1}{2kT} \iint d\boldsymbol{x} d\boldsymbol{x}' \rho^{(2)}(\boldsymbol{x}, \boldsymbol{x}') \delta \phi(\boldsymbol{x}, \boldsymbol{x}'), \tag{6·8′}$$

where

$$\rho(\boldsymbol{x}) \equiv \rho_\nu(\boldsymbol{r}) \quad \text{and} \quad \rho^{(2)}(\boldsymbol{x}, \boldsymbol{x}') \equiv \rho_{\nu\nu'}^{(2)}(\boldsymbol{r}, \boldsymbol{r}'). \tag{6·9}$$

Eq. (6·8′) gives

$$\rho(\boldsymbol{x}) = \frac{\delta \ln \Xi}{\delta \ln z^*(\boldsymbol{x})} \tag{6·10}$$

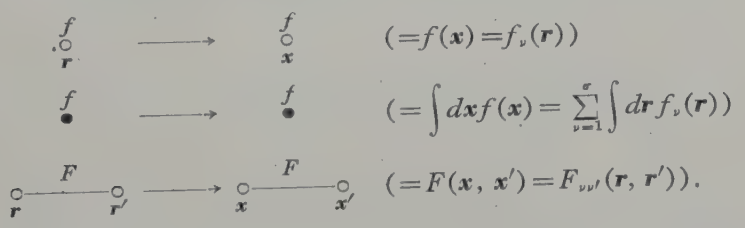
$$\rho^{(2)}(\boldsymbol{x}, \boldsymbol{x}') = -2kT \frac{\delta \ln \Xi}{\delta \phi(\boldsymbol{x}, \boldsymbol{x}')}. \tag{6·11}$$

These expressions are formally identical with those in the case of one-component system (see (3·22) and (3·19)). It can also be shown that the situation is similar for the other thermodynamic functions appearing in § 3. Especially the expression for the Helmholtz free energy is

$$A/kT = \sum_{\nu=1}^{\sigma} \bar{N}_\nu \ln z_\nu \lambda_\nu^3 - \ln \Xi = \int d\boldsymbol{x} \rho(\boldsymbol{x}) \ln z(\boldsymbol{x}) \lambda(\boldsymbol{x})^3 - \ln \Xi, \tag{6·12}$$

where $z(\boldsymbol{x}) = z_\nu$ and $\lambda(\boldsymbol{x}) = \lambda_\nu$ (see (3·10) and (3·12)).

After all, the expressions in the case of one-component system can be translated into those in the case of σ-component system in the following way. The coordinate $\boldsymbol{r}$ is interpreted as $\boldsymbol{x}(=(\boldsymbol{r}, \nu))$ and $\int d\boldsymbol{r}$ as $\int d\boldsymbol{x}(=\sum_{\nu=1}^{\sigma} \int d\boldsymbol{r})$. The diagrams are correspondingly interpreted as follows:



It is to be noted that the lemmas 1∼6 can be applied to the diagrams obtained by means of the above interpretation. Therefore the arguments in the case of one-component system are applied to the present case as they are.

For example, we obtain from (4·6)

$$\ln \Xi = \int d\boldsymbol{x} \rho(\boldsymbol{x}) \ln z^*(\boldsymbol{x}) - \int d\boldsymbol{x} \rho(\boldsymbol{x}) \{\ln \rho(\boldsymbol{x}) - 1\}$$
$$+ \{\text{Sum of all the more than singly connected diagrams which are composed of black circles } \overset{\rho}{\bullet} \text{ and bonds } \overset{b}{\text{——}}\}, \tag{6·13}$$

which is rewritten explicitly as

$$\ln \Xi = \sum_{\nu=1}^{\sigma} \int d\boldsymbol{r}\, \rho_\nu(\boldsymbol{r}) \ln z_\nu^*(\boldsymbol{r}) - \sum_{\nu=1}^{\sigma} \int d\boldsymbol{r}\, \rho_\nu(\boldsymbol{r}) \{\ln \rho_\nu(\boldsymbol{r}) - 1\}$$

$$+ \{\text{Sum of all the more than singly connected diagrams which are composed of black circles } \overset{\rho_\nu}{\bullet} \ (\nu=1, 2, \cdots, \sigma) \text{ and bonds } \overset{b_{\nu\nu'}}{\text{——}} \ (\nu, \nu'=1, 2, \cdots, \sigma)\}, \tag{6·13′}$$

where

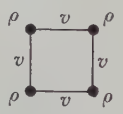
$$b(\boldsymbol{x}, \boldsymbol{x}') = b_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}') = e^{-\phi_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}')/kT} - 1. \tag{6·14}$$

Eq. (6·13′) corresponds to the usual density expansion in the case of multi-component systems. The Helmholtz free energy A is expressed from (4·23) as

$$A/kT = \int d\boldsymbol{x} \rho(\boldsymbol{x}) \psi(\boldsymbol{x})/kT - \frac{1}{2} \iint d\boldsymbol{x}\, d\boldsymbol{x}' \rho(\boldsymbol{x})\, \rho(\boldsymbol{x}') \{1 + v(\boldsymbol{x}, \boldsymbol{x}')\} \ln \{1 + b(\boldsymbol{x}, \boldsymbol{x}')\}$$

$$+ \int d\boldsymbol{x} \rho(\boldsymbol{x}) \{\ln \lambda^3 \rho(\boldsymbol{x}) - 1\}$$

$$+ \frac{1}{2} \iint d\boldsymbol{x}\, d\boldsymbol{x}' \rho(\boldsymbol{x})\, \rho(\boldsymbol{x}') [\{1 + v(\boldsymbol{x}, \boldsymbol{x}')\} \ln \{1 + v(\boldsymbol{x}, \boldsymbol{x}')\} - v(\boldsymbol{x}, \boldsymbol{x}')]$$

$$- \quad \text{  } \quad + \quad \text{  } \quad - + \cdots\cdots$$

$$- \{\text{Sum of all the more than doubly connected diagrams which are composed of black circles } \overset{\rho}{\bullet} \text{ and bonds } \overset{v}{\text{——}}\}, \tag{6·15}$$

where

$$v(\boldsymbol{x}, \boldsymbol{x}') = v_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}') = \frac{\rho_{\nu\nu'}{}^{(2)}(\boldsymbol{r}, \boldsymbol{r}')}{\rho_\nu(\boldsymbol{r}) \rho_{\nu'}(\boldsymbol{r}')} - 1. \tag{6·16}$$

In (6·14)  , for example, is rewritten explicitly as

$$= \frac{1}{6} \quad \text{  }$$

$$= \frac{1}{6} \sum_{\nu_1=1}^{\sigma} \sum_{\nu_2=1}^{\sigma} \sum_{\nu_3=1}^{\sigma} \iiint d\boldsymbol{r}_1\, d\boldsymbol{r}_2\, d\boldsymbol{r}_3 \rho_{\nu_1}(\boldsymbol{r}_1) \rho_{\nu_2}(\boldsymbol{r}_2) \rho_{\nu_3}(\boldsymbol{r}_3)$$

$$\times v_{\nu_1\nu_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) v_{\nu_2\nu_3}(\boldsymbol{r}_2, \boldsymbol{r}_3) v_{\nu_3\nu_1}(\boldsymbol{r}_3, \boldsymbol{r}_1).$$

It can be verified that Eq. (6·15) is reduced to the expression derived in paper II[2] if $\psi_\nu(\boldsymbol{r})=0$, $b_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}')=b_{\nu\nu'}(|\boldsymbol{r}-\boldsymbol{r}'|)$ and $v_{\nu\nu'}(\boldsymbol{r}, \boldsymbol{r}')=v_{\nu\nu'}(|\boldsymbol{r}-\boldsymbol{r}'|)$.

It will be needless to say that theorems 1 and 2 in § 5 are valid also in the case of multicomponent systems if the above-mentioned interpretation is adopted. It can be shown that the situation is similar for the variational principles for the Helmholtz free energy, (5·25) and theorems 3 and 4, provided that one understands the restriction $\sum\int P_t(N, \boldsymbol{r}^N)N=\bar{N}$ to be replaced by the set of $\sum\int P_t(N, \boldsymbol{x}^N)N_\nu=\bar{N}_\nu$ $(\nu=1, 2, \cdots, \sigma)$. For instance, theorem 4 is transformed as follows: A/kT is given by the minimum of A_t/kT, which is obtained from the right-hand side of (6·15) by attaching subscript t to $\rho(\boldsymbol{x})$ and $v(\boldsymbol{x}, \boldsymbol{x}')$, when $\rho_t(\boldsymbol{x})$ and $v_t(\boldsymbol{x}, \boldsymbol{x}')$ are varied while $\int d\boldsymbol{r}\rho_{\nu t}(\boldsymbol{r})=\bar{N}_\nu(\nu=1, 2, \cdots, \sigma)$ is kept constant. The functions $\rho(\boldsymbol{x})$ and $v(\boldsymbol{x}, \boldsymbol{x}')$ are equal to $\rho_t(\boldsymbol{x})$ and $v_t(\boldsymbol{x}, \boldsymbol{x}')$, respectively, that minimize A_t/kT.

§ 7. Discussion

In the foregoing sections we have considered a system which is enclosed in a volume Ω. Naturally, the results which have been obtained there are valid for the system of any finite extent. However, if one is concerned with a macroscopic system as usual, Ω must be very large. In fact, in order to discuss the behavior of the macroscopic system, it is often convenient to consider the system to be of infinite extent. In the following, by comparing with the empirically known behavior of the macroscopic system, we shall investigate a situation in the limit of $\Omega\to\infty$. For the sake of simplicity, we shall confine ourselves to the case of one-component system in which the potential energy is a sum of two-particle interaction potentials depending only upon the distance of two particles so that $\psi(\boldsymbol{r})=0$, $z^*(\boldsymbol{r})=z$ and $b(\boldsymbol{r}, \boldsymbol{r}')=b(|\boldsymbol{r}-\boldsymbol{r}'|)$ (see (3·1), (3·17) and (3·20)).

In the first place we shall take notice of the following facts which are known empirically on the behavior on the macroscopic system in thermal equilibrium. As the fugacity z increases from zero at sufficiently low temperatures, the state of the system changes from gaseous phase into liquid phase and then into solid phase. In the gaseous and liquid phases, the one-particle distribution function, $\rho(\boldsymbol{r})$, does not depend on $\boldsymbol{r}$ and the function $v(\boldsymbol{r}, \boldsymbol{r}')$, which is related to the two-particle distribution function (see (4·9)), depends on $|\boldsymbol{r}-\boldsymbol{r}'|$ alone. In the solid phase $\rho(\boldsymbol{r})$ is periodic in $\boldsymbol{r}$. Figs. 1 and 2 show schematically the behavior of the pressure p and the number density* ρ as the functions of z respectively. Fig. 3 shows the relation between p and ρ.

Now, in the limit of $\Omega\to\infty$, (4·22) and (4·18) become

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* The number density is to be defined, in the present case, by $\rho=\lim_{\Omega\to\infty}\frac{1}{\Omega}\int d\boldsymbol{r}\,\rho(\boldsymbol{r})$.

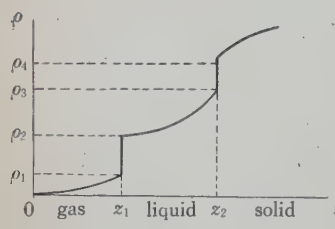


Fig. 1. Relation between the number density and the fugacity at a given low temperature.

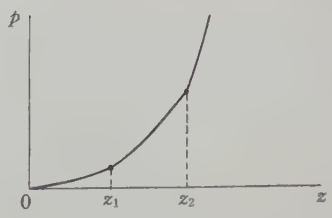


Fig. 2. Relation between the pressure and the fugacity at a given low temperature.

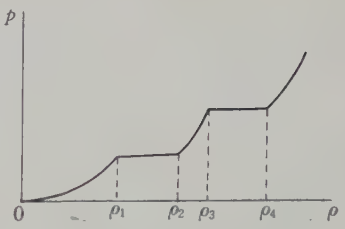
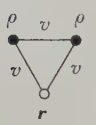
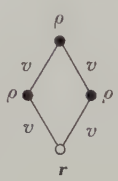


Fig. 3. Relation between the pressure and the number density at a given low temperature.

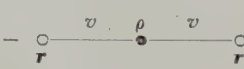
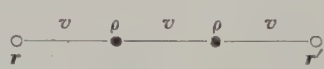
$$\ln z=\ln\rho(\boldsymbol{r})-\int d\boldsymbol{r}'\rho(\boldsymbol{r}')[\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+b(|\boldsymbol{r}-\boldsymbol{r}'|)\}$$
$$-\{1+v(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}+v(\boldsymbol{r},\boldsymbol{r}')]$$

$-$  $+$  $-+\cdots\cdots$

$$-F^{(1)}[\rho(\boldsymbol{r}),v(\boldsymbol{r},\boldsymbol{r}')] \qquad (7\cdot1)$$

and

$$\ln\{1+b(|\boldsymbol{r}-\boldsymbol{r}'|)\}=\ln\{1+v(\boldsymbol{r},\boldsymbol{r}')\}$$

$-$  $+$  $-+\cdots\cdots$

$$-F^{(2)}[\rho(\boldsymbol{r}),v(\boldsymbol{r},\boldsymbol{r}')], \qquad (7\cdot2)$$

where $F^{(1)}$ and $F^{(2)}$, being functionals of $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r},\boldsymbol{r}')$, are defined by

$F^{(1)}[\rho(\boldsymbol{r}),v(\boldsymbol{r},\boldsymbol{r}')]=\lim_{\Omega\to\infty}$ {Sum of all the connected diagrams which are composed of one white circle $\circ_{\boldsymbol{r}}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\frown}$ and which have neither articulation circles nor articulation pairs of circles}

(7·3)

and

$F^{(2)}[\rho(\boldsymbol{r}),v(\boldsymbol{r},\boldsymbol{r}')]=\lim_{\Omega\to\infty}$ {Sum of all the connected diagrams which are composed of two white circles $\circ_{\boldsymbol{r}}$, $\circ_{\boldsymbol{r}'}$, black circles $\overset{\rho}{\bullet}$ and bonds $\overset{v}{\frown}$ and which have no articulation circles, no s-circles and no articulation pairs of circles, and in which the bond connecting $\circ_{\boldsymbol{r}}$ and $\circ_{\boldsymbol{r}'}$ is not allowed to appear}. (7·4)

In the above expressions the integrations over the coordinates are to be performed within an infinite volume.

Since a set of (7·1) an (7·2) is an exact integral equation for $\rho(\boldsymbol{r})$ and $v(\boldsymbol{r}, \boldsymbol{r}')$ in the macroscopic system, its solution ought to describe rigorously the behavior of the system which has been explained above. Namely we should find ourselves in the following situation: (For the meaning of z_1, z_2, ρ_1, ρ_2, ρ_3, and ρ_4, see Figs. 1~3.)

(i) When $z > z_2$, the integral equation has such a solution that $\rho(\boldsymbol{r})$ is periodic in $\boldsymbol{r}$. This case corresponds to the solid phase.

(ii) When $z_2 > z > z_1$, the integral equation has such a solution that $\rho(\boldsymbol{r})$, being independent of $\boldsymbol{r}$, is equal to the number density ρ and $v(\boldsymbol{r}, \boldsymbol{r}')$, being of the form $v(|\boldsymbol{r}-\boldsymbol{r}'|)$, satisfies the following equation,

$$\ln\{1+b(|\boldsymbol{r}-\boldsymbol{r}'|)\} = \ln\{1+v(|\boldsymbol{r}-\boldsymbol{r}'|)\} - \frac{1}{(2\pi)^3}\frac{1}{\rho}\int d\boldsymbol{k}\frac{\{\rho V(k)\}^2}{1+\rho V(k)}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} - F_l^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)], \tag{7·5}$$

where $V(k)$ is the Fourier transform of $v(\boldsymbol{r})$, namely

$$V(k) = \int d\boldsymbol{r}\, v(r)\, e^{i\boldsymbol{k}\boldsymbol{r}}, \tag{7·6}$$

and $F_l^{(2)}$ is a function of ρ and functional of $v(|\boldsymbol{r}-\boldsymbol{r}'|)$ which is defined by

$$F_l^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)] = F^{(2)}[\rho(\boldsymbol{r}), v(\boldsymbol{r}, \boldsymbol{r}')] \quad \text{if} \quad \rho_3 > \rho(\boldsymbol{r}) = \rho > \rho_2 \quad \text{and} \quad v(\boldsymbol{r}, \boldsymbol{r}') = v(|\boldsymbol{r}-\boldsymbol{r}'|). \tag{7·7}$$

(iii) When $z_1 > z > 0$, the integral equation has such a solution that $\rho(\boldsymbol{r}) = \rho$ and $v(\boldsymbol{r}, \boldsymbol{r}') = v(|\boldsymbol{r}-\boldsymbol{r}'|)$ which satisfies the equation

$$\ln\{1+b(|\boldsymbol{r}-\boldsymbol{r}'|)\} = \ln\{1+v(|\boldsymbol{r}-\boldsymbol{r}'|)\} - \frac{1}{(2\pi)^3}\frac{1}{\rho}\int d\boldsymbol{k}\frac{\{\rho V(k)\}^2}{1+\rho V(k)}e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')} - F_g^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)], \tag{7·8}$$

where $F_g^{(2)}$ is defined by

$$F_g^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)] = F^{(2)}[\rho(\boldsymbol{r}), v(\boldsymbol{r}, \boldsymbol{r}')] \quad \text{if} \quad \rho_1 > \rho(\boldsymbol{r}) = \rho > 0 \quad \text{and} \quad v(\boldsymbol{r}, \boldsymbol{r}') = v(|\boldsymbol{r}-\boldsymbol{r}'|). \tag{7·9}$$

Case (ii) corresponds to the liquid phase and case (iii) to the gaseous phase. Two quantities, $F_l^{(2)}$ and $F_g^{(2)}$, are not necessarily same with each other.*

As proved by Yang and Lee,[11] the volume dependence of the cluster integrals may be ignored for the case (iii) so that the usual density expansion

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* For example, if $F^{(2)}(\rho)$ is of the form $\lim_{\Omega\to\infty}\frac{1}{e^{\Omega(c-\rho)}+1}$ where $\rho_2 > c > \rho_1$, we have $F_l^{(2)}(\rho) \equiv 1$ and $F_g^{(2)}(\rho) \equiv 0$.

formulas are valid. Since such expansion formulas were the starting point of the investigations in paper I,[1] the results obtained in paper I ought to be valid at least for the case (iii), namely for the gaseous phase. Therefore, comparing (7·8) with (6·3)~(6·6) in I, we obtain

$$F_g^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)]=x(|\boldsymbol{r}-\boldsymbol{r}'|) \text{ which was used in I}$$
$$=\text{The one which is derived from (7·4) in the present paper by interchanging the order of the limit and the summation.} \quad (7\cdot10)$$

Similarly, when the limit of $\Omega\to\infty$ is taken in the expressions for the thermodynamic functions in § 4, thus derived expressions which are valid for the gaseous phase, where the order of the limit $\Omega\to\infty$ and the summation may be interchanged, are in agreement with the expressions that were derived in I. In particular, the pressure p for the gaseous phase is expressed as (see (4·19))

$$\begin{aligned} p/kT &= \lim_{\Omega\to\infty}\frac{1}{\Omega}\ln\Xi \\ &= \rho+\rho(\ln z-\ln\rho)+\frac{\rho^2}{2}\int d\boldsymbol{r}[\{1+v(r)\}\ln\{1+b(r)\} \\ &\qquad -\{1+v(r)\}\ln\{1+v(r)\}+v(r)] \\ &\quad +\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\left[\ln\{1+\rho V(k)\}-\rho V(k)-\frac{\{\rho V(k)\}^2}{2}\right] \\ &\quad +\left(\lim_{\Omega\to\infty}\frac{1}{\Omega}\right)\{\text{Sum of all the more than doubly connected diagrams} \\ &\qquad \text{which are composed of black circles } \overset{\rho}{\bullet} \text{ and bonds } \overset{v}{\text{---}} \text{ where} \\ &\qquad \rho(\boldsymbol{r})=\rho \text{ and } v(\boldsymbol{r},\boldsymbol{r}')=v(|\boldsymbol{r}-\boldsymbol{r}'|)\}, \end{aligned} \quad (7\cdot11)$$

where $(\lim_{\Omega\to\infty}(1/\Omega))$ means merely that the integration for one black circle chosen arbitrarily in each diagram is not carried out but its coordinates are fixed at an arbitrary position. With the aid of (7·1) for the gaseous phase, it is easy to see that (7·11) is equivalent to (5·6) in I.

We shall next give our careful consideration to the phase change between the gaseous and liquid phases. As is well known, the pressure is expressed in terms of the cluster integrals $b_l(\Omega, T)$, being in general the functions of Ω as well as of T, as[6]

$$p/kT=\lim_{\Omega\to\infty}\sum_{l=1}^{\infty}b_l(\Omega, T)z^l. \quad (7\cdot12)$$

Assuming that

$$\lim_{\Omega\to\infty}\sum_{l=1}^{\infty}b_l(\Omega, T)z^l=\sum_{l=1}^{\infty}b_l^{(0)}(T)z^l, \quad (7\cdot13)$$

where

$$b_l^{(0)}(T)=\lim_{\Omega\to\infty}b_l(\Omega, T),$$

Mayer[12] has concluded that the condensation of gas is to occcur at such a number density ρ_m that it is the lowest value on the positive real axis of ρ for which either

$$\sum_{k=1}^{\infty} k\beta_k\rho^k=\text{singular} \qquad (7\cdot14a)$$

or

$$1-\sum_{k=1}^{\infty} k\beta_k\rho^k=0, \qquad (7\cdot14b)$$

where β_k's are the so-called irreducible integrals. (7·14a) and (7·14b) correspond to the singularity of the right-hand side of (7·13). Although Mayer's theory has been of much importance in the development of the theory of condensation, there is some doubt about the validity of Mayer's conclusions on condensation. Katsura and Fujita[13] have pointed out that, as z increases from zero, the first irregular point of the (non-analytic) function $\lim_{\Omega\to\infty}\sum b_l z^l$ does not necessarily agree with the first singular point of the analytic functions $\sum b_l^{(0)} z^l$ (see (7·13)) so that the condition (7·14) does not necessarily give the condensation point. Ikeda[14] and Katsura[15] have pointed out also that Yang-Lee's theory[11] does not justify Mayer's theory.* Katsura and Fujita[16),13] and Katsura[15] have made the conjecture that the condition (7·14) or the first singular point of $\sum b_l^{(0)} z^l$ does not give the condensation point but gives the end point of the metastable gaseous state. In other words the condition (7·14) will be assumed to be satisfied by $\rho=\rho_1'$ in Fig. 4, not by $\rho=\rho_1$. They have given this conjecture based on the fact that it is valid for a special case.**

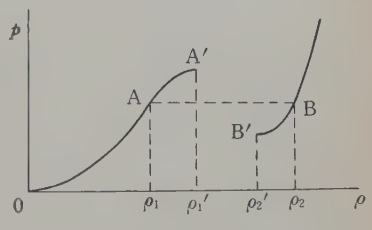


Fig. 4. An isotherm which is obtained with the restraint of uniform density (compare with Fig. 3). Portions AA′ and BB′ represent the metastable gaseous and liquid states respectively.

It is needless to say that a comprehensive investigation is necessary for clarifying whether Mayer's conclusions on condensation are valid or not. Such an investigation is not attempted here. However, the present authors wish to accept the above conjecture of Katsura and Fujita, mainly based upon the following very natural assumption: If one introduces the restraint of uniform (macroscopic) density into a theory but no other approximation, then the theory rigorously describes the metastable states of the system as well as the stable states. As have been mentioned above, in Mayer's theory the order of the limit $\Omega\to\infty$ and the summation is interchanged (see (7·13)). If the same procedure is done in the expression of $\rho(\boldsymbol{r})$ (see (4·3))

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* Note added in proof. The situation has been much clarified by the recent investigation of Saito about the lattice gas. (N. Saito; preprint, to appear in J. Chem. Phys.) The authors are grateful to Dr. Saito for sending them his preprint.

** Note added in proof. Saito's investigation (*loc. cit.*) also supports this conjecture.

$$\rho(\boldsymbol{r}) = z[1 + \lim_{\Omega\to\infty} \{\text{Sum of all the connected diagrams which are composed of one white circle } \circ_{\boldsymbol{r}}, \text{ black circles } \overset{z}{\bullet} \text{ and bonds } \overset{b}{\text{---}}\}], \quad (7\cdot15)$$

we obtain the result that the one-particle distribution function, $\rho(\boldsymbol{r})$, does not depend on $\boldsymbol{r}$ so that the system is uniform. Therefore we may say that Mayer's theory is the one in which the restraint of uniform density is introduced. Then the above conjecture follows. The situation is similar also for the Ikeda's discussions[17] of condensing systems based on Mayer's theory.

The above conjecture may be considered to be based upon the following assumption: The metastable states and the adjacent equilibrium regions are described by the same branch of one analytic function.* This implies that (7·8) and (7·11) are useful until the value of ρ reaches ρ_1'(not ρ_1). It is to be noted here that the possibility of the metastable states must be generally considered. In some cases, ρ_1' may be equal to ρ_1, but whether it is so or not can be determined either by performing the calculations based on the correct limiting process or by obtaining ρ_1 by the method stated below. Ikeda[14] has given a criterion on the former method.

Similar considerations will be applied also to the metastable liquid state. The integral equation (7·5) will be useful not only for $\rho > \rho_2$ but also for $\rho_2 > \rho > \rho_2'$ (see Fig. 4).

It will follow, from the above considerations, that the condensation point cannot be found so far as we restrict ourselves to the study of gaseous phase alone,** because in such a case ρ_1 is not any particular value of ρ. Similarly, the value of ρ_2 will be unable to be found so far as only the liquid phase is considered.** In order to find the values of ρ_1 and ρ_2 from the rigorous theory with the restraint of uniform density, we must consider both of the gaseous and liquid phases and find two values of ρ at which the chemical potentials of two phases are equal.

It will be of interest to examine what a form the condition (7·14) takes in our expressions. The isothermal compressibility, κ, is expressed in terms of irreducible integrals as[8]

$$\frac{1}{\kappa} = \rho kT\left[1 - \sum_{k=1}^{\infty} k\beta_k \rho^k\right]. \quad (7\cdot16)$$

On the other hand, (3·15) is rewritten for the case of the fluid state as

$$1/\kappa = \rho kT \frac{1}{1+\rho V(0)}, \quad (7\cdot17)$$

* If this assumption were not true, there would appear a phase change between a metastable and the adjacent equilibrium region, but such has not been observed.

** These statements never mean that one cannot know, for example, the functional $F^{(2)}[\rho(\boldsymbol{r}), v(\boldsymbol{r}, \boldsymbol{r}')]$ in which ρ_1 and ρ_2 are the particular values of ρ (see (7·4), (7·7) and (7·9)). If one is concerned with $F_g^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)]$ or $F_l^{(2)}[\rho, v(|\boldsymbol{r}-\boldsymbol{r}'|)]$ only, ρ_1 (or ρ_2) does not play a special role.

where use has been made of Fourier transform of $v(r)$ (see (7·6)). Therefore (7·14b) is written in the form

$$\frac{1}{1+\rho V(0)}=0 \quad \text{or} \quad \rho V(0)=+\infty. \tag{7·18b}$$

It is difficult to examine the condition (7·14a) exactly. This condition will be replaced, at least in part, by

$$\text{The last term on the right-hand side of } (7\cdot11)=\text{singular}, \tag{7·18a}$$

since the last sum on (7·11) is related with a partial sum of $\sum k_l \beta_{k_l} \rho^k$. Another possibility to realize (7·14a) may be that

$$1+\rho V(k)<0 \quad \text{for some values of } k \tag{7·18a′}$$

since (7·11) then becomes singular. However, perhaps (7·18a′) will never be realized, because, as has been pointed out by Meeron,[10] it is known that the function $z(r)$ defined by*

$$z(r)=\frac{1}{(2\pi)^3}\frac{1}{\rho}\int d\boldsymbol{k}\left[1-\frac{1}{1+\rho V(k)}\right]e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{7·19}$$

does always exist and is of short range for the fluid state.** As the consequence, (7·14) will be replaced by (7·18a) and (7·18b).

The above considerations on the phase change between the gaseous phase and liquid phase will be applied also to that between the liquid and solid phases.

Throughout the present paper we have been concerned with a rigorous theory. In concluding the paper we shall mention some remarks on the hyper-netted chain approximation which was proposed by Morita[18),4)] and independently by Meeron[19] who called it the convolution approximation. The approximation was originally proposed in order to consider, by means of Fourier transformation, as many terms as possible in the usual density expansion formulas for the two-particle distribution function and the Helmholtz free energy. As was pointed out in paper I, (7·5) or (7·8) and (7·11) are reduced to the expressions in the hyper-netted chain approximation if one neglects the last terms on the right-hand side. Therefore one may understand the hyper-netted chain approximation in such a way that it is an approximation in which the last terms on the right-hand side of (4·18), (4·19), (4·22), (4·23) and (4·25) are neglected. It is clear that the hyper-netted chain approximation, which has been understood in such a way, is significant independently of the density expansion formulas. In other words the approximation has a meaning in the solid and liquid phases as well as in the gaseous phase. It is another question whether this approximation is useful in the solid and liquid phases or not. However, we expect that for the hyper-netted chain approximation

* The function $z(r)$ is the same one as in paper I.

** It is noted that $1+\rho V(0)\geqq 0$ and $\lim_{k\to\infty} V(k)=0$ or $\lim_{k\to\infty}[1+\rho V(k)]=1$.

an isotherm near the liquid-vapour coexistence region is of the form of Fig. 5.* In Fig. 5, points A′ and B′ correspond to singularities of $\{1+\rho V(k)\}$ (see (7·11)). Condensation point A and vaporization point B are to be determined in such a way that the pressure and the chemical potential are equal at those points. Point A″ is a maximum of the pressure for the gaseous phase. It is to be noted that, in contrast with the rigorous theory, an unstable region such as A″A′ may appear in the case of approximate theories. From the standpoint of Mayer's theory of condensation, Ikeda[21] has concluded that the condensation point for the hyper-netted chain approximation is given by A″. But we cannot accept his conclusion that point A″ is the condensation point, since we do not agree with Mayer's theory of condensation. On the other hand Katsura and Harumi[22] have concluded, by analogy with the ring approximation, that the hyper-netted chain approximation will not describe the liquid phase. The reasons why we expect the appearance of the liquid phase for the hyper-netted chain approximation against Katsura and Harumi are as follows:

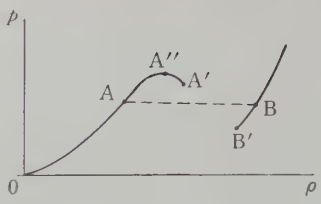


Fig. 5. Relation between the pressure and the number density at a given low temperature for the hyper-netted chain approximation (compare with Fig. 4).

i) In the ring approximation, if the singularity once appears at some number density, it always appears for larger number densities. However, such is not necessarily the case with the hyper-netted chain approximation, since $V(k)$ itself is dependent on the number density.

ii) It has been shown[4),23] that the integral equation for the pair distribution function in the hyper-netted chain approximation is of a form similar to the non-linear Born-Green integral equation. The latter equation is known[24] to have the solutions corresponding to the liquid phase.

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Appendix 1

Proofs of lemmas 1 and 2

In the first place we shall prove lemma 1. According to definition 3 of

* In general, we do not know whether a maximum A″ appears or not. We have written it in Fig. 5, following the calculations of Rodriguez[20] and Ikeda et al.[21]

diagrams, Γ will be expressed in the from

$$\Gamma=\frac{1}{m!}\{\Gamma_1(\underset{\boldsymbol{r}_1}{\bullet},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\Gamma_2(\underset{\boldsymbol{r}_1}{\bullet},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\cdots+\Gamma_\sigma(\underset{\boldsymbol{r}_1}{\bullet},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})\},\quad\text{(A1·1)}$$

where $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_\sigma\}$ is a set of all the diagrams which are topologically different from each other but are reduced to Γ if all the suffices $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m$ are removed. We shall take a functional derivative of the right-hand side of (A1·1) with respect to $f(\boldsymbol{r})$ that is attached to $\underset{\boldsymbol{r}_1}{\bullet}$. The result is written, with the aid of (2·3), as

$$\frac{1}{m!}\{\Gamma_1^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\Gamma_2^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\cdots+\Gamma_\sigma^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})\},$$

where $\Gamma_i^{(1)}(i=1, 2, \cdots, \sigma)$ denotes a diagram which is obtained from Γ_i by changing the black circle $\overset{f}{\underset{\boldsymbol{r}_1}{\bullet}}$ into $\underset{\boldsymbol{r}}{\circ}$. Some of $\Gamma_i^{(1)}$'s will in fact be zero because the one-particle function attached to $\underset{\boldsymbol{r}_1}{\bullet}$ is not always $f(\boldsymbol{r})$. Functional derivatives with respect to $f(\boldsymbol{r})$ attached to the other black circles are taken in the same way. Then we obtain

$$\frac{\delta\Gamma}{\delta f(\boldsymbol{r})}=\frac{1}{m!}\sum_{j=1}^{m}\sum_{i=1}^{\sigma}\Gamma_i^{(j)}(\underset{\boldsymbol{r}_1}{\bullet},\cdots,\underset{\boldsymbol{r}_{j-1}}{\bullet},\underset{\boldsymbol{r}}{\circ},,\underset{\boldsymbol{r}_{j+1}}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet}).\quad\text{(A1·2)}$$

On the other hand each of $\underset{\boldsymbol{r}_j}{\bullet}$'s plays an equivalent role in the right-hand side of (A1·1), so that the sum $\sum_{j=1}^{m}$ on the right-hand side of (A1·2) may be replaced by m times the contribution from $j=1$. Namely we have

$$\frac{\delta\Gamma}{\delta f(\boldsymbol{r})}=\frac{1}{(m-1)!}\{\Gamma_1^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\Gamma_2^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})+\cdots+\Gamma_\sigma^{(1)}(\underset{\boldsymbol{r}}{\circ},\underset{\boldsymbol{r}_2}{\bullet},\cdots,\underset{\boldsymbol{r}_m}{\bullet})\}\quad\text{(A1·3)}$$

Now $\Gamma_i^{(1)}$'s appearing in (A1·3) can be grouped together according to the types of diagrams to which they are reduced if the suffices $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_m$ are removed. A sum, divided by $(m-1)!$, of all the $\Gamma_i^{(1)}$'s belonging to one type is expressed by one diagram which follows definition 3. When a set of such diagrams is denoted by $\{\gamma_1, \gamma_2, \cdots, \gamma_\tau\}$, (A1·3) is reduced to

$$\frac{\delta\Gamma}{\delta f(\boldsymbol{r})}=\gamma_1+\gamma_2+\cdots+\gamma_\tau.\quad\text{(A1·4)}$$

It is clear that the set of γ_i's is included in a set of all the topologically different diagrams which are obtained from Γ by changing a black circle $\overset{f}{\bullet}$ into $\underset{\boldsymbol{r}}{\circ}$. It is easy to prove that, conversely, the latter set is included in the former. Therefore (A1·4) can be expressed as

$$\frac{\delta\Gamma}{\delta f(\boldsymbol{r})}=\{\text{Sum of all the topologically different diagrams which are obtained from } \Gamma \text{ by changing a black circle } \overset{f}{\bullet} \text{ into a white circle } \underset{\boldsymbol{r}}{\circ}\}.\quad\text{(A1·5)}$$

Lemma 1 immediately follows if both sides of (A1·5) are multiplied by $y(\boldsymbol{r})$.

The proof of lemma 2 is similar to that of lemma 1. When we take a functional derivative of the right-hand side of (A1·1) with respect to $F(\boldsymbol{r}, \boldsymbol{r}')$, we may take a functional derivative with respect to only $F(\boldsymbol{r}, \boldsymbol{r}')$ connecting $\bullet_{\boldsymbol{r}_1}$ and $\bullet_{\boldsymbol{r}_2}$, and multiply the result by $m(m-1)/2$ which is the number of pairs. Namely we obtain

$$\frac{\delta\Gamma}{\delta F(\boldsymbol{r}, \boldsymbol{r}')} = \frac{1}{2}\frac{1}{(m-2)!}\{\Gamma_1^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m}) + \Gamma_2^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m}) + \cdots + \Gamma_\sigma^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m})\}, \tag{A1·6}$$

where $\Gamma_i^{(1,2)}(i=1, 2, \cdots, \sigma)$ denotes a diagram which is obtained from Γ_i by changing the bond $\overset{f}{\bullet}\!\!-\!\!\overset{F}{-}\!\!-\!\!\overset{f'}{\bullet}$ $\begin{matrix}\\ \boldsymbol{r}_1\end{matrix}$ $\begin{matrix}\\ \boldsymbol{r}_2\end{matrix}$ into $({}^{f}\circ_{\boldsymbol{r}}\ {}^{f'}\circ_{\boldsymbol{r}'})$. Some of $\Gamma_i^{(1,2)}$ will in fact be zero. Strictly speaking, $\Gamma^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m})$ must be replaced by $\{\Gamma_i^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots \bullet_{\boldsymbol{r}_m}) + \Gamma_i^{(1,2)}(\circ_{\boldsymbol{r}'}, \circ_{\boldsymbol{r}}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m})\}/2$ if it is not symmetric in $\boldsymbol{r}$ and $\boldsymbol{r}'$ (see (2·3)). However, in such a case there exists surely another term $\Gamma_j^{(1,2)}(\circ_{\boldsymbol{r}}, \circ_{\boldsymbol{r}'}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m})$, on the right-hand side of (A1·6), which is equal to $\Gamma_i^{(1,2)}(\circ_{\boldsymbol{r}'}, \circ_{\boldsymbol{r}}, \bullet_{\boldsymbol{r}_3}, \cdots, \bullet_{\boldsymbol{r}_m})$, so that (A1·6) does not alter as a whole. Grouping $\Gamma_i^{(1,2)}$'s according to the types of diagrams to which they are reduced when the suffices $\boldsymbol{r}_3, \cdots, \boldsymbol{r}_m$ are removed, we obtain lemma 2.

Appendix 2

Proofs of lemmas 3 and 4

Let us assume that the set $\boldsymbol{N}$ consists of N diagram $\gamma_1, \gamma_2, \cdots, \gamma_N$ where N may be an infinity. According to the hypothesis, any diagram belonging to $\boldsymbol{M}$ is uniquelly characterized by the numbers of $\gamma_1, \gamma_2, \cdots, \gamma_N$ from which it is built up, so that it will be written as $\Gamma(n_1, n_2, \cdots, n_N)$ where $n_1, n_2, \cdots$ and n_N denote the numbers of $\gamma_1, \gamma_2, \cdots$ and γ_N respectively. $\Gamma(n_1, n_2, \cdots, n_N)$ is not equal to $\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_N^{n_N}$. The symmetry numbers must be taken account of (see definition 3′). The symmetry number of $\Gamma(n_1, n_2, \cdots, n_N)$ is

$$n_1!\, n_2! \cdots n_N! \prod_{i=1}^{N} [\text{symmetry number of } \gamma_i]^{n_i}.$$

Therefore we obtain

$$\Gamma(n_1, n_2, \cdots, n_N) = \frac{1}{n_1!\, n_2! \cdots n_N!}\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_N^{n_N}. \tag{A2·1}$$

Summing up over all the sets of the integers including zero, $n_1, n_2, \cdots$, and n_N, except for $n_1=n_2=\cdots=n_N=0$, we obtain

$$\sum \Gamma(n_1, n_2, \cdots, n_N) = \exp[\gamma_1+\gamma_2+\cdots+\gamma_N]-1,$$

which is the relation to be proved.

Appendix 3

Proofs of lemmas 5 and 6

In the first place we shall prove lemma 5. Let us assume that there are m black circles in Γ and they are identical ones. According to the hypothesis, any diagram belonging to $\boldsymbol{N}$ is characterized by the diagrams, belonging to $\boldsymbol{M}$, which are attached to the m black circles. Accordingly we shall write a diagram belonging to N as $\Lambda(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ where $\gamma_i (i=1, 2, \cdots, m)$ denotes a diagram, belonging to $\boldsymbol{M}$, which is attached to the i-th black circle of Γ. It is noted that two Λ's corresponding to two different sets of γ_i's are not necessarily different diagrams. Namely lemma 5 is to be expressed in terms of Λ's as

$$\sum_{\{\gamma_1, \gamma_2, \cdots, \gamma_m\}} \Lambda(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m) = \{\text{A diagram which is obtained from } \Gamma \text{ by attaching } g(\boldsymbol{r}) \text{ to each black circle}\}, \tag{A3·1}$$

where $\sum_{\{\gamma_1, \gamma_2, \cdots, \gamma_m\}}$ means that the summation is to be taken over all sets of γ_i's which give topologically different diagrams. Now $\Lambda(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ is expressed from definition 3′ of diagrams as

$$\Lambda(\Gamma; \gamma_1, \gamma_2 \cdots, \gamma_m) = \frac{1}{S(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)} \Lambda(\Gamma'; \gamma_1', \gamma_2', \cdots, \gamma_m'), \tag{A3·2}$$

where $S(\Gamma; \gamma_1, \gamma_2 \cdots, \gamma_m)$ is the symmetry number of the diagram and $\Lambda(\Gamma'; \gamma_1', \gamma_2', \cdots, \gamma_m')$ is a diagram which is obtained from $\Lambda(\Gamma; \gamma_1, \cdots, \gamma_m)$ by attaching coordinates to all the black circles.* By definition, $S(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ is the number of permutations, of the coordinates on the black circles of $\Lambda(\Gamma'; \gamma_1', \gamma_2, \cdots, \gamma_m')$, that give a diagram topologically identical with $\Lambda(\Gamma'; \gamma_1', \gamma_2', \cdots, \gamma_m')$. We shall group these $S(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ permutations according to the permutations of only the coordinates on the black circles of Γ. Let $S^*(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ be the number of permutations, of the coordinates on the black circles of $\Lambda(\Gamma''; \gamma_1, \gamma_2, \cdots, \gamma_m)$, that give a diagram topologically identical with $\Lambda(\Gamma''; \gamma_1, \gamma_2, \cdots, \gamma_m)$ where $\Lambda(\Gamma''; \gamma_1, \gamma_2, \cdots, \gamma_m)$ denotes a diagram which is obtained from $\Lambda(\Gamma'; \gamma_1', \gamma_2', \cdots, \gamma_m')$ by removing the coordinates attached to the black circles of $\gamma_1, \gamma_2, \cdots, \gamma_m$. In $\Lambda(\Gamma'; \gamma_1', \gamma_2', \cdots, \gamma_m')$, for each of these $S^*(\Gamma; \gamma_1, \gamma_2, \cdots, \gamma_m)$ permutations, there are $\prod_{i=1}^{m} S(\gamma_i)$ permutations of the coordinates on the black circles of $\gamma_1, \gamma_2, \cdots, \gamma_m$ that give topologically identical diagrams

* The prime on Γ or γ_i means that the coordinates are attached in some way to all the black circles of Γ or γ_i.

where $S(\gamma_i)$ is the symmetry number of γ_i. As the consequence we obtain the relation

$$S(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)=S^*(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)\prod_{i=1}^{m}S(\gamma_i). \qquad (A3\cdot3)$$

Next we shall pay attention to the symmetry number of Γ, $S(\Gamma)$, which is also the symmetry number of the diagram of the right-hand side of (A3·1). $S(\Gamma)$ is the number of permutations, of the coordinates on the black circles of Γ', that give a diagram topologically identical with Γ', where Γ' denotes a diagram which is obtained from Γ by attaching the coordinates to the black circles. It is, of course, not equal to $S^*(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ unless all the γ_i's are the same diagrams, but is closely related to S^* in the following way. We shall group $S(\Gamma)$ permutations by operating them on $\Lambda(\Gamma';\gamma_1,\gamma_2,\cdots,\gamma_m)$. Let $n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ be the number of diagrams which are topologically different from each other but are reduced to $\Lambda(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ if the coordinates on the black circles are removed. Then a set of $S(\Gamma)$ diagrams which are obtained from $\Lambda(\Gamma';\gamma_1,\gamma_2,\cdots,\gamma_m)$ by operating $S(\Gamma)$ permutations on it is divided into such $n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ groups that the diagrams belonging to different groups are topologically different from each other. Each of these $n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ groups consists of $S^*(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ diagrams which are topologically identical with each other. Therefore we obtain the relation

$$S(\Gamma)=n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)S^*(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m), \qquad (A3\cdot4)$$

which gives, being combined with (A3·3),

$$S(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)=S(\Gamma)\prod_{i=1}^{m}S(\gamma_i)/n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m). \qquad (A3\cdot5)$$

This relation is essential to the proof of lemma 5, as will be seen in the following. The left-hand side of (A3·1) is rewritten, with the aid of (A3·2) and (A3·5), as

$$\sum_{\{\gamma_1,\gamma_2,\cdots,\gamma_m\}}\frac{n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)}{S(\Gamma)\prod_{i=1}^{m}S(\gamma_i)}\Lambda(\Gamma';\gamma_1',\gamma_2',\cdots,\gamma_m')$$

$$=\sum_{\{\gamma_1,\gamma_2,\cdots,\gamma_m\}}\frac{n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)}{S(\Gamma)}\Lambda(\Gamma';\gamma_1,\gamma_2,\cdots,\gamma_m), \qquad (A3\cdot6)$$

where definition 3′ of diagrams has been used for γ_i's. As stated before, $\sum_{\{\gamma_1,\gamma_2,\cdots,\gamma_m\}}$ means that the summation is to be taken over all sets of γ_i's which give topologically different $\Lambda(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$'s. On the other hand, $n(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ is the number of the diagrams, $\Lambda(\Gamma';\gamma_1,\gamma_2,\cdots,\gamma_m)$'s, which are topologically different from each other but are reduced to $\Lambda(\Gamma;\gamma_1,\gamma_2,\cdots,\gamma_m)$ if the coordinates on the black circles are removed. Therefore we can write (A3·6) as

$$\sum_{\gamma_1, \gamma_2, \cdots, \gamma_m} \frac{1}{S(\Gamma)} \Lambda(\Gamma'; \gamma_1, \gamma_2, \cdots, \gamma_m), \tag{A3·7}$$

where the summation over $\gamma_1, \gamma_2, \cdots$, and γ_m are to be taken independently of each other. It is clear that (A3·7) is equal to a diagram, divided by $S(\Gamma)$, which is obtained from Γ' by attaching $g(\boldsymbol{r})$ to each black circle, so that (A3·7) is equal to the right-hand side of (A3·1) if one remembers definition 3′ of diagrams. Consequently (A3·1) or lemma 5 has been proved.

The proof of lemma 6 is quite similar to that of lemma 5. If one considers the bonds in Γ instead of the black circles, the above arguments are applied to the case of lemma 6 as they are.

Appendix 4

Proof of theorem 2

The proof of theorem 2 is quite similar to that of theorem 1, so that we shall give here the proof in a concise form as possible.

A trial probability function will be assumed to be of the form

$$P_t(N, \boldsymbol{r}^N) = \frac{1}{\Xi_t^*} \frac{z^N}{N!} \exp\left\{-\sum_{i=1}^{N} \psi_t(\boldsymbol{r}_i)/kT - \sum_{N \geq i > j \geq 1} \phi_t(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}, \tag{A4·1}$$

where Ξ_t^*, being considered to be the true grand partition function of a model system in which its potential energy is of the form (5·18), is defined by

$$\Xi_t^* = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int d\boldsymbol{r}^N \exp\left\{-\sum_{i=1}^{N} \psi_t(\boldsymbol{r}_i)/kT - \sum_{N \geq i > j \geq 1} \phi_t(\boldsymbol{r}_i, \boldsymbol{r}_j)/kT\right\}. \tag{A4·2}$$

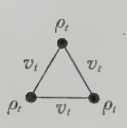
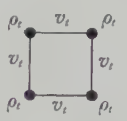
The functions $\psi_t(\boldsymbol{r})$ and $\phi_t(\boldsymbol{r}, \boldsymbol{r}')$ are now regarded as the trial functions to be determined so as to maximize $\ln \Xi_t$. Then the variational principle (5·6) ought to give the rigorous results by the same reason as for (5·9). The relation of $\ln \Xi_t$ and $\ln \Xi_t^*$ is given, with the aid of (3·6)—(3·9) and (5·1)—(5·4), by

$$\begin{aligned} \ln \Xi_t = \ln \Xi_t^* &+ \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\psi_t(\boldsymbol{r}) - \psi(\boldsymbol{r})\}/kT \\ &+ \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho_t^{(2)}(\boldsymbol{r}, \boldsymbol{r}') \{\phi_t(\boldsymbol{r}, \boldsymbol{r}') - \phi(\boldsymbol{r}, \boldsymbol{r}')\}/kT, \end{aligned} \tag{A4·3}$$

where $\rho_t(\boldsymbol{r})$ and $\rho_t^{(2)}(\boldsymbol{r}, \boldsymbol{r}')$ are the one- and two-particle distribution functions for the model system under consideration respectively. On the other hand the analysis in § 4, being applied to the model system, gives (see (4·19))

$$\begin{aligned} \ln \Xi_t^* = &\int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln z - \psi_t(\boldsymbol{r})/kT\} - \int d\boldsymbol{r}\, \rho_t(\boldsymbol{r}) \{\ln \rho_t(\boldsymbol{r}) - 1\} \\ &- \frac{1}{2} \iint d\boldsymbol{r}\, d\boldsymbol{r}'\, \rho_t(\boldsymbol{r}) \rho_t(\boldsymbol{r}') \{1 + v_t(\boldsymbol{r}, \boldsymbol{r}')\} \phi_t(\boldsymbol{r}, \boldsymbol{r}')/kT \end{aligned}$$

$$-\frac{1}{2}\iint d\boldsymbol{r}\,d\boldsymbol{r}'\,\rho_t(\boldsymbol{r})\rho_t(\boldsymbol{r}')[\{1+v_t(\boldsymbol{r},\boldsymbol{r}')\}\ln\{1+v_t(\boldsymbol{r},\boldsymbol{r}')\}-v_t(\boldsymbol{r},\boldsymbol{r}')]$$

$+$  $-$  $+-\cdots\cdots$

$+\{$Sum of all the more than doubly connected diagrams which are composed of black circles $\overset{\rho_t}{\bullet}$ and bonds $\overset{v_t}{\text{———}}\}$, (A4·4)

where $v_t(\boldsymbol{r},\boldsymbol{r}')$ is defined by

$$\rho_t^{(2)}(\boldsymbol{r},\boldsymbol{r}')=\rho_t(\boldsymbol{r})\rho_t(\boldsymbol{r}')[1+v_t(\boldsymbol{r},\boldsymbol{r}')]. \qquad \text{(A4·5)}$$

Introducing (A4·4) into (A4·3), we obtain the expression for $\ln\Xi_t$, (5·19). Furthermore, correspondingly to (4·3) and (4·10) or (4·22) and (4·18), there exist the formulas relating $\psi_t(\boldsymbol{r})$ and $\phi_t(\boldsymbol{r},\boldsymbol{r}')$ to $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r},\boldsymbol{r}')$. These formulas show that the variations with respect to $\psi_t(\boldsymbol{r})$ and $\phi_t(\boldsymbol{r},\boldsymbol{r}')$ are equivalent to those with respect to $\rho_t(\boldsymbol{r})$ and $v_t(\boldsymbol{r},\boldsymbol{r}')$. As the consequence, the original variational principle (5·6) is reduced to that given by theorem 2.

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Geometrical Study of the Pair Distribution Function in the Many-Body Problem

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The properties of pair distribution functions which are independent of physical natures as statistics, temperature or potential were studied. We found that the inequalities (23) and (55) are always satisfied by the pair distribution function. Furthermore, the inequalities (9) and (45) are satisfied when the particles have hard-cores. These inequalities will serve as the subsidiary conditions in the variational treatments of many-body problems.

§ 1. Introduction

The pair distribution function, which gives us information on the probabilities of finding two particles separated by certain distances, is very important in the many-body problem. Not only is it measurable by, for example, X-ray diffraction, but also it is all we have to know in order to calculate the potential energy from known (or assumed) two-body potential as far as we neglect the single- and many-body forces.

In treating many-body problems there are variational methods which use the pair distribution function in the expression of the potential energy. The typical ones are those of Mayer[1] and of Jastrow, Iwamoto, Yamada et al.[2] In these methods the energy minima are sought by varying the pair distribution functions. Of course, the kinetic energy cannot be calculated from the pair distribution function only. In order to include the kinetic energy Mayer[1] used the one-body density matrix as the variational function in addition to the pair distribution function. Instead, Jastrow et al.[2] assumed the form of the wave function of the system from the beginning. By using the cluster development method the kinetic energy, the potential energy and the pair distribution function were expressed in terms of the correlation functions, which were regarded as the variational functions. Therefore, the pair distribution function is an indirect variational function in this case.

In most variational problems there must be subsidiary conditions to ensure the existence of extrema. The necessity of subsidiary conditions in the above-mentioned methods was noticed by several authors.[3] The purpose of this paper is to give geometrical conditions to the pair distribution function motivated by these situations. Although the effect of these conditions on the Mayer method

is not yet known, they are expected to be quite helpful to the Jastrow-Iwamoto-Yamada method,[2] and may be interesting in themselves.

Let the system be composed of N identical particles in a space of volume V. As is customary in the treatment of an infinitely large system we make the following assumptions.

$$\rho = N/V \tag{1}$$

is generally finite. The space is infinitely large in any direction and the effect of the surface can be neglected. In the space all the points are assumed to be equivalent and all the directions equivalent, too. In other words, if we consider a region in the space, the distribution of particles in this region depends only on the shape and size of the region and not on its position and direction in the space.

Let $F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$ be the probability of finding the centers of the particle 1 at $\boldsymbol{r}_1$, of the particle 2 at $\boldsymbol{r}_2$, $\cdots$, and of the particle N at $\boldsymbol{r}_N$ respectively. Quite generally, we can take this distribution function, $F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$, as a symmetric function with respect to all the variables, $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N$.* In quantum mechanics

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = \sum_\alpha |\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N; \alpha_1, \alpha_2, \cdots, \alpha_N)|^2, \tag{2}$$

where $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N; \alpha_1, \alpha_2, \cdots, \alpha_N)$ is the normalized wave function and $\sum_\alpha$ means the sum over all the internal variables, α_i's. $F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N)$ is normalized, i.e.

$$\int_V d\boldsymbol{r}_1 \int_V d\boldsymbol{r}_2 \cdots \int_V d\boldsymbol{r}_N F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = 1. \tag{3}$$

We define the pair distribution function $F(r)$ as

$$F(|\boldsymbol{r}_2 - \boldsymbol{r}_1|) = V^2 \int_V d\boldsymbol{r}_3 \int_V d\boldsymbol{r}_4 \cdots \int_V d\boldsymbol{r}_N F(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N). \tag{4}$$

That the pair distribution function depends only on the relative distance follows from our assumption that the local distribution is invariant under translation and rotation. Some distributions as crystal structures are not invariant under rotation. In such cases our theory can be applied only after averaging the distribution over all directions. In our normalization** the pair distribution function, $F(r)$, approaches to unity as r increases; or more precisely,

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* For non-identical particles the distribution function may not be symmetric. In such a case, however, we can symmetrize the distribution function. Then, the arguments in the text can be applied to this symmetrized distribution function if we interpret d as the smallest hard-core diameter. It should be noted that this symmetrization is completely different from the symmetrization of wave functions and can be applied to systems of any statistics.

** In this normalization, $F(r)$ is usually called the radial distribution function.

$$\lim_{R\to\infty}\frac{3}{R^3}\int_0^R F(r)r^2dr=1, \tag{5}$$

where $R\to\infty$ means to make R very large keeping it much smaller than the dimension of the total space. Since $F(r)$ is a probability function, it must be non-negative;

$$F(r)\geq 0. \tag{6}$$

All the functions satisfying (5) and (6) cannot be pair distribution functions. In order that a function of r be a possible pair distribution function it must satisfy conditions other than (5) and (6). In this paper we seek these conditions with particular emphasis on the system of hard-core particles. The conditions we are seeking are geometrical ones, i.e. they depend only on the density and the hard-core radius. No other physical nature as statistics, temperature or potential is concerned. In other words, every pair distribution function, whatever the physical nature may be, must satisfy the conditions. Probably the conditions obtained in this paper are merely necessary conditions. The necessary and sufficient conditions seem to be too difficult to pursue.

§ 2. Geometrical conditions to the pair distribution function

1) *Condition* 1

We start from the condition that is easiest to obtain. Let us consider a system of particles with hard-cores of diameter d. d is the shortest distance to which the centers of two particles can approach. It is often called the hard-core radius, but we call it the hard-core diameter in this paper. $\delta(\boldsymbol{r})+\rho F(r)$ is the average density in the coordinate system in which there is a center of particle at the origin. As the particles have hard-cores, only one center of particle (the "center of" is frequently omitted hereafter) can enter in a sphere of diameter d. This gives an upper bound for an integrated $F(r)$ as

$$\int_v[\delta(\boldsymbol{r})+\rho F(r)]d\boldsymbol{r}\leq 1, \tag{7}$$

where $\int_v$ means the integration over a sphere of diameter d. (7) must be satisfied for any position of the sphere. When the sphere includes the origin we get

$$F(r)=0 \qquad \text{for } r<d. \tag{8}$$

When the sphere does not include the origin we get

$$2\pi\rho\int_{R-d/2}^{R+d/2}F(r)\frac{(d/2)^2-(r-R)^2}{2R}rdr\leq 1, \tag{9}$$

where R $(R>d/2)$ is the radial coordinate of the center of the sphere. Inequality

(9) must be satisfied for any R greater than $d/2$.

It should be noted that Eq. (8) is almost equivalent to the existence of hard-cores. Although a system of particles without hard-cores can take a distribution satisfying (8), the same distribution can be taken by a system of hard-core particles. Therefore, a distribution satisfying (8) must also satisfy (9) in any way.

2) *Condition* 2

In order to obtain further conditions we consider a region (called the region A (or A) hereafter) of volume v. As is already known, the average number of particle pairs in the region A can be expressed by using the pair distribution function. Assuming that a particular particle, say the particle 1, lies in A, we consider the average density of other particles in A in the coordinate system in which the particle 1 is located at the origin. We further average out this average density over the uniform distribution of the particle 1 in A and over all directions of A. Then, this average density becomes a function of the distance r from the particle 1. By making use of the function $\xi(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r})$ which is unity when both $\boldsymbol{r}_1$ and $\boldsymbol{r}_1+\boldsymbol{r}$ lie in A and is zero otherwise, the average density is written as

$$\rho F_1(r)=\rho F(r)\cdot\frac{1}{4\pi v}\mathfrak{A}\iint\xi(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r})\,d\boldsymbol{r}_1 d\Omega_r. \tag{10}$$

Here, $|\boldsymbol{r}|=r$, the integration, $\int d\boldsymbol{r}_1$, is done over the region A (or equivalently over the whole space) and $\int d\Omega_r$ over all directions of $\boldsymbol{r}$, and $\mathfrak{A}$ means the averaging over all directions of A. If we put

$$\rho F_1(r)=\rho F(r)w(r\,;\alpha), \tag{11}$$

where

$$w(r\,;\alpha)=\frac{1}{4\pi v}\mathfrak{A}\iint\xi(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r})\,d\boldsymbol{r}_1 d\Omega_r, \tag{12}$$

then, $w(r\,;\alpha)$ is determined from the shape and size, α, of the region A. For example, if the region A is a sphere of radius a, $w(r\,;\alpha)$ is

$$w(r\,;\text{sphere})=\begin{cases}1-\dfrac{3r}{4a}+\dfrac{r^3}{16a^3}, & \text{for } r\leq 2a,\\ 0, & \text{for } r\geq 2a.\end{cases} \tag{13}$$

In the region A the average number of pairs in which the particle 1 takes part is $4\pi\int_0^\infty\rho F_1(r)r^2dr$ times the probability of finding the particle 1 in A, v/V. The average number of all pairs in A on which no restriction is imposed can be obtained by a further multiplication of $N/2$ as

$$2\pi\rho^2 v\int_0^\infty F(r)w(r\,;\alpha)r^2\,dr. \tag{14}$$

In the following we shall get lower bounds for this quantity. The first lower bound is related to the average density. Let p_n be the probability of finding n particles in the region A. They satisfy the following relations:

$$p_n \geq 0, \tag{15}$$

$$\sum_{n=0}^{\infty} p_n = 1, \tag{16}$$

$$\sum_{n=1}^{\infty} n p_n = \rho v. \tag{17}$$

The average number of pairs in A is

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} p_n = 2\pi\rho^2 v \int_0^{\infty} F(r) w(r\,;\alpha) r^2 dr. \tag{18}$$

For sets of p_n's satisfying (15), (16) and (17), the quantity (18) has a minimum. The minimum is attained when the fluctuation of the density is minimum. Namely, if we define an integer n_1, by

$$n_1 \leq \rho v < n_1 + 1, \tag{19}$$

then the distribution,

$$p_n = 0, \qquad \text{for } n \neq n_1,\ n_1+1, \tag{20}$$

gives the minimum. In this case we write the values of p_{n_1} and p_{n_1+1} as $p_{n_1}^{(0)}$ and $p_{n_1+1}^{(0)}$ respectively. They are determined from the following equations:

$$p_{n_1}^{(0)} + p_{n_1+1}^{(0)} = 1, \tag{21}$$

$$n_1 p_{n_1}^{(0)} + (n_1+1) p_{n_1+1}^{(0)} = \rho v. \tag{22}$$

With these $p_n^{(0)}$'s we get a lower bound of (18) as

$$2\pi\rho^2 v \int_0^{\infty} F(r) w(r\,;\alpha) r^2 dr \geq \frac{1}{2} n_1 (2\rho v - n_1 - 1). \tag{23}$$

This condition must be satisfied for any shape and size of the region A. A special example is discussed in Appendix A.

3) *Condition* 3

The second lower bound is related not only to the average density but also to the fluctuation of the density or the localization of particles. Let us consider two regions which do not overlap each other. For simplicity, we assume that the particles have hard-cores of diameter d, and take one of the regions as a sphere of diameter d. We call it the region B (or B) and the other the region A (or A). Only one center of particle can enter in the region B. The average number of particles in B is $(\pi/6)\rho d^3$, which is also the probability of finding a particle in this region. Let p_{1n} be the probability of finding a particle in B and

n particles in A simultaneously, and p_{0n} be the probability of finding no particle in B and n particles in A simultaneously. They satisfy the following relations:

$$p_{1n} \geq 0, \tag{24}$$

$$p_{0n} \geq 0, \tag{25}$$

$$\sum_{n=0}^{\infty} p_{1n} = \frac{\pi}{6}\rho d^3, \tag{26}$$

$$\sum_{n=0}^{\infty} p_{0n} = 1 - \frac{\pi}{6}\rho d^3, \tag{27}$$

$$\sum_{n=1}^{\infty} n(p_{1n} + p_{0n}) = \rho v, \tag{28}$$

where v is the volume of A.

$\sum_{n=1}^{\infty} n p_{1n}$ can be expressed by the pair distribution function. $\sum_{n=1}^{\infty} n p_{1n} / [(\pi/6)\rho d^3]$ is the average number of particles in A for the case in wich there is definitely a particle in B. In this case, similarly to the procedure leading to Eq. (10), we consider the average density of particles in A in the coordinate system in which the particle in B is located at the origin. We further average out this average density over the uniform distribution of the particle in B and over all directions of the system of two regions A and B. Then, this average density is a function of the distance r from the particle in B. By making use of the function $\zeta(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r})$ which is unity when $\boldsymbol{r}_1$ lies in B and $\boldsymbol{r}_1+\boldsymbol{r}$ in A and is zero otherwise, the average density is written as

$$\rho F_2(r) = \rho F(r) \frac{3}{2\pi^2 d^3} \mathfrak{A} \iint \zeta(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r}) d\boldsymbol{r}_1 d\Omega_r. \tag{29}$$

Here, $|\boldsymbol{r}_1| = r$, the integration, $\int d\boldsymbol{r}_1$, is done over the region B (or equivalently over the whole space) and $\int d\Omega_r$ over all directions of $\boldsymbol{r}$, and $\mathfrak{A}$ means the averaging over all directions of the system of A and B. If we put

$$\rho F_2(r) = \rho F(r) u(r\,;\, \alpha, \lambda), \tag{30}$$

where

$$u(r\,;\, \alpha, \lambda) = \frac{3}{2\pi^2 d^3} \mathfrak{A} \iint \zeta(\boldsymbol{r}_1, \boldsymbol{r}_1+\boldsymbol{r}) d\boldsymbol{r}_1 d\Omega_r, \tag{31}$$

then $u(r\,;\, \alpha, \lambda)$ is determined from the shape and size, α, of the region A and the position of A relative to B, λ.

For example, if the region A is a sphere of radius a ($a \geq d/2$) whose center is apart from the center of B by R, $u(r\,;\, \alpha, \lambda)$ is

$$u(r\,;\,\text{sphere},\ R)$$
$$= \frac{1}{d^3 r R} \left\{ -\frac{1}{5}\left[a + \frac{d}{2}\right]^3 \left[a^2 - 3\frac{d}{2}a + \left(\frac{d}{2}\right)^2\right] + \frac{3}{4}\left[a + \frac{d}{2}\right]^2 \left[a - \frac{d}{2}\right]^2 [R - r] -\right.$$

$$-\left[a+\frac{d}{2}\right]\left[a^2-\frac{d}{2}a+\left(\frac{d}{2}\right)^2\right][R-r]^2+\frac{1}{2}\left[a^2+\left(\frac{d}{2}\right)^2\right][R-r]^3-\frac{1}{20}[R-r]^5\Big\},$$

$$\text{for } R-a-\frac{d}{2}\leq r\leq R-a+\frac{d}{2},$$

$$=\frac{1}{d^3rR}\left\{\frac{2}{5}\left(\frac{d}{2}\right)^3\left[5a^2-\left(\frac{d}{2}\right)^2\right]-2\left(\frac{d}{2}\right)^3[R-r]^2\right\},$$

$$\text{for } R-a+\frac{d}{2}\leq r\leq R+a-\frac{d}{2},$$

$$=\frac{1}{d^3rR}\left\{-\frac{1}{5}\left[a+\frac{d}{2}\right]^3\left[a^2-3\,\frac{d}{2}\,a+\left(\frac{d}{2}\right)^2\right]\right.$$

$$-\frac{3}{4}\left[a+\frac{d}{2}\right]^2\left[a-\frac{d}{2}\right]^2[R-r]-\left[a+\frac{d}{2}\right]\left[a^2-\frac{d}{2}a+\left(\frac{d}{2}\right)^2\right][R-r]^2$$

$$\left.-\frac{1}{2}\left[a^2+\left(\frac{d}{2}\right)^2\right][R-r]^3+\frac{1}{20}[R-r]^5\right\},$$

$$\text{for } R+a-\frac{d}{2}\leq r\leq R+a+\frac{d}{2},$$

$$=0, \quad \text{otherwise.} \tag{32}$$

The average number of particles in A for the case in which there is definitely a particle in B is obtained by integrating (30) as

$$\sum_{n=1}^{\infty} np_{1n}\Big/\left(\frac{\pi}{6}\rho d^3\right)=4\pi\rho\int_d^{\infty}F(r)u(r\,;\,\alpha,\,\lambda)r^2\,dr. \tag{33}$$

From (28) and (33), the average number of particles in A for the case in which there is no particle in B is

$$\sum_{n=1}^{\infty} np_{0n}\Big/\left(1-\frac{\pi}{6}\rho d^3\right)=\left[\rho v-\frac{2\pi^2}{3}\rho^2 d^3\int_0^{\infty}F(r)u(r\,;\,\alpha,\,\lambda)r^2\,dr\right]\Big/\left(1-\frac{\pi}{6}\rho d^3\right). \tag{34}$$

For simplicity, we put

$$4\pi\rho\int_0^{\infty}F(r)u(r\,;\,\alpha,\,\lambda)r^2\,dr=\rho' v, \tag{35}$$

$$\left[\rho v-\frac{2\pi^2}{3}\rho^2 d^3\int_0^{\infty}F(r)u(r\,;\,\alpha,\,\lambda)r^2\,dr\right]\Big/\left(1-\frac{\pi}{6}\rho d^3\right)=\rho'' v. \tag{36}$$

ρ' and ρ'' may be called the partial densities. They satisfy

$$\frac{\pi}{6}\rho d^3\rho' v+\left(1-\frac{\pi}{6}\rho d^3\right)\rho'' v=\rho v. \tag{37}$$

(33) and (34) can be written as

$$\sum_{n=1}^{\infty} n p_{1n} = \frac{\pi}{6} \rho d^3 \rho' v, \tag{38}$$

and

$$\sum_{n=1}^{\infty} n p_{0n} = \left(1 - \frac{\pi}{6} \rho d^3\right) \rho'' v, \tag{39}$$

respectively.

For sets of p_{1n}'s and p_{0n}'s satisfying (24), (25), (26), (27), (38) and (39) the average number of pairs in A, $\sum_{n=2}^{\infty} \frac{n(n-1)}{2} (p_{1n} + p_{0n})$, has a minimum. This minimum is the weighted average of the minima for the distributions having the partial densities, ρ' and ρ''. Therefore, arguments similar to those following Eq. (18) can be used. Define integers n_1' and n_1'' by

$$n_1' \leq \rho' v < n_1' + 1, \tag{40}$$

$$n_1'' \leq \rho'' v < n_1'' + 1. \tag{41}$$

Then, we get

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} p_{1n} \Big/ \left(\frac{\pi}{6} \rho d^3\right) \geq \frac{1}{2} n_1' (2\rho' v - n_1' - 1), \tag{42}$$

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} p_{0n} \Big/ \left(1 - \frac{\pi}{6} \rho d^3\right) \geq \frac{1}{2} n_1'' (2\rho'' v - n_1'' - 1), \tag{43}$$

and finally

$$\sum_{n=2}^{\infty} \frac{n(n-1)}{2} (p_{1n} + p_{0n})$$

$$\geq \frac{1}{2} \cdot \frac{\pi}{6} \rho d^3 n_1' (2\rho' v - n_1' - 1) + \frac{1}{2} \left(1 - \frac{\pi}{6} \rho d^3\right) n_1'' (2\rho'' v - n_1'' - 1). \tag{44}$$

If we define $w(r\,;\alpha)$ for the region A as before by Eq. (12), we get

$$2\pi \rho^2 v \int_0^{\infty} F(r) w(r\,;\alpha) r^2 dr$$

$$\geq \frac{1}{2} \cdot \frac{\pi}{6} \rho d^3 n_1' (2\rho' v - n_1' - 1) + \frac{1}{2} \left(1 - \frac{\pi}{6} \rho d^3\right) n_1'' (2\rho'' v - n_1'' - 1). \tag{45}$$

4) *Condition* 4

The condition (45) is certainly stronger than the condition (23). However, the applicability of (45) is narrower since the existence of hard-cores is necessary

to derive (45), while (23) can be applied to any particles. Therefore, we try to get a condition similar to (45) by removing the restriction of the existence of hard-cores.

We consider two regions (A and B) which do not overlap each other. Let $p(n_a, n_b)$ be the probability of finding n_a particles in the region A and n_b particles in B simultaneously. They satisfy

$$\sum_{n_a=0}^{\infty}\sum_{n_b=0}^{\infty} p(n_a, n_b)=1. \tag{46}$$

If we denote the volumes of A and B as v_a and v_b respectively,

$$\sum_{n_a=1}^{\infty}\sum_{n_b=0}^{\infty} n_a p(n_a, n_b)=\rho v_a, \tag{47}$$

$$\sum_{n_a=0}^{\infty}\sum_{n_b=1}^{\infty} n_b p(n_a, n_b)=\rho v_b. \tag{48}$$

The average numbers of pairs in A and B can be expressed by the expressions similar to (18). As before, let us define the functions $w(r\,;\alpha)$ for A and $w(r\,;\beta)$ for B. Then,

$$\sum_{n_a=2}^{\infty}\sum_{n_b=0}^{\infty} \frac{n_a(n_a-1)}{2} p(n_a, n_b)=2\pi\rho^2 v_a\int_0^{\infty} F(r)w(r\,;\alpha)r^2\,dr\equiv K_{aa}, \tag{49}$$

$$\sum_{n_a=0}^{\infty}\sum_{n_b=2}^{\infty} \frac{n_b(n_b-1)}{2} p(n_a, n_b)=2\pi\rho^2 v_b\int_0^{\infty} F(r)w(r\,;\beta)r^2\,dr\equiv K_{bb}. \tag{50}$$

Similarly to (31), we define $u_1(r\,;\alpha,\beta,\lambda)$ as

$$u_1(r\,;\alpha,\beta,\lambda)=\frac{1}{4\pi v_b}\mathfrak{A}\iint\zeta(\mathbf{r}_1, \mathbf{r}_1+\mathbf{r})\,d\mathbf{r}_1\,d\Omega_r. \tag{51}$$

For example, if the regions A and B are spheres of radii a and b respectively (we assume $a\geq b$ for simplicity), and if their centers are apart from each other by R, $u_1(r\,;\alpha,\beta,\lambda)$ is

$$\begin{aligned}
&u_1\ (r\,;\ \text{sphere, sphere, } R)\\
&=\frac{1}{8b^3rR}\left\{-\frac{1}{5}(a+b)^3(a^2-3ab+b^2)\right.\\
&\quad+\frac{3}{4}(a+b)^2(a-b)^2(R-r)-(a+b)(a^2-ab+b^2)(R-r)^2\\
&\quad\left.+\frac{1}{2}(a^2+b^2)(R-r)^3-\frac{1}{20}(R-r)^5\right\},\\
&\qquad\qquad\text{for } R-a-b\leq r\leq R-a+b,
\end{aligned}$$

$$= \frac{1}{8b^3rR}\left\{\frac{2}{5}\,b^3(5a^2-b^2)-2b^3(R-r)^2\right\},$$

$$\text{for } R-a+b\leq r\leq R+a-b,$$

$$= \frac{1}{8b^3rR}\left\{-\frac{1}{5}\,(a+b)^3(a^2-3ab+b^2)\right.$$

$$-\frac{3}{4}(a+b)^2(a-b)^2(R-r)-(a+b)(a^2-ab+b^2)(R-r)^2$$

$$\left.-\frac{1}{2}(a^2+b^2)(R-r)^3+\frac{1}{20}(R-r)^5\right\},$$

$$\text{for } R+a-b\leq r\leq R+a+b,$$

$$=0, \quad \text{otherwise.} \tag{52}$$

The average density (fully averaged in the sense explained in the previous subsections) of particles in A in the coordinate system in which a particle in B is located at the origin is

$$\rho F_3(r)=\rho F(r)u_1(r\,;\,\alpha,\,\beta,\,\lambda). \tag{53}$$

Multiplying (53) by ρv_b and integrating the product, we get the average number of pairs of particles between A and B. Namely,

$$\sum_{n_a=1}^{\infty}\sum_{n_b=1}^{\infty} n_a n_b p(n_a,\,n_b)=4\pi v_b\rho^2\int_0^{\infty}F(r)u_1(r\,;\,\alpha,\,\beta,\,\lambda)r^2dr\equiv K_{ab}=K_{ba}. \tag{54}$$

Now we can set up a mathematical problem as follows: What is the minimum value of K_{aa} for $p(n_a,\,n_b)$'s satisfying (46), (47), (48), (50) and (54) with given values of K_{bb} and K_{ab}? We could not solve this problem because of the complexity arising from the integrality of n_a's and n_b's. However, a lower bound of K_{aa} could be obtained by allowing n_a's and n_b's to take non-integral values. As is proved in Appendix B, this lower bound of K_{aa} is expressed as

$$2K_{aa}\geq\rho v_a(\rho v_a-1)+\frac{(K_{ab}-\rho^2 v_a v_b)^2}{2K_{bb}-\rho v_b(\rho v_b-1)}. \tag{55}$$

This inequality is meaningless if the denominator is zero. In this case, $p(n_a,\,n_b)$'s are different from zero only if $n_b=\rho v_b$. Then, no condition other than (23) can be obtained. Except for this special case the denominator of (55) is always positive. Thus, (55) can also be written in a symmetric form as*

$$[2K_{aa}-\rho v_a(\rho v_a-1)][2K_{bb}-\rho v_b(\rho v_b-1)]\geq(K_{ab}-\rho^2 v_a v_b)^2. \tag{56}$$

* Very recently, Percus[4] obtained the inequality (56) and some other conditions by a different procedure.

§ 3. Discussion

There is a qualitative feature common to all the conditions obtained above. Let us use the geographical words, valley, mountain and plateau, to describe the behavior of the pair distribution function. The general feature of the conditions is that the valleys cannot be too large. The magnitude of a valley is determined from its depth and width, the heights and widths of the mountains or the plateaus around, and the average particle density. It should be noted that in speaking of the magnitude of the valley the pair distribution function should be somewhat smoothened. A narrow peak at the edge of the valley generally decreases the magnitude of the valley.

If the particles have hard-cores, there is a valley around the origin. When the density is high, there is a peak at the edge of the valley in order to decrease the magnitude of the valley. This peak is not very wide since, if so, it will make the valley rather large. Accordingly, there will be the second valley next to the first peak. If the mathematical treatment of the pair distribution function is complete, it ought to reproduce the oscillatory nature of the function at high densities as is seen in the crystal structure. How the treatment of the previous section is near this completeness is not yet known.

The appearance of the peak just outside the hard-core increases the kinetic energy quantum-mechanically. This effect seems to be very important for the saturation of binding energies of hard-core particle systems, and for keeping the system from collapsing to the closely packed state. Even though the density as a whole may be low, the conditions 3 and 4 produce the peak just outside the hard-core if there is a region of high local density. Therefore, these conditions will be effective for keeping the system from collapsing into heavy and dense clusters. It will be interesting to study these problems more quantitatively with practical examples. In treating the dynamical parts of these problems we shall be able to use an appropriate version* of the Jastrow-Iwamoto-Yamada method.[2)]

We assumed the identity of particles and the directional independence of the pair distribution functions. If more than one kind of particles are mixed, we can deal with either one kind of particles or an assembly of different kinds of particles. In each case we have the conditions given in § 2. Inclusion of possible directional dependence of the pair distribution function seems to be more complicated, although there will be no essential difficulty.

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* If we could use all the terms in the cluster development, there should be no restriction to the variational functions, the correlation functions (not the same as the pair distribution function). Actually, however, we can use only few terms. It seems to be natural to restrict the correlation functions so that the pair distribution function which is approximated by the few terms may satisfy the subsidiary conditions.

Appendix A

In this appendix we apply the condition 2 to the simple pair distribution function,*

$$F(r)=\begin{cases}0 & \text{for } r\leq d,\\ 1 & \text{for } r>d.\end{cases} \tag{A·1}$$

In applying (23) we use (13) for $w(r;\alpha)$. We define r_0 by

$$\rho=\frac{3}{4\pi r_0^3}. \tag{A·2}$$

Since

$$v=\frac{4\pi a^3}{3}, \tag{A·3}$$

(19) can be written as

$$n_1\leq a^3/r_0^3<n_1+1. \tag{A·4}$$

Then, for $a\geq d/2$, (23) becomes

$$1-\left(\frac{d^3}{a^3}-\frac{9d^4}{16a^4}+\frac{d^6}{32a^6}\right)\geq\left(n_1\frac{r_0^3}{a^3}\right)\left(2-n_1\frac{r_0^3}{a^3}-\frac{r_0^3}{a^3}\right). \tag{A·5}$$

Both sides of (A·5) are shown in Fig. 1. The inequality (A·5) gives a restriction to r_0, and consequently to ρ. In order that (A·1) be a possible pair distribution function, r_0 must satisfy (A·5). We put

$$n_1=\frac{a^3}{r_0^3}-\varepsilon, \tag{A·6}$$

where

$$0\leq\varepsilon<1. \tag{A·7}$$

Then, (A·5) can be transformed into

$$\frac{d^3}{a^3}-\frac{9d^4}{16a^4}+\frac{d^6}{32a^6}\leq\frac{r_0^3}{a^3}-\frac{r_0^6}{a^6}\varepsilon(1-\varepsilon). \tag{A·8}$$

(A·8) must be satisfied for any value of a greater than $d/2$. For $a\geq\sqrt{2}\,d$ we easily see that

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* Mayer's equation,[1]

$$4\pi\rho\int_0^\infty[1-F(r)]\,r^2\,dr=1,$$

is not satisfied by (A·1) except for $d=[3/(4\pi\rho)]^{1/3}$. However, since the domain of the above integration extends to infinity, we can modify (A·1) by infinitesimal amounts for infinite r values so as to satisfy the above equation.

$$r_0 \geq d \qquad \text{(A·9)}$$

is the necessary and sufficient condition for (A·8). As is clear from Fig. 1, the more detailed computation shows that (A·9) is sufficient for (A·8) even in the case of $a<\sqrt{2}\,d$.

The above argument indicates that in the case of $r_0<d$ the pair distribution function has at least one maximum in order to diminish the magnitude of the valley around the origin (cf. § 3). It is interesting to compare r_0 and d which occur in the actual examples.[5),6)] They are shown in Table I. Since the values of d are not well known and even not well defined, the values in Table I are only approximate values. In spite of this crudeness the observed first maxima in the pair distribution functions of liquid noble gases may be interpreted as due to the above geometrical effect, while this effect is not expected in the nuclear matter.

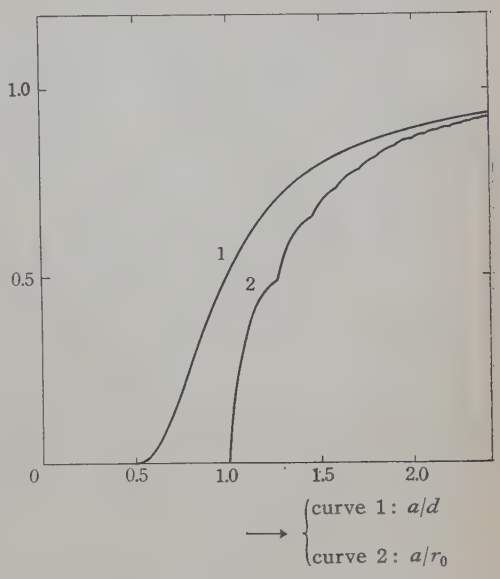


Fig. 1.
Curve 1: The left-hand side of (A·5)
Curve 2: The right-hand side of (A·5)

Table I. The radius of the average volume per particle, r_0, and the hard-core diameter, d. For the noble gases reference 5) gives the Lennard-Jones (6−12) potentials, $V=4\epsilon[(\sigma/r)^{12}-(\sigma/r)^{6}]$. The hard-core diameters are taken to be 0.85σ in this table. r_0 values of the liquid noble gases are given for the normal boiling points.

	r_0 (cm)	d (cm)
Liquid He	2.27×10^{-8}	$\sim2.2\times10^{-8}$
Liquid Ne	1.83×10^{-8}	$\sim2.4\times10^{-8}$
Liquid A	2.19×10^{-8}	$\sim2.9\times10^{-8}$
Liquid Kr	2.42×10^{-8}	$\sim3.1\times10^{-8}$
Liquid Xe	2.51×10^{-8}	$\sim3.4\times10^{-8}$
Nuclear Matter	1.07×10^{-13}	$\sim0.5\times10^{-13}$

Appendix B

Proof of (55)

We will prove the inequality (60) for $p(n_a, n_b)$'s satisfying (46), (47), (48), (50) and (54). Originally, only integral values were allowed for n_a's and n_b's. However, (55) is valid for any positive values of n_a's and n_b's. In the follow-

ing we assume that n_a's and n_b's can take continuous values.

We use the mathematical induction. First, we consider the case in which $p(n_a, n_b)$'s are different from zero only for two pairs of n_a and n_b values. We denote them as $p(n_{a1}, n_{b1})$ and $p(n_{a2}, n_{b2})$. They satisfy

$$p(n_{a1}, n_{b1})+p(n_{a2}, n_{b2})=1, \tag{B·1}$$

$$n_{a1}p(n_{a1}, n_{b1})+n_{a2}p(n_{a2}, n_{b2})=\rho v_a, \tag{B·2}$$

$$n_{b1}p(n_{a1}, n_{b1})+n_{b2}p(n_{a2}, n_{b2})=\rho v_b, \tag{B·3}$$

$$n_{b1}(n_{b1}-1)p(n_{a1}, n_{b1})+n_{b2}(n_{b2}-1)p(n_{a2}, n_{b2})=2K_{bb}, \tag{B·4}$$

$$n_{a1}n_{b1}p(n_{a1}, n_{b1})+n_{a2}n_{b2}p(n_{a2}, n_{b2})=K_{ab}. \tag{B·5}$$

Therefore, for given ρ, v_a, v_b, K_{bb} and K_{ab}, only one of the six variables, n_{a1}, n_{b1}, n_{a2}, n_{b2}, $p(n_{a1}, n_{b1})$ and $p(n_{a2}, n_{b2})$ is independent. If we express the first five variables by the last one, the expression of $2K_{aa}$,

$$2K_{aa}=n_{a1}(n_{a1}-1)p(n_{a1}, n_{b1})+n_{a2}(n_{a2}-1)p(n_{a2}, n_{b2}), \tag{B·6}$$

would be written with $p(n_{a2}, n_{b2})$ only. Actually, the expression of $2K_{aa}$ thus obtained does not include $p(n_{a2}, n_{b2})$ either. It is

$$2K_{aa}=\rho v_a(\rho v_a-1)+\frac{(K_{ab}-\rho^2 v_a v_b)^2}{2K_{bb}-\rho v_b(\rho v_b-1)}. \tag{B·7}$$

Here, we assume that the denominator does not vanish for the reason mentioned in the text. The denominator is non-negative, so that we assumed its positiveness.

Now, suppose that (55) was proved for the case in which $p(n_a, n_b)$'s are different from zero for m pairs of n_a's and n_b's. When $p(n_a, n_b)$'s are different from zero for $m+1$ pairs of n_a's and n_b's (written as n_{ai} and n_{bi}; $i=1, 2, \cdots, m+1$), we define $p_0(n_{ai}, n_{bi})$ as

$$p_0(n_{ai}, n_{bi})=\frac{p(n_{ai}, n_{bi})}{\sum_{j=1}^{m} p(n_{aj}, n_{bj})}. \tag{B·8}$$

Then,

$$\sum_{i=1}^{m} p_0(n_{ai}, n_{bi})=1. \tag{B·9}$$

We put

$$\sum_{i=1}^{m} n_{ai}p_0(n_{ai}, n_{bi})=\rho v_{a0}, \tag{B·10}$$

$$\sum_{i=1}^{m} n_{bi}p_0(n_{ai}, n_{bi})=\rho v_{b0}, \tag{B·11}$$

$$\sum_{i=1}^{m} \frac{n_{ai}(n_{ai}-1)}{2}p_0(n_{ai}, n_{bi})=K_{aa0}, \tag{B·12}$$

$$\sum_{i=1}^{m} \frac{n_{bi}(n_{bi}-1)}{2} p_0(n_{ai}, n_{bi}) = K_{bb0}, \tag{B·13}$$

$$\sum_{i=1}^{m} n_{ai} n_{bi} p_0(n_{ai}, n_{bi}) = K_{ab0}. \tag{B·14}$$

Then, (46), (47), (48), (49), (50) and (54) can be written as

$$\sum_{i=1}^{m} p(n_{ai}, n_{bi}) + p(n_{am+1}, n_{bm+1}) = 1, \tag{B·15}$$

$$\rho v_{a0} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + n_{am+1} p(n_{am+1}, n_{bm+1}) = \rho v_a, \tag{B·16}$$

$$\rho v_{b0} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + n_{bm+1} p(n_{am+1}, n_{bm+1}) = \rho v_b, \tag{B·17}$$

$$K_{aa0} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + \frac{n_{am+1}(n_{am+1}-1)}{2} p(n_{am+1}, n_{bm+1}) = K_{aa}, \tag{B·18}$$

$$K_{bb0} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + \frac{n_{bm+1}(n_{bm+1}-1)}{2} p(n_{bm+1}, n_{bm+1}) = K_{bb}, \tag{B·19}$$

$$K_{ab0} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + n_{am+1} n_{bm+1} p(n_{am+1}, n_{bm+1}) = K_{ab}. \tag{B·20}$$

Our problem is to get the minimum value of K_{aa} for given ρv_a, ρv_b, K_{bb} and K_{ab}. When K_{aa} takes its minimum value, $2K_{aa0}$ also takes the minimum value for given ρv_{a0}, ρv_{b0}, K_{bb0} and K_{ab0}. Owing to our assumption, this minimum value is

$$\rho v_{a0}(\rho v_{a0}-1) + \frac{(K_{ab0}-\rho^2 v_{a0} v_{b0})^2}{2K_{bb0}-\rho v_{b0}(\rho v_{b0}-1)}. \tag{B·21}$$

Here, we assume that the denominator does not vanish. If the denominator vanishes, the numerator also vanishes, and the minimum of $2K_{aa0}$ is $\rho v_{a0}(\rho v_{a0}-1)$. In this case, the situation becomes similar to that in the case in which $p(n_a, n_b)$'s are different from zero only for two pairs of n_a's and n_b's. Thus, (60) is immediately proved. Except for this special case the minimum of $2K_{aa}$ is the minimum of

$$\left\{\rho v_{a0}(\rho v_{a0}-1) + \frac{(K_{ab0}-\rho^2 v_{a0} v_{b0})^2}{2K_{bb0}-\rho v_{b0}(\rho v_{b0}-1)}\right\} \sum_{i=1}^{m} p(n_{ai}, n_{bi}) + n_{am+1}(n_{am+1}-1) p(n_{am+1}, n_{bm+1}). \tag{B·22}$$

Among the eight variables, ρv_{a0}, ρv_{b0}, K_{bb0}, K_{ab0}, $\sum_{i=1}^{m} p(n_{ai}, n_{bi})$, n_{am+1}, n_{bm+1} and $p(n_{am+1}, n_{bm+1})$, three are independent because of the five subsidiary conditions, (B·15), (B·16), (B·17), (B·19) and (B·20). After some manipulation (B·22) becomes

$$\rho v_a(\rho v_a-1)+(\rho v_a-n_{am+1})^2 q+\frac{[K_{ab}-\rho^2 v_a v_b-(\rho v_a-n_{am+1})(\rho v_b-n_{bm+1})q]^2}{2K_{bb}-\rho v_b(\rho v_b-1)-(\rho v_b-n_{bm+1})^2 q}, \tag{B·23}$$

where we put

$$q=p(n_{am+1}, n_{bm+1})/[1-p(n_{am+1}, n_{bm+1})]. \tag{B·24}$$

The denominator of (B·23) is $1-p(n_{am+1}, n_{bm+1})$ times the denominator of (B·21). Therefore, it is non-negative, and also is not zero owing to our assumption. $2K_{bb}-\rho v_b(\rho v_b-1)$ is positive for the reason given in the text. q is also positive. The minimum of (B·23) is attained when

$$[2K_{bb}-\rho v_b(\rho v_b-1)][\rho v_a-n_{am+1}]=[K_{ab}-\rho^2 v_a v_b][\rho v_b-n_{bm+1}]. \tag{B·25}$$

Substituting (B·25) in (B·23) we get the minimum of $2K_{aa}$ as

$$\rho v_a(\rho v_a-1)+\frac{(K_{ab}-\rho^2 v_a v_b)^2}{2K_{bb}-\rho v_b(\rho v_b-1)}. \tag{B·26}$$

Thus, our proof has been completed.

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Spin Waves in MnF_2 and the Inelastic Scattering of Neutrons

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The spin wave dispersion relation in the case of MnF_2 is studied for various assumed values of the ratio of the exchange coupling constants J_2 to J_1. The inelastic scattering cross-section of neutrons by spin waves in this crystal is also calculated, and, particularly, using the value of $J_2=1.08J_1$, which seems to be supported experimentally, distinctive features of the scattering cross-section are discussed.

§ 1. Introduction

In this paper, we look into the spin wave dispersion relation in the case of MnF_2 and calculate the inelastic scattering cross-section of neutrons by the spin waves in this crystal.

In MnF_2, there are three conceivable kinds of superexchange coupling, with constants J_1, J_2 and J_3, among the spins, as illustrated in Fig. 1. We shall assume J_3 to be so small compared with J_1 and J_2 that we can neglect it. On the other hand, J_1 and J_2 must have the same sign and the same order of magnitude, since the ratio of the paramagnetic Curie temperature θ to the Néel temperature T_N differs appreciably from unity ($\theta/T_N=1.57$). The value of $J_2/J_1=\alpha$ has an important bearing on the spin wave energy spectrum, and, furthermore, it has been shown[1] that this parameter defines the stability range of the MnF_2-type spin superstructure. It is suggested that the value of α is

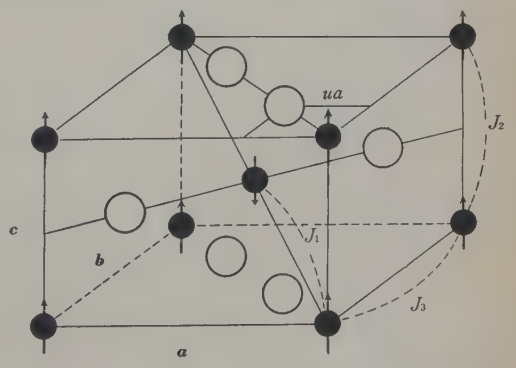


Fig. 1. The unit cell of MnF_2. Black circles are Mn^{2+} ions and white circles F^- ions. The spin arrangement at absolute zero is shown by arrows, and conceivable superexchange couplings are indicated by J_1, J_2 and J_3.
$|\boldsymbol{a}|=|\boldsymbol{b}|=a=4.8734$ A, $|\boldsymbol{c}|=c=3.3103$ A and $u=0.310$.

deduced to be 1.08[2] from the measurements of the sublattice magnetization by Jaccarino and Walker[3] at low temperatures. Precise measurements of other thermodynamic quantities such as specific heat, parallel susceptibility, etc., at low temperatures may give us certain information on the value of α, but it appears that data available at the present time are too poor for this purpose. Experiments of the inelastic scattering of neutrons by spin waves to find the dispersion relation for the spin wave frequencies have been performed with success for magnetite by Brockhouse[4] and by Riste, Blinowski and Janik.[5] These experiments have proved the validity of the theoretically expected quadratic law for the spin wave energy versus wave vector for ferromagnetics. Similar experiments may be of interest for antiferromagnetics; the frequencies of the spin waves of infinite wave length in antiferromagnetics are usually not very small compared with $k_B T_N/\hbar$, and since they are proportional to $(2H_E H_A)^{1/2}$, H_E and H_A being the exchange field and anisotropy field, respectively, the measurements of these frequencies by inelastic collisions will give valuable information on H_A. Furthermore, the spin wave spectrum may be complex (and so worth investigating) in certain antiferromagnetics, such as MnF_2, which we study in the present paper, and MnO which will be treated in a forthcoming paper. In § 3, we shall give formula for the differential cross-section of the inelastic scattering of neutrons by spin waves in MnF_2 and discuss distinctive features of the cross-section in the case of $\alpha=1.08$.

§ 2. The spin wave energy and its dispersion relation

We divide the magnetic body-centered tetragonal lattice of MnF_2 into two simple tetragonal sublattices. Each lattice point, i, of one sublattice, which is made up of corner points, is occupied by an up-spin and each lattice point, k, of the other sublattice, which is made up of body-center points, is occupied by a down-spin. The Hamiltonian of the whole spin system will consist of the three kinds of exchange interaction mentioned in § 1 and the effective anisotropy field acting on each of them. We know that dipole-dipole interaction constitutes the major part of the anisotropy energy in MnF_2, but for simplicity we shall replace them by an appropriate anisotropy energy of the one-ion type. For our purpose, this simplification would be permissible, since the wave vector dependence of the spin wave energy originating from dipolar interaction is smaller than that originating from exchange interaction.[6]

Thus, we write

$$H=2J_1\sum_{\langle i,k\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_k+2J_2\Big(\sum_{\langle i,i'\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_{i'}+\sum_{\langle k,k'\rangle}\boldsymbol{S}_k\cdot\boldsymbol{S}_{k'}\Big)$$
$$+2J_3\Big(\sum_{\langle i,i'\rangle}\boldsymbol{S}_i\cdot\boldsymbol{S}_{i'}+\sum_{\langle k,k'\rangle}\boldsymbol{S}_k\cdot\boldsymbol{S}_{k'}\Big)-2D\Big(\sum_i S_i^{z2}+\sum_k S_k^{z2}\Big), \qquad (1)$$

where J_1, J_2 and J_3 are as defined in Fig. 1, and D is the anisotropy energy

constant. Taking the $+c$-direction to be the z-direction, we introduce spin deviation operators, a_i^*, b_k^*, a_i and b_k by

$$S_i^z = S - a_i^* a_i, \quad S_i^+ = \sqrt{2S}\, a_i, \quad S_i^- = \sqrt{2S}\, a_i^*,$$
$$S_k^z = -S + b_k^* b_k, \quad S_k^+ = \sqrt{2S}\, b_k^* \quad \text{and} \quad S_k^- = \sqrt{2S}\, b_k,$$

where $S_i^{\pm} = S_i^x \pm i S_i^y$ and $S_k^{\pm} = S_k^x \pm i S_k^y$.

If we write (1) in terms of these operators and retain second order terms with respect to these operators, we can obtain the spin wave Hamiltonian. Then, introducing the Fourier transformed spin wave creation and annihilation operators, α_q^*, β_q^*, α_q and β_q, we can diagonalize it. When constant terms are omitted, the diagonalized Hamiltonian is written as follows:

$$H = \sum_q \hbar\omega(q)\,(\alpha_q^* \alpha_q + \beta_q^* \beta_q), \tag{2}$$

where $\hbar\omega(q)$ denotes the spin wave energy with spin wave vector q, and

$$\hbar\omega(q) = 2SJ_1\gamma_{10}[\{1 - \alpha(\gamma_{20} - \gamma_{2q})/\gamma_{10} - \beta(\gamma_{30} - \gamma_{3q})/\gamma_{10} + (2D/J_1\gamma_{10})\}^2 - (\gamma_{1q}/\gamma_{10})^2]^{1/2}, \tag{3}$$

where

$$\gamma_{1q} = \sum_{r_i - r_k} \exp[iq(r_i - r_k)], \quad \gamma_{10} = 8,$$
$$\gamma_{2q} = \sum_{r_i - r_{i'}} \exp[iq(r_i - r_{i'})], \quad \gamma_{20} = 2,$$
$$\gamma_{3q} = \sum_{r_i - r_{i'}} \exp[iq(r_i - r_{i'})], \quad \gamma_{30} = 4,$$
$$\alpha = J_2/J_1, \quad \beta = J_3/J_1.$$

Therefore,

$$\hbar\omega(q) = 16SJ_1[\{1 - (\alpha/2)\sin^2(cq_z/2) - (\beta/2)(\sin^2(aq_x/2) + \sin^2(aq_y/2)) + \varepsilon_A\}^2 - \{\cos(aq_x/2)\cos(aq_y/2)\cos(cq_z/2)\}^2]^{1/2}, \tag{4}$$

where $\varepsilon_A = 2D/8J_1$. In the above calculation, new coordinates α_q and β_q were defined by

$$a_i = \sqrt{\frac{2}{N}} \sum_q (\cosh\theta_q \alpha_q \exp(iqr_i) + \sinh\theta_q \beta_q^* \exp(-iqr_i)),$$
$$b_k = \sqrt{\frac{2}{N}} \sum_q (\sinh\theta_q \alpha_q^* \exp(-iqr_k) + \cosh\theta_q \beta_q \exp(iqr_k)), \tag{5}$$

where N is the total number of spins and θ_q satisfies the following conditions:

$$\theta_q = \theta_{-q} = \theta_q^*,$$
$$\tanh 2\theta_q = -\frac{\gamma_{1q}}{\gamma_{10}}\left[1 - \frac{\alpha}{\gamma_{10}}(\gamma_{20} - \gamma_{2q}) - \frac{\beta}{\gamma_{10}}(\gamma_{30} - \gamma_{3q}) + \varepsilon_A\right]^{-1}.$$

Keffer[7] estimated the effective exchange field $H_E = 2SJ_1\gamma_{10}/g\mu_B$ to be

5.4×10^5 oe using the observed perpendicular susceptibility at 0°K, assuming $g=2$ (μ_B is the Bohr magneton $=0.927\times10^{-20}$ erg oe^{-1}). This value is close to that calculated from the Néel temperature by putting $T_N=4J_1S(S+1)/k_B=67$°K.[8),9)] He also estimated the anisotropy field at 0°K to be 8800 oe and showed that it consists of a crystalline anisotropy field 500 oe and a dipolar anisotropy field of 8300 oe. Using these numerical values, one obtains $\varepsilon_A=2D/8J_1=0.0163$. It is interesting to note that, with the use of this value of ε_A and the value of J_1, which corresponds to $H_E=5.4\times10^5$ oe, we obtain from (4) a value of 12.7°K for $\hbar\omega(0)/k_B$, which is not small compared with T_N. (The value of $\hbar\omega(0)/k_B$ deduced from the measurements of antiferromagnetic resonance in MnF_2[10),11)] under application of a static magnetic field is (12.1 ± 1.1)°K.) For long waves, the spin wave energy can be expanded in powers of q as follows:

$$\left(\frac{\hbar\omega(q)}{2SJ_1\gamma_{10}}\right)^2-2\varepsilon_A=\frac{1-\beta}{4}a^2(q_x^2+q_y^2)+\frac{1-\alpha}{4}c^2q_z^2$$
$$+\frac{1-(\alpha\beta/2)}{16}c^2a^2q_z^2(q_x^2+q_y^2)+\frac{1}{16}(\alpha+(4/3)(\alpha-1))c^4q_z^4 \qquad (6)$$
$$+\cdots.$$

As can be seen from Eq. (6) (or (4)), the spin wave frequencies as functions of the direction of $\boldsymbol{q}$ are not isotropic. As mentioned before, we assume $\beta(=J_3/J_1)$ to be so small that we can neglect it. The parameter α affects the frequency of the spin waves travelling in the z-direction. The $\hbar\omega(q)$ vs. q_z curves for various values of α are illustrated in Fig. 2 ($q_x=q_y=0$); curves with values of α larger than 1 have a valley at a non-zero value of q_z, and, furthermore, for $\alpha>1.2$, $\hbar\omega(q)$ becomes imaginary in a certain region of the $\boldsymbol{q}$-space. Actually, it was shown by Yoshimori[1)] that for larger values of α than this critical value, $\alpha_c=1.2$, the spin superstructure of the screw type is more stable than that of the MnF_2 type. If actually MnF_2 has a value of α which is between 1 and 1.2, so that $\hbar\omega(q)$ has a valley at a non-zero q_z, neutrons may excite or absorb spin waves in the region of this valley with relatively small changes in their energies. Therefore one will be able to observe in the inelastic diffuse scattering of neutrons a certain conspicuous peak due to this effect. In Fig. 3, we illustrate constant spin wave

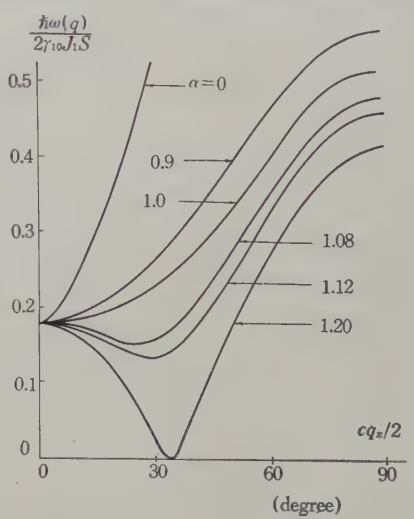


Fig. 2. $\hbar\omega(q)$ versus q_z curves for various values of α ($q_x=q_y=0$).

energy surfaces on the q_x-q_z plane ($q_y=0$) for several assumed values of α.

A few comments will be given on the value of α. If the actual value of α is close to unity, the temperature dependence of the thermodynamic quantities are insensitive to the precise value of α at temperatures higher than $\hbar\omega(0)/k_B=12.7°K$. The calculations of the specific heat, parallel susceptibility, and the magnitude of the sublattice magnetizations in the case of $\alpha=1$ actually show good agreement between theory and experiment in this temperature region. However, at temperatures appreciably lower than 12.7°K, the temperature dependence of these quantities is sensitively affected by the value of α, and it is found from comparison between calculated and observed sublattice magnetizations that by putting $\alpha=1.08$ a satisfactory agreement can be obtained in this temperature region.

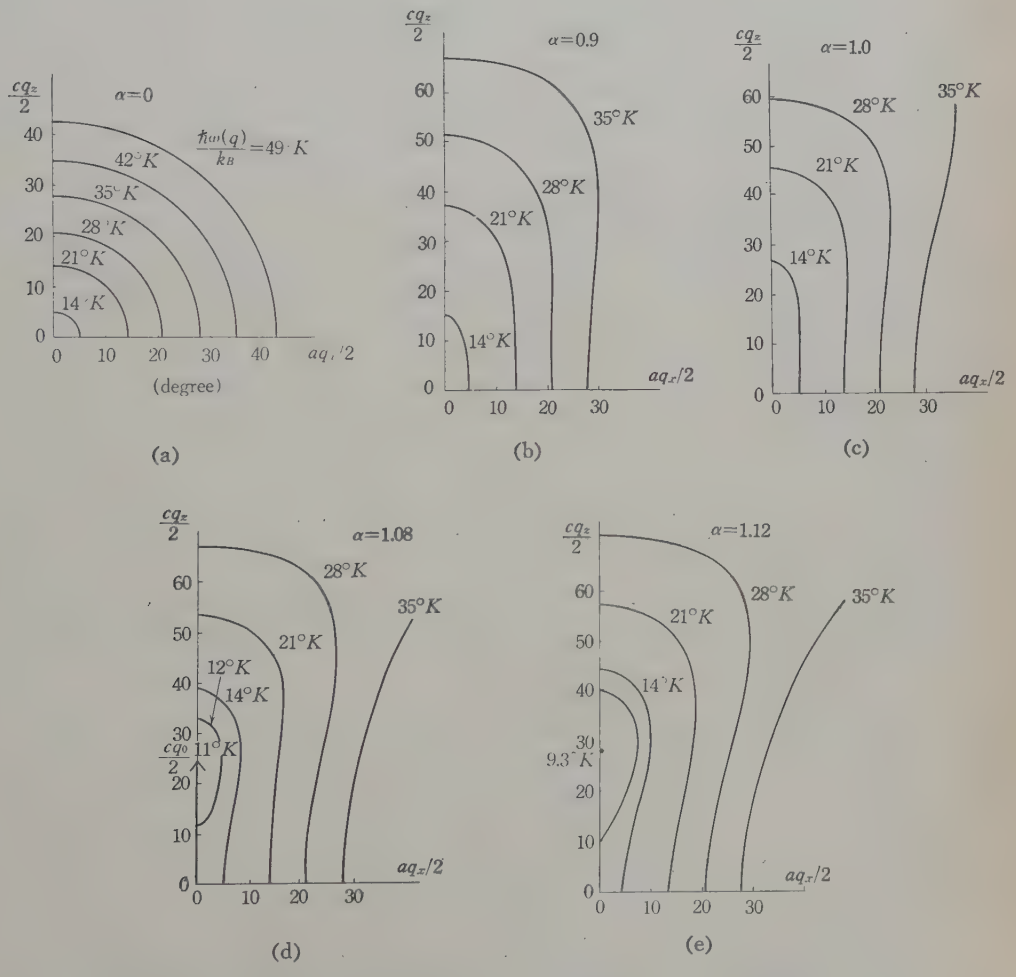


Fig. 3. The constant spin wave energy surfaces on the q_x-q_z plane ($q_y=0$).

§ 3. The scattering cross-section

Having found the q-dependence of the spin wave frequency for MnF_2, we shall now briefly discuss the magnetic inelastic scattering of neutrons by this crystal.

In the process of magnetic scattering of neutrons, the conservation of energy and of momentum in the neutron and magnon system must hold. If $\hbar\boldsymbol{k}_0$ and $\hbar\boldsymbol{k}$ are respectively the in-going and out-going neutron momenta, and if m_0 is the neutron mass, then

$$\boldsymbol{k}_0-\boldsymbol{k}+2\pi\boldsymbol{\tau}\mp\boldsymbol{q}=0, \tag{7}$$

$$\frac{\hbar^2}{2m_0}(k_0^2-k^2)=\pm\hbar\omega(q), \tag{8}$$

where the upper and lower signs correspond to the process of a magnon emission and that of a magnon absorption, respectively, and $\boldsymbol{\tau}$ is a reciprocal lattice vector.

Formula for the differential cross-section of this inelastic scattering has been given by van Hove.[12] It can be written as

$$\begin{aligned}\frac{d^2\sigma}{d\Omega d(\hbar^2k^2/2m_0)}&=\left(\frac{\gamma e^2}{mc^2}\right)^2\frac{1}{\hbar}\frac{k}{k_0}|f(\kappa)|^2\sum_{\alpha,\beta}\left(\delta_{\alpha\beta}-\frac{\kappa_\alpha\kappa_\beta}{\kappa^2}\right)\\&\times\sum_{n_0}p_{n_0}\sum_n\Big[\sum_R S_R^{\alpha}\exp(-i\kappa R)\Big]_{n_0n}\Big[\sum_R S_R^{\beta}\exp(i\kappa R)\Big]_{nn_0}\\&\times\delta\left(\frac{\hbar}{2m_0}(k_0^2-k^2)+\frac{1}{\hbar}(E_{n_0}-E_n)\right),\end{aligned} \tag{9}$$

where e and m are the electron charge and mass, respectively, $\gamma=1.91$ is the neutron magnetic moment in nuclear magnetons, c the light velocity, $\boldsymbol{\kappa}=\boldsymbol{k}_0-\boldsymbol{k}$ the scattering vector, E_n the eigenvalue of the n-th eigenstate of the spin system in the crystal, p_n means the statistical weight of the n-th eigenvalue E_n, indices $\alpha, \beta=x, y, z$ refer to the rectangular coordinates in space, and $f(\kappa)$ is the magnetic form factor.

Writing S_R^{α}, etc., in (9) in terms of a_i, b_k, etc., and substituting (5) in the result, we obtain

$$\begin{aligned}\frac{d^2\sigma}{d\Omega d(\hbar^2k^2/2m_0)}&=\left(\frac{\gamma e^2}{mc^2}\right)^2\frac{1}{h}\frac{k}{k_0}|f(\kappa)|^2\left(1+\frac{\kappa_z^2}{\kappa^2}\right)\\&\times\frac{NS}{2}\Big[\cosh(2\theta_q)+\sinh(2\theta_q)\cos(\boldsymbol{w}\cdot(\boldsymbol{\kappa}\mp\boldsymbol{q}))\Big]\\&\times\delta\left(\frac{\hbar}{2m_0}(k_0^2-k^2)\mp\omega(q)\right)\binom{n_q+1}{n_q},\end{aligned} \tag{10}$$

with $\boldsymbol{\kappa}\mp\boldsymbol{q}+2\pi\boldsymbol{\tau}=0$. Here n_q is the Bose factor, i.e. $[\exp(\hbar\omega(q)/k_BT)-1]^{-1}$, and $\boldsymbol{w}=(1/2)(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})$ is the vector connecting the origin to the body-center

site of the unit cell. Taking the argument of the δ-function as a new variable of integration, we can perform the above integration over $(\hbar^2k^2/2m_0)$. Thus, we put

$$\frac{\hbar}{2m_0}(k_0^2-k^2)\mp\omega(q)=-x,$$

so that

$$d\left(\frac{\hbar k^2}{2m_0}\right)=\frac{dx}{1\pm(m_0/\hbar^2k^2)\boldsymbol{k}\cdot\nabla_q\hbar\omega(q)},$$

and integrate (10) over x. Then we obtain the following final result:

$$\begin{aligned}\frac{d\sigma}{d\Omega}=&\left(\frac{\gamma e^2}{mc^2}\right)^2\frac{k}{k_0}|f(\kappa)|^2\frac{NS}{2}\left(1+\frac{\kappa_z^2}{\kappa^2}\right)\\&\times\left[\exp(-2\theta_q)\sin^2(\pi\boldsymbol{\tau}\cdot\boldsymbol{w})+\exp(2\theta_q)\cos^2(\pi\boldsymbol{\tau}\cdot\boldsymbol{w})\right]\\&\times\left[1\pm\left(\frac{m_0}{\hbar k^2}\right)\boldsymbol{k}\cdot\nabla_q\omega(q)\right]^{-1}\binom{n_q+1}{n_q}.\end{aligned}\tag{11}$$

To illustrate this formula, let us consider the inelastic scattering with $\tau=(0,\,0,\,1/c)$; for τ parallel to the c-axis, no elastic magnetic scattering should appear. In this case, only the $\sin^2$ term in the formula (11) of the scattering cross-section is left and the $\cos^2$ term vanishes. The factor $[1\pm(m_0/\hbar k^2)\boldsymbol{k}\cdot\nabla_q\omega(q)]^{-1}$ can be calculated using (6) and is found to be very close to 1 for any value of α in which we are interested. The main variation of $d\sigma/d\Omega$ as a function of q arises from $\exp(-2\theta_q)$, which can be written

$$\begin{aligned}\exp(-2\theta_q)=&(\gamma_{10}+\gamma_{1q}-\alpha(\gamma_{20}-\gamma_{2q})+\varepsilon_A)^{1/2}\\&\times(\gamma_{10}-\gamma_{1q}-\alpha(\gamma_{20}-\gamma_{2q})+\varepsilon_A)^{-1/2}.\end{aligned}$$

This function is illustrated in Fig. 4 against q_z, putting $q_x=q_y=0$, for several values of α. With this, we can calculate $(d\sigma/d\Omega)$ for known k_0 and q, calculating k with the use of the energy and momentum conservation laws. If we have a small $\alpha(<1)$, the peak of the cross-section for $q=(0,\,0,\,q_z)$ will appear at $q_z=0$, while the cross-section for scattering by other spin wave modes with $q_z\neq0$ will be smaller; if, on the other hand, we have a large $\alpha(>1)$, the cross-section for the scattering by modes with $q=(0,\,0,\,q_z)$ will have a maximum at non-vanishing q_z. Putting, for example, $2\pi/k_0(=\lambda)=1.5$ A and $T=T_N/3$ and

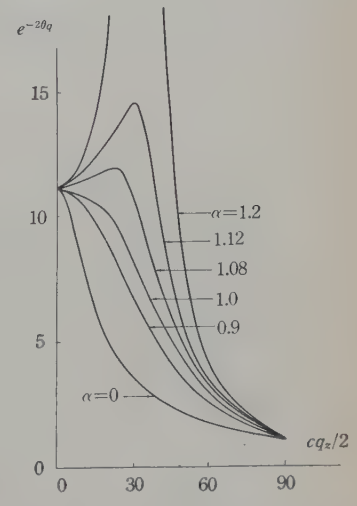


Fig. 4. $\exp(-2\theta_q)$ versus $cq_z/2$ curves for various values of α $(q_x=q_y=0)$.

considering a magnon emission or absorption process, we obtain $(d\sigma/d\Omega) = 110\times10^{-24}$ (for emission) 66×10^{-24} (for absorption) $\mathrm{cm}^2/4\pi$ steradian per atom for the mode of $q=0$; assuming further $\alpha=1.08$, the ratio of $(d\sigma/d\Omega)_{q=(0,0,q_0)}$ to $(d\sigma/d\Omega)_{q=0}$ is calculated to be 1.18 (for emission) 1.33 (for absorption).

In the above treatment, we have omitted the effect of the thermal vibrations of magnetic atoms on the cross-section for the inelastic scattering. To take this effect into account, we have to add the Debye-Waller factor, $\exp(-2W)$, to the right-hand side of (11).

In the method of Riste et al., quoted in § 1, white neutron beam was used. After integrating over the whole scattering surface, which is obtained from Eqs. (7) and (8) by eliminating q, they obtained the cross-section in terms of the Bragg angle ϑ_B and the missetting $(\vartheta-\vartheta_B)$. Their method will thus apply only in the case where the $\omega(q)-q$ relation is simple and therefore the integration over the scattering surface can be carried out without much trouble. On the other hand, Brockhouse uses monochromatized incident neutron beams and measures the energy and momentum of the scattered neutrons precisely. Therefore, his method may be more convenient for the present case.

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Characteristic Functional for Plasma Turbulence

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The functional analytical description of magnetohydrodynamic turbulence of an incompressible plasma flow is investigated by the method introduced by Hopf for the ordinary hydrodynamic turbulence. An exact stationary solution of the functional differential equation for the characteristic functional is given for the special case of inviscid and infinitely conducting plasma. This solution shows the equi-partition of the kinetic and magnetic energies. A formal general solution is also obtained by assuming the possibility of the functional Fourier transformation.

§ 1. Introduction

Not only the possibility but also the difficulty of the turbulence problem arises from the non-linearity of the flow equation. If the flow state at the initial instance is given definitely, the flow state at any later instance is determined by the flow equation. But in the problem of the turbulence the initial state is given in terms of an arbitrary function called " fonction aléatoire ". If the flow equation were linear, we could obtain the general solution and discuss the problem with arbitrary initial condition. But actually the equation of fluid flow is non-linear and here arises the difficulty of the problem of turbulence.

For the physical purposes we are sometimes satisfied with the knowledge of the averaged quantities, such as correlation functions, which describe the statistical property of the field. The equations for the correlation functions derived from the flow equation are linear, but they are not closed but coupled with each other : The two-point correlations are related with those of three-point, the three-point ones with the four-point ones, and the sequence goes on indefinitely. This is in fact a reflection of the non-linearity of the flow equation. But Hopf[1] introduced a method of describing the field of turbulence in terms of a linear functional differential equation for the characteristic functional. If the characteristic functional is determined, all correlation functions are derived by the functional differentiation.

Now it was shown by Batchelor[2] and Chandrasekhar[3] that the phenomenological theory of turbulence in magnetohydrodynamics can be developed to the same extent as the corresponding theories in ordinary hydrodynamics developed by Taylor, von Karman and Howarth, Robertson, Kolmogoroff, Batchelor, and Heisenberg. Here we will show how the method introduced by Hopf is applied

to the magnetohydrodynamic turbulence.

The functional differential equation for the characteristic functional describing the magnetohydrodynamic turbulence in an incompressible plasma is derived by the method of Hopf (§ 2, § 3). An exact stationary solution of the equation is given and it is shown that we have, as a special case, an isotropic Gaussian distribution and the equipartition of the kinetic and magnetic energies (§ 4). A formal general solution is also given by assuming the possibility of the functional Fourier transformation (§ 5).

§ 2. The functional description of the field of turbulence in a plasma

The method of Hopf for describing the field of turbulence in terms of the characteristic functional was first constructed for the turbulence in ordinary hydrodynamics. But this method is applicable to any type of turbulent field, and here we will apply it to the magnetohydrodynamic turbulence in an incompressible plasma.

In magnetohydrodynamic approximation the macroscopic flow of incompressible plasma is described by the following equations,

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_k}(u_i u_k-h_i h_k)=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p+\frac{1}{2}\rho|\boldsymbol{h}|^2\right)+\nu\nabla^2 u_i, \tag{2·1}$$

$$\frac{\partial h_i}{\partial t}+\frac{\partial}{\partial x_k}(h_i u_k-u_i h_k)=\lambda\nabla^2 h_i, \tag{2·2}$$

$$\frac{\partial u_i}{\partial x_i}=0, \tag{2·3}$$

$$\frac{\partial h_i}{\partial x_i}=0, \tag{2·4}$$

where ν, λ, p and u_i denote the viscosity, the magnetic diffusivity, the pressure and the velocity respectively, and h_i is a quantity of the dimension of the velocity defined by

$$h_i=\left(\frac{\mu}{4\pi\rho}\right)^{1/2}H_i, \tag{2·5}$$

μ being the magnetic permeability and H_i the magnetic field strength. The other quantities such as the electric field strength $\boldsymbol{E}$ and the electric current $\boldsymbol{j}$ may be derived from u_i and h_i using appropriate formulas.

Now in order to describe a given flow field $(u_1, u_2, u_3, h_1, h_2, h_3)$ we introduce a functional

$$\Phi(\xi, \eta, t)=\int e^{i\int(\xi_j u_j+\eta_j h_j)dx}P^t(du\,dh), \tag{2·6}$$

where the arguments $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ are arbitrary continuous functions of

the coordinates and $P^t(du\,dh)$ is the differential of the probability distribution P^t of ensemble at time t. The functional $\Phi(\xi, \eta)$ is called the characteristic functional for the (u, h) field.

When the field (u, h) satisfies the flow equations (2·1) and (2·2), the characteristic functional Φ must satisfy the following functional differential equation:

$$\frac{\partial \Phi}{\partial t}=\int\Big[\xi_\alpha(x)\Big\{i\frac{\partial}{\partial x_\beta}\Big(\frac{\delta^2\Phi}{\delta\xi_\beta\delta\xi_\alpha}-\frac{\delta^2\Phi}{\delta\eta_\beta\delta\eta_\alpha}\Big)+\nu\nabla^2\Big(\frac{\delta\Phi}{\delta\xi_\alpha}\Big)-\frac{\partial\Pi}{\partial x_\alpha}\Big\}$$
$$+\eta_\alpha(x)\Big\{i\frac{\partial}{\partial x_\beta}\Big(\frac{\delta^2\Phi}{\delta\eta_\alpha\delta\xi_\beta}-\frac{\delta^2\Phi}{\delta\xi_\alpha\delta\eta_\beta}\Big)+\lambda\nabla^2\Big(\frac{\delta\Phi}{\delta\eta_\alpha}\Big)\Big\}\Big]dx, \qquad (2\cdot7)$$

where Π is an auxiliary functional to be determined by the simultaneous functional Φ just as $\pi=p+(1/2)\rho|\boldsymbol{h}|^2$ is determined by the simultaneous field u and h. Equations (2·3) and (2·4) give the following condition for the functional Φ:

$$\Phi\Big(\xi_j+\frac{\partial\varphi}{\partial x_j},\ \eta_j+\frac{\partial\psi}{\partial x_j}\Big)=\Phi(\xi_j, \eta_j), \qquad (2\cdot8)$$

where φ and ψ are arbitrary continuous functions. Equations (2·7) and (2·8) may be easily proved by a straightforward application of Hopf's method.

The pressure functional Π is eliminated from (2·7) by introducing a function ξ_j^* such as

$$\xi_j=\xi_j^*+\frac{\partial\varphi}{\partial x_j}, \qquad (2\cdot9)$$

with

$$\frac{\partial\xi_j^*}{\partial x_j}=0 \text{ in the domain,}\quad \xi_j^*=0 \text{ on the boundary.} \qquad (2\cdot10)$$

Then Eq. (2·7) becomes, through partial integration,

$$\frac{\partial\Phi}{\partial t}=\int\Big[\xi_\alpha^*\Big\{i\frac{\partial}{\partial x_\beta}\Big(\frac{\delta^2\Phi}{\delta\xi_\beta\delta\xi_\alpha}-\frac{\delta^2\Phi}{\delta\eta_\beta\delta\eta_\alpha}\Big)+\nu\nabla^2\Big(\frac{\delta\Phi}{\delta\xi_\alpha}\Big)\Big\}$$
$$+\eta_\alpha\Big\{i\frac{\partial}{\partial x_\beta}\Big(\frac{\delta^2\Phi}{\delta\eta_\alpha\delta\xi_\beta}-\frac{\delta^2\Phi}{\delta\xi_\alpha\delta\eta_\beta}\Big)+\lambda\nabla^2\Big(\frac{\delta\Phi}{\delta\eta_\alpha}\Big)\Big\}\Big]dx. \qquad (2\cdot11)$$

The functional equation (2·11) involves all knowledge about the magnetohydrodynamic turbulence in an incompressible plasma. It is equivalent to the infinitely coupled equations for correlation functions. To solve it is equivalent to solving the whole system of coupled equations for correlation functions taking into account all orders of correlations. When the characteristic functional Φ is obtained, all correlation functions are derived by the functional differentiation:

$$\overline{u_\alpha(x_1)\cdots u_\beta(x_l)\,h_\gamma(x_{l+1})\cdots h_\delta(x_n)}=\Big|\frac{1}{i^n}\ \frac{\delta^n\Phi}{\delta\xi_\alpha(x_1)\cdots\delta\xi_\beta(x_l)\delta\eta_\gamma(x_{l+1})\cdots\delta\eta_\delta(x_n)}\Big|_{\xi=0,\ \eta=0}, \qquad (2\cdot12)$$

because

$$\int u_\alpha \cdots u_\beta h_\gamma \cdots h_\delta P^t(du\,dh)$$
$$= \left| \frac{1}{i^n} \frac{\delta^n}{\delta\tilde{\xi}_\alpha \cdots \delta\tilde{\xi}_\beta\, \delta\eta_\gamma \cdots \delta\eta_\delta} \int e^{i\int (\xi_\alpha u_\alpha + \eta_\alpha h_\alpha)\,dx} P^t(du\,dh) \right|_{\xi=0,\,\eta=0}. \tag{2·13}$$

§ 3. The wave-number space representation

The wave-number space representation is sometimes useful for the physical application. The flow field $u(x)$ and $h(x)$ may be expressed in terms of their Fourier transforms $\tilde{u}(k)$ and $\tilde{h}(k)$ defined by

$$u_\alpha(x) = \int \tilde{u}_\alpha(k) e^{ikx}\,dk, \qquad h_\alpha(x) = \int \tilde{h}_\alpha(k) e^{ikx}\,dk.$$

Then the basic equations (2·1) to (2·4) for $u(x)$ and $h(x)$ are converted into the following equations for $\tilde{u}(k)$ and $\tilde{h}(k)$:

$$\frac{\partial \tilde{u}_\alpha(k)}{\partial t} = -i\int k_\beta' \{\tilde{u}_\alpha(k')\tilde{u}_\beta(k-k') - \tilde{h}_\alpha(k')\tilde{h}_\beta(k-k')\}\,dk'$$
$$+ i\int k_\alpha \frac{k_\gamma k_\beta'}{k^2} \{\tilde{u}_\gamma(k')\tilde{u}_\beta(k-k') - \tilde{h}_\gamma(k')\tilde{h}_\beta(k-k')\}\,dk' - \nu k^2 \tilde{u}_\alpha(k), \tag{3·1}$$

$$\frac{\partial \tilde{h}_\alpha(k)}{\partial t} = -i\int k_\beta' \{\tilde{h}_\alpha(k')\tilde{u}_\beta(k-k') - \tilde{u}_\alpha(k')\tilde{h}_\beta(k-k')\}\,dk' - \lambda k^2 \tilde{h}_\alpha(k), \tag{3·2}$$

with

$$k_\alpha \tilde{u}_\alpha(k) = 0 \text{ and } k_\alpha \tilde{h}_\alpha(k) = 0, \tag{3·3}$$

where the pressure term has been eliminated by the usual way of taking the divergence of Eq. (2·1).

The characteristic functional in the wave-number space representation may be defined by

$$\Phi(\tilde{\xi}, \tilde{\eta}, t) = \int e^{i\int (\tilde{\xi}(k)\tilde{u}(-k) + \tilde{\eta}(k)\tilde{h}(-k))\,dk} P(d\tilde{u}\,d\tilde{h}),$$

where $\tilde{\xi}(k)$ and $\tilde{\eta}(k)$ are argument functions. When the flow field $\tilde{u}$ and $\tilde{h}$ satisfy Eqs. (4·1) and (4·2), the functional $\Phi(\tilde{\xi}, \tilde{\eta}, t)$ must satisfy the following functional differential equation:

$$\frac{\partial \Phi}{\partial t} = \iint \Big\{ k_\beta' \cdot \pi_{\alpha\gamma}(k)\tilde{\xi}_\gamma(k) \Big(\frac{\delta^2\Phi}{\delta\tilde{\xi}_\beta(k-k')\,\delta\tilde{\xi}_\alpha(k')} - \frac{\delta^2\Phi}{\delta\tilde{\eta}_\beta(k-k')\,\delta\tilde{\eta}_\alpha(k')} \Big)$$
$$+ k_\beta' \tilde{\eta}_\alpha(k) \Big(\frac{\delta^2\Phi}{\delta\tilde{\eta}_\alpha(k')\,\delta\tilde{\xi}_\beta(k-k')} - \frac{\delta^2\Phi}{\delta\tilde{\xi}_\alpha(k')\,\delta\tilde{\eta}_\beta(k-k')} \Big) \Big\}\,dk\,dk'$$
$$- \int \Big\{ \nu k^2 \tilde{\xi}_\alpha(k) \frac{\delta\Phi}{\delta\tilde{\xi}_\alpha(k)} + \lambda k^2 \tilde{\eta}_\alpha(k) \frac{\delta\Phi}{\delta\tilde{\eta}_\alpha(k)} \Big\}\,dk, \tag{3·4}$$

where

$$\pi_{\alpha\gamma}(k)=\delta_{\alpha\gamma}-\frac{k_\alpha k_\gamma}{k^2}. \tag{3·5}$$

The relation (2·8) may be written, in k-space, as

$$\Phi(\tilde{\xi}_\alpha+\tilde{\varphi}(k)k_\alpha,\ \tilde{\eta}_\beta+\tilde{\psi}(k)k_\beta)=\Phi(\tilde{\xi},\ \tilde{\eta}). \tag{3·6}$$

If $\tilde{\xi}_\alpha(k)$ is the Fourier transform of $\xi_\alpha(x)$, then that of $\xi_\alpha{}^*(x)$, defined by (2·9), becomes

$$\tilde{\xi}_\alpha{}^*(k)=\tilde{\xi}_\alpha(k)-\frac{k_\alpha k_\gamma}{k^2}\tilde{\xi}_\gamma(k)=\pi_{\alpha\gamma}(k)\,\tilde{\xi}_\gamma(k). \tag{3·7}$$

This can be shown by taking the Fourier transform of (2·9) and eliminating φ by the divergence condition of $\xi_\alpha{}^*$. If we substitute $\tilde{\xi}_\alpha{}^*(k)$ for $\pi_{\alpha\gamma}(k)\tilde{\xi}_\gamma(k)$ in (3·4), we may see that Eq. (3·4) corresponds exactly to Eq. (2·11). When we introduce

$$\tilde{\eta}_\alpha{}^*(k)=\tilde{\eta}_\alpha(k)+\tilde{\psi}(k)k_\alpha=\pi_{\alpha\gamma}(k)\,\tilde{\eta}_\gamma(k),\left(\text{with }\tilde{\psi}(k)=-\frac{k_\gamma\tilde{\eta}_\gamma(k)}{k^2}\right) \tag{3·8}$$

and replace the differentiation with respect to $\tilde{\xi}$ and $\tilde{\eta}$ by

$$\frac{\partial}{\partial\tilde{\xi}_\alpha}=\pi_{\alpha\gamma}\frac{\partial}{\partial\tilde{\xi}_\gamma{}^*},\quad \frac{\partial}{\partial\tilde{\eta}_\alpha}=\pi_{\alpha\gamma}\frac{\partial}{\partial\tilde{\eta}_\gamma{}^*}, \tag{3·9}$$

we have, by virtue of (3·6),

$$\begin{aligned}\frac{\partial\Phi}{\partial t}=&\iint k_\beta{}'\pi_{\beta\gamma}(k'')\,\pi_{\alpha\delta}(k')\Big[\tilde{\xi}_\alpha{}^*(k'+k'')\left\{\frac{\delta^2\Phi}{\delta\tilde{\xi}_\gamma{}^*(k'')\,\delta\tilde{\xi}_\delta{}^*(k')}-\frac{\delta^2\Phi}{\delta\tilde{\eta}_\gamma{}^*(k'')\,\delta\tilde{\eta}_\delta{}^*(k')}\right\}\\&+\tilde{\eta}_\alpha{}^*(k'+k'')\left\{\frac{\delta^2\Phi}{\delta\tilde{\eta}_\delta{}^*(k')\,\delta\tilde{\xi}_\gamma{}^*(k'')}-\frac{\delta^2\Phi}{\delta\tilde{\xi}_\delta{}^*(k')\,\delta\tilde{\eta}_\gamma{}^*(k'')}\right\}\Big]dk'\,dk''\\&-\int\left\{\nu k^2\tilde{\xi}_\alpha{}^*(k)\frac{\delta\Phi}{\delta\tilde{\xi}_\alpha{}^*(k)}+\lambda k^2\tilde{\eta}_\alpha(k)\frac{\delta\Phi}{\delta\tilde{\eta}_\alpha(k)}\right\}dk,\end{aligned} \tag{3·10}$$

where $k-k'$ has been replaced by k''.

When the characteristic functional in the wave-number space representation $\Phi(\tilde{\xi},\ \eta,\ t)$ is obtained, the correlation functions may be derived by the functional differentiation:

$$\overline{\tilde{u}_\alpha(-k_1)\cdots\tilde{u}_\beta(-k_l)\tilde{h}_\gamma(-k_{l+1})\cdots\tilde{h}_\delta(-k_n)}=\left|\frac{1}{i^n}\frac{\delta^n\Phi}{\delta\tilde{\xi}_\alpha(k_1)\cdots\delta\tilde{\xi}_\beta(k_l)\delta\tilde{\eta}_\gamma(k_{l+1})\cdots\delta\tilde{\eta}_\delta(k_n)}\right|_{\tilde{\xi}=0,\,\tilde{\eta}=0}, \tag{3·11}$$

and

$$\begin{aligned}&\overline{u_\alpha(x_1)\cdots u_\beta(x_l)\,h_\gamma(x_{l+1})\cdots h_\delta(x_n)}\\&=\left|\frac{1}{i^n}\int\frac{\delta^n\Phi}{\delta\tilde{\xi}_\alpha(k_1)\cdots\delta\tilde{\xi}_\beta(k_l)\,\delta\tilde{\eta}_\gamma(k_{l+1})\cdots\delta\tilde{\eta}_\delta(k_n)}e^{-i(k_1x_1+\cdots+k_nx_n)}\,dk_1\cdots dk_n\right|_{\tilde{\xi}=0,\,\tilde{\eta}=0}.\end{aligned} \tag{3·12}$$

When h tends to zero, Eq. (2·11) or (3·9) for the characteristic functional reduces to the corresponding equation of Hopf for ordinary turbulence. We will try to obtain some solutions of the equation in the succeeding sections.

§ 4. An exact stationary solution in the case $\nu=\lambda=0$

Let us consider the case $\nu=0$ and $\lambda=0$. For a stationary case Eq. (3·10) can be expressed as follows:

$$I\equiv\iiint k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\Big[\tilde{\xi}_\alpha^*(-k)\Big\{\frac{\partial^2\Phi}{\partial\tilde{\xi}_\gamma^*(k'')\,\partial\tilde{\xi}_\delta^*(k')}-\frac{\partial^2\Phi}{\partial\tilde{\eta}_\gamma^*(k'')\,\partial\tilde{\eta}_\delta^*(k')}\Big\}$$
$$+\tilde{\eta}_\alpha^*(-k)\Big\{\frac{\partial^2\Phi}{\partial\tilde{\eta}_\delta^*(k')\,\partial\tilde{\xi}_\gamma^*(k'')}-\frac{\partial^2\Phi}{\partial\tilde{\xi}_\delta^*(k')\,\partial\tilde{\eta}_\gamma^*(k'')}\Big\}\Big]$$
$$\times\delta(k+k'+k'')\,dk\,dk'\,dk''=0, \tag{4·1}$$

where $\tilde{\xi}_\alpha^*$, $\tilde{\eta}_\alpha^*$ and $\pi_{\alpha\gamma}$ satisfy the following relations, by definition,

$$k_\alpha\tilde{\xi}_\alpha^*(k)=0,\quad k_\alpha\tilde{\eta}_\alpha^*(k)=0, \tag{4·2}$$

$$\pi_{\alpha\gamma}(k)\,\tilde{\xi}_\gamma^*(k)=\tilde{\xi}_\alpha^*(k),\quad \pi_{\alpha\gamma}(k)\,\tilde{\eta}_\gamma^*(k)=\tilde{\eta}_\alpha^*(k) \tag{4·3}$$

and

$$k_\alpha\pi_{\alpha\gamma}(k)=0. \tag{4·4}$$

Now it is shown that

$$\Phi=F\Big(a\int\tilde{\xi}_\alpha^*(k)\,\tilde{\xi}_\alpha^*(-k)\,dk+b\int\tilde{\eta}_\alpha^*(k)\,\tilde{\eta}_\alpha^*(-k)\,dk\Big)$$
$$\text{with } a=b \text{ (constants)} \tag{4·5}$$

is a solution of (4·1) if $F(\zeta)$ is an arbitrary function of a single variable which has a continuous second derivative.

The integrand of the left side of (4·1) consists of four terms and we will denote the integrals involving them by I_1, I_2, I_3 and I_4 respectively. Inserting (4·5) into (4·1), the first integral becomes

$$I_1=\iiint k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\tilde{\xi}_\alpha^*(-k)\frac{\delta}{\delta\tilde{\xi}_\gamma^*(k'')}(2a\tilde{\xi}_\delta^*(-k')F')\delta(k+k'+k'')\,dk\,dk'\,dk''$$
$$=\iiint k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\,\tilde{\xi}_\alpha^*(-k)\{2aF'\delta_{\gamma\delta}\delta(k'+k'')$$
$$+4a^2\tilde{\xi}_\delta^*(-k')\tilde{\xi}_\gamma^*(-k'')F''\}\delta(k+k'+k'')\,dk\,dk'\,dk''. \tag{4·6}$$

The first term of the right side of (4·6) vanishes, because

$$k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\tilde{\xi}_\alpha^*(-k)\delta_{\gamma\delta}\delta(k'+k'')$$
$$=-[k_\beta''\pi_{\beta\gamma}(k'')]\pi_{\alpha\gamma}(k')\tilde{\xi}_\alpha^*(-k)\delta(k'+k'')$$
$$=0. \qquad \text{(by (4·4))} \tag{4·7}$$

The second term of the right side of (4·6) vanishes, because

$$\begin{aligned}
&k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\tilde{\xi}_\alpha^*(-k)\tilde{\xi}_\delta^*(-k')\tilde{\xi}_\gamma^*(k'')\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=k_\beta'\tilde{\xi}_\beta^*(k'')\tilde{\xi}_\alpha^*(-k)\tilde{\xi}_\alpha^*(-k')\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=k_\beta\tilde{\xi}_\beta^*(k'')\tilde{\xi}_\alpha^*(-k')\tilde{\xi}_\alpha^*(-k)\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&\qquad\qquad\text{(by exchanging } k \text{ and } k') \\
&=\frac{1}{2}(k_\beta+k_\beta')\tilde{\xi}_\beta^*(k'')\xi_\alpha^*(-k')\tilde{\xi}_\alpha^*(-k)\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=-\frac{1}{2}[k_\beta''\tilde{\xi}_\beta^*(k'')]\tilde{\xi}_\alpha^*(-k')\tilde{\xi}_\alpha^*(-k)\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=0. \qquad\qquad\text{(by (4·3))} \qquad\qquad (4\cdot8)
\end{aligned}$$

Thus the integral I_1 vanishes.

The third integral on the left side of (4·1) also vanishes in the same way as (4·8):

$$\begin{aligned}
I_3&=\iiint k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\tilde{\eta}_\alpha^*(-k)\tilde{\eta}_\delta^*(-k')\tilde{\xi}_\gamma^*(k'')4abF''\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=\iiint k_\beta'\tilde{\xi}_\beta^*(k'')\tilde{\eta}_\alpha^*(-k)\tilde{\eta}_\alpha^*(-k')\,4abF''\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
&=-\frac{1}{2}\iiint[k_\beta''\tilde{\xi}_\beta^*(k'')]\tilde{\eta}_\alpha^*(-k)\tilde{\eta}_\alpha^*(-k')\,4abF''\delta(k+k'+k'') \\
&=0. \qquad\qquad (4\cdot9)
\end{aligned}$$

The second and the fourth integrals in (4·1) become

$$\begin{aligned}
I_2+I_4=&-\iiint k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\Big[\tilde{\xi}_\alpha^*(-k)\Big\{2bF'\delta_{\gamma\delta}\delta(k'+k'') \\
&+\tilde{\eta}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')4b^2F''\Big\}+\tilde{\eta}_\alpha^*(-k)\tilde{\xi}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')4abF''\Big] \\
&\times\delta(k+k'+k'')\,dk\,dk'\,dk'' \\
=&-\iiint\Big[4ab\,k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\Big\{\tilde{\xi}_\alpha^*(-k)\tilde{\eta}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'') \\
&+\tilde{\eta}_\alpha^*(-k)\tilde{\xi}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')\Big\}-4(a-b)\,bk_\beta'\pi_{\beta\gamma}(k'') \\
&\times\pi_{\alpha\delta}(k')\tilde{\xi}_\alpha^*(-k)\tilde{\eta}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')\Big]F''\delta(k+k'+k'')\,dk\,dk'\,dk''. \\
&\qquad\qquad\text{(by (4·7))} \qquad\qquad (4\cdot10)
\end{aligned}$$

The first integral of (4·10) vanishes, because

$$\begin{aligned}
&k_\beta'\pi_{\beta\gamma}(k'')\pi_{\alpha\delta}(k')\{\tilde{\xi}_\alpha^*(-k)\tilde{\eta}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')+\tilde{\eta}_\alpha^*(-k)\tilde{\xi}_\delta^*(-k')\tilde{\eta}_\gamma^*(-k'')\} \\
&\qquad\times\delta(k+k'+k'')\,dk\,dk'\,dk''
\end{aligned}$$

$$= \{k_\beta' \tilde{\eta}_\beta^*(-k'') \tilde{\tilde{\xi}}_\alpha^*(-k) \tilde{\eta}_\alpha^*(-k') + k_\beta' \tilde{\eta}_\beta^*(-k'') \tilde{\tilde{\xi}}_\alpha^*(-k') \tilde{\eta}_\alpha^*(k)\}$$
$$\times \delta(k+k'+k'')\, dk\, dk'\, dk''$$
$$= \{k_\beta' \tilde{\eta}_\beta^*(-k'') \tilde{\tilde{\xi}}_\alpha^*(-k) \tilde{\eta}_\alpha^*(-k') + k_\beta \tilde{\eta}_\beta^*(-k'') \tilde{\tilde{\xi}}_\alpha^*(-k) \tilde{\eta}_\alpha^*(-k')\}$$
$$\times \delta(k+k'+k'')\, dk\, dk'\, dk''$$
$$= -[k_\beta'' \tilde{\eta}_\beta^*(-k'')] \tilde{\tilde{\xi}}_\alpha^*(-k) \tilde{\eta}_\alpha^*(-k')\, \delta(k+k'+k'')\, dk\, dk'\, dk''$$
$$= 0. \qquad (4\cdot11)$$

Summarizing the results, we have

$$I = I_1 + I_2 + I_3 + I_4$$
$$= 4(a-b)b \iiint k_\beta' \pi_{\beta\gamma}(k'') \tilde{\tilde{\xi}}_\alpha^*(-k) \tilde{\eta}_\alpha^*(-k') \tilde{\eta}_\gamma^*(-k'') F''$$
$$\times \delta(k+k'+k'')\, dk\, dk'\, dk''. \qquad (4\cdot12)$$

This vanishes for arbitrary $\tilde{\tilde{\xi}}^*$ and $\tilde{\eta}^*$ when $a=b$. Thus (4·5) is an exact solution of (4·1).

As a special case, it is found that

$$\Phi = \exp\left(-\frac{\kappa_u^2}{2}\int \tilde{\tilde{\xi}}_\alpha^*(k) \tilde{\tilde{\xi}}_\alpha^*(-k)\, dk - \frac{\kappa_m^2}{2}\int \tilde{\eta}_\alpha^*(k) \tilde{\eta}_\alpha^*(-k)\, dk\right)$$
$$\text{with } \kappa_u^2 = \kappa_m^2 \qquad (4\cdot13)$$

is a particular solution and this is the characteristic functional of an isotropic Gaussian distribution.[1] The condition $\kappa_u^2 = \kappa_m^2$ shows that the equi-partition of the kinetic and magnetic energies holds in this case.

§ 5. A formal general solution by the functional Fourier transformation

The functional differential equation (2·11) can be written in the following form:

$$\frac{\partial \Phi(\tilde{\xi}^*, \eta)}{\partial t} = i\int \left\{\tilde{\xi}_\alpha^* Q_\alpha\left(\frac{\delta}{i\delta\tilde{\xi}^*}, \frac{\delta}{i\delta\eta}\right) + \eta_\beta R_\beta\left(\frac{\delta}{i\delta\tilde{\xi}^*}, \frac{\delta}{i\delta\eta}\right)\right\} \Phi(\tilde{\xi}^*, \eta)\, dx. \qquad (5\cdot1)$$

The forms of Q_α and R_β may be easily given, but in the following discussion we need not use any knowledge of their form.

Now we assume that the functional Fourier transformation is possible for Φ:

$$\Phi(\tilde{\xi}^*, \eta) = \int \chi(\theta, \tau)\, e^{i\int(\xi_\alpha^*\theta_\alpha + \eta_\beta\tau_\beta)\, dx}\, \delta\theta\, \delta\tau, \qquad (5\cdot2)$$

where $\delta\theta$ and $\delta\tau$ are certain measures of the volume elements of the function space of $\theta(x, t)$ and $\tau(x, t)$, and the functional integration by such measures

are assumed to exist. If we substitute (5·2) in (5·1), we have the following equation for χ:

$$\int i\left(\xi_\alpha^* \frac{\partial\theta_\alpha}{\partial t}+\eta_\beta \frac{\partial\tau_\beta}{\partial t}\right)\chi\, dx+\frac{\partial\chi}{\partial t}=i\int\{\xi_\alpha^* Q_\alpha(\theta,\tau)+\eta_\beta R_\beta(\theta,\tau)\}\chi\, dx. \quad (5\cdot3)$$

Thus we have

$$\chi=\chi_0 \exp\left[i\iint\left\{\xi_\alpha^*\left(-\frac{\partial\theta_\alpha}{\partial t}+Q_\alpha(\theta,\tau)\right)+\eta_\beta\left(-\frac{\partial\tau_\beta}{\partial t}+R_\beta(\theta,\tau)\right)\right\}dx dt\right], \quad (5\cdot4)$$

where χ_0 is the Fourier transform of $\Phi(\xi^*,\eta)$ at the initial instant and does not involve t. Inserting (5·4) into (5·2) and through partial integration, we have

$$\Phi(\xi^*,\eta)=\int\chi_0 \exp\left[i\iint\left\{\theta_\alpha\frac{\partial\xi_\alpha^*}{\partial t}+\xi_\alpha^* Q_\alpha(\theta,\tau)+\tau_\beta\frac{\partial\eta_\beta}{\partial t}+\eta_\beta R_\beta(\theta,\tau)\right\}dx dt\right]\delta\theta\,\delta\tau. \quad (5\cdot5)$$

For the correlation functions we have

$$\begin{aligned}\langle u_\alpha\cdots u_\beta h_\gamma\cdots h_\delta\rangle&=\int\theta_\alpha\cdots\theta_\beta\tau_\gamma\cdots\tau_\delta\chi\,\exp\left\{i\int(\xi_\alpha^*\theta_\alpha+\eta_\beta\tau_\beta)dx\right\}\delta\theta\,\delta\tau\Big|_{\xi=0,\eta=0}\\&=\int\theta_\alpha\cdots\theta_\beta\tau_\gamma\cdots\tau_\delta\chi_0\exp\left[i\iint\left\{\theta_\alpha\frac{\partial\xi_\alpha^*}{\partial t}+\xi_\alpha^* Q_\alpha(\theta,\tau)+\tau_\beta\frac{\partial\eta_\beta}{\partial t}\right.\right.\\&\quad\left.\left.+\eta_\beta R_\beta(\theta,\tau)\right\}dx dt\right]\delta\theta\,\delta\tau\Big|_{\xi=0,\eta=0}. \quad (5\cdot6)\end{aligned}$$

Thus we have obtained the formal general solution in a form of functional integral. For practical calculation some approximate method must be employed such as the lattice point method or the generalized W-K-B method which are popular in quantum field theory.[4] But it is not yet clear in which case and to what extent such approximate calculations are better than the ordinary treatment that takes into account only a small part of infinite coupled equations for correlation functions. The exact solution of the functional differential equation may not be easily obtained and it may sometimes be useful to derive some general properties of the equation mathematically.

§ 6. The correlation functions involving odd number of h's

It was assumed by Chandrasekhar (1955) in an investigation of the spectrum of magnetohydrodynamic turbulence that the correlation functions involving odd number of components of h should vanish:

$$\overline{u_i h_j'}=\overline{u_i h_j h_k' h_l'}=\cdots\cdots=0. \quad (6\cdot1)$$

This assumption is natural because the flow equations allow the solutions (u, h) and $(u, -h)$ at the same time, so that h and $-h$ are equally possible for a given u.

Here we will give some remarks about this assumption in terms of the characteristic functional.

First it is shown that such correlations vanish if the characteristic functional is even with respect to η : $\Phi(\xi^*, -\eta)=\Phi(\xi^*, \eta)$. If we put

$$G(\xi^*, \eta)=\frac{\delta}{i\delta\xi_\alpha^*}\cdots\frac{\delta}{i\delta\xi_\beta^*}\frac{\delta}{i\delta\eta_\gamma}\cdots\frac{\delta}{i\delta\eta_\delta}\Phi(\xi, \eta),$$

we have

$$G(\xi^*, -\eta)=-G(\xi^*, \eta)$$

for an odd number of $\frac{\delta}{\delta\eta}$'s. Putting $\xi=\eta=0$, we have

$$G(0, 0)=\langle u_\alpha\cdots u_\beta h_\gamma\cdots h_\delta\rangle=0. \tag{6·2}$$

Secondly it is shown that if $\Phi(\xi^*, \eta)$ is even with respect to η at a certain instant this is always even. When $\Phi(\xi^*, \eta)$ is even at the initial instant, its Fourier transform χ_0 is also even with respect to τ. Now we see that the operators Q and R have the symmetry property:

$$Q(\theta, -\tau)=Q(\theta, \tau),\ R(\theta, -\tau)=-R(\theta, \tau). \tag{6·3}$$

Then we have

$$\begin{aligned}\Phi(\xi^*, -\eta)&=\int\chi_0\exp\Big[i\iint\Big\{\theta_\alpha\frac{\partial\xi_\alpha^*}{\partial t}+\xi_\alpha^* Q_\alpha(\theta, \tau)+\tau_\beta\frac{\partial(-\eta_\beta)}{\partial t}\\&\qquad+(-\eta_\beta)R_\beta(\theta, \tau)\Big\}dx\,dt\Big]\delta\theta\,\delta\tau\\&=\int\chi_0\exp\Big[i\iint\Big\{\theta_\alpha\frac{\partial\xi_\alpha^*}{\partial t}+\xi_\alpha^* Q_\alpha(\theta, \tau)+\tau_\beta'\frac{\partial\eta_\beta}{\partial t}+\eta_\beta R_\beta(\theta, \tau')\Big\}dx\,dt\Big]\delta\theta\delta\tau'\\&\qquad(\tau'=-\tau)\\&=\Phi(\xi^*, \eta).\end{aligned} \tag{6·4}$$

Thus, the correlations involving an odd number of components of h vanish if they vanish at an instant of time.

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This paper treats the method of kernel functional with the results corresponding to that given in § 5.

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Antisymmetric Treatment of Deuteron-Stripping Reaction†

——*Stripping and Heavy-Particle-Stripping Reactions*——

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The deuteron stripping reaction is reformulated by using completely antisymmetric wave functions: The amplitude of the reaction will be represented as a sum of two parts, stripping and heavy-particle-stripping amplitudes.

Quite analogous to the case of the "ordinary" stripping, the amplitude of the heavy-particle-stripping can be represented in terms of three reduced width amplitudes of bound nucleons, e.g. in (d, p) reaction, those of captured neutron and proton in the residual nucleus and that of the emitted proton in the target nucleus, or alternatively, in terms of two reduced width amplitudes of the emitted proton and captured deuteron in bound states. The latter formulation is natural generalization of Owen and Madansky's.

By making use of present formalism, one can calculate not only angular distribution but also the "magnitude" of heavy-particle-stripping cross section.

For simplicity, the Butler approximation, i.e. the cutoff Born approximation, is employed, though its generalization can be performed easily, as will be discussed in the final section.

Some qualitative discussions are given. Numerical calculations and more detailed comparison between the stripping and heavy-particle-stripping cross sections will appear in our subsequent paper.

§ 1. Introduction

When a deuteron is bombarded to a target nucleus to form the final state of a residual nucleus and free proton, one may immediately imagine the following two sorts of direct-interaction-mechanism. First the neutron in the deuteron is captured by the target nucleus, whereas the proton in the deuteron does not interact with the target nucleus, appearing at infinity as free proton. In other words, the neutron in the deuteron is "stripped" by the target nucleus to form the residual nucleus. This mechanism is called the deuteron stripping, which has been dealt with by many authors after the first work of Butler.[1] On the other hand, the target nucleus may be "stripped" by the incident deuteron. Namely, if the target nucleus is assumed to be composed of one proton, say $p2$, and the remaining core, the deuteron may capture the core to form the residual nucleus, while the proton is emitted in the final state. This is the so-called heavy-particle-stripping reaction proposed by Owen and Madansky.[2]

† This work is included in a thesis submitted by T. Honda to Rikkyo University, in partial fulfillment of the requirements for the degree of Doctor of Science.

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In some experiments of (d, p) and (d, n) reactions, there appear peaks of angular distribution in rather backward directions, which have usually been explained in terms of heavy-particle-stripping reaction.

The theories of heavy-particle-stripping reaction so far employed, however, were concerned only with the shape of angular distribution and the magnitude of the cross section has been adjusted so as to fit the experimental data.

The main contents of the present paper may be expressed as follows. By using completely antisymmetric wave function the stripping and the heavy-particle-stripping amplitudes will be treated on an equal footing, which will be given in the next section. In sections 3 and 4, the amplitude of heavy-particle-stripping will be represented in terms of three reduced width amplitudes of bound nucleons, i.e. those of captured neutron and proton in the residual nucleus and that of the emitted proton in the target nucleus, or alternatively, in terms of two reduced width amplitudes of the emitted proton and of the captured deuteron in bound states. The latter formulation is natural generalization of Owen and Madansky's. The nuclear interaction appearing in the matrix element will be eliminated by using the Schroedinger equation of target nucleus. On the other hand, the amplitude of "ordinary" stripping is expressed, as usual, in terms of a reduced width amplitude of the captured neutron in the residual nucleus. Then one can calculate not only the angular distribution but also the magnitude of the heavy-particle-stripping reaction without explicit knowledge of the nuclear interaction. In section 5, the cross section for (d, p) and (d, n) reactions will be presented, which consists of three parts: stripping, heavy-particle-stripping and their interference terms. Finally, in section 6, some discussions will be added concerning qualitative nature between stripping and heavy-particle-stripping reactions. It is pointed out that the energy-dependence of the cross section of the heavy-particle-stripping reaction is stronger than that of the stripping reaction, so that the former reaction will not play an important role at higher energy of incident deuteron.

Detailed qualitative and quantitative discussions will be presented in a subsequent paper, in which it will be discussed especially in what condition, i.e. incident energy, Q-value, etc., the heavy-particle-stripping mechanism might be preferential to (d, p) or (d, n) reactions in comparison with the "ordinary" stripping one. The result is almost always unfavourable to the heavy-particle-stripping mechanism except the special case of low incident energy and some appropriate combination of reaction Q-value and the binding energy of the emitted proton in the target nucleus, *provided the Butler approximation is adopted.*

Other reactions, for example (α, p), may be treated in quite a similar way with slight modification which will be discussed in the derivation of the formula.

After completing the present work, it called authors' attention that quite recently Nagarajan and Banarjee have performed an antisymmetric treatment of (d, p) and (d, n) reactions. Their formulation is almost the same as our "alternative" formulation mentioned previously. If needed, the differences be-

tween their and our formulations will be noted in the footnotes of the present paper.* Their conclusion obtained through the analysis of experimental data, however, seems to be essentially different from ours; this may be due to the fact that our analysis is more systematic than theirs.**

§ 2. Antisymmetrization

As is well known, the amplitude of a nuclear rearrangement collision such as $A(a, b)B$ may be obtained simply as a coefficient of the asymptotic wave function in the final channel b of the total wave function which should obey the Schroedinger equation with appropriate boundary conditions.

In the antisymmetric treatment, we must further impose an additional requirement on the total wave function, i.e. the total wave function should be antisymmetric with respect to the exchange of the coordinates of arbitrary two nucleons in the system. Then, there appear various emitted particles (if distinguishable) in the channel b. The amplitude in the antisymmetric treatment, therefore, can be obtained as a linear combination of the coefficients of the asymptotic wave functions in the channel b.

In the present section, an elementary method will be employed, though completely the same result has been obtained by the general method mentioned above.

1. *Simple Example*

First, we shall deal with a simple example; antisymmetrization between the proton $p1$ of one of the constituent of the incident deuteron and a proton $p2$ in the target nucleus, which is assumed to be composed of proton $p2$ and remaining core. Further, the spins of incident and emitted particles will be disregarded.

For the sake of convenience, the interaction between the incident deuteron and the target nucleus will be denoted by

$$V(p2, C; d),$$

where the notation $p2, C$ stands for the target nucleus, i.e. $p2+\text{core}$, which may be represented as the sum of interaction between the target nucleus and the proton $p1$ and that between the target nucleus and neutron in the deuteron:

$$V(p2, C; d) = V(p2, C; n) + V(p2, C; p1).$$

More explicitly,

$$V(p2, C; n) = V(p2; n) + V(C; n),$$
$$V(p2, C; p1) = V(p2; p1) + V(C; p1).$$

Here, $V(C; n)$ stands for the interaction between the core and the neutron in the incident deuteron.

* M. A. Nagarajan and M. K. Banarjee, Nuclear Physics **17** (1960), 341. Their paper will be referred to as N-B in the following footnotes.

** Details will be discussed in the subsequent paper.

The wave function of the final state, i.e. residual nucleus+one free proton, may be expressed as

$$(1/\sqrt{2})\{e^{i\boldsymbol{k}\cdot\boldsymbol{r}_1}\psi_R(n,\ p2,\ C)-e^{i\boldsymbol{k}\cdot\boldsymbol{r}_2}\psi_R(n,\ p1,\ C)\}, \tag{2·1}$$

where $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the coordinates of proton $p1$ and $p2$, respectively, $\psi_R(n,\ p2,\ C)$ denotes the wave function of the residual nucleus which is composed of the core, proton $p2$ and the neutron.

On the other hand, the wave function of the initial state may be written, in the antisymmetric treatment, in two forms:

$$(1/\sqrt{2})\phi_d(\boldsymbol{r}_n-\boldsymbol{r}_1)e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_n+\boldsymbol{r}_1)/2}\psi_T(p2,\ C), \tag{2·2a}$$

and

$$-(1/\sqrt{2})\phi_d(\boldsymbol{r}_n-\boldsymbol{r}_2)e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_n+\boldsymbol{r}_2)/2}\psi_T(p1,\ C), \tag{2·2b}$$

where $\psi_T(p2,\ C)$ denotes the wave function of the target nucleus which is composed of the proton $p2$ and the core, ϕ_d the wave function of the internal motion of deuteron, $\boldsymbol{r}_n$ the coordinate of the neutron and $\boldsymbol{K}$ being the wave number vector of the incident deuteron. The interactions in the initial state should be taken to be

$$V(p2,\ C;\ p1)+V(p2,\ C;\ n) \tag{2·3a}$$

and

$$V(p1,\ C;\ p2)+V(p1,\ C;\ n) \tag{2·3b}$$

corresponding to the forms of the wave function of initial state (2·2a) and (2·2b), respectively.

After simple calculations, we obtain the transition matrix element in the antisymmetric treatment:

$$T=\int d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{C}\{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1}\psi_R^*(n,\ p2,\ C)-e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}\psi_R^*(n,\ p1,\ C)\}$$
$$\times\{V(p2,\ C;\ p1)+V(p2,\ C;\ n)\}\psi_T(p2,\ C)e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n), \tag{2·4}^{*,**}$$

where $d\boldsymbol{C}$ stands for the integration over all coordinates of the core. The range of integration with respect to $\boldsymbol{r}_n$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ will be discussed later. The expression (2·4) has already been obtained by Owen and Madansky.[2]

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* Another, but equivalent expression has been obtained by French,[3] using the final state interaction:

$$T\equiv\int d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{C}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1}\psi_R^*(n,\ p2,\ C)V(n,\ p2,\ C;\ p1)$$
$$\times\{\psi_T(p2,\ C)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)-\psi_T(p1,\ C)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_2+\boldsymbol{r}_n)/2}\phi_d(\boldsymbol{r}_2-\boldsymbol{r}_n)\}.$$

** Strictly speaking, the expression (2·4) will remain correct, only if the target nucleus is infinitely heavy and the coordinate of emitted proton is measured from its center of mass. As a result, this expression includes the error of order $1/A$ (1/(mass number of nucleus)), which leads sometimes to rather serious difference in the angular distribution of heavy-particle-stripping. The improvement will be performed in the next subsection.

Next, it may be convenient to divide the T into two parts:

$$T = T\ (\text{direct}) + T\ (\text{exchange}), \tag{2·5}$$

where

$$T\ (\text{direct}) \equiv \int d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{C}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1} \psi_R^*(n,\ p2,\ C)$$
$$\times \{V(p2,\ C\,;\ p1) + V(p2,\ C\,;\ n)\} \psi_T(p2,\ C) \phi_d(\boldsymbol{r}_n - \boldsymbol{r}_1)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}, \tag{2·5a}$$

and

$$T\ (\text{exchange}) \equiv -\int d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{C} \cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2} \psi_R^*(n,\ p1,\ C)$$
$$\times \{V(p2,\ C\,;\ p1) + V(p2,\ C\,;\ n)\} \cdot \psi_T(p2,\ C) \phi_d(\boldsymbol{r}_n - \boldsymbol{r}_1)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}. \tag{2·5b}$$

The first term, (2·5a), represents the direct part, which has been dealt with by many authors[1] in the "ordinary" stripping reaction. As is well known, the initial-state-interaction $V(p2,\ C\,;\ p1) + V(p2,\ C\,;\ n)$ in T(direct) can be replaced by the final-state-interaction, i.e.

$$V(p2,\ n,\ C\,;\ p1) = V(p2,\ C\,;\ p1) + V(n\,;\ p1), \tag{2·6}$$

if the Born approximation be adopted. In the case of the Butler approximation, i.e. the cutoff Born approximation, the same relation holds also exactly, provided the cutoff radii in the initial and final states be taken properly.* The net interaction in (2·6) which produces the stripping reaction is known to be $V(n\,;\ p1)$ and the corresponding initial state interaction is given by $V(p2,\ C\,;\ n)$; the interaction between the target nucleus and the neutron in the incident deuteron.

We shall deal only with the following matrix element in T(direct), while the interaction $V(p2,\ C\,;\ p1)$ will be disregarded hereafter:

$$M\ (\text{stripping}) \equiv \int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 d\boldsymbol{C} \cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1} \psi_R^*(n,\ p2,\ C)$$
$$\times V(n\,;\ p1) \psi_T(p2,\ C) \phi_d(\boldsymbol{r}_1 - \boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}, \tag{2·7a}$$
(by means of final state interaction),

$$\equiv \int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 d\boldsymbol{C} \cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1} \psi_R^*(n,\ p2,\ C)$$
$$\times V(p2,\ C\,;\ n) \psi_T(p2,\ C) \phi_d(\boldsymbol{r}_1 - \boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2} \tag{2·7b}$$
(by means of initial state interaction),

where the symbol $\int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1$ denotes the integration over $\boldsymbol{r}_n$ and $\boldsymbol{r}_1$ with the follow-

* The relation between the cutoff radii R_i and R_f in the initial and final states can be shown to be determined uniquely by requirement of "reciprocity".

ing restriction of radial integrals ; $r_n \geq R_n$ and $r_1 \geq R_1$.* This is nothing but the Butler approximation, which may be verified from the short mean free path of deuteron in nuclear matter. For simplicity, we shall assume $R_n = R_1 \equiv R$, R being approximately equal to nuclear radius.

In the second term of (2·5), T(exchange), the interaction $V(p2, C; n) + V(p2, C; p1)$ can also be proved to be replaced by the final state interaction, i.e.

$$V(p1, n, C; p2) \equiv V(p1; p2) + V(n; p2) + V(C; p2), \qquad (2\cdot8)$$

in a similar way as in the direct part, though the proof will not be quoted here. The interaction which produces the heavy-particle-stripping mechanism of Owen and Madansky may be safely ascribed to $V(C; p2)$ in the final state interaction and the corresponding interaction in the initial state is easily shown to be $V(C; p1) + V(C; n)$; the remaining interactions $V(n; p1) + V(p2; p1)$ will not be dealt with in the present paper. It should be added here, however, that the contribution of the interaction $V(p1; n)$ has been calculated in some detail by French,[3] obtaining the result that the angular distribution arising from this interaction shows the peaks in rather forward directions similarly to that of " ordinary " stripping reaction. The authors will not always believe that those interaction might produce such a large net contribution in the reaction because their matrix element may correspond to the so-called knock-on process, which seems to have rather small cross section contrary to the case of α-induced reaction due to very small internal energy of deuteron.

Hereafter, we shall deal with the following matrix element in T(exchange), which describes heavy-particle-stripping reaction as was previously discussed :

$$M\,(\text{H. P. Stripping}) = -\int_{ext} d\boldsymbol{r}_n\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \int d\boldsymbol{C}\cdot e^{-i\boldsymbol{K}\cdot\boldsymbol{r}_2}\psi_R^*(n, p1, C)$$
$$\times V(C; p2)\psi_T(p2, C)\ \phi_d(\boldsymbol{r}_1 - \boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}, \qquad (2\cdot9a)$$
(when we use the final state interaction),
$$= -\int_{ext} d\boldsymbol{r}_n\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2 \int d\boldsymbol{C}\cdot e^{-i\boldsymbol{K}\cdot\boldsymbol{r}_2}\psi_R^*(n, p1, C)$$
$$\times \{V(C; p1) + V(C; n)\}\psi_T(p2, C)\phi_d(\boldsymbol{r}_1 - \boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}. \qquad (2\cdot9b)$$
(when we use the initial state interaction).

Similarly to the stripping reaction, $\int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2$ denotes the integration over $\boldsymbol{r}_n$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ with the restriction of $r_n \geq R$, $r_1 \geq R$ and $r_2 \geq R$. The first two restric-

* The restriction with respect to one radial coordinate (say, r_1) is automatically satisfied from that of another (say, r_n), only if the zero-range nuclear force is adopted for $V(p1; n)$ as is apparent from Eq. (3·6). Even for the nuclear force with finite range, the error in the differential cross section arising from the approximate restriction $r_n \leq R_n$ and $\infty > r_1 > 0$ will amount at most to 10% numerically, which was checked up by Nagasaki and one of the present authors (T. H.)[4].

tions, $r_n \geq R$ and $r_1 \geq R$ may be safely applicable as was discussed in M(Stripping). But, for $r_2 \geq R$, it may not always be verified since the mean free path of proton $p2$ in nuclear matter may be expected to be long in comparison with that of deuteron. Nevertheless, we adopt the restriction $r_2 \geq R$, though we may obtain somewhat underestimated value of M(H.P. Stripping), since the integration of r_2 over all space, i.e. $\infty > r_2 \geq 0$, leads sometimes to unusually very large value of amplitude which can be easily understood by using the bound state wave function of $p2$ in square well potential. It seems to the authors that we should use the more reliable approximation, e.g. the distorted Born approximation in order to adopt the integration of r_2 without the restriction, i.e. $\infty > r_2 \geq 0$.

2. *Improvement arising from the correct choice of coordinate system —— Effect of recoil ——*

As was noticed previously, the T-matrix element (2·4) is not correct, including an error of order $1/A$, since the vector displacements of proton $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and neutron $\boldsymbol{r}_n$ have been taken with respect to the centre of mass of the target nucleus. On the contrary, the coordinate of the emitted proton in the final state should be taken with respect to the centre of mass of the residual nucleus, which consists of the neutron and the target nucleus. In the " ordinary " stripping reaction, therefore the wave function of the emitted proton $\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_1)$ should be correctly taken to be $\exp[-i\boldsymbol{k}\{\boldsymbol{r}_1-(M/M_f)\boldsymbol{r}_n\}]$, where M_f and M denote the reduced mass of residual nucleus and that of nucleon, respectively. Similarly, in the matrix element of heavy-particle-stripping, $\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_2)$ should be replaced by $\exp[-i\boldsymbol{k}\cdot\{\boldsymbol{r}_2-(M_d(\boldsymbol{r}_1+\boldsymbol{r}_n)/2+M_C\boldsymbol{r}_C)/M_f\}]$, where M_d and M_C are, respectively, the reduced masses of deuteron and the core, $\boldsymbol{r}_C$ the coordinate of the core. After appropriate transformation of integral variables, we can obtain just similar expressions to (2·7) of M(Stripping) and (2·9) of M(H.P. Stripping), in which the difference appears only in the replacement of $\boldsymbol{k}\rightarrow(M_i/M_f)\boldsymbol{k}$ in M(Stripping) and $\boldsymbol{K}\rightarrow\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}$ in M(H.P. Stripping). Therefore, it will be sufficient to perform the following replacement, i.e.

$$\boldsymbol{k}\rightarrow(M_i/M_f)\boldsymbol{k} \qquad \text{in } M\text{(Stripping)}$$

$$\boldsymbol{K}\rightarrow\boldsymbol{K}+(M_d/M_f)\boldsymbol{k} \qquad \text{in } M\text{(H. P. Stripping)}$$

in the explicit expressions of matrix elements obtained in later sections.

The replacement in M(Stripping) can be proved to produce no serious error numerically for both magnitude and angular dependence of the differential cross section. On the other hand, that in M(H.P. Stripping) will play an important role for angular distribution, which will be discussed later.

3. *Generalization of the simple example*

Hitherto, the antisymmetrization has been performed between the proton $p1$ in the incident deuteron and a proton $p2$ in the target nucleus. Further, the spins are disregarded of the emitted and incident particles. Therefore, the re-

sults previously obtained are only the antisymmetrization with respect to space coordinate variables of two protons. When the spin coordinate is taken into account, the expression (2·4) of total amplitude remains correct, only if the spin state of $p1$ and $p2$ is symmetric, i.e. in triplet spin state. For antisymmetric spin state, i.e. singlet spin state, we must use the wave function which is symmetric in space coordinate variables. Namely, minus signs appearing in (2·1), (2·2b) and, consequently, the final expression (2·4) should be replaced by plus signs. Then, T(direct) and T(exchange) should be simply multiplied by factors α(direct) and α(exchange), respectively, which are determined by relative spin states. When we apply the single particle model without spin-dependent force or the L-S coupling shell model to target and residual nuclei, we can easily obtain these multiplication factors:

$$\alpha(\text{direct}) = \text{weight of triplet spin wave function} + \text{that of singlet one},$$

$$\alpha(\text{exchange}) = \text{weight of triplet spin wave function} - \text{that of singlet one}.$$

In the case of j-j coupling shell model, the situation may be rather involved, since the z-component of proton spin in the target nucleus remains no longer to be a good quantum number. Nevertheless, we might calculate the multiple factors, obtaining the same result, provided the approximation* previously employed be adopted.

Finally, we should generalize the antisymmetrization to the whole system. For this purpose, it may be sufficient to use completely antisymmetric wave functions of target and residual nuclei. Adopting the shell model, this procedure will be performed simply by means of "coefficient of fractional parentage," or shortly "c.f.p.".[5] For example, in the case of ideal j-j coupling shell model, it may be sufficient to multiply square of "c.f.p.",

$$\langle j_p^N(\alpha J)\,\{|\, j_p^{N-1}(\beta J')\, j_p\rangle$$

to T(exchange), and to perform summation over all possible states of j_p and eigen-energy. Here, we assume that the proton $p2$ is in the definite state with the total angular momentum j_p, in which the proton exists in the target nucleus. The angular momentum J' appearing in c.f.p. is just equal to the spin of the core for even-A nuclei of zero spin, since all other shells are filled up so as to form the resultant angular momentum zero. For odd-A nuclei, the spin of the core should be determined by the vector addition of angular momentum J and

* Strictly speaking, the rigorous treatment of antisymmetrization is impossible between nucleons in nuclear bound state and in positive energy state, if the Born or cutoff Born approximation be adopted. It is because the same set of quantum numbers derived from the Schroedinger equation can not be attributed to these nucleons.

the resultant angular momentum of all protons in the most outer-shell for ground states of nuclei.

Similarly, if there are several neutrons in the state which is occupied by the neutron in the incident deuteron, the following " c.f.p. " should be multiplied to T(direct),

$$\langle j_n^m(\alpha J_n)\{|j_n^{m-1}(\beta J_n')\, j_n\rangle.$$

The applications to L-S coupling or intermediate coupling shell model, and further, more complex cases with configuration interaction will be carried out straightforwardly.

§ 3. Amplitudes of stripping and heavy-particle-stripping reactions. I

——Elimination of nuclear interaction——

In this section, the nuclear interaction appearing in the matrix elements of (2·7) and (2·9) will be eliminated. As a result, each matrix element will be represented as the products of reduced width amplitudes and some sorts of overlapping integrals, the calculation of which will be carried out in the next section.

Meanwhile we shall deal with the simple example discussed in the first part of the preceding section. The matrix elements of stripping and heavy-particle-stripping reaction have been given by (2·7a) and (2·9a), in terms of final state interactions :

$$M\,(\text{Stripping}) = \int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 d\boldsymbol{C}\cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_1}\psi_R^*(n,\ p2,\ C)$$
$$\times V(n\,;\ p1)\psi_T(p2,\ C)\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}, \qquad (2\cdot7a)$$

and

$$M\,(\text{H. P. Stripping}) = -\int_{ext} d\boldsymbol{r}_n d\boldsymbol{r}_1 d\boldsymbol{r}_2 \int d\boldsymbol{C}\cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}\psi_R^*(n,\ p1,\ C)$$
$$\times V(C\,;\ p2)\psi_T(p2,\ C)\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)\, e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}. \qquad (2\cdot9a)$$

First, it is necessary to present the explicit expressions of wave functions of target and residual nuclei in the external region.*

1. *Notation*

For this purpose we shall use the following notation of various angular momenta, together with each z-component being written in subsequent brackets :

(i) Nuclear spins
target nucleus $I_T(M_T)$, residual nucleus $I_R(M_R)$, core $I_C(M_C)$.

(ii) Angular momenta of nucleons

* " External region " denotes the region of $r \gtrsim R$.

	orbital angular momentum	intrinsic spin	total angular momentum
captured neutron	$l_n(m_n)$	$1/2(\nu_n)$	$j_n(\mu_n)$
proton $p1$	$l_{p1}(m_{p1})$	$1/2(\nu_{p1})$	$j_{p1}(\mu_{p1})$
proton $p2$	$l_{p2}(m_{p2})$	$1/2(\nu_{p2})$	$j_{p2}(\mu_{p2})$
captured deuteron	$l_d(m_d)$	$1(\nu_d)$	$j_d(\mu_d)$

(iii) Channel spins

$$\boldsymbol{J}_n=\boldsymbol{I}_T+\boldsymbol{1/2}, \quad (M_n=M_T+\nu_n)$$

$$\boldsymbol{J}_{p2}=\boldsymbol{I}_C+\boldsymbol{1/2}, \quad (M_{p2}=M_C+\nu_{p2})$$

$$\boldsymbol{J}_d=\boldsymbol{I}_C+\boldsymbol{1}. \quad (M_d=M_C+\nu_d)$$

(iv) Spin wave functions

neutron	$\chi_{1/2\,\nu_n}(n)$
protons	$\chi_{1/2\,\nu_{p1}}(p1)$, $\chi_{1/2\,\nu_{p2}}(p2)$
deuteron	$\chi_{1\nu_d}(d)$; $\nu_d=\nu_n+\nu_{p1}$

Further, the binding energy of nucleon and deuteron will be denoted by

$B_n=(\hbar^2\kappa_n^2/2M)$; the binding energy of the neutron in the residual nucleus.

$B_{p2}=(\hbar^2\kappa_{p2}^2/2M)$; the binding energy of the proton $p2$ in the target nucleus.

$B_{p1}=(\hbar^2\kappa_{p1}^2/2M)$; the binding energy of the proton $p1$ in the residual nucleus.

$B_D=(\hbar^2\kappa_d^2/2M_d)$; the binding energy of the deuteron in the residual nucleus.

Since the state of residual nucleus formed by heavy-particle-stripping reaction should be the same as that formed by "ordinary" stripping reaction, the subscripts 1 and 2 in $p1$ and $p2$ need not be written explicitly, e.g. $B_{p1}=B_{p2}\equiv B_p$, $\kappa_{p1}=\kappa_{p2}\equiv\kappa_p$ etc.

2. *Nuclear wave functions*

By using the above notation, the wave function of residual nucleus appearing in (2·7a) may be decomposed into the product of wave functions of captured neutron and that of target nucleus in the external region:

$$\begin{aligned}\psi_R(n, p2, C)=&\sum_{l_n m_n \nu_n j_n \mu_n}(l_n\,1/2\;m_n\nu_n|j_n\mu_n)(j_n I_T\mu_n M_T|I_R M_R)\\&\times\psi_T(p2, C)\cdot(2M/\hbar^2 R)^{1/2}\cdot\gamma_{l_n}\cdot[h_{l_n}^{(1)}(i\kappa_n r_n)/h_{l_n}^{(1)}(i\kappa_n R)]\\&\times Y_{l_n m_n}(\Omega_{r_n})\cdot\chi_{1/2\,\nu_n}(n),\end{aligned}\qquad (3\cdot1)^{6)}$$

where $h_l^{(1)}(i\kappa r)$ are spherical Hankel functions or, for (d, n) reaction, their Coulomb analogues. γ_{l_n} is reduced width amplitude of the bound neutron, defined by

$$\tilde{\gamma}_{l_n}=(\hbar^2 R/2M)^{1/2}\,\tilde{\mathcal{R}}_{l_n}(R), \tag{3·2a}$$

where $\tilde{\mathcal{R}}_{l_n}(r)$ is the radial wave function of the neutron normalized through the internal region. It may sometimes be convenient to express γ_{l_n} in terms of dimensionless parameter θ_{l_n}, that is, in the unit of Teichman and Wigner's sum rule limit:

$$\gamma=(3\hbar^2/2MR^2)^{1/2}\theta. \tag{3·2b}$$

It should be noted, however, that the sum rule limit has not a strict meaning especially for the bound state; even for order-of-magnitude estimation, one may preferably use the formula (3·2a) to (3·2b).

Eq. (3·1) is expressed in *j-j* coupling scheme, because it will be convenient to the direct application of *j-j* coupling shell model to the nucleus. The transformation of *j-j* coupling scheme to other representations can be performed simply by means of the so-called transformation brackets, which is, for the transformation from *j-j* to channel-spin representations, given by

$$\langle 1/2\; I_C(J)\,l\,I|1/2\;l(j)I_C I\rangle=[(2J+1)(2j+1)]^{1/2}W(I_C J j l;\,1/2\;I).$$

Similarly, $\psi_T(p2\cdot C)$ in M(H.P. Stripping) may be represented as a product of the wave functions of bound proton $p2$ and the core in the external region.

$$\begin{aligned}\psi_T(p2,C)=&\sum_{l_{p2}m_{p2}\nu_{p2}j_{p2}\mu_{p2}I_C M_C}(l_{p2}\,1/2\;m_{p2}\nu_{p2}|j_{p2}\mu_{p2})(j_{p2}I_C\mu_{p2}M_C|I_T M_T)\\&\times\psi_C(C)(2M/\hbar^2R)^{1/2}\cdot\gamma_{l_{p2}}[h^{(1)}_{l_{p2}}(i\kappa_{p2}r_2)/h^{(1)}_{l_{p2}}(i\kappa_{p2}R)]\\&\times Y_{l_{p2}m_{p2}}(\Omega_{r2})\chi_{1/2\,\nu_{p2}}(p2).\end{aligned} \tag{3·3}$$

Further, $\psi_R(n,\,p1,\,C)$ appearing in the matrix element of heavy-particle-stripping reaction may be represented as the product of three wave functions, i.e. those of the core, the proton $p1$ and the neutron. It can be obtained directly by inserting (3·3) into the expression (3·1), in which all suffix $p2$ in (3·3) should be replaced by the suffix $p1$. Namely,

$$\begin{aligned}\psi_R(n,\,p1,\,C)=&\sum_{l_{p1}m_{p1}\tilde{\nu}_{p1}j_{p1}\mu_{p1}\tilde{l}_n\tilde{m}_n\tilde{\nu}_n\tilde{j}_n\tilde{\mu}_n\tilde{I}_C\tilde{M}_C}(l_{p1}\,1/2\;m_{p1}\tilde{\nu}_{p1}|j_{p1}\mu_{p1})(j_{p1}\tilde{I}_C\mu_{p1}\tilde{M}_C|I_T M_T)\\&\times(\tilde{l}_n\,1/2\;\tilde{m}_n\tilde{\nu}_n|\tilde{j}_n\tilde{\mu}_n)(\tilde{j}_n I_T\tilde{\mu}_n M_T|I_R M_R)\cdot\psi_C(C)\\&\times(2M/\hbar^2R)^{1/2}\gamma_{l_{p1}}(2M/\hbar^2R)^{1/2}\cdot\gamma_{\tilde{l}_n}\\&\times[h^{(1)}_{l_{p1}}(i\kappa_{p1}r_1)/h^{(1)}_{l_{p1}}(i\kappa_{p1}R)]\cdot[h^{(1)}_{\tilde{l}_n}(i\kappa_n r_n)/h^{(1)}_{\tilde{l}_n}(i\kappa_n R)]\\&\times Y_{l_{p1}m_{p1}}(\Omega_{r1})\,Y_{\tilde{l}_n\tilde{m}_n}(\Omega_{r_n})\cdot\chi_{1/2\,\tilde{\nu}_{p1}}(p1)\chi_{1/2\,\tilde{\nu}_n}(n).\end{aligned} \tag{3·4}$$

On the other hand, the wave function of residual nucleus may also be decomposed into the product of core and "interacting"* two-nucleon system:

* Not necessarily interacting. When the residual nucleus can be described by ideal single particle model, this is nothing but coordinate transformation.

$$\psi_R(n, p1, C) = \sum_{l_d m_d \nu_{d'} j_d \mu_d \tilde{I}_C \tilde{M}_C} (l_d s_{d'} m_d \nu_{d'} | j_d \mu_d)(j_d \tilde{I}_C \mu_d \tilde{M}_C | I_R M_R)$$
$$\times \psi_C(C) \cdot \phi_{np}(\boldsymbol{\omega}) \cdot (2M_d/\hbar^2 R')^{1/2} \cdot \gamma_{l_d}$$
$$\times [h_{l_d}^{(1)}(i\kappa_d \rho)/h_{l_d}^{(1)}(i\kappa_d R')] \cdot Y_{l_d m_d}(\Omega_\rho) \cdot \chi_{s_{d'} \nu_{d'}}(d), \qquad [3\cdot5]^*$$

where $\boldsymbol{\omega}=\boldsymbol{r}_1-\boldsymbol{r}_n$, $\boldsymbol{\rho}=(\boldsymbol{r}_1+\boldsymbol{r}_n)/2$. Here, $s_{d'}(\nu_{d'})$ is the spin of the correlating two-nucleon system, which is either 0 or 1. $\phi_{np}(\boldsymbol{\omega})$ is the wave function of correlating two nucleons near the nuclear surface.

Similar expression may be quite suitable to the case of α-induced reactions, though some (trivial) improvements, of course, are necessary for α-induced reactions. But, for deuteron-induced-reactions, it seems convenient in the present time to use the expression (3·4) rather than [3·5] because of the following reason. Firstly, when the shell model is applied to the nucleus, it may be almost impossible to show explicitly that the residual nuclear states formed by both "ordinary" stripping and heavy-particle-stripping reactions are the same. Secondly, we have not yet any knowledge of the reduced width amplitude γ_{ld}. Thirdly, we have not yet an adequate method to obtain correlating two-nucleon function $\phi_{np}(\boldsymbol{\omega})$ near the nuclear surface: The theories of nuclear many-body problems so far developed especially by Brueckner[7] have been concerned mainly with infinite nuclear matter.** Moreover, it seems likely to the authors that the nuclear reactions induced by loosely bound system, such as deuteron, should be treated in principle along the line of Eq. (3·4).*** Namely, intuitively speaking, capture of deuteron by a nucleus may be preferably considered to be a two-step process: First, one of the constituent of deuteron enters the nucleus and then another of constituent enters to form a residual nucleus. This has already been fully discussed and formulated in detail in the previous paper by one of the present authors (H.U.).[9]

3. *Elimination of nuclear interactions*

Next, we shall eliminate nuclear interactions appearing in the matrix elements of stripping and heavy-particle-stripping reactions, (2·7a) and (2·9a).

For stripping reaction, $V(p1\,;\,n)$ appearing in M(Stripping) may be eliminated simply by using the Schroedinger equation obeyed by the internal motion of deuteron:

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* We shall use square brackets [ ] instead of curly ones ( ) to denote the formula derived using Eq. [3·5] hereafter.

** It may be expected from the work of Weisskopf and his coworkers[8] that the correlation of two nucleons will be more important in nuclear surface than in its interior, because the "healing distance" will become longer in nuclear surface due to the weakness of the effect of Pauli principle near nuclear surface. Moreover, it seems likely to the authors that in nuclear surface three- or four-particle correlation rather than two-particle correlation may be important, which could be inferred from the careful analysis of α-induced and t-induced reactions or their reciprocal reactions.

*** Unfortunately, however, we can not analytically calculate the matrix element of heavy-particle-stripping reaction by making use of (3·4), that will be discussed in detail in §4 and §5.

$$V(\boldsymbol{r}_1-\boldsymbol{r}_n)\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)=-(\hbar^2/M)(8\pi\alpha)^{1/2}\delta(\boldsymbol{r}_1-\boldsymbol{r}_n), \tag{3·6}$$

where α is defined by

$$B_d=\hbar^2\alpha^2/M=2.23\ \text{Mev}\ ;\ \text{binding energy of deuteron.}$$

Eq. (3·6) is valid only for the deuteron wave function of Yukawa type, i.e.

$$\phi_d(r)=\alpha^{3/2}/(2\pi)^{1/2}\cdot(\alpha r)^{-1}\cdot\exp(-\alpha r).$$

If we use more realistic wave function $\phi_d(r)$, e.g. Hulthén type or phenomenological one derived by the shape independent formula, the result (3·6) will not be much altered numerically, say, at most within the difference of 10%.

On the other hand, the interaction $V(p2, C)$ appearing in the matrix element M(H.P. Stripping) may be eliminated similarly by means of the Schroedinger equation of target nucleus,

$$H_T\psi_T(p2, C)=E_T\psi_T(p2, C),$$

where H_T is the total Hamiltonian of target nucleus which may be expressed as

$$H_T=H_C+T_2+V(p2\,;\,C),$$

where H_C is the Hamiltonian of the core, T_2 kinetic energy operator of the proton $p2$. Then,

$$\begin{aligned} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}V(p2\,;\,C)\psi_T(p2, C)&=e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}\cdot[E_T-H_C-T_2]\psi_T(p_2, C)\\ &\doteqdot-[B_{p2}+E_p]\cdot e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}\cdot\psi_T(p2, C) \end{aligned} \tag{3·7}$$

where

$$E_p=\hbar^2k^2/2M,$$

and

E_T(total energy of target nucleus) $-E_C$(total energy of the core) $=-B_{p2}$.

Here we use (3·3) for $\psi_T(p2, C)$ since in M(H. P. Stripping) it may be sufficient to use the wave function $\psi_T(p2, C)$ only in the external region.

4. *Expressions of* M(Stripping) *and* M(H. P. Stripping)

Inserting (3·1) and (3·6) into M(Stripping) and (3·3), (3·4) and (3·7) into M(H.P. Stripping), we can express those matrix elements in terms of reduced width amplitudes of bound nucleons and overlapping integrals :

$$\begin{aligned} M\,(\text{Stripping})=&\sum_{l_n m_n \nu_n j_n \mu_n}(1/2\ 1/2\ \nu_{p1}\nu_n|1\nu_d)(l_n\ 1/2\ m_n\nu_n|\ j_n\mu_n)\\ &\times(j_nI_T\mu_nM_T|I_RM_R)(-)(\hbar^2/M)\cdot(8\pi\alpha)^{1/2}\cdot(2M/\hbar^2R)^{1/2}\cdot\gamma_{l_n}\\ &\times\int_{ext}d\boldsymbol{r}_n\cdot[h_{l_n}^{(1)}(i\kappa_n r_n)/h_{l_n}^{(1)}(i\kappa_n R)]\cdot Y_{l_nm_n}^{*}(\Omega_{r_n})\,e^{i(K-k)\cdot r_n}, \end{aligned} \tag{3·8}$$

and

$$M\,(\text{H. P. Stripping}) = \sum_{\substack{l_{p1}\,m_{p1}\,\tilde{\nu}_{p1}\,j_{p1}\mu_{p1}\cdot\tilde{l}_n\,\tilde{m}_n\,\tilde{\nu}_n\,\tilde{j}_n\,\tilde{\mu}_n \\ l_{p2}\,m_{p2}\,j_{p2}\,\mu_{p2}\,I_C\,M_C}} (1/2\;1/2\;\tilde{\nu}_{p1}\tilde{\nu}_n|1\nu_d)\,(l_{p1}\;1/2\;m_{p1}\tilde{\nu}_{p1}|j_{p1}\mu_{p1})$$

$$\times (j_{p1}\,I_C\mu_{p1}\,M_C|I_T\,M_T)\,(\tilde{l}_n\;1/2\;\tilde{m}_n\tilde{\nu}_n|\tilde{j}_n\tilde{\mu}_n)\,(\tilde{j}_n\,I_T\tilde{\mu}_n\,M_T|I_R\,M_R)$$

$$\times (l_{p2}\;1/2\;m_{p2}\nu_{p1}|j_{p2}\mu_{p2})\,(j_{p2}\,I_C\mu_{p2}\,M_C|I_T\,M_T)\,(B_{p2}+E_p)$$

$$\times (2M/\hbar^2 R)^{1/2}\gamma_{l_{p2}}\int_{ext} d\boldsymbol{r}_2\cdot[h^{(1)}_{l_{p2}}(i\kappa_{p2}\,r_2)/h^{(1)}_{l_{p2}}(i\kappa_{p2}\,R)]\cdot Y_{l_{p2}\,m_{p2}}(\Omega_{r_2})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}$$

$$\times (2M/\hbar^2 R)\cdot\gamma_{l_{p1}}\gamma_{\tilde{l}_n}\int_{ext} d\boldsymbol{r}_1\,d\boldsymbol{r}_n\,\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)\cdot[h^{(1)}_{\tilde{l}_n}(i\kappa_n\,r_n)/h^{(1)}_{\tilde{l}_n}(i\kappa_n R)]$$

$$\times[h^{(1)}_{l_{p1}}(i\kappa_{p1}\,r_1)/h^{(1)}_{l_{p1}}(i\kappa_{p1}\,R)]\cdot Y^*_{\tilde{l}_n\,\tilde{m}_n}(\Omega_{r_n})\,Y_{l_{p1}\,m_{p1}}(\Omega_{r_1})\,e^{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2}. \qquad (3\cdot9)^*$$

If the wave function of residual nucleus be expressed as the product of the core and the correlating two-nucleon-system as was given in [3·5], the matrix element M(H.P. Stripping) is represented as**

$$M\,(\text{H. P. Stripping}) = \sum_{\substack{l_d\,m_d\,j_d\,\mu_d\,l_{p2}\,m_{p2}\,j_{p2}\,\mu_{p2}\,I_C\,M_C}} (l_d 1 m_d\nu_d|j_d\mu_d)\,(j_d\,I_C\mu_d\,M_C|I_R\,M_R)$$

$$\times (l_{p2}\;1/2\;m_{p2}\nu_{p1}|j_{p2}\mu_{p2})\,(j_{p2}\,I_C\mu_{p2}\,M_C|I_T\,M_T)\cdot(B_{p2}+E_p)$$

$$\times (2M/\hbar^2 R)^{1/2}\cdot\gamma_{l_{p2}}\int_{ext} d\boldsymbol{r}_2\cdot[h^{(1)}_{l_{p2}}(i\kappa_{p2}\,r_2)/h^{(1)}_{l_{p2}}(i\kappa_{p2}\,R)]\cdot Y_{l_{p2}\,m_{p2}}(\Omega_{r_2})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_2}$$

$$\times (2M_d/\hbar^2 R')^{1/2}\gamma_{l_d}\int d\boldsymbol{\omega}\cdot\phi^*_{np}(\boldsymbol{\omega})\,\phi_d(\boldsymbol{\omega})$$

$$\times\int_{ext} d\boldsymbol{\rho}\cdot[h^{(1)}_{l_d}(i\kappa_d\rho)/h^{(1)}_{l_d}(i\kappa_d R')]\cdot Y^*_{l_d\,m_d}(\Omega_\rho)\,e^{i\boldsymbol{K}\cdot\boldsymbol{\rho}}. \qquad [3\cdot10]$$

§ 4. Amplitudes of stripping and heavy-particle-stripping reactions. II

——Evaluation of overlapping integrals——

The overlapping integrals appearing in (3·8) and (3·9) (or in [3·10]) will be calculated in this section.

1. *Preparations*

For this purpose, it seems convenient to present the following formula of integral:

$$\int_{ext} d\boldsymbol{r}\,[h^{(1)}_l(i\kappa r)/h^{(1)}_l(i\kappa R)]\,Y^*_{lm}(\Omega_r)\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$$

$$= 4\pi i^l Y^*_{lm}(\Omega_k)[R/(k^2+\kappa^2)]\,J_l(k,\,\kappa,\,R), \qquad (4\cdot1)$$

---

* If we use the matrix element (2·7b) and (2·9b), which have been expressed by means of initial-state-interactions, we can obtain also the same result by a similar procedure.

** This expression can be applied straightforwardly to the case of α-induced reaction, the result of which will be published later in a separate paper together with connected problems. See also the footnote of the previous page.

in which the quantization axis of angular momenta is taken to be an arbitary direction $\boldsymbol{n}$, Ω_K being solid angle of $\boldsymbol{k}$ measured from the axis $\boldsymbol{n}$, or in a slightly simpler case, in which z-direction is taken to be that of $\boldsymbol{k}$, (4·1) becomes

$$\sqrt{4\pi(2l+1)}\,i^l[R/(k^2+\kappa^2)]J_l(k,\kappa,R). \tag{4·2}$$

Here

$$J_l(k,\kappa,R)\equiv[kRj_{l-1}(kR)+c_l(\kappa R)\,j_l(kR)], \tag{4·3a}$$

and

$$c_l(\kappa R)=-i\kappa R[h_{l-1}^{(1)}(i\kappa R)/h_l^{(1)}(i\kappa R)], \tag{4·3b}$$

with the following properties:

(i) $c_l(\alpha)$ is real and positive for a real variable α.

(ii) Asymptotic form,

$$\lim_{\alpha\to\infty}c_l(\alpha)=\alpha, \quad \text{for all values of } l.$$

(iii) Recurrence relation,

$$c_{l+1}(\alpha)=\alpha^2/[c_l(\alpha)+(2l+1)], \tag{4·3c}$$

with $$c_0(\alpha)=\alpha.$$

These relations can be derived directly from the property of spherical Hankel functions with imaginary arguments.

2. *Calculations*

By the aid of the above formula, we can easily carry out the following integrations* appearing in Eqs. (3·8) and (3·9):

$$\int_{ext}d\boldsymbol{r}_n[h_{l_n}^{(1)}(i\kappa_n r_n)/h_{l_n}^{(1)}(i\kappa_n R)]\cdot Y_{l_n m_n}{}^*(\Omega_{r_n})\exp\{i(\boldsymbol{K}-\boldsymbol{k})\cdot\boldsymbol{r}_n\}$$
$$=4\pi i^{l_n}Y_{l_n m_n}{}^*(\Omega_{\boldsymbol{K}-\boldsymbol{k}})[R/\{|\boldsymbol{K}-\boldsymbol{k}|^2+\kappa_n{}^2\}]J_{l_n}(|\boldsymbol{K}-\boldsymbol{k}|,\kappa_n,R), \tag{4·4}$$

and

$$\int_{ext}d\boldsymbol{r}_2[h_{l_{p2}}^{(1)}(i\kappa_{p2}r_2)/h_{l_{p_2}}^{(1)}(i\kappa_{p2}R)]\cdot Y_{l_{p2}m_{p2}}(\Omega_{r_2})\exp(-i\boldsymbol{k}\cdot\boldsymbol{r}_2)$$
$$=4\pi i^{-l_{p2}}Y_{l_{p2}m_{p2}}(\Omega_k)[R/(k^2+\kappa_{p2}^2)]\cdot J_{l_{p2}}(k,\kappa_{p2},R). \tag{4·5}$$

Further, in the expression [3·10] of M (H. P. Stripping),

$$\int_{ext}d\boldsymbol{\rho}\cdot[h_{l_d}^{(1)}(i\kappa_d\rho)/h_{l_d}^{(1)}(i\kappa_d R')]\cdot Y_{l_d m_d}{}^*(\Omega_\rho)\exp(i\boldsymbol{K}\cdot\boldsymbol{\rho})$$
$$=\sqrt{4\pi(2l_d+1)}\,i^{l_d}\cdot[R'/(K^2+\kappa_d{}^2)]\cdot J_{l_d}(K,\kappa_d,R'). \tag{4·6}$$

---

* Hereafter, the quantization axis of the angular momenta will be taken to be the direction of $\boldsymbol{K}$. ($\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}$ in the case including the recoil effect.) Of course, it is more convenient to use the axis $\boldsymbol{K}$–$\boldsymbol{\kappa}$, ($\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}$) if we are concerned only with the stripping reaction.

On the other hand, the integration over $\boldsymbol{r}_1$ and $\boldsymbol{r}_n$ in the expression (3·9) of M(H.P. Stripping) seems to be rather complicated, since we can not carry out the integration over $\boldsymbol{r}_1$ and $\boldsymbol{r}_n$ separately due to the appearance of $\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)$. In order to perform the integration separately, it might be necessary to expand $\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)$ in terms of Legendre polynomials $P_L(\cos\theta)$, where θ stands for the angle between $\boldsymbol{r}_1$ and $\boldsymbol{r}_n$. Unfortunately, however, the convergence of this series is rather poor in this case, because the ranges of r_1, r_n and θ which are effective in the integral are $r_1\simeq R$, $r_n\simeq R$ and $\theta=0$, as is easily seen from the integrand of (3·9). Therefore, we shall adopt the steepest decent method to handle the integral. Namely, $\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)$ will be taken out of the integral, in which the value of $\phi_d(r)$ will be fixed to $\phi_d(0)$, or more strictly speaking, an average value of $\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)$ near $r_1=R$, $r_n=R$ and $\theta=0$. Then, the integration over $\boldsymbol{r}_1$ and $\boldsymbol{r}_n$ appearing in (3·9) can be performed separately by using the formula (4·2), that is,

$$\begin{aligned}
&\int_{ext} d\boldsymbol{r}_1\, d\boldsymbol{r}_n\cdot\phi_d(\boldsymbol{r}_1-\boldsymbol{r}_n)\cdot[h_{l_n}^{(1)}(i\kappa_n r_n)/h_{l_n}^{(1)}(i\kappa_n R)]\cdot[h_{l_{p1}}^{(1)}(i\kappa_{p1} r_1)/h_{l_{p1}}^{(1)}(i\kappa_{p1} R)]\\
&\quad\times Y_{l_n m_n}{}^{*}(\Omega_{\boldsymbol{r}_n})\cdot Y_{l_{p1} m_{p1}}{}^{*}(\Omega_{\boldsymbol{r}_1})\exp\{i\boldsymbol{K}\cdot(\boldsymbol{r}_1+\boldsymbol{r}_n)/2\}\\
&\quad=\phi_d(0)\cdot\sqrt{4\pi(2l_n+1)}\cdot\sqrt{4\pi(2l_{p1}+1)}\cdot i^{l_n+l_{p1}}\\
&\quad\times[R/\{(K/2)^2+\kappa_n^2\}]\cdot J_{l_n}(K/2,\ \kappa_n,\ R)[R/\{(K/2)^2+\kappa_{p1}^2\}]\cdot J_{l_{p1}}(K/2,\ \kappa_{p1},\ R).
\end{aligned}\tag{4·7}$$

Using the above results, we can analytically calculate the matrix element of the stripping and the heavy-particle-stripping reactions:

$$\begin{aligned}
M\ (\text{Stripping}) &= \sum (1/2\ 1/2\ \nu_{p1}\nu_n|1\nu_d)\,(l_n\ 1/2\ m_n\nu_n|j_n\mu_n)\\
&\quad\times(j_n I_T\mu_n M_T|I_R M_R)\,(-)\,(\hbar^2/M)\,(8\pi\alpha)^{1/2}\,(2M/\hbar^2 R)^{1/2}\gamma_{l_n}\\
&\quad\times 4\pi i^{l_n} Y_{l_n m_n}{}^{*}(\Omega_{\boldsymbol{K}-\boldsymbol{k}})\cdot[R/\{|\boldsymbol{K}-\boldsymbol{k}|^2+\kappa_n^2\}]\cdot J_{l_n}(|\boldsymbol{K}-\boldsymbol{k}|,\ \kappa_n,\ R).
\end{aligned}\tag{4·8}$$

$$\begin{aligned}
M\ (\text{H. P. Stripping}) &= \sum (1/2\ 1/2\ \tilde{\nu}_{p1}\tilde{\nu}_n|1\nu_d)\,(l_{p1}\ 1/2\ m_{p1}\tilde{\nu}_{p1}|j_{p1}\mu_{p1})\\
&\quad\times(j_{p1} I_C\mu_{p1} M_C|I_T M_T)\,(\tilde{l}_n\ 1/2\ \tilde{m}_n\tilde{\nu}_n|\tilde{j}_n\tilde{\mu}_n)\,(\tilde{j}_n I_T\tilde{\mu}_n M_T|I_R M_R)\\
&\quad\times(l_{p2}\ 1/2\ m_{p2}\nu_{p1}|j_{p2}\mu_{p2})\,(j_{p2} I_C\mu_{p2} M_C|I_T M_T)\,(B_{p2}+E_p)\cdot(2M/\hbar^2 R)^{1/2}\gamma_{l_{p2}}\\
&\quad\times 4\pi i^{-l_{p2}}\cdot[R/(k^2+\kappa_{p2}^2)]\cdot J_{l_{p2}}(k,\ \kappa_{p2},\ R)\cdot(2M/\hbar^2 R)\gamma_{l_{p1}}\gamma_{\tilde{l}_n}\cdot\phi_d(0)\\
&\quad\times\sqrt{4\pi(2l_{p1}+1)}\cdot\sqrt{4\pi(2\tilde{l}_n+1)}\cdot i^{l_{p1}+\tilde{l}_n}\cdot[R/\{(K/2)^2+\kappa_n^2\}]\cdot J_{\tilde{l}_n}(K/2,\ \kappa_n,\ R)\\
&\quad\times[R/\{(K/2)^2+\kappa_{p1}^2\}]\cdot J_{l_{p1}}(K/2,\ \kappa_{p1},\ R)\,Y_{l_{p2} m_{p2}}(\Omega_k)\,\delta_{\tilde{m}_n 0}\,\delta_{m_{p1} 0},
\end{aligned}\tag{4·9}$$

or in the alternative expression of M(H.P. Stripping), Eq. [3·10], in which γ_{l_d} is utilized,

$$M\,(\text{H. P. Stripping}) = \sum (l_d 1 m_d \nu_d | j_d \mu_d)\,(j_d I_C \mu_d M_C | I_R M_R)$$
$$\times (l_{p2}\, 1/2\; m_{p2}\nu_{p1} | j_{p2}\mu_{p2})\,(j_{p2} I_C \mu_{p2} M_C | I_T M_T)\cdot (B_{p2}+E_p)$$
$$\times (2M/\hbar^2 R)^{1/2}\cdot\gamma_{l_{p2}}\cdot 4\pi i^{-l_{p2}}\cdot[R/(k^2+\kappa_{p2}^2)]\cdot J_{l_{p2}}(k,\ \kappa_{p2},\ R)$$
$$\times (2M_d/\hbar^2 R')^{1/2}\gamma_{l_d}\int d\boldsymbol{\omega}\cdot\phi_{np}^*(\boldsymbol{\omega})\,\phi_d(\boldsymbol{\omega})\cdot\sqrt{4\pi(2l_d+1)}\,i^{l_d}$$
$$\times[R'/(K^2+\kappa_d^2)]\cdot J_{l_d}(K,\ \kappa_d,\ R')\,Y_{l_{p2}m_{p2}}(\Omega_k)\,\delta_{m_d 0}. \qquad [4\cdot 10]$$

§ 5. Cross sections*, **

The cross section of (d, p) or (d, n) reaction can be obtained by inserting M(Stripping) and M(H.P. Stripping) into the following formula,

$$\frac{d\sigma}{d\Omega}(d,\ p) = \frac{MM_d}{(2\pi\hbar^2)^2}\cdot\frac{k}{K}\cdot\frac{1}{3(2I_T+1)}\cdot\sum_{spin}|M\,(\text{Stripping})$$
$$+M\,(\text{H. P. Stripping})|^2, \qquad (5\cdot 1)$$

where the notation $\sum_{spin}$ denotes the summation over all possible states of initial and final spin states. Before performing actual calculations, it may be convenient to divide the cross section into three parts : stripping, heavy-particle-stripping and their interference terms, that is,

$$\frac{d\sigma}{d\Omega}(d,\ p) = \frac{d\sigma}{d\Omega}(\text{Stripping}) + \frac{d\sigma}{d\Omega}(\text{H. P. Stripping})$$
$$+\frac{d\sigma}{d\Omega}(\text{Interference}), \qquad (5\cdot 2)$$

where

$$\frac{d\sigma}{d\Omega}(\text{Stripping}) = \frac{MM_d}{(2\pi\hbar^2)^2}\cdot\frac{k}{K}\cdot\frac{1}{3(2I_T+1)}\sum_{spin}|M\,(\text{Stripping})|^2, \qquad (5\cdot 3a)$$

$$\frac{d\sigma}{d\Omega}(\text{H. P. Stripping}) = \frac{MM_d}{(2\pi\hbar^2)^2}\cdot\frac{k}{K}\cdot\frac{1}{3(2I_T+1)}\sum_{spin}|M\,(\text{H. P. Stripping})|^2, \qquad (5\cdot 3b)$$

$$\frac{d\sigma}{d\Omega}(\text{Interference}) = \frac{MM_d}{(2\pi\hbar^2)^2}\cdot\frac{k}{K}\cdot\frac{1}{3(2I_T+1)}$$
$$\times\sum_{spin}2\{\text{Real part of } M\,(\text{Stripping})\,M^*\,(\text{H. P. Stripping})\}. \qquad (5\cdot 3c)$$

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* The formula presented in this section are more general than those of "simple example", though the geometrical factors in §2-3 are not explicitly written. On the other hand, N-B contains no expressions of cross section, but those of matrix elements.

** The recoil effect discussed in detail in §2-2 will be explicitly included in the formula of the present section, whereas it was not written in the formula of N-B.

The summation over initial and final states combined with that over various z-components of angular momenta appearing in the marix elements can be performed by means of Racah's technique, obtaining the following results: Inserting (4·8) into (5·3a),

$$\frac{d\sigma}{d\Omega}(\text{Stripping}) = 24\cdot(\alpha R)\cdot R^2\cdot(2I_R+1)/(2I_T+1)\cdot k/K \times \sum_{l_n} 1/\{(|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|^2+\kappa_n^2)R^2\}^2\cdot J_{l_n}^2(|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|,\ \kappa_n,\ R)\theta_{l_n}^2, \tag{5·4a}$$

and (4·9) into (5·3b),

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\text{H. P. Stripping}) = {}& \frac{9}{2}\{(1+x)(2+x)\}\cdot(\alpha R)^3\cdot R^2\cdot\frac{1}{2I_T+1}\cdot\frac{k}{K}\\
&\times\sum i^{l_{p1}-l_{p1'}+\tilde{l}_n-\tilde{l}_{n'}-l_{p2}+l_{p2'}}\cdot(2\tilde{l}_n+1)^{1/2}(2\tilde{l}_n'+1)^{1/2}(2l_{p1}+1)^{1/2}(2l_{p1}'+1)^{1/2}\\
&\times(2l_{p2}+1)^{1/2}(2l_{p2}'+1)^{1/2}(2j_{p2}+1)^{1/2}(2j_{p2}'+1)^{1/2}J_{l_{p2}}(k,\ \kappa_{p2},\ R)\\
&\times J_{l_{p2'}}(k,\ \kappa_{p2}',\ R)\theta_{l_{p2}}\theta_{l_{p2'}}\\
&\times 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2\}^2R^2+\kappa_n^2R^2]\cdot 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2\}^2R^2+\kappa^2{}_{n'}R^2]\\
&\times J_{\tilde{\tau}_n}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_n,\ R)\cdot J_{\tilde{\tau}_{n'}}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_{n'},\ R)\theta_{\tilde{\tau}_n}\theta_{\tilde{\tau}_{n'}}\\
&\times 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2\}^2R^2+\kappa_{p1}^2R^2]\cdot 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2\}^2R^2+\kappa^2{}_{p1}'R^2]\\
&\times J_{l_{p1}}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_{p1},\ R)\cdot J_{l_{p1'}}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_{p1'},\ R)\theta_{l_{p1}}\theta_{l_{p1'}}\\
&\times\sum_L(l_{p2}l_{p2}'00|L0)\,W(l_{p2}j_{p2}l_{p2'}j_{p2'};\,1/2\ L)P_L(\cos\Omega_k)\\
&\times\sum(-1)^{1/2+\tilde{\nu}_{p1}}(j_{p2}j_{p2'}-\tilde{\nu}_{p1}\tilde{\nu}_{p1}|L0)\,(1/2\ 1/2\ \tilde{\nu}_{p1}\tilde{\nu}_n|1\,\tilde{\nu}_{p1}+\tilde{\nu}_n)^2\\
&\times(l_{p1}\ 1/2\ 0\ \tilde{\nu}_{p1}|j_{p1}\tilde{\nu}_{p1})\,(l_{p1}'\ 1/2\ 0\ \tilde{\nu}_{p1}|j_{p1}'\tilde{\nu}_{p1})\,(j_{p1}I_C\tilde{\nu}_{p1}M_C|I_TM_T)\\
&\times(j_{p1}'I_C'\tilde{\nu}_{p1}M_C|I_TM_T)\,(\tilde{l}_n\ 1/2\ 0\ \tilde{\nu}_n|\tilde{j}_n\tilde{\nu}_n)\,(\tilde{l}_n'\ 1/2\ 0\ \tilde{\nu}_n|\tilde{j}_n'\tilde{\nu}_n)\,(\tilde{j}_nI_T\tilde{\nu}_nM_T|I_RM_R)\\
&\times(\tilde{j}_n'I_T\tilde{\nu}_nM_T|I_RM_R)\,(j_{p2}I_C\tilde{\nu}_{p1}M_C|I_TM_T)\,(j_{p2}'I_C'\tilde{\nu}_{p1}M_C|I_TM_T),\end{aligned} \tag{5·4b}$$

and (4·8) and (4·9) into (5·3c),

$$\begin{aligned}\frac{d\sigma}{d\Omega}(\text{Interference}) = {}& (-)24\pi\cdot\{(1+x)(2+x)\}^{1/2}\cdot(\alpha R)^2\cdot R^2\cdot\frac{1}{2I_T+1}\cdot\frac{k}{K}\\
&\times\sum i^{l_{n'}+l_{p2}-l_{p1}-\tilde{l}_n}\cdot(2l_{p1}+1)^{1/2}\cdot(2\tilde{l}_n+1)^{1/2}J_{l_{p2}}(k,\ \kappa_{p2},\ R)\cdot\theta_{l_{p2}}\\
&\times 1/\{|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|^2R^2+\kappa_n^2R^2\}\cdot J_{l_n}(|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|,\ \kappa_n,\ R)\theta_{l_n}\\
&\times 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|R/2\}^2+\kappa_n^2R^2]\cdot J_{\tilde{\tau}_n}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_n,\ R)\theta_{\tilde{\tau}_n}\\
&\times 1/[\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|R/2\}^2+\kappa_{p1}^2R^2]\cdot J_{l_{p1}}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|/2,\ \kappa_{p1},\ R)\theta_{l_{p1}}\\
&\times\sum(1/2\ 1/2\ \nu_{p1}\nu_n|1\,\nu_d)\,(l_n\ 1/2\ m_n\nu_n|j_n\tilde{\nu}_n)\,(j_nI_T\tilde{\nu}_nM_T|I_RM_R)\\
&\times(1/2\ 1/2\ \tilde{\nu}_{p1}\tilde{\nu}_n|1\,\nu_d)\,(l_{p1}\ 1/2\ 0\ \tilde{\nu}_{p1}|j_{p1}\tilde{\nu}_{p1})\,(j_{p1}I_C\tilde{\nu}_{p1}M_C|I_TM_T)\end{aligned}$$

$$\times(\tilde{l}_n\, 1/2\, 0\, \tilde{\nu}_n|\tilde{j}_n\tilde{\nu}_n)(\tilde{j}_n I_T\tilde{\nu}_n M_T|I_R M_R)(l_{p2}\, 1/2\, m_{p2}\nu_{p1}|j_{p2}\tilde{\nu}_{p1})$$

$$\times(j_{p2} I_C\tilde{\nu}_{p1} M_C|I_T M_T)\, Y_{l_n m_n}^{*}(\Omega_{\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}})\, Y_{l_{p2} m_{p2}}^{*}(\Omega_k), \qquad (5\cdot 4c)$$

which does not depend on the azimuthal angle φ, since two vectors $\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}$ and $\boldsymbol{k}$ appearing in spherical harmonics and the vector $\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}$, i.e. z-axis, lie in the same plane.

Here we have adopted the Hulthén-type wave function of deuteron:

$$\phi_d(r)=\beta^{-1}\{\alpha(\alpha+\beta)(2\alpha+\beta)/(2\pi)\}^{1/2}\cdot\{e^{-\alpha r}-e^{-(\alpha+\beta)r}\}/r,$$

therefore, $$\phi_d(0)=\alpha^{3/2}/(2\pi)^{1/2}\cdot\{(1+x)(2+x)\}^{1/2}$$

with $$\beta=x\alpha.$$

On the other hand, using the alternative expression of M(H.P. Stripping), [4·10], we obtain the following formula:

$$\frac{d\sigma}{d\Omega}(\text{H. P. Stripping})=6\cdot(R'/R)\cdot R'^2\cdot(2I_R+1)\cdot\frac{k}{K}$$

$$\times\{\int\phi_{np}^{*}(\boldsymbol{\omega})\phi_d(\boldsymbol{\omega})\,d\boldsymbol{\omega}\}^2\cdot\sum i^{l_d-l_d'-l_{p2}+l_{p2}'}(-)^{I_R+1/2-I_T-1}$$

$$\times(2l_d+1)^{1/2}(2l_d'+1)^{1/2}(2j_d+1)^{1/2}(2j_d'+1)^{1/2}(2l_{p2}+1)^{1/2}(2l_{p2}'+1)^{1/2}$$

$$\times(2j_{p2}+1)^{1/2}(2j_{p2}'+1)^{1/2}J_{l_{p2}}(k,\kappa_{p2},R)J_{l_{p2}'}(k,\kappa_{p2}',R)\theta_{l_{p2}}\theta_{l_{p2}'}$$

$$\times 1/\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|^2R'^2+\kappa_d^2R'^2\}\cdot 1/\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|^2R'^2+\kappa_d'^2R'^2\}$$

$$\times J_{l_d}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|,\kappa_d,R')\cdot J_{l_d'}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|,\kappa_d',R')\theta_{l_d}\theta_{l_d'}$$

$$\times\sum_L(l_{p2}l_{p2}'00|L0)(l_dl_d'00|L0)W(l_dj_dl_d'j_d';1L)W(j_dI_Cj_d'I_C';I_RL)$$

$$\times W(l_{p2}j_{p2}l_{p2}'j_{p2}';1/2\,L)W(j_{p2}I_Cj_{p2}'I_C';I_TL)P_L(\cos\Omega_k), \qquad [5\cdot 5b]$$

and

$$\frac{d\sigma}{d\Omega}(\text{Interference})=(-)32\cdot\sqrt{6}\,\pi\cdot(\alpha R)^{1/2}\cdot(R'/R)^{1/2}\cdot RR'$$

$$\times\frac{1}{2I_T+1}\frac{k}{K}\{\int\phi_{np}^{*}(\boldsymbol{\omega})\phi_d(\boldsymbol{\omega})\,d\boldsymbol{\omega}\}$$

$$\times\sum i^{l_n+l_{p2}-l_d}\cdot(2l_d+1)^{1/2}\cdot J_{l_{p2}}(k,\kappa_{p2},R)\cdot\theta_{l_{p2}}$$

$$\times 1/\{|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|^2R^2+\kappa_n^2R^2\}\cdot J_{l_n}(|\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}|,\kappa_n,R)\theta_{l_n}$$

$$\times 1/\{|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|^2R'^2+\kappa_d^2R'^2\}\cdot J_{l_d}(|\boldsymbol{K}+(M_d/M_f)\boldsymbol{k}|,\kappa_d,R')\theta_{l_d}$$

$$\times\sum(1/2\,1/2\,\nu_{p1}\nu_n|1\nu_d)(l_n\,1/2\,m_n\nu_n|j_n\mu_n)(j_nI_T\mu_nM_T|I_RM_R)$$

$$\times(l_d10\nu_d|j_d\nu_d)(j_dI_C\nu_dM_C|I_RM_R)(l_{p2}\,1/2\,m_{p2}\nu_{p1}|j_{p2}\mu_{p2})$$

$$\times(j_{p2}I_C\mu_{p2}M_C|I_TM_T)\,Y_{l_nm_n}^{*}(\Omega_{\boldsymbol{K}-(M_i/M_f)\boldsymbol{k}})\,Y_{l_{p2}m_{p2}}^{*}(\Omega_k). \qquad [5\cdot 5c]$$

As the validity of the "steepest decent method" employed in (4·9) is not necessarily clear, we have performed a number of numerical calculations of cross sections (5·4b, c) and [5·5b, c], obtaining the result that the angular distributions derived from both (5·4b, c) and [5·5b, c] do not coincide with each other in the case of a low incident energy of deuteron and a large value of B_n and B_p. Therefore, we should use the formula [5·5b, c] rather than (5·4b, c) to calculate the angular distribution for such a case. The magnitude of the cross sections should, however, be estimated by means of (5·4b, c), because there are some ambiguities in (5·5b, c) such as γ_{l_d} and the overlapping integral over $\boldsymbol{\omega}$ as was discussed in detail in § 3—2.

It seems necessary to treat the integral in (3·9) without the use of "steepest decent method".

§ 6. Discussions

In this section, we shall first briefly discuss qualitative properties of the cross sections of the stripping and the heavy-particle-stripping reactions, which can be inferred directly from the formula obtained in the previous sections. Detailed discussions, especially quantitative ones, will be left to the subsequent paper.

From the angular distribution of (d, p) and (d, n) reactions, it has been conjectured that the heavy-particle-stripping mechanism becomes not to be important in comparison with the ordinary stripping one in higher incident energy of deuteron, say, above 10 Mev, since the angular distributions can be well explained by the theory of stripping reaction at those energies. On the other hand, the heavy-particle-stripping mechanism was proposed by Owen and Madansky[2] to explain the angular distributions at backward directions in low incident energies of deuteron such as $E_d=1\sim4$ Mev. In view of the above situations, therefore, the energy-dependence of heavy-particle-stripping reaction may be supposed to be stronger than that of the "ordinary" stripping reaction. Such circumstances will really be true; it can be inferred directly from the expressions of the matrix elements presented in § 4. Namely, in the matrix element of the stripping reaction (4·8) there appears one overlapping integral between an exponentially decaying wave function of nuclear bound state and an oscillating wave function of positive energy state, whereas the matrix element of the heavy-particle-stripping reaction, (4·9) or [4·10], contains two or three overlapping integrals of the same kind. As the energies of those positive energy state become high, these integrals decrease in general, because the higher the energy becomes, the rapider the wave function oscillates; this fact leads to larger cancelation in the integral. Numerical calculations will be performed in the subsequent paper.

Next, we shall discuss the angular distribution of heavy-particle-stripping reaction. It may be easily shown from Eqs. (5·4b) and [5·5b] that the angular distribution of heavy-particle-stripping will be isotropic in the case of $l_d=0$, or

symmetric about 90° direction for $l_d \neq 0$, provided the approximation $(1/A)=0$ be adopted. On the other hand, the actual calculation of Owen and Madansky were confined only to the case of $l_d=0$, obtaining the peak at backward directions. Consequently, we may expect that the improvement of the approximation $(1/A)$, which was discussed in § 2–2, will play an important role in the angular distribution of the heavy-particle-stripping reaction at backward directions.

Finally, it seems necessary to add some remarks concerning the two papers: The antisymmetric treatment of (d, p) reaction has already been made by French[8] and Soga and Nakumura.[10] The former author has employed quite a similar method to that in § 2–2, whereas the latter ones have used the technique of the second quantization. Unfortunately, however, they have not divided the matrix element of (d, p) reaction clearly into the "ordinary" stripping, the heavy-particle-stripping and the extra terms, but into rather intricated parts. In consequence, they have not practically treated the matrix element corresponding to the heavy-particle-stripping reaction.

§ 7. Final comment

The formulation presented in the previous sections has been performed within the restriction of the cutoff Born approximation.

It is well-known[1] that the cutoff Born approximation may not always be reliable in the stripping reaction especially for low incident energies of deuteron. In fact, Tobocman[11] has employed "distorted wave approximation", obtaining better agreement with experiments. Consequently, it seems likely that the distortion of the plane waves plays an important role in the heavy-particle-stripping reaction. Here we shall briefly discuss the use of "distorted wave approximation" to our problem. When the distorted wave approximation is adopted in our formulation, it may be easily shown that the modification will be confined to the overlapping integrals appearing in the matrix elements, all other terms including numerical factors remaining unchanged.

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Deuteron Stripping and Heavy-Particle-Stripping Reactions†

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The formulation of the preceding paper is applied to the analysis of experimental differential cross sections of (d, n) reactions; especially, their dependences on the incident energies of deuteron are compared with the results of previous workers', in which the ratio (λ_1/λ_2) between the matrix element of the stripping and that of the heavy-particle-stripping reactions has been taken to be an adjustable parameter.

$B^{11}(d, n)C^{12}$, $Be^{9}(d, n)B^{10}$, and $C^{12}(d, n)N^{13}$ reactions are treated in detail. Quantitative discussions are also added concerning the dependence of the cross sections on the reaction Q-value, on the binding energy of the emitted nucleon in target nucleus, and on the orbital angular momentum of the captured deuteron.

It is pointed out that the energy-dependences of (λ_1/λ_2) obtained phenomenologically by previous workers are rather unreasonable: The experimental differential cross sections and, especially, their energy-dependences can hardly be well explained by merely the stripping and the heavy-particle-stripping mechanisms, provided the cutoff Born approximation be adopted in the calculation of the cross sections.

It is emphasized that the calculations by means of the "distorted wave method" will be highly required.

§ 1. Introduction and summary

The formulation presented in the preceding paper*,[1] will be applied to the analysis of experimental (d, p) and (d, n) reactions, special emphasis being devoted to the re-examination of the previous analyses by many authors.

We shall use the expression of the cross section given by [I, 5·5] rather than (I, 5·4), because, beside the reasons discussed in detail in I, § 5, its form is especially convenient to compare the previous results of Owen and Madansky[2] and others. In fact, [I, 5·5] is the natural generalization of Owen and Madansky's. The cross section will be presented in a compact form, i.e.

$$\frac{d\sigma}{d\Omega}(d, n) = \frac{d\sigma}{d\Omega}(\text{Stripping}) + \frac{d\sigma}{d\Omega}(\text{H. P. Stripping}) + \frac{d\sigma}{d\Omega}(\text{Interference}), \tag{1}$$

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* Hereafter, the preceding paper will be referred to as I, while its subsections and equations will be denoted such as I, § 2–1 and (I, 2·1), respectively.

where

$$\frac{d\sigma}{d\Omega}(\text{Stripping}) = C_S \cdot \mathcal{G}_S \sum_{l_p, j_p} \{|\boldsymbol{K} - (M_t/M_f)\cdot\boldsymbol{k}|^2 + \kappa_p^2\}^{-2} R^{-4}$$
$$\times J_{l_p}^2(|\boldsymbol{K} - (M_t/M_f)\cdot\boldsymbol{k}|, \kappa_p, R)\cdot\theta^2_{l_p\cdot j_p}, \tag{2·a}$$

$$\frac{d\sigma}{d\Omega}(\text{H.P. Stripping}) = C_H \sum \mathcal{G}_H \cdot O_d^2 \cdot J_{l_{n2}}(k, \kappa_{n2}, R)\cdot\theta_{l_{n2}\cdot j_{n2}}\cdot J_{l'_{n2}}(k, \kappa'_{n2}, R)$$
$$\times \theta_{l'_{n2}\cdot j'_{n2}}\cdot R'^{-2}\{|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|^2 + \kappa_d^2\}^{-1}$$
$$\times J_{l_d}(|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|, \kappa_d, R')\cdot\theta_{l_d\cdot j_d}\cdot R'^{-2}$$
$$\times \{|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|^2 + \kappa_d'^2\}^{-1}\cdot J_{l'_d}(|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|, \kappa_d', R')\theta_{l'_d\cdot j'_d}, \tag{2·b}$$

and

$$\frac{d\sigma}{d\Omega}(\text{Interference}) = C_I \sum \mathcal{G}_I \cdot O_d \cdot J_{l_{n2}}(k, \kappa_{n2}, R)\cdot\theta_{l_{n2}\cdot j_{n2}}$$
$$\times R^{-2}\{|\boldsymbol{K} - (M_t/M_f)\cdot\boldsymbol{k}|^2 + \kappa_p^2\}^{-1}\cdot J_{l_p}(|\boldsymbol{K} - (M_t/M_f)\cdot\boldsymbol{k}|, \kappa_p, R)\cdot\theta_{l_p\cdot j_p}$$
$$\times R'^{-2}\{|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|^2 + \kappa_d^2\}^{-1}\cdot J_{l_d}(|\boldsymbol{K} + (M_d/M_f)\cdot\boldsymbol{k}|, \kappa_d, R')\cdot\theta_{l_d\cdot j_d}. \tag{2·c}$$

Here, the constant factors C's are given by

$$C_S = 24(\alpha R)\cdot R^2\cdot(k/K), \tag{3·a}$$

$$C_H = 6(R'/R)\cdot R'^2\cdot(k/K), \tag{3·b}$$

$$C_I = -36\sqrt{6}\cdot\pi(\alpha R)^{1/2}\cdot(R'/R)^{1/2}\cdot RR'(k/K), \tag{3·c}$$

and the factors $\mathcal{G}$'s, which are mainly geometrical, are given by

$$\mathcal{G}_S = [(2I_R+1)/(2I_T+1)], \tag{4·a}$$

$$\mathcal{G}_H(l_{n2}, l_{n2}', j_{n2}, j_{n2}', l_d, l_d', j_d, j_d', I_T, I_R, I_C, I_C', 1/2, 1; L, \Omega_k)$$
$$= (2I_R+1)(-)^{I_R+1/2-I_T-1}\sum i^{l_d-l'_d-l_{n2}-l'_{n2}}$$
$$\times[(2l_d+1)(2l_d'+1)(2j_d+1)(2j_d'+1)(2l_{n2}+1)(2l_{n2}'+1)(2j_{n2}+1)(2j_{n2}'+1)]^{1/2}$$
$$\times\sum_L (l_{n2}l_{n2}'00|L0)(l_dl_d'00|L0)\cdot W(l_d, j_d, l_d', j_d'; 1, L)\cdot W(j_d, I_C, j_d', I_C'; I_R, L)$$
$$\times W(l_{n2}, j_{n2}, l_{n2}', j_{n2}'; 1/2, L)\cdot W(j_{n2}, I_C, j_{n2}', I_C'; I_T, L)\cdot P_L(\cos\Omega_k), \tag{4·b}$$

$$\mathcal{G}_I(l_p, j_p, l_{n2}, j_{n2}, l_d, j_d, I_R, I_T, I_C, 1/2, 1; \Omega_{K-(M_t/M_f)k}, \Omega_k)$$
$$= (2I_T+1)^{-1}\cdot\sum i^{l_p+l_{n2}-l_d}(2l_d+1)^{1/2}$$
$$\times\sum(1/2\ 1/2\ \nu_{n2}\ \nu_p|1\nu_d)(l_p\ 1/2\ m_p\ \nu_p|j_p\mu_p)(j_p\ I_T\ \mu_p M_T|I_R M_R)$$

$$\times (l_d 1 0 \nu_d | j_d \nu_d)\,(j_d I_C \nu_d M_C | I_R M_R)\,(l_{n2}\, 1/2\; m_{n2}\nu_{n2} | j_{n2}\mu_{n2})$$

$$\times (j_{n2} I_C \mu_{n2} M_C | I_T M_T) \cdot Y^*_{l_p m_p}(\Omega_{K-(M_i/M_f)\cdot k}) \cdot Y^*_{l_{n2}\cdot m_{n2}}(\Omega_k). \qquad (4\cdot c)$$

Further, O_d denotes the overlapping integral between the wave function of internal motion of deuteron $\phi_d(\boldsymbol{\omega})$ and that of two-nucleon system near nuclear surface $\phi_{np}(\boldsymbol{\omega})$:

$$O_d = \int \phi^*_{np}(\boldsymbol{\omega}) \cdot \phi_d(\boldsymbol{\omega})\, d\boldsymbol{\omega}, \qquad (5)$$

and $J_l(k, \kappa, R)$ has been defined in (I, 4·3a), namely,

$$J_l(k, \kappa, R) \equiv [kR j_{l-1}(kR) + C_l(\kappa R)\, j_l(kR)], \qquad (\text{I}, 4\cdot 3a)$$

$$C_l(\kappa R) = -i\kappa R [h^{(1)}_{l-1}(i\kappa R)/h^{(1)}_l(i\kappa R)]. \qquad (\text{I}, 4\cdot 3b)$$

Other notations have already been defined in I, § 3–1.*,**

There are various triangle relations between angular momenta, which are implicitly contained in the geometrical factors $\mathcal{G}$. Among them, it seems convenient for the discussions of subsequent sections to present the following relations :

$$\boldsymbol{l}_{n2} + (\mathbf{1/2}) = \boldsymbol{j}_{n2}, \quad \boldsymbol{j}_{n2} + \boldsymbol{I}_C = \boldsymbol{I}_T, \qquad (6\cdot a)$$

$$\boldsymbol{l}_d + \mathbf{1} = \boldsymbol{j}_d, \qquad \boldsymbol{j}_d + \boldsymbol{I}_C = \boldsymbol{I}_R, \qquad (6\cdot b)$$

$$\boldsymbol{l}_p + (\mathbf{1/2}) = \boldsymbol{j}_p, \quad \boldsymbol{j}_p + \boldsymbol{I}_T = \boldsymbol{I}_R. \qquad (6\cdot c)$$

On the other hand, as has been discussed in I, the above expression of cross section involves several ambiguities about the magnitude of the cross section such as O_d and θ_d, so we shall use another expression of the cross section (I, 5·4) to check its magnitude.

In analyzing experiments, the reduced width amplitudes appearing in the cross section will all be taken to be the sum rule limit[1] of Teichman and Wigner. Further, the overlapping integral O_d between the internal wave function of deuteron $\phi_d(\boldsymbol{\omega})$ and that of two-nucleon system $\phi_{np}(\boldsymbol{\omega})$ will be assumed to be unity, i.e. $O_d = 1$. Consequently, the cross section calculated in the present paper may correspond to the most preferable case to the heavy-particle-stripping mechanism.

Since (2·a, b, c) does not contain unknown energy-dependent parameters, we can directly calculate the energy-dependence of cross section by using the present formalism. On the contrary, in the previous treatments, the ratio of cross sections of stripping and heavy-particle-stripping has been taken to be an adjust-

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* The expression of cross sections presented here has been represented for (d, n) reaction, while that in I, § 5 was written for (d, p) reaction. Therefore, the subscripts n and p in I have been interchanged in (2·a, b, c).

** Notice the footnote in I, § 5.

able parameter, which has been determined in each energy of incident deuteron so as to fit an experiment at this energy. Consequently, we can examine the reliability of the previous treatments through the energy-dependence of differential cross sections.

It should be noted that quite recently Nagarajan and Banerjee[3] have developed an antisymmetric treatment of (d, p) and (d, n) reactions, which is very similar to ours. They have also analyzed experimental differential cross sections in such a way as to fit the experiments at a fixed energy without imposing the restriction on the value of the reduced widths. Moreover, they have not treated the energy dependence of the cross sections. Their conclusion, therefore, seems to be essentially different from ours.

In the next section, $B^{11}(d, n)C^{12}$ reaction is analyzed, which was first treated by Owen and Madansky, obtainting fairly good agreement with experiment through the method mentioned previously. As can be seen from Figs. 1 and 2, we cannot reproduce the energy-dependence of the cross sections.

Next, $Be^{9}(d, n)B^{10}$ reaction will be treated in § 3, which may be considered to be the most favourable case to the heavy-particle-stripping mechanism due to very small binding energy of the outer neutron in Be^{9}. Though we can obtain rather large possibility of the heavy-particle-stripping reaction at low incident energy of deuteron such as $E_d=1$ Mev, the agreement with experimental energy-dependence is again rather poor at higher energies, contrary to the result by Hasegawa and Ichikawa.[4]

In view of the above situations, therefore, the magnitudes of the adjustable parameter, with which the fairly good agreement with experiment was obtained by previous authors, seems to be rather unreasonable.

In § 4 we shall treat $C^{12}(d, n)N^{13}$ as an example of an unfavourable reaction to the heavy-particle-stripping mechanism, i.e. since the binding energy of neutron in C^{12} is very large, while that of the captured proton in N^{13} is rather small, it seems to have a large cross section of " ordinary " stripping reaction. In fact, it has been shown by our calculation that the cross section of the heavy-particle-stripping reaction is very small compared with that of the stripping reaction.

In summary, the conclusion of the present paper may be expressed as follows: The experimental angular distribution of (d, p) and (d, n) reactions at larger angles in low energy region of incident deuteron, especially its energy-dependence, can hardly be well explained by conventional theory of heavy-particle-stripping reaction, provided the value of adjustable parameter previously adopted be reasonably revised.

It should be noted that in our theory as well as in the conventional one the cutoff Born approximation has been adopted. Our conclusion, therefore, will not immediately convey the meaning of the unimportance of heavy-particle-stripping mechanism; this will be discussed in detail in the final section.

In § 5, some discussions will be added concerning the dependence of the cross sections on the reaction Q-value and on the binding energy of the emitted nucleon in the target nucleus.

§ 2. $B^{11}(d, n)C^{12}$ reaction

The reaction $B^{11}(d, n)C^{12}$ was first analyzed by Owen and Madansky[2] in terms of both stripping and heavy-particle-stripping reactions. In this section, we shall re-examine their result by making use of the method mentioned in the preceding section.

The quantities used in our calculation will be summarized as follows:

Spin of target nucleus B^{11}: $I_T=3/2$ $(-)$,

That of residual nucleus C^{12}: $I_R=0$ $(+)$,

Reaction Q-value $=13.731$ Mev.

Binding energies of neutron, proton and captured deuteron in nuclei,

$B_n=11.464$ Mev $(B^{11}\rightarrow B^{10}+n-B_n\cdot\text{Mev})$,

$B_p=15.958$ Mev $(C^{12}\rightarrow B^{10}+p-B_p\cdot\text{Mev})$,

$B_D=25.195$ Mev $(C^{12}\rightarrow B^{10}+d-B_D\cdot\text{Mev})$

(or $B_D=B_n+B_p-$B.E. of deuteron),

κ_n, κ_p and κ_d are calculated by means of these binding energies.

From the nuclear shell model, it may be reasonable to assign the orbital angular momentum of the captured proton l_p as 1 and that of the emitted neutron l_{n1} as 1:

$$l_p=1, \text{ and } l_{n1}=1.$$

On the other hand, the orbital angular momentum of captured deuteron l_d and the spin of core I_c can be assigned in various way so as to satisfy the conservation law presented in $(6\cdot\text{a,b,c})$.

First, we shall assume $l_d=0$; Consequently, the spin of core is determined uniquely as $I_c=1(+)$. These assignment is the same as those of Owen and Madansky. The other choices of l_d and I_c will be discussed later.

In Fig. 1, the result of Owen and Madansky[2] is presented together with the experimental angular distributions.[2] Fig 2 is the result of our calculation, in which we take the cutoff radii of initial and final states, R and R', to be $R=3.80\times10^{-13}$ cm and $R'=5.00\times10^{-13}$ cm, respectively. The calculations are also performed in terms of other values of the cutoff radii; the results are almost the same as that in Fig 2.

As has been mentioned in § 1, in the analysis of Owen and Madansky, the ratio of probabilities of stripping and heavy-particle-stripping reaction was taken

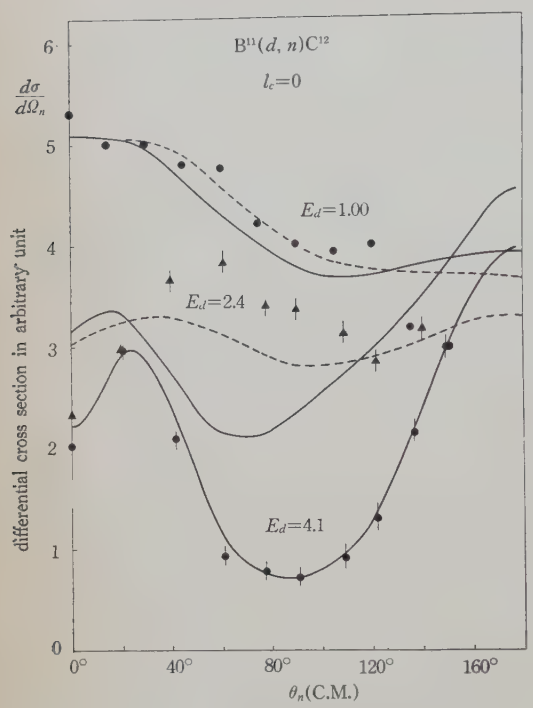


Fig. 1. $B^{11}(d, n)C^{12}$. The solid curves are the result of the analysis by Owen and Madansky at deuteron energies of 1.0, 2.4 and 4.1 Mev, in which they assumed $l_p=1$, $l_d=0$ (in their notation, $l_c=0$) $R=R'=4.5\times10^{-13}$ cm. Dotted curves are that for $R=R'=3.8\times10^{-13}$ cm. Points represent the experimental data of Ames, Owen and Swartz quoted in Owen and Madansky's.

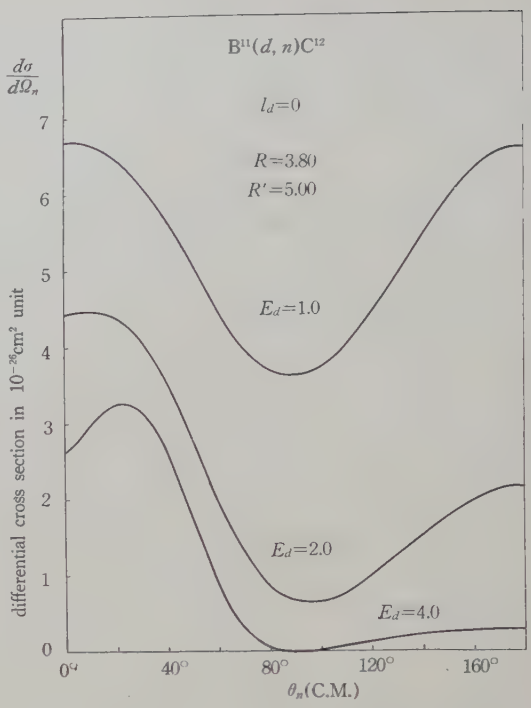


Fig. 2. $B^{11}(d, n)C^{12}$. Our calculated results for E_d=1, 2 and 4 Mev, in which we adopted $l_p=1$, $l_d=0$, $R=3.8\times10^{-13}$cm and $R'=5.0\times10^{-13}$cm.

to be an adjustable parameter (λ_1/λ_2), which was determined so as to fit the experiment at each energy. On the other hand, in our analysis, the cross section is represented in terms of energy-independent parameters, i.e. reduced width amplitudes, the other quantities needed in the calculation being almost the same as those of Owen and Madansky. From Figs. 1 and 2, the value of their parameter (λ_1/λ_2) and, especially, its energy-dependence can be seen to be rather unreasonable.

In the calculations, we have assumed $O_d=1$ and $\theta_d=\theta_n=1$. In order to check the validity of the above assumptions, we have performed the calculation by making use of another expression of cross section (I, 5·4), in which we assume $\theta_n=\theta_p=1$. In this case, several values of core-spin are available. Assuming all c.f.p. appearing in the cross section to be unity,* we have performed the

* Accurate treatment, based on the shell model, of the wave function of B^{11} may be quite involved due to its basic configuration $(1s)^4(1p)^7$.

calculation by taking into account all available combinations of angular momenta. Then, the resulting magnitude of the cross section does not differ from that derived by (4) in order-of-magnitude: the latter is smaller than the former by a factor of about 1/2. The factor 1/2 seems to have little meaning because an ambiguous approximation, i.e. the steepest-descent-method, has been adopted in (I, 5·4).

Next, we shall present in Fig. 3 the calculated cross sections of $B^{11}(d, n)C^{12}$ for the case of $l_d \neq 0$.

Nagarajan and Banerjee[3] have already mentioned qualitatively that it may be sufficient to take into account the smallest l_d which is determined by the conservation law of angular momentum and parity. In the $B^{11}(d, n)C^{12}$ reaction, this is $l_d=0$. In order to assure their argument quantitatively, calculations are performed for the cases of $l_d=1$ and $l_d=2$. In the former case, $l_d=1$, the spin of core I_c should be taken to be $I_c=1(-)$ and $2(-)$, while the neutron should be emitted from the $(1s)$ state because of the conservation of parity. Although κ_d and κ_{n2} must be taken to be larger than those used in the case of $l_d=0$, we adopt the same values as those in $l_d=0$, since the purpose of the calculation is to show the dependence of magnitude of cross section upon the orbital angular momentum of captured deuteron l_d. In the same way, there are three available values of core-spin $I_c=1(+)$, $2(+)$ and $3(+)$ for $l_d=2$. In Fig. 3, the calculated cross scetions are presented for $l_d=0$, $l_d=1$, and $l_d=2$, in which all available I_c are taken into account, but the c.f.p.'s are assumed to be unity. From the figure, it seems to be sufficient to take only the smallest l_d in the calculation of the heavy-particle-stripping reaction. Such circumstances may merely be due to the effect of the penetrability for different l_d's.

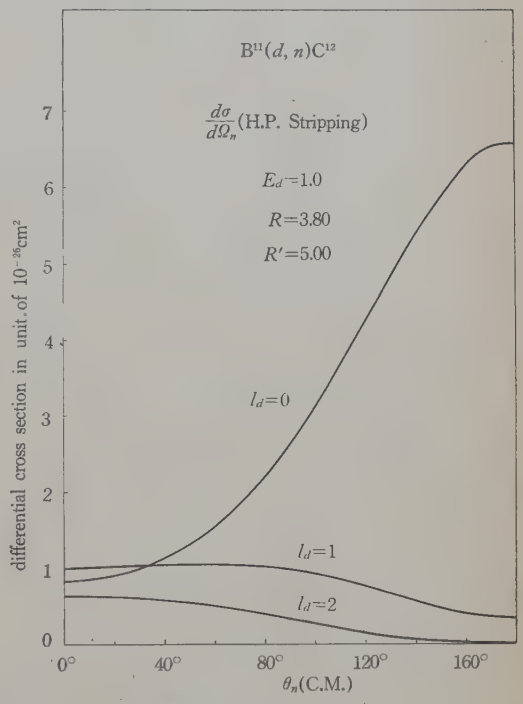


Fig. 3. $B^{11}(d, n)C^{12}$. The calculated cross section at 1 Mev of the heavy-particle-stripping reaction for $l_d=0$, 1 and 2. The figure is presented as an illustration of the dependence of the cross section upon the orbital angular momentum of captured deuteron l_d.

§ 3. $Be^9(d, n)B^{10}$ reaction

Spin of target nucleus Be^9: $I_T=3/2$ $(-)$,
Spin of residual nucleus B^{10}: $I_R=3$ $(+)$,
Reaction Q-value $=4.358$ Mev,
Binding energy of the proton in B^{10}: $B_p=6.585$ Mev,
Binding energy of the neutron in Be^9: $B_n=1.667$ Mev,
Binding energy of captured deuteron in B^{10}: $B_D=6.025$ Mev.

$Be^9(d, n)B^{10}$ reaction may be considered to be the most favourable case to the heavy-particle-stripping mechanism, since the binding energy of outer neutron in Be^9 is exceptionally small.

We can safely take the orbital angular momenta, l_p, l_{n1} and l_d to be $l_p=1$, $l_{n1}=1$ and $l_d=0$. Consequently, I_c is determined as $I_c=2(+)$ and $3(+)$. Further, we adopt $R=4.24\times10^{-13}$ cm and $R'=5.23\times10^{-13}$ cm.

In Fig. 4, the calculated cross sections of stripping and heavy-particle-stripping reactions are presented for the incident energies of 1 Mev, 2 Mev and 4 Mev, the interference term being omitted in the figure for convenience of illustration. When the interference term be added in the figure, the shapes of differential cross sections do not change drastically.

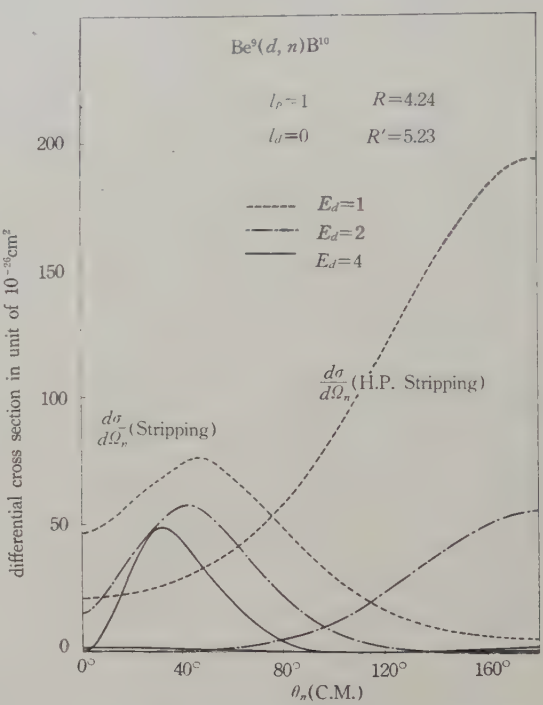


Fig. 4. $Be^9(d, n)B^{10}$. Calculated cross section of the stripping and the heavy-particle-stripping reactions at 1, 2 and 4 Mev. The parameters adopted are $l_p=1$, $l_d=0$, $R=4.24\times10^{-13}$ cm and $R'=5.23\times10^{-13}$ cm.

Then the heavy-particle-stripping mechanism may play an important role in this reaction at low incident energy of deuteron such as $E_d=1$ Mev.*

* It should be noticed that the spin of ground state of Be^8 is $0(+)$. Therefore, if accurate wave function of Be^9 be decomposed into a neutron+core, its main configuration may correspond to the core with $0(+)$, others being probably smaller. In order to take account of the core $0(+)$, the larger l_d will be needed, that leads to small cross section of heavy-particle-stripping reaction. Strictly speaking, therefore, the calculated cross section presented in Fig. 4 may not correspond to actual situations of $Be^9(d, n)B^{10}$. In the experimental angular distributions[5] at low incident energy of deuteron, the cross section at backward direction seems to be not so much pronounced as expected from Fig. 4. Careful analysis will be presented elsewhere.

It should be noted that $Be^9(d, n)B^{10}$ reaction has already been analyzed by Hasegawa and Ichikawa.[4] They have improved the original theory of Owen and Madansky in such a way as to factorize the matrix elements of the stripping and the heavy-particle-stripping reactions, respectively, into two terms: energy-dependent dynamical term and nuclear matrix element which is independent of the incident energy. These nuclear matrix elements would be represented in terms of reduced widths, if they could perform the derivation. Their result of analysis seems to be very similar to ours at low incident energy of deuteron up to about $E_d=1$ Mev. At higher energies, however, their results do not coincide with ours.

§ 4. $C^{12}(d, n)N^{13}$ reaction

Spin of target nucleus C^{12}: $I_T=0$ (+),
Spin of residual nucleus N^{13}: $I_R=1/2$ (−),
Reaction Q-value: $Q=-0.286$ Mev,
Binding energy of proton in N^{13}: $B_p=1.941$ Mev,
Binding energy of neutron in C^{12}: $B_n=18.722$ Mev,
Binding energy of captured deuteron: $B_D=18.436$ Mev,

The reaction $C^{12}(d, n)N^{13}$ is represented as an example of the opposite case of $Be^9(d, n)B^{10}$ reaction, since the relation between the binding energies B_n and B_p is just opposite to that of $Be^9(d, n)B$.[10] Then, the cross section of the stripping reaction may be expected to be large due to the small B_p, while that of the heavy-particle-stripping reaction is likely to be very small.[6] Such circumstances may be clearly seen from our calculations in Fig. 5, in which we have assumed $l_p=l_n=1$, $l_d=0$, $R=4.49\times10^{-13}$ cm and $R'=5.00\times10^{-13}$ cm.

Experiment of this reaction has been performed by Elwyn, Kane, Ofer and Wilkinson[6] in the range of deuteron energies from 1.45 to 2.95 Mev. They have analyzed their data by making use of the simple stripping theory of

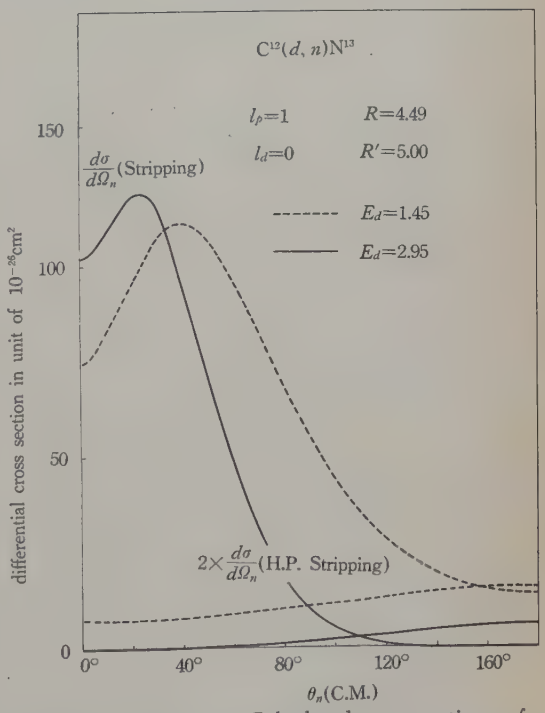


Fig. 5. $C^{12}(d, n)N^{13}$. Calculated cross sections of the stripping and the heavy-particle-stripping reaction at 1.45 and 2.95 Mev. The parameters adopted are $l_p=1$, $l_d=0$, $R=4.49\times10^{-13}$ cm and $R'=5.00\times10^{-13}$ cm.

Butler, obtaining the result that, even at low incident energy of deuteron such as $E_d=2$ Mev, the differential cross sections at angles smaller than 90° can be fitted by the Butler theory and that there appear little evidences of the heavy-particle-stripping reaction in the angular distributions of backward directions, though some rises have been observed at larger angles.

§ 5. Dependence of cross sections on the binding energies

In this section, we shall briefly discuss the dependence of the cross sections on the binding energy B_p of the captured proton in a residual nucleus and on the binding energy B_n of the emitted neutron in a target nucleus. In all reactions treated in the previous sections, we have adopted the same set of the orbital angular momenta l_p and l_d, i.e. $l_p=1$ and $l_d=0$. Further, the cutoff radii R and R' have been taken to be similar to each other for these reactions. Therefore, the dependence of cross sections on B_n and B_p can be inferred directly from Figs. 2, 4 and 5, since the calculated cross sections have all been written in the absolute scale in the figures.

Then, the magnitude of the cross section of the heavy-particle-stripping reaction depends rather sensitively on the binding energy B_n of the emitted neutron in a target, while that of the stripping reaction is rather sensitive on the binding energy B_p of the captured proton in a residual nucleus. Consequently, the heavy-particle-stripping reaction will not play an important role when $B_n \gg B_p$. In this connection, it may be expected that the heavy-particle-stripping mechanism will become more unimportant, if the residual nucleus be left to the higher excited states.

§ 6. Discussions

As has been mentioned in § 1, in the conventional theory of the heavy-particle-stripping reaction as well as in our formalism, the cutoff Born approximation has been adopted.

As energies of incident deuterons become lower, the cutoff Born approximation becomes, as is well known, rather unreliable even in treating " ordinary " stripping reactions. By making use of the " distorted wave approximation ", one has been able to explain, at least, the average behavior of angular distribution of the stripping reaction. Actually, when this approximation be applied to the calculation of the stripping cross section, the differential cross section at smaller angles is appreciably reduced usually, the forward peaks being displaced toward larger angles, in comparison with the result of the cutoff Born approximation. In fact, if the distorting potential be taken to be of a Coulomb type, the angular distribution can be calculated analytically[7] for the case of $l_n=0$ in the limit of zero energy of incident deuteron, and the result shows that the angular distribution has a maximum at 180°, monotonously decreasing to mini-

mum value of zero at 0°, while it shows the well-known Butler curve in the cutoff Born approximation. Consequently, it seems to be quite likely that, when the distorted wave approximation be applied to our problem, the angular distribution at smaller angles due to the stripping mechanism will be strongly reduced, whereas its backward parts arising from the heavy-particle-stripping mechanism may not so much be reduced as in the forward parts. Such circumstances as mentioned above are evidently a preferable tendency for the explanation of the angular distributions in the energy range dealt with in the present paper.

In view of the above situations, therefore, calculations by means of the distorted wave approximation seem to be definitely necessary,* by which we might explain the angular distribution over the whole angles and its energy-dependence by the stripping and the heavy-particle-stripping mechanisms without introducing other more complicated mechanisms. Then, we are allowed to hope that more informations of nuclear structure of lighter nuclei will be accumulated through the determination of the reduced widths and, further, of the behaviors of two-body correlations at nuclear surface through O_d.

In applying the distorted wave approximation, we should pay attention to the cutoff assumption of the radial coordinate in the overlapping integrals: the cutoff $R \leq r < \infty$ may be safely adopted for the integral over r and r_1, i.e. (I, 3·9), due to rather short mean free path of deuteron in nuclear matter, while in the integral with respect to the radial coordinate of emitted nucleon r_2 the cutoff may not be available due to long mean free path of nucleon in nuclear matter, that has already been discussed in I, § 2–1. Just the same integrals of the latter type always appear in the theory of direct excitation of single particle mode in the case of the inelastic scattering of nucleon by nuclei.[8] In fact, in this case, Levinson and Banerjee[9] have performed most elavorated calculation by making use of the distorted wave approximation without cutoff, obtaining fairly good agreement with experiment.

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* The formal generalization of our formalism to include the distorted wave approximation has already been discussed in I, § 7.

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$KK\pi\pi$-Interaction and Nucleon Core

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The contribution from K meson cloud to s-wave pion-nucleon scattering is estimated under the assumption that there exists strong $KK\pi\pi$-interaction. Some discussion about the effect of this interaction on high energy phenomena is made.

§ 1. Introduction

It may be expected that some important problems such as the so-called divergence difficulty inherent in the current field theory will be able to be solved by the realization of the correct dynamics in the region of nucleon core. With the object of pursuing some knowledge about the dynamics we have previously examined both the problems of s-wave pion-nucleon scattering[1] and those of s-wave K meson-nucleon scattering.[2] And we have obtained some different kind of knowledge about nucleon core by the study of these processes, because in the former process the nucleon core does not undergo any essential change in the course of collision in the meaning of perturbation, while in the latter process it undergoes some remarkable change in the course of collision. Particularly, the following character of pion-nucleon interaction has been pointed out. If the practical effect brought by the application of the correct dynamics to the phenomena in the region of nucleon core is expressed in terms of the change of the value of coupling constant, the value of coupling constant averaged over the spread of nucleon core should be reduced in appearance to $(\mu/2m)g=f$ in spite of the fact that its value in the neighborhood of pion cloud is g. (μ and m are the masses of pion and nucleon respectively, and g is the renormalized (unrationalized) pseudoscalar coupling constant.)

But, so far as the description for pion-nucleon scattering is concerned, the K meson cloud may be regarded as a part of nucleon core. We think it worthwhile to study the contribution from this K meson cloud to pion-nucleon scattering in order to discuss the interactions in nucleon core. As our first approach to the interactions we should like to examine how the phenomena in nucleon core are described by a possible model within the framework of current field theory. Recently K meson-nucleon scattering has been investigated by taking into account the effect of $KK\pi\pi$-interaction.[3] This type of interaction may be regarded as one of the suitable forms to consider the property of pion-nucleon interaction

within the range shorter than $1/2m_K \sim 1/m$, because pions interact with nucleon through the exchange of two K mesons. (Where m_K is the mass of K meson.) The purpose of this paper is to examine how the K meson cloud has an effect on the phenomena in the region of nucleon core, if this $KK\pi\pi$-interaction plays the most important role. In § 2 we try to estimate the contribution from the $KK\pi\pi$-interaction to the amplitude of s-wave pion-nucleon scattering by means of perturbation theory. In § 3 some discussions about the effect of this interaction on high energy phenomena are made. In § 4 a method to examine the existence of strong $KK\pi\pi$-interaction is suggested.

§ 2. $KK\pi\pi$-interaction and s-wave pion-nucleon scattering

When there is some kind of direct $KK\pi\pi$-interaction, Fig. 1* shows the diagram corresponding to the lowest order perturbation. In the case where $\pi^{\pm}$-p scatterings take place through such a process as Fig. 1, the interaction range will be of the order of $1/(2m_K) \simeq 1/m$ because of the exchange of two K mesons. Therefore such a process as this may be regarded as one of the phenomena in the region of nucleon core. In our previous paper[1] we have mentioned that the effects due to the internal structure of nucleon core are almost cancelled out in $D^{(2)}(\mu)=1/2[D_-(\mu)-D_+(\mu)]$, while the effects play an important role in $D^{(1)}(\mu)=1/2[D_-(\mu)+D_+(\mu)]$, where $D_-(\mu)$ and $D_+(\mu)$ are the real parts of the forward scattering amplitudes for π^--p and π^+-p scattering respectively at the energy $\omega=\mu$. Of course we now pay attention to the problem of $D^{(1)}(\mu)$ because of our approach to the interaction in nucleon core.

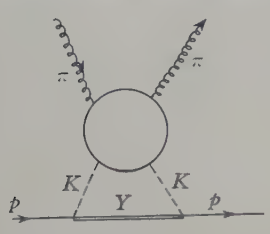


Fig. 1.

As the possible types of $KK\pi\pi$-interaction, there are the following three kinds of interactions as was mentioned by Igi.[3]

$$H_1=\lambda_1 K_i^* K_i \pi_j \pi_j, \tag{1}**$$

$$H_2=\lambda_2 i(K^* \tau_\alpha \partial_\mu K - \partial_\mu K^* \tau_\alpha K)(\pi \times \partial_\mu \pi)_\alpha, \tag{2}$$

$$H_3=\lambda_3 (K_i^* \partial_\mu K_i)(\pi_j \partial_\mu \pi_j). \tag{3}$$

But we here adopt the H_1-type interaction among them on account of the following reasons.

(i) There is no divergent quantity in the calculation for pion-nucleon scattering when this type of interaction is used (cf. Fig. 1).***

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* The notation of Y in Fig. 1 indicates Λ or Σ-particle.

** With regard to the notation, see reference 3).

*** When H_2- or H_3-type interaction is adopted we are obliged to use the cutoff theory. The cutoff theory is not so useful in describing the phenomena in nucleon core. This is the main reason why H_1-type interaction is adopted here.

(ii) The effects of this $KK\pi\pi$-interaction on $D_-(\mu)$ are equal to those on $D_+(\mu)$ if the scatterings take place through the process shown in Fig. 1. In other words, only $D^{(1)}(\mu)$ is affected by the introduction of this H_1-type interaction. Since $D^{(2)}(\mu)$ can be explained by the current pion field theory, this $KK\pi\pi$-interaction may be one of the suitable types in order to examine the phenomena in the region of nucleon core.

As the K meson-hyperon-nucleon interactions, the usual interaction Hamiltonians are used, that is,

$$H_{N\Lambda K}=g_\Lambda(\bar{\psi}_\Lambda O\psi_n\phi_{K^0}{}^*+\bar{\psi}_\Lambda O\psi_p\phi_{K^+}{}^*)+\text{h.c.}, \tag{4}$$

$$H_{N\Sigma K}=g_\Sigma(\sqrt{2}\,\bar{\psi}_{\Sigma^+}O\psi_p\phi_{K^0}{}^*+\sqrt{2}\,\bar{\psi}_{\Sigma^-}O\psi_n\phi_{K^+}{}^*+\bar{\psi}_{\Sigma^0}O\psi_p\phi_{K^+}{}^*-\bar{\psi}_{\Sigma^0}O\psi_n\phi_{K^0}{}^*)+\text{h.c.}, \tag{5}$$

$$O=\begin{cases} i\gamma_5, \\ 1, \end{cases} \tag{6}$$

where $\phi_K^*{}^+$ corresponds to emission of K^+meson or absorption of K^-meson, and similarly for $\phi_{K^0}^*$. The subscripts (Λ, Σ, p and n) in ψ are used in order to discriminate the baryons. g_Λ and g_Σ are the rationalized $KN\Lambda$ and $KN\Sigma$ coupling constants respectively.

The straightforward calculation gives the following results.

(1) When both $KN\Lambda$ and $KN\Sigma$ interactions are of pseudoscalar type ($O=i\gamma_5$ in Eq. (6)),

$$D_+(\mu)_{K\pi}=D_-(\mu)_{K\pi}=\frac{1}{m+\mu}\frac{\lambda_1}{4\pi}\left[\frac{g_\Lambda{}^2}{4\pi}\frac{A_\Lambda}{2\pi}+\frac{g_\Sigma{}^2}{4\pi}\frac{3A_\Sigma}{2\pi}\right], \tag{7}$$

where the suffix $(K\pi)$ in $D_\pm(\mu)$ stands for the contribution from the process via the $KK\pi\pi$-interaction to scattering amplitudes. A_Λ and A_Σ are obtained by replacing hyperon mass M in the following expression for A by M_Λ and M_Σ respectively (M_Λ and M_Σ are the masses of Λ and Σ respectively).

$$\frac{A}{m}=\int_0^1 dx\int_0^x dy\,\frac{(M-m)+m(x-y)}{M^2(x-y)-m^2\{(x-y)-(x-y)^2\}+m_K{}^2(1-x+y)} \tag{8}$$

$$=\frac{1}{m}\left[-1+\frac{1}{2m^2}(M^2+m^2-Mm-m_K{}^2)\log(M/m_K)^2\right.$$
$$+\frac{1}{2m^4C}\left\{-(M^4+m^4+m_K{}^4)+2m_K{}^2(M^2+m^2)+Mm(M^2+m^2-m_K{}^2)\right\}$$
$$\left.\times\left\{\tan^{-1}\frac{(M^2+m^2-m_K{}^2)}{2m^2C}-\tan^{-1}\frac{(M^2-m^2-m_K{}^2)}{2m^2C}\right\}\right], \tag{9}$$

where

$$C=\frac{\sqrt{\{M^2-(m-m_K)^2\}\{(m+m_K)^2-M^2\}}}{2m^2}. \tag{10}$$

(2) When both $KN\Lambda$ and $KN\Sigma$ interactions are of scalar type ($O=1$ in Eq. (6)),

$$D_+(\mu)_{K\pi}=D_-(\mu)_{K\pi}=\frac{1}{m+\mu}\frac{\lambda_1}{4\pi}\left[\frac{g_\Lambda^2}{4\pi}\frac{B_\Lambda}{2\pi}+\frac{g_\Sigma^2}{4\pi}\frac{3B_\Sigma}{2\pi}\right], \tag{11}$$

where B_Λ and B_Σ are obtained by replacing M in the following expression for B by M_Λ and M_Σ respectively.

$$\frac{B}{m}=\int_0^1 dx\int_0^x dy\,\frac{-(M+m)+m(x-y)}{M^2(x-y)-m^2\{(x-y)-(x-y)^2\}+m_K^2(1-x+y)}. \tag{12}$$

In order to get the explicit form of B, we have only to replace M in the expression of Eq. (9) by $-M$.

Putting the values of $M_\Lambda=1115$ Mev, $M_\Sigma=1190$ Mev, $m=938$ Mev and $m_K=495$ Mev into Eqs. (9) and (12), we obtain

$$A_\Lambda=0.529,\ A_\Sigma=0.559,\ B_\Lambda=-2.325,\ B_\Sigma=-2.207. \tag{13}$$

At the present stage it will be difficult to estimate correctly the values of $D_\pm(\mu)_{K\pi}$ because we have not yet the reliable values of coupling constants $\lambda_1/4\pi$, $g_\Lambda^2/4\pi$ and $g_\Sigma^2/4\pi$. With the object of obtaining some knowledge about the order of magnitude of $D_\pm(\mu)_{K\pi}$, however, let us tentatively adopt the values of

$$\lambda_1/4\pi\cong-0.1,\ g_\Lambda^2/4\pi\cong 2.5,\ g_\Sigma^2/4\pi\cong 4.5 \tag{14}$$

which have been given by Igi[3] in the case of pseudoscalar coupling. Then we get the following result from Eq. (9),

$$D_-(\mu)_{K\pi}=D_+(\mu)_{K\pi}=D^{(1)}(\mu)_{K\pi}=-0.123/m. \tag{15}$$

This value of $|D^{(1)}(\mu)_{K\pi}|$ is much smaller than that of $|D^{(1)}(\mu)_p|$ which was obtained by the usual perturbation theory,

$$D^{(1)}(\mu)_p=-g^2/m\cong-14.5/m. \tag{16}*$$

If $D^{(1)}(\mu)_{K\pi}$ is regarded as the correction for $D^{(1)}(\mu)_p$ in Eq. (16), the magnitude of this correction will be so small that it may be neglected. But it must be noted that the value of Eq. (15) is nearly equal to

$$\begin{aligned} D^{(1)}(\mu)&=-f^2/m \ \text{ or } \ -2f^2/m \\ &=-0.08/m \ \text{ or } \ -0.16/m \end{aligned} \tag{17}**$$

which have been obtained previously.[1] Since the values*** of Eq. (14) may be

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* Note that g in Eq. (16) and f in Eq. (17) are the unrationalized coupling constants.

** S-wave phase shifts for pion-nucleon scattering could be explained satisfactorily by using the results.

*** The values of coupling constants in nucleon core may differ from those in the neighborhood of pion cloud or K meson cloud. With regard to this problem some discussion will be made in another place.

considerably modified in future, our result mentioned in Eq. (15) should not be interpreted seriously. Moreover we must pay attention to the fact that the sign of $D^{(1)}(\mu)_{K\pi}$ depends on that of λ_1.

§ 3. The effects of $KK\pi\pi$-interaction on high energy phenomena

It may be expected that the pion cloud and K meson cloud spread over the region of $\sim 1/\mu$ and $\sim 1/m_K$ respectively. Therefore pion and K meson will be produced within the range of $1/\mu$ and $1/m_K$ respectively in the course of collision. If there exists some kind of strong $KK\pi\pi$-interaction, we may expect the large probability with which pions are produced through the mechanism via $KK\pi\pi$-interaction even the case where $K\bar{K}$-pairs are considerably emitted within the range of $1/m_K$. The pions produced through this process will undergo both π-π interaction and π-N interaction until they will be observed.* In such cases it may be difficult to observe directly the $K\bar{K}$-pairs which are produced in nucleon core.

The experimental results for proton-antiproton annihilation in the low energy region have shown that the observed number of produced K meson is much smaller than the theoretical one predicted by the statistical theory. According to the Koba-Takeda model,[4] the number of $K\bar{K}$-pairs produced in nucleon core is nearly equal to that of the observed ones.[5),6)] Although some investigation about this problem has been made by Frautschi,[7] we want to point out the following possibility. Even if $K\bar{K}$-pairs are produced in the region of nucleon core with considerably large probability, the number of $K\bar{K}$-pairs may turn out to be small owing to the strong $KK\pi\pi$-interaction.

Moreover there is an important problem to be discussed, that is, the energy dependence of the $KK\pi\pi$-interaction. Let us consider some K meson phenomena at very high energy. When a considerable part of the available energy is supplied to $K\bar{K}$-pairs production, we may suppose the following two possible cases with respect to the *unobserved* K and $\bar{K}$ mesons which will hereafter undergo various kinds of interactions until they will be observed.

(i) Mean energy of the K and $\bar{K}$ mesons is low, that is, the multiplicity of $K\bar{K}$-pairs is large in the region of nucleon core.

(ii) Mean energy of them is high, that is, the multiplicity of $K\bar{K}$-pairs is small in the region of nucleon core.**

In the case (i), we may expect the similar situation with that discussed in

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* Although there may be the inverse process through which two pions are absorbed and $K\bar{K}$-pair is produced within the range of $1/m_K$, this pair will mostly be changed again into two pions through the $KK\pi\pi$-interaction in the region between $1/m_K$ and $1/\mu$ until the $K\bar{K}$-pair will be observed. It must be noted that under our assumption $K\bar{K}$-pair may not be produced with large probability in the region between $1/2m_K$ and $1/\mu$.

** If there exists some strong K-K interaction, for example, $\phi_K{}^4(x)$-type interaction, we shall not be able to expect such a case as this.

the problem of proton-antiproton annihilation. In the case (ii), the observed value for the number of $K\bar{K}$-pairs may be affected by energy dependence of the $KK\pi\pi$-interaction. If the magnitude of the cross section for pion production due to the $KK\pi\pi$-interaction turns out to be small with increase of energies of the K and $\bar{K}$ mesons,* it may be possible to observe directly the $K\bar{K}$-pairs of high energy which are produced in the region of nucleon core. Thus we may say as follows. If the $K\bar{K}$-pairs of high energy are observed in experiment with large probability, the experimental results may be consistent with the above model and may be used to determine what kind of $KK\pi\pi$-interaction among the three types mentioned in § 2 is appropriate in describing high energy phenomena.

§ 4. On the existence of strong $KK\pi\pi$-interaction

In order to examine whether there is some kind of strong $KK\pi\pi$-interaction or not, we want to suggest a method, that is, the measurement of transverse momentum $p_\perp$ of nucleon in the reaction $K+N\to K+N+\pi$, for example $K^-+p\to\bar{K}^0+p+\pi^-$ or $K^++p\to K^0+p+\pi^+$. If the $K+N\to K+N+\pi$ reaction is induced mainly through the $KK\pi\pi$-interaction as is shown in Fig. 2a,

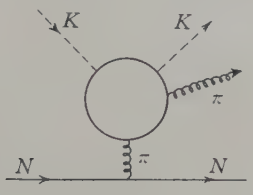


Fig. 2a.

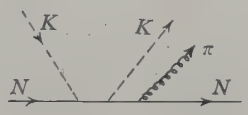


Fig. 2b.

the collision in periphery of the nucleon will play the most important role. Such a model as this may predict that during a collision the nucleon does not change its direction remarkably and that the mean value of $p_\perp$ will be small. On the contrary, if the $K+N\to K+N+\pi$ reaction is induced mainly by Yukawa-type interactions (for example, the process mentioned in Fig. 2b), the situation is very different from that in periphery collision. In this case there are odd number of γ_5-type interaction** independently of the type of K meson-hyperon-nucleon interaction. This model may predict that the mean value of $p_\perp$ will be considerably large. Thus we may conclude as follows. If the experimental data for this reaction show the small mean value of $p_\perp$, then it may be said that the results are consistent with the existence of strong $KK\pi\pi$-interaction.***

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* Whether such a situation as this may be realized or not will be examined with respect to some high energy phenomena, for instance, π-N collision. Another note will be given on this point.

** Only the pseudoscalar-type of pion-nucleon interaction is now taken into consideration, because the pseudovector-type can be derived from the pseudoscalar-type by the well-known transformation.

*** The situation in this reaction will be similar to that in π-2π process.[8]

The author would like to express his thanks to Prof. M. Koshiba for his helpful comments.

After completion of this paper Igi kindly informed me that the following values of coupling constants had been derived from his latest study,

$$\lambda_1/4\pi \simeq -0.5, \quad \lambda_2/4\pi \simeq -0.15.$$

It will now be natural for us to regard the value of coupling constant as an open question.

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On the Stability of the Hartree-Fock Solution in Many-Body Problem

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A necessary condition for the stability of the Hartree-Fock solution in many-body problem is presented. If this condition is not satisfied, then the solution becomes no longer stable as was discussed by Overhauser in connection with the one-dimensional spin wave model. A variational method is proposed in this case to construct a stable solution which has definitely a lower energy than the Hartree-Fock value. The workability of our method is tested in some realistic examples like the B.C.S.-Bogolyubov theory of superconductivity. Then the method is applied to field theory which seems to be inconsistent in view of the presence of "ghost states". A canonical transformation leads to a new vacuum state with lower energy, but the high momentum part of the coupling is not damped.

§ 1. Introduction

Recently, a serious question has been raised concerning the stability of the Hartree-Fock solution by Overhauser[1] who treated the one-dimensional system of interacting fermions and found a new solution which gives a lower energy than the Hartree-Fock value. In such a system, the free ground state is not connected with the true one adiabatically, but rather with some kind of excited state. The B.C.S. theory of superconductivity[2] is another example where the ground state is far from the Fermi distribution.

Now in the Overhauser model, one can show that the propagator for a pair of particle and hole with spin either parallel or antiparallel according to the sign of force has an imaginary pole, the corresponding excitation being completely unstable. In the B.C.S. theory, the propagator for a pair of particle or of hole with opposite momenta and spins has also an imaginary pole at the momentum close to the Fermi value.[3] One may expect that this difficulty is intimately connected with the instability of the Fermi distribution, that is, the Hartree-Fock solution against the deformation in accordance with such an excitation.

Following this observation, one will introduce in § 2 an approximate collective operator by applying the method of normal mode,[4] calculate the variational energy with respect to the deformed state corresponding to thier collective excitation, and then show that the second derivative of this energy at the Hartree-Fock state is actually positive if the eigenfrequency is real, but is zero if this is imaginary. In the former case the Hartree-Fock state is stable within the family of variations employed and in the latter case it is unstable.

Then a family of variational functions are suggested which would make the energy lower than the Hartree-Fock value. In order to assure the absolute stability, one will in principle have to look for a complete set of these excited modes, but from physical grounds lower excited states will usually play an important role.

In the following three sections, the criterion of instability is applied to the three-dimensional model of Overhauser, to the B.C.S. theory of superconductivity, and to the Bogolyubov Hamiltonian of liquid He^4 with negative scattering length.[5] It is then shown that our new variational solution will actually turn out to be the lower energy state. The new excited eigenmodes, phonon spectrum, for instance, based on this ground state now have real frequencies.

In § 6, the method is applied to field theory by taking the meson-nucleon system as an illustration. The meson spectrum now includes an imaginary frequency which is made use of to derive a new ground state with lower energy. The canonical transformation leading to this state has two effects. One is to change the effective coupling between the meson and nucleon which becomes weaker than the original one. The high frequency part of the coupling is however not altered. The other effect is that the eigenfrequencies of the meson are now all real.

§ 2. Stability condition for the Hartree-Fock solution of a Fermi gas

As an orientation of our basic idea, let us first consider a Fermi gas in which particles are interacting through ordinary two-body forces. The Hamiltonian of our system is given in the second quantized form as

$$H=\Sigma_p C_p^* C_p \frac{p^2}{2m}+\frac{1}{2}\Sigma_{p,p',q} v(q) C_p^* C_{p'}^* C_{p'+q} C_{p-q}, \qquad (2\cdot1)$$

where $\boldsymbol{p}$'s are the momentum and spin of a particle, C_p's and C_p^*'s are annihilation and creation operators of a particle, respectively, and $v(q)$ the Fourier transform of the potential. In the hole theory formulation in which the Fermi distribution is taken as a vacuum, C_p's are replaced by

$$C_p=\begin{cases} a_p, & \text{for } |\boldsymbol{p}|>p_F, \\ b_p^*, & \text{for } |\boldsymbol{p}|<p_F, \end{cases} \qquad (2\cdot2)$$

where a_p and b_p are the annihilation operators of a "particle" and a "hole", respectively, and p_F the Fermi momentum.

It is one of the most important objects in the recent many-body problem to find some kind of approximate normal modes in the interacting system which are usually called collective oscillations. The condition for the normal mode is given by the presence of operators A_c and A_c^* which satisfy an equation

$$[A_c^*,\ H]_- = -\omega_c A_c^*, \tag{2·3}$$

where ω_c is the frequency of the collective oscillation. These A_c's are a function of original operators a's and b's, and obey the canonical commutation relation, plus or minus according to the Fermion and Boson type collective motion. One of the typical examples is the plasma oscillation at high density discussed by us and others.[4] In some cases, however, the collective frequency ω_c becomes imaginary as will be shown in the following sections, which turns out to be connected with the instability of the Hartree-Fock solution.

A theorem concerning this instability can be formulated as follows. Suppose there is an operator S which is an n-th order ordered polynormial of a's and b's and satisfies

$$[S^*,\ H]_- = -\omega S^* + C \tag{2·4}$$

where C is composed of products of more than n ordered operators.* ω may be real or imaginary. Denoting the free vacuum, that is, the Fermi distribution by Φ_0 which is known to be the Hartree-Fock solution of our system, we will consider a family of variational functions defined by

$$\Psi(\lambda) = e^{i\lambda(S^*+S)}\Phi_0,\ \Psi(0) = \Phi_0. \tag{2·5}$$

Then the variational energy is given by

$$E(\lambda) = \langle \Psi(\lambda),\ H\Psi(\lambda)\rangle = \langle \Phi_0,\ e^{-i\lambda(S^*+S)} H e^{i\lambda(S^*+S)}\Phi_0\rangle, \tag{2·6}$$

which is of course equal to the Hartree-Fock energy for $\lambda=0$. If one takes the first derivative of Eq. (2·5) with respect to λ and makes λ tend to zero, one has

$$\frac{\partial}{\partial\lambda}E(\lambda)\Big|_{\lambda=0} = (-i)\langle \Phi_0, [S^*+S,\ H]_-\Phi_0\rangle \tag{2·7}$$

which is zero on account of Eq. (2·4). This is the ordinary statement that the Hartree-Fock solution is the variational one. If one takes a_p or b_p itself as S, one will have the Hartree-Fock single particle energy as ω.

In order to see whether the solution is really stable, one has to look into the sign of the second derivative of $E(\lambda)$ at $\lambda=0$ which is given by

$$\begin{aligned} -\frac{\partial^2}{\partial\lambda^2}E(\lambda)\Big|_{\lambda=0} &= (-i)^2\langle \Phi_0, [S^*+S[S^*+S,\ H]_-]_-\Phi_0\rangle \\ &= (\omega+\omega^*)\langle \Phi_0, [S,\ S^*]_-\Phi_0\rangle, \end{aligned} \tag{2·8}$$

since the commutators of S's and C's have the zero expectation value. In the case where all ω's are real and positive,[6] one can show that the expectation value of $[S,\ S^*]_-$ is positive and one has

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* S and C are assumed to have no constant terms.

$$\frac{\partial^2}{\partial\lambda^2}E(\lambda)\Big|_{\lambda=0}>0, \quad \text{for } \omega>0. \tag{2·9}$$

The Hartree-Fock solution satisfies, therefore, the minimum condition within the family of variations employed here. However, if some of ω's are imaginary ($\omega=i\Gamma$), then one has

$$\frac{\partial^2}{\partial\lambda^2}E(\lambda)\Big|_{\lambda=0}=0, \quad \text{for } \omega=i\Gamma, \tag{2·10}$$

and the minimum condition is usually violated.* It will be shown in the following sections that there exists in this case another ground state solution with lower energy which is to be obtained by the variational principle by making use of trial functions similar to Eq. (2·5):

$$\Psi(\alpha)=e^{i(S^*(\alpha)+S(\alpha))}\Phi_0, \tag{2·11}$$

where the function $S(\alpha)$ has the same structure as S with respect to operators, a's and b's, but the coefficients contain some variational parameters.

§ 3. Spin wave model of Overhauser—*Three-dimensional case*—

As an application of our stability condition, let us consider in this section a model discussed by Overhauser[1] where the potential is assumed to be of the δ-function. For the sake of convenience, one will separate the spin index and employ the two-component operator defined by

$$\boldsymbol{C}_p=\begin{pmatrix}C_{p\uparrow}\\ C_{p\downarrow}\end{pmatrix}. \tag{3·1}$$

Then the total Hamiltonian is written as

$$H=\Sigma_p \boldsymbol{C}_p^* \boldsymbol{C}_p E_p+\frac{\gamma}{2}\Sigma_{pp'q}\boldsymbol{C}_p^*(\boldsymbol{C}_{p'}^*\boldsymbol{C}_{p'+q})\boldsymbol{C}_{p-q},$$

$$E_p=p^2/2m, \tag{3·2}$$

where γ is the strength of the potential and is positive for repulsion and negative for attraction.

Now one takes as the operator S^* a form suggested by the study of collective oscillations in a Fermi gas,[4] in particular the plasma oscillation, and puts

$$S^*=\Sigma_p(\Phi^\sigma_{p;q}\boldsymbol{a}^*_{p+q}\sigma\boldsymbol{b}_p^*+\Theta^\sigma_{p;q}\boldsymbol{b}_{-p}\sigma\boldsymbol{a}_{-p-q}+\chi^\sigma_{p;q}\boldsymbol{a}^*_{p+q}\sigma\boldsymbol{a}_p+\Xi^\sigma_{p;q}\boldsymbol{b}_{-p}\sigma\boldsymbol{b}^*_{-p-q}), \tag{3·3}$$

where σ is 1 or any of the Pauli matrices, σ_x, σ_y and σ_z, and q is an arbitrary

* We will put the special case out of consideration in which $E^{III}(0)=0$ and $E^{IV}(0)>0$.

momentum. The meaning of $\boldsymbol{a}$ and $\boldsymbol{b}$ is clear from Eqs. (2·2) and (3·1). The coefficients Φ's are to be determined from Eq. (2·4), where the operator C now consists of products of more than three ordered operators. Putting

$$\omega_{p;q}=E_{p+q}-E_p, \tag{3·4}$$

and taking $\sigma=1$ one has the following coupled integral equations for Φ's;

$$\begin{aligned}
&(\omega-\omega_{p;q})\Phi^1_{p;q}=\gamma\Sigma_{p'}(\Phi^1_{p';q}-\Theta^1_{p';q}),\\
&(\omega+\omega_{p;q})\Theta^1_{p;q}=\ \text{,,}\quad\text{,,}\quad\text{,,}\quad\text{,}\\
&(\omega-\omega_{p;q})\chi^1_{p;q}=\ \text{,,}\quad\text{,,}\quad\text{,,}\quad\text{,}\\
&(\omega-\omega_{p;q})\Xi^1_{p;q}=\ \text{,,}\quad\text{,,}\quad\text{,,}\quad\text{,}
\end{aligned} \tag{3·5}$$

from which the secular equation for ω is obtained as

$$1-\gamma\sum_{\substack{p\\|p+q|>p_F\\|p|<p_F}}\left(\frac{1}{\omega-\omega_{p;q}}-\frac{1}{\omega+\omega_{p;q}}\right)=0. \tag{3·6}$$

If one takes $\sigma=\sigma_x$, σ_y or σ_z, the sign in front of γ is changed and one has, instead of (3·6), an equation

$$1+\gamma\sum_{\substack{p\\|p+q|>p_F\\|p|<p_F}}\left(\frac{1}{\omega-\omega_{p;q}}-\frac{1}{\omega+\omega_{p;q}}\right)=0. \tag{3·7}$$

From the stability conditions, Eqs. (2·8) and (2·9), one can conclude that if the coupling is sufficiently large, either Eq. (3·6) or Eq. (3·7) according to the sign of γ has an imaginary root for ω and the Hartree-Fock solution becomes unstable. In the one-dimensional case, this instability always occurs for q equal to $2P_F$ irrespective of the strength and sign of γ as pointed out by Overhauser. More specifically, for the repulsive case ($\gamma>0$), Eq. (3·6) leads only to real ω, whereas Eq. (3·7) has always an imaginary solution in the one-dimensional problem. For the attractive case ($\gamma<0$), the role of Eqs. (3·6) and (3·7) is interchanged.[7]

In order to get the Overhauser solution for the one-dimensional repulsive case ($\gamma>0$) according to our procedure, Eq. (2·11), one puts

$$S=-i\sum_k C_k^*\frac{\sigma_x+i\sigma_y}{2}C_{k+q}\alpha_k, \tag{3·8}$$

which generates the transformations

$$\begin{aligned}
&e^{i(S+S^*)}C_{k\uparrow}e^{-i(S+S^*)}=\cos\alpha_k\cdot C_{k\uparrow}-\sin\alpha_k\cdot C_{k+q\downarrow},\\
&e^{i(S+S^*)}C_{k\downarrow}e^{-i(S+S^*)}=\cos\alpha_{k-q}\cdot C_{k\downarrow}+\sin\alpha_{k-q}\cdot C_{k-q\uparrow}.
\end{aligned} \tag{3·9}$$

The minimum condition for the expectation value with respect to $\Psi(\alpha)$ determines the function α_k by

$$2\alpha_k=\tan^{-1}\frac{\beta_q}{\omega_{k,q}};\ 1=\gamma\sum_l((|l|<p_F)-(|l+q|<p_F))\frac{\omega_{l;q}/|\omega_{l;q}|}{\sqrt{\omega_{l;q}^2+\beta_q^2}}. \quad (3\cdot10)$$

The transformed Hamiltonian takes the form, disregarding the constant term which is actually lower than the Hartree-Fock energy,

$$H_{Overhauser}=\sum_k E_{k\uparrow}C_{k\uparrow}^*C_{k\uparrow}+\sum_k E_{k\downarrow}C_{k\downarrow}^*C_{k\downarrow}+(4\text{ operators ordered}), \quad (3\cdot11)$$

with

$$E_{k\uparrow}=\frac{E_k+E_{k+q}}{2}-\frac{\sqrt{\omega_{k;q}^2+\beta_q^2}}{2}\frac{\omega_{k;q}}{|\omega_{k;q}|},$$

$$E_{k\downarrow}=\frac{E_k+E_{k+q}}{2}+\frac{\sqrt{\omega_{k;q}^2+\beta_q^2}}{2}\frac{\omega_{k;q}}{|\omega_{k;q}|}. \quad (3\cdot12)$$

There is another form of $S(\alpha)$ in this case given by

$$S=i\sum_k \boldsymbol{C}_k^*\sigma_z\boldsymbol{C}_{k+q}\alpha_k, \quad (3\cdot13)$$

which however leads to a higher ground state energy than the value obtained above.

The above result is identical with what Overhauser has already obtained, but our criterion seems to work in more general cases in finding the variational trial functions. It should be noticed that the Hartree-Fock solution will be stable in the three dimensional case within the family of variations employed here, if the coupling is sufficiently weak.[8]

§ 4. B.C.S. theory of superconductivity

Let us now turn to the difficulty which one encounters in dealing with the B.C.S. model Hamiltonian of superconductors. In this case, one can sum up all ladder type diagrams up to infinite order following the technique of Gell-Mann and Brueckner,[3] the remaining diagrams being inversely proportional to the total volume to be neglected, but the propagator for a pair of electrons has an imaginary pole.[3] This situation can be easily reproduced in our language and is shown, as expected, to lead to the instability of the Hartree-Fock solution.

The B.C.S. Hamiltonian is written as

$$H_{BCS}=\sum_{k,s}a_{k,s}^*a_{k,s}E_k-\sum_{k,s}b_{k,s}^*b_{k,s}E_k$$
$$+\gamma\sum_k(a_{k\uparrow}^*+b_{k\uparrow})(a_{-k\downarrow}^*+b_{-k\downarrow})\cdot\sum_{k'}(a_{-k'\downarrow}+b_{-k'\downarrow}^*)(a_{k'\uparrow}+b_{k'\uparrow}^*), \quad (4\cdot1)$$

where E_k's ($\lesseqgtr0$ according as $|\boldsymbol{k}|\lesseqgtr p_F$) are measured from E_F. Consider the operator S defined by

$$S=\sum_k(A_k a_{k\uparrow}^*a_{-k\downarrow}^*+B_k b_{k\uparrow}b_{-k\downarrow}). \quad (4\cdot2)$$

Then the coefficients A's and B's are to be determined from Eq. (2·4), where the operator C now consists of products of four ordered operators. One has

in this way

$$(\omega-2E_k)A_k=\gamma\sum_{k'}(A_{k'}-B_{k'}),$$
$$(\omega-2E_k)B_k=\gamma\sum_{k'}(A_{k'}-B_{k'}). \tag{4·3}$$

From which follows the eigenvalue equation for ω:

$$1=\gamma\left(\sum_{k(>|p_F|)}\frac{1}{\omega-2E_k}-\sum_{k(<|p_F|)}\frac{1}{\omega-2E_k}\right). \tag{4·4}$$

This result is identical with what was obtained in the diagram method.[10]

For the repulsive case ($\gamma>0$), all the eigenvalues ($\omega\gtrless0$) shift from zero to the outer side with respect to zero and are of course real, but for the attractive case ($\gamma<0$), this shift occurs to toward zero and the eigenvalue adjacent to zero goes over into an imaginary value. According to our criterion, the Hartree-Fock solution Φ_0 becomes unstable in the latter case and the new trial functions should be looked for in the form of Eq. (2·10), namely

$$\Psi(\lambda)=e^{i(\Sigma_k h_k(\lambda)\, c_{k\uparrow}{}^* c_{-k\downarrow}{}^*+\text{H.C.})}\Phi_0, \tag{4·5}$$

which generates the Bogolyubov transformation.[2] Thus one sees that our criterion works also in this case.

It should be noticed at this point that the collective mode discussed here is a special kind of scattering mode for a pair of opposite momenta and spins. The general treatment of scattering modes for the full Hamiltonian (2·1) has been presented elsewhere by one of the authors (K.S.),[11] which will serve as the instability criterion for the Hartree-Fock solution when the interaction includes an appreciable attractive part.

§ 5. Liquid helium with negative scattering length

Another example to which our instability criterion can be applied will be found in the model Hamiltonian derived by Bogolyubov and others for the discussion of a dilute Boson gas.[5] This is written as

$$H=\sum_k \beta_k{}^*\beta_k E_k+\frac{1}{2}\, n_0{}^2\alpha+\sum_q n_q 2\alpha n_0+\left(\frac{1}{2}\sum_q \beta_q{}^*\beta_{-q}^*\alpha\beta_0\beta_0+\text{H.C.}\right), \tag{5·1}$$

where β_q and $\beta_q{}^*$ are the annihilation and creation operators for a Boson with momentum $\boldsymbol{q}$, respectively, and the constant α is proportional to the scattering length. The sum in Eq. (5·1) does not include $\boldsymbol{q}=0$. The Hartree solution is in this case the state in which all the particles are degenerate in the state $\boldsymbol{K}=0$. If this solution is stable, then one will obtain, following Bogolyubov, the excitation energy of the form

$$\sqrt{E_k{}^2+2\alpha NE_k}=\omega_k, \tag{5·2}$$

where N is the total number of the particles.

In the language of normal mode, one can equivalently obtain the result by choosing the operators S in Eq. (2·4) as[10]

$$S_k^* = A_k \beta_k^* + B_k \beta_{-k}. \tag{5·3}$$

Then, putting $n_0 = N - \sum_q n_q$ and $\beta_0^* \beta_0 = N$, one has, omitting $(\sum_q n_q)^2$ term compared to N,

$$\begin{aligned}[S_k^*, H]_- &= A_k\{-(E_k+\alpha N)\beta_k^* - \alpha N \beta_{-k}\} + B_k\{(E_k+\alpha N)\beta_{-k} + \alpha N \beta_k^*\} \\ &= -\omega_k (A_k \beta_k^* + B_k \beta_{-k}),\end{aligned} \tag{5·4}$$

from which follow the equations to determine the coefficients A and B:

$$\begin{aligned}&\{\omega_k - (E_k+\alpha N)\} A_k + \alpha N B_k = 0, \\ &-\alpha N A_k + \{\omega_k + (E_k+\alpha N)\} B_k = 0.\end{aligned} \tag{5·5}$$

The solubility condition for this leads to the eigenfrequency ω_k given by Eq. (5·2).

If the scattering length (α) is negative, then ω_q becomes imaginary for low momentum $\boldsymbol{q} \sim 0$ and the Hartree solution is unstable. One may expect in this case to obtain a stable solution by adopting the variational trial functions,

$$\Psi(\lambda) = e^{i\sum_k (\alpha_k(\lambda)\beta_k^* \beta_0 + \text{H.C.})} \Phi_0. \tag{5·6}$$

One of the authors (K.S.) and Vasudevan[9] have recently determined the parameter α from the variational principle and actually obtained the ground state energy lower than the Hartree value. It was shown moreover that the real excitation energy was derived in this way.

§ 6. Application to field theory

It is anticipated also in field theory that some of the approximate eigenmodes of strongly interacting elementary particles may be imaginary and the free vacuum state becomes unstable according to our criterion. Then one may choose some appropriate variational trial functions to obtain the stable vacuum state with the real eigenmodes for excitation. The effective coupling will be changed in this case.

Let us consider, as a typical example, a system of the nucleon and the pseudoscalar neutral meson which are interacting each other through pseudoscalar coupling. The Hamiltonian is given by

$$\begin{aligned}H = &\sum_k \varphi_k^* \varphi_k \nu_k + \sum_{p;s} C_p^{*s} C_p^s \varepsilon_p \\ &+ \sum_k g_k (\varphi_k + \varphi_{-k}^*) \sum_{pss'} C_p^{*s} (u_p^{*s} \rho_2 u_{p-k}^{s'}) C_{p-k}^{s'},\end{aligned} \tag{6·1}$$

where φ_k's are the meson operators and C's the nucleon operators with positive

and negative energy, s being the spin index.* The function u is the ordinary Dirac spinor and

$$g_k=\frac{g}{(2\nu_k\Omega)^{1/2}},$$
$$\nu_k=\sqrt{\boldsymbol{k}^2+\mu^2},\quad \varepsilon_p=\pm\sqrt{\boldsymbol{p}^2+m^2}, \tag{6·2}$$

where Ω is the normalization volume, and μ and m the masses of meson and nucleon, respectively. In the hole theory, one usually puts

$$C_p^{\ s}=\begin{cases} a_p^{\ s} & \text{for } \varepsilon_p>0, \\ b_p^{s*} & \text{for } \varepsilon_p<0, \end{cases} \tag{6·3}$$

where a is the annihilation operator for the nucleon and b for the anti-nucleon. In order to avoid the divergence difficulty, one will occasionally introduce the high momentum cutoff.

Let us first look for approximate eigenmodes for the meson by choosing the function S in Eq. (2·4) as

$$S^*=\sum_k A_k\varphi_k{}^*+\sum_k B_k\varphi_{-k}+\sum_{pkss'} D_{p,p-k}^{s,s'}C_p^{*s}C_{p-k}^{s'}. \tag{6·4}$$

Then the coefficients A's are to be determined from Eq. (2·4) where C now consists of products of three ordered operators. One gets in this way

$$(\omega-\nu_k)A_k=-g_k\sum_{p,s,s'}D_{p,p-k}^{s,s'}\{(u_{p-k}^{*s'}\rho_2 u_p^{\ s-})-(u_{p-k}^{*s'+}\rho_2 u_p^{\ s})\},$$
$$(\omega+\nu_k)B_k=-g_k\sum_{p,s,s'}\ \text{(same as above)},$$
$$(\omega-(\varepsilon_p-\varepsilon_{p-k}))\,D_{p,p-k}^{s,s'}=g_k(u_p^{*s}{}^+\rho_2 u_{p-k}^{s'})(A_k+B_k), \tag{6·5}$$

where $\boldsymbol{p}^-$'s (or $\boldsymbol{p}^+$'s) refer to the negative (or positive) energy state hereafter. The eigenvalue equation for ω becomes

$$\omega^2-\nu_k{}^2+2g_k{}^2\nu_k\sum_{p,s,s'}\left\{\frac{|(u_p^{s*-}\rho_2 u_{p-k}^{s'+})|^2}{\omega+|\varepsilon_p|+|\varepsilon_{p-k}|}+\frac{|(u_{p^+}^{s*}\rho_2 u_{p-k}^{s'-})|^2}{-\omega+|\varepsilon_p|+|\varepsilon_{k-k}|}\right\}=0,$$

namley,

$$1=-\frac{4g_k{}^2\nu_k}{\omega^2-\nu_k{}^2}\sum_p\frac{2(|\varepsilon_{p+(k/2)}|+|\varepsilon_{p-(k/2)}|)\{|\varepsilon_p|^2-(1/4)(|\varepsilon_{p+(k/2)}|-|\varepsilon_{p-(k/2)}|)^2\}}{\{-\omega^2+(|\varepsilon_{p+(k/2)}|+|\varepsilon_{p-(k/2)}|)^2\}\,|\varepsilon_{p+(k/2)}|\,|\varepsilon_{p-(k/2)}|}, \tag{6·6}$$

from which one sees that if the cutoff for the sum is large enough, an imaginary solution is obtained and the free vacuum becomes unstable. This situation will be seen clearly in the Appendix for the special case $\boldsymbol{k}=0$.

The following variational trial functions are now tenable from our criterion to obtain the stable solution:

---

* The unit $\hbar=c=1$ is taken. The Dirac operator ρ_2 is equal to $i\gamma_4\gamma_5$ in terms of the ordinary γ-matrices.

$$\Psi=\exp[\textstyle\sum_k(A_k\varphi_k{}^*-A_k{}^*\varphi_k)+i\sum_{p,p',s,s'}D^{s,s'}_{p,p'}C^{*s}_{p}C^{s'}_{p'}]\cdot\Phi_0 \qquad (6\cdot7)$$

which generates the transformations

$$\varphi_k{}^*\to\varphi_k{}^*+A_k{}^*,\quad \varphi_k\to\varphi_k+A_k,$$
$$C^{*s}_{p}\to\textstyle\sum_{p',s'}C^{*s'}_{p'}(e^{-iD})^{s'}_{p'}{}^{s}_{,p}. \qquad (6\cdot8)$$

The new Hamiltonian has the form

$$\begin{aligned}
&H\to H_{transf.}=H_1+H_2+E_0,\\
&H_1=\textstyle\sum_k\varphi_k{}^*\varphi_k\nu_k+\sum_{\substack{p',p''\\ s',s''}}:C^{*s'}_{p'}\{\sum_{ps}(e^{-iD})^{s's}_{p'p}\varepsilon_p(e^{iD})^{ss''}_{pp''}\\
&\quad+\textstyle\sum_k g_k(A_k+A^*_{-k})\sum_{pts}(e^{-iD})^{s's}_{p'p}(u^{*s}_{p}\rho_2u^{t}_{p-k})(e^{iD})^{t}_{p-k}{}^{s''}_{p''}\}C^{s''}_{p''}:\\
&\quad+\textstyle\sum_k g_k(\varphi_k+\varphi^*_{-k})\sum_{\substack{p'p''p\\ s's''st}}:C^{*s'}_{p'}(e^{-iD})^{s's}_{p'p}(u^{*s}_{p}\rho_2u^{t}_{p-k})(e^{iD})^{t}_{p-k}{}^{s''}_{p''}C^{s''}_{p''}:\\
&H_2=\textstyle\sum_k\varphi_k{}^*\{\nu_kA_k+g_k\sum_{p's'ps't}(e^{-iD})^{s's}_{p'-p}(u^{*s}_{p}\rho_2u^{t}_{p+k})(e^{iD})^{t}_{p+k}{}^{s'}_{p'-}\}+\text{H.C.}\\
&E_0=\textstyle\sum_k A_k{}^*A_k\nu_k+\sum_{\substack{pp'\\ ss'}}(e^{-iD})^{s's}_{p'-p}\varepsilon_p(e^{iD})^{ss'}_{pp'}-\\
&\quad+\textstyle\sum_k g_k(A_k+A^*_{-k})\sum_{p's'pst}(e^{-iD})^{s's}_{p'-p}(u^{*s}_{p}\rho_2u^{t}_{p-k})(e^{iD})^{t}_{p-k}{}^{s'}_{p'-}. \qquad (6\cdot9)
\end{aligned}$$

The stationary condition for E_0 with respect to A and A^* leads to the equations

$$\begin{aligned}
&\nu_kA_k+g_k\textstyle\sum_{p's'pst}(e^{-iD})^{s's}_{p'-p}(u^{*s}_{p}\rho_2u^{t}_{p+k})(e^{iD})^{t}_{p+k}{}^{s'}_{p'-}=0,\\
&\nu_{-k}A^*_{-k}+g_{-k}\textstyle\sum_{p's'pst}\ \text{(same as above)}=0, \qquad (6\cdot10)
\end{aligned}$$

which makes H_2 vanish. To make variation with respect to D, one must take into account the condition

$$\textstyle\sum_{p,s}(e^{-iD})^{s's}_{p'p}(e^{iD})^{ss'}_{pp'}=1. \qquad (6\cdot11)$$

By introducing the Lagrange indefinite multiplier, one obtains another equation

$$(\delta_{p'p'}-\varepsilon_p-\lambda^{s'}_{p'})(e^{iD})^{ss'}_{pp'}+\delta_{p'-p'-}-\textstyle\sum_k g_k(A_k+A^*_{-k})\sum_t(u^{*s}_{p}\rho_2u^{t}_{p-k})(e^{iD})^{t}_{p-k}{}^{s'}_{p'}=0, \qquad (6\cdot12)$$

from which it follows that

$$\lambda^{s'}_{p'}=0,\quad (e^{iD})^{s}_{pp'}{}^{s'}_{-}=\text{arbitrary}. \qquad (6\cdot13)$$

For the sake of convenience, one may therefore replace Eq. (6·12) by

$$(\varepsilon_p-\lambda^{s'}_{p''})(e^{iD})^{ss'}_{pp'}+\textstyle\sum_k g_k(A_k+A^*_{-k})\sum_t(u^{*s}_{p}\rho_2u^{t}_{p-k})(e^{iD})^{t}_{p-k}{}^{s'}_{p'}=0. \qquad (6\cdot14)$$

Multiplying this with $(e^{-iD})^{s'''s}_{p'''p}$ and taking sum over s and p, one gets

$$\begin{aligned}
&\textstyle\sum_{ps}(e^{-iD})^{s'''s}_{p'''p}\varepsilon_p(e^{iD})^{ss'}_{pp'}-\lambda^{s'}_{p''}\delta_{s's'''}\delta_{p'p'''}\\
&\quad+\textstyle\sum_k g_k(A_k+A^*_{-k})\sum_{pss''}(e^{-iD})^{s'''s}_{p'''p}(u^{*s}_{p}\rho_2u^{s''}_{p-k})(e^{iD})^{s''}_{p-k}{}^{s'}_{p'}=0. \qquad (6\cdot15)
\end{aligned}$$

One can now rewrite the transformed Hamiltonian, Eq. (6·9), and Eqs. (6·10) and (6·14) in terms of the function

$$v_q^t(\boldsymbol{p}) = \sum_{s(\pm\varepsilon_p)} u_p^s (e^{iD})_{pq}^{st}, \tag{6·16}$$

where $(\pm\varepsilon_p)$ means the sum over the sign of the energy with momentum $\boldsymbol{p}$. Namely, one has

$$\begin{aligned} H_{transf.} &= \sum_k \varphi_k^* \varphi_k \nu_k + \sum_{ps} : C_p^{*s} C_p^s : \lambda_p^s \\ &+ \sum_k g_k (\varphi_k + \varphi_{-k}^*) \sum_{\substack{ps \\ p's'}} : C_p^{*s} \sum_{p''} (v_p^{*s}(\boldsymbol{p}'') \rho_2 v_{p'}^{s'}(\boldsymbol{p}''-\boldsymbol{k})) C_{p'}^{s'} : \\ &+ \sum_k A_k^* A_k \nu_k + \sum_{p,s} \lambda_{p^-}^s ; \end{aligned} \tag{6·17}$$

$$\nu_k A_k + g_k \sum_{pp's} (v_{p^-}^{*s}(\boldsymbol{p}') \rho_2 v_{p^-}^s(\boldsymbol{p}'+\boldsymbol{k})) = 0, \tag{6·18}$$

$$(\boldsymbol{\alpha p} + \beta m - \lambda_q^t) v_q^t(\boldsymbol{p}) + \sum_k g_k (A_k + A_{-k}^*) \rho_2 v_q^t(\boldsymbol{p}-\boldsymbol{k}) = 0. \tag{6·19}$$

In deriving Eq. (6·17), use has been made of Eq. (6·15). The last two equations lead to a non-linear equation for v, the general solution of which is difficult to obtain, and one is obliged to content oneself with the following particular solution. Let us put

$$v_q^t(\boldsymbol{p}) = \delta_{q,p} w_q^t. \tag{6·20}$$

Then Eqs. (6·18) and (6·19) become

$$\begin{aligned} &\nu_k A_k + \delta_{k;0} g_k \sum_{qt} (w_{q^-}^{*t} \rho_2 w_{q^-}^{\bar{t}}) = 0, \\ &(\boldsymbol{\alpha q} + \beta m + 2g_0 A_0 \rho_2 - \lambda_q^t) w_q^t = 0, \end{aligned} \tag{6·21}$$

which has a solution

$$w_q^t = \frac{1}{(1 + \{2g_0 A_0/(\lambda_q + \varepsilon_q)\}^2)^{1/2}} \left(1 + \frac{2g_0 A_0}{\lambda_q + \varepsilon_q} \rho_2\right) u_q^t, \tag{6·22}$$

where u_q^t is the ordinary Dirac amplitude and

$$\begin{aligned} \lambda_q &= \sqrt{\varepsilon_q^2 + (2g_0 A_0)^2}, \quad \text{for } \varepsilon_q > 0, \\ &= -\sqrt{\varepsilon_q^2 + (2g_0 A_0)^2}, \quad \text{for } \varepsilon_q < 0. \end{aligned} \tag{6·23}$$

The function A_k is zero for $\boldsymbol{k} \neq 0$ and A_0 satisfies

$$1 + \frac{4g_0^2}{\nu_0} \sum_q \frac{(-1)}{|\lambda_q|} = 0, \tag{6·24}$$

which has the solution if the cutoff is sufficiently large.[12] Our transformed Hamiltonian (6·17) has now the form

$$\begin{aligned} H_{transf.} &= \sum_k \varphi_k^* \varphi_k \nu_k + \sum_{ps} : C_p^{*s} C_p^s : \pm \sqrt{\varepsilon_p^2 + (2g_0 A_0)^2} \\ &+ \sum_k g_k (\varphi_k + \varphi_{-k}^*) \sum_{pss'} : C_p^{*s} \cdot \frac{1}{(1 + \{2g_0 A_0/(\lambda_p + \varepsilon_p)\}^2)^{1/2}} \end{aligned}$$

$$\times\frac{1}{(1+\{2g_0A_0/(\lambda_{p-k}+\varepsilon_{p-k})\}^2)^{1/2}}\cdot\left(u_p^{*s}\cdot\left\{\rho_2\left(1+\frac{(2g_0A_0)^2}{(\lambda_p+\varepsilon_p)(\lambda_{p-k}+\varepsilon_{p-k})}\right)\right.\right.$$

$$\left.\left.+2g_0A_0\left(\frac{1}{\lambda_p+\varepsilon_p}+\frac{1}{\lambda_{p-k}+\varepsilon_{p-k}}\right)\right\}u_{p-k}^{s'}\right)\times C_{p-k}^{s'}:-2\sum_p|\varepsilon_p|+\Delta E_0, \quad (6\cdot25)$$

with

$$\Delta E_0=-\sum_p\frac{(\sqrt{\varepsilon_p^2+(2g_0A_0)^2}-|\varepsilon_p|)^2}{\sqrt{\varepsilon_p^2+(2g_0A_0)^2}}\ (<0), \quad (6\cdot26)$$

which is the energy decrease on account of our transformation.

It should be noticed here that the solubility of Eq. (6·24) leads to the condition for the cutoff momentum and the coupling constant, which is consistent with the condition that Eq. (6·6) has an imaginary solution. The reasonable values for these quantities adopted in the cutoff theory give

$$\frac{4g_0^2}{\nu_0}\sum_q\frac{1}{|\varepsilon_q|}>1, \quad (6\cdot27)$$

which is the condition that Eq. (6·6) with $\boldsymbol{k}=0$ has an imaginary solution for ω (see the Appendix) and is at the same time the condition that Eq. (6·24) has a real solution for $2g_0A_0$. Thus our criterion seems really to work. The effective coupling in Eq. (6·26) has now a scalar term in marked contrast to the original coupling in Eq. (6·1), but these interaction terms becomes always smaller with respect to the positive-negative component and vanishes in the strong coupling limit.

After the transformation, one can construct the eigen-value equation corresponding to Eq. (6·6) again, but in this case on account of the deformation of vacuum (or mathematically due to Eq. (6·24)) there is no imaginary root (see the Appendix), and the free vacuum defined by our new Hamiltonian Eq. (6·25) is now stable against the deformation considered.

§ 7. Conclusion

A criterion has been presented to see the stability of the Hartree-Fock solution for the given dynamical system. If one can construct any kind of approximate collective modes which may have an imaginary frequency, then the Hartree-Fock solution becomes unstable in the sense that a state deformed in accordance with this collective oscillation will give a lower ground state energy. A method of choosing appropriate variational trial functions is proposed in this case and is actually shown to give stable solutions in some examples, that is, the three-dimensional spin wave model of Overhauser, the theory of Liquid He^4 with negative scattering length and the B.C.S. theory of superconductivity. This method is then applied to field theory, in particular a meson-nucleon system which in fact shows the instability mentioned above, and one sees that the

"zeroth order" vacuum is deformed into another which now includes virtual pairs and mesons and that the effective coupling is changed in such a way that the new eigenfrequencies of the meson and nucleon become real. The high frequency part of the original coupling is not altered, but there might be some collective modes, e.g. nucleonic level strucutre, arising from this part, just like the plasma oscillation which screens the long-range tail of the Coulomb force.

Appendix

The eigenvalues of the meson

The eigenvalue equation, Eq. (6·6), can be evaluated explicity. As an illustration, let us consider the special case $\boldsymbol{k}=0$. Then one has to solve the equation

$$1=-\frac{4g_0^2\nu_0}{\omega^2-\nu_0^2}\sum_{\boldsymbol{p}}\frac{|\mathcal{E}_{\boldsymbol{p}}|}{|\mathcal{E}_{\boldsymbol{p}}|^2-\omega^2/4}\,.$$

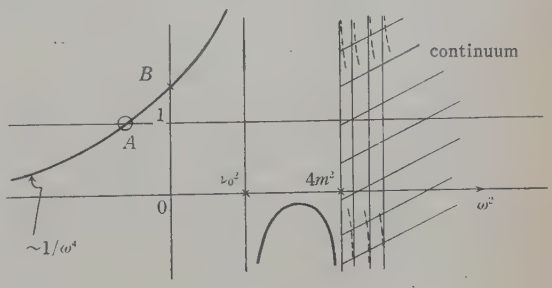


Fig. 1.

The right-hand side is shown as a function of ω^2 in Fig. 1, on the assumption that the instability condition Eq. (6·27) is satisfied, which means the point B lies above 1. (ν_0 is, of course, smaller than m.) The point A corresponds to an imaginary eigenvalue. If the stability condition is satisfied, one has Fig. 2 in which the point B lies below 1 and a collective mode of excitation associated with the point C appears which lies between 0 and $Min(\nu_0^2, 4m^2)$. But as stated in the text, any reasonable cutoff momentum and coupling strength can hardly satisfy the stability condition.

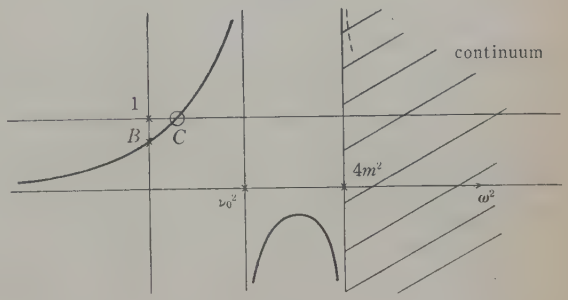


Fig. 2.

In the system which is described by the Hamiltonian Eq. (6·25), we can again construct (approximate) normal modes corresponding to Eq. (6·4), and the eigenvalue equation has a similar form to Eq. (6·6), the only difference being that u's and $\mathcal{E}$'s are now replaced by ω's and λ's which satisfy Eq. (6·21), namely

$$\omega^2-\nu_k^2+2g_k^2\nu_k\sum_{\boldsymbol{p}ss'}\left\{\frac{|(w_{\boldsymbol{p}}^{s*}\rho_2 w_{\boldsymbol{p}^+-\boldsymbol{k}}^{s'})|^2}{\omega+|\lambda_{\boldsymbol{p}}|+|\lambda_{\boldsymbol{p}-\boldsymbol{k}}|}+\frac{|(w_{\boldsymbol{p}^+}^{s*}\rho_2 w_{\boldsymbol{p}^--\boldsymbol{k}}^{s'})|^2}{-\omega+|\lambda_{\boldsymbol{p}}|+|\lambda_{\boldsymbol{p}-\boldsymbol{k}}|}\right\}=0.$$

This leads to the eigenvalue equation (for $\boldsymbol{k}=0$)

$$1=-\frac{4g_0^2\nu_0}{\omega^2-\nu_0^2}\sum_p \frac{|\varepsilon_p|^2}{(|\lambda_p|^2-\omega^2/4)|\lambda_p|} .$$

The right hand side is an increasing function of ω^2 ranging from $-\infty$ to 0, and at $\omega^2=0$ we have

$$\text{right hand side}=\frac{4g_0^2}{\nu_0}\sum_p \frac{1}{|\lambda_p|}\frac{|\varepsilon_p|^2}{|\lambda_p|^2}<1,$$

owing to the eigenvalue equation for A_0, Eq. (6·24). One has therefore no imaginary solution for ω. There appears instead a solution for ω which lies between 0 and $Min(\nu_0, 2|\lambda_0|)$ and is to be connected with the solutions for $\boldsymbol{k}\neq 0$ to make a continuous spectrum.

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10) This equation is given in the second reference in (2).
11) The last reference in (4).
12) Note that Eq. (6·24) is relativistically invariant, because $\sum_q 1/|\lambda_q|$ is scalar ($|\lambda_q|$ is the energy of the nucleon, see Eq. (6·25)) if one does not introduce the cutoff.

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Pion-Nucleon Interaction and the Multiple-Production Experiment†

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A method is proposed to investigate the pion-nucleon interaction with recourse to the multiple-production experiments at extremely high energy. The method consists of calculating theoretically the characteristics of the collision event with small multiplicity of meson production and comparing the results with the extrapolation from the experimental data for the multiple production. Along this line of thought we have performed the perturbation calculation for the single-pion production process. Although the fit of the results with the extrapolations is not so satisfactory, we might still conclude that the pion-nucleon coupling of the pseudo-scalar type is rather preferable to that of the pseudo-vector type.

§ 1. Introduction and summary

One of the most challenging problems in the physics today is to explore the behaviour of elementary particles in the higher and higher energy regions.[1] Now, are we well equipped with experimental and theoretical methods for this exploration? As to the experimental methods, this question may be answered affirmatively to some extent: Although we have no means to measure accurately the energy of an extremely relativistic particle,* to discriminate particle species and so on, the experimental materials are being accumulated and the information increased on the extremely high energy collision (by this term we mean in this paper the collision with $E_{Lab} \gtrsim 10^{11}$ ev in the laboratory system) with production of particles. In addition to the evidence for the multiple production of mesons, we have data, not sufficient but much, on the multiplicity, the angular distribution and the composition of the collision products.[2] On

†The preliminary reports of this paper were submitted to the Conference on the Cosmic Ray Physics at Moscow and the Conference on the High Energy Nuclear Physics at Kiev, both in 1959, under the title "Impact Parameter and Perturbation Treatment of the Distant Collision of Nucleons with Pion Emission" and "To deduce the Type of Meson-Nucleon Interaction from the Multiple-Production Experiments", respectively.

* In the course of the preparation for the publication of this paper, we received a preprint from N. A. Dobrotin and S. A. Slavatinsky, which presents us some interesting results of their "calorimetric" measurement of the particle energy.

the contrary, no affirmative answer might be given to the theoretical side of the question, since it does not seem satisfactory that the investigations hitherto made are based, more or less, on the intuitive models, the theoretical bases of which are still left obscure. Indeed, this obscurity makes it difficult to deduce what the experimental facts imply about the behaviour of elementary particles in the extremely high energy region.

When Heisenberg[3] pointed out the possibility of the multiple particle production, his interest was concentrated around the possible role of the interactions of the second kind.

Since then, the main theme of our exploration into the higher energy region has been to detect the effects of the second kind interactions, or, when the meson-nucleon interaction is concerned, to solve the problem whether it belongs to the second kind; this problem remains still unsolved. One of the authors (H.E.), in collaboration with Umezawa and Tomozawa,[4] has once attempted to solve this problem by examining the equation of state of the meson assembly which plays a central role in the Landau theory of the multiple production.[5] The second kind interaction, if existed among mesons and nucleons, was proved to be detectable through the measurement of multiplicity n as a function of the energy E of colliding nucleons in the centre-of-mass system (C.M.S). Contrary to their expectation, however, experiments have revealed the fact that the multiplicity n widely fluctuates collision by collision even when we restrict ourselves to a narrow range of the energy E,[6),7] thus making it difficult to proceed along the line described above.

It is this situation that compelled us to invent an idea which we call "extrapolation philosophy" (§ 2). This philosophy enables us to deduce some characteristics of the collisions with the smaller multiplicity from those with the larger multiplicity. The point is that the theoretical calculations are expected to be easier for the former while the experimental materials are much ampler for the latter. One of the authors (H.E.) presented an argument which made it plausible that the collision with the smaller multiplicity has correspondingly the larger impact-parameter.[7] This argument has stimulated us to try a perturbation calculation for the nucleon-nucleon collision producing a small number of mesons, because the distant collision is expected to disturb the nucleon proper field very little.

As a conclusion from our calculation for single pion production process the pseudo-scalar interaction between nucleon and pion is proved to yield results satisfactory to some extent (§ 3). That the nucleon recoil is in fact small confirms our anticipation of distant collision. On the contrary, the pseudo-vector interaction fails at all (§ 4), as far as the perturbation calculation of the lowest order is concerned. As for this coupling, however, the predominance of the large nucleon recoil makes the applicabililty of the perturbation method dubious. In spite of the uncertainties thus resulting, we might still

conclude that the meson-nucleon coupling is preferably of the pseudo-scalar type. This conclusion of ours is in conformity with the suggestion once made by Umezawa and others[8] that all the strong interaction should belong to the first kind. This is, however, not in accordance with the conclusion recently obtained by N. Hoshizaki and S. Machida.[9]

We may note again that the above conclusion has been obtained through our extrapolation philosophy. Moreover, the extrapolation is not so ambiguous as it may appear at first sight, and therefore it will help us also in the further investigations.

§ 2. Extrapolation philosophy

As described in the introduction, our aim in investigating the phenomena of multiple-particle production is to explore the behaviour of elementary particles in the extremely high energy region. In view of this fact, it does not seem satisfactory that the investigations hitherto made are based, more or less, on the intuitive models.[10]

Thus it is highly desirable to attack the problem through a direct application of our field theory. We have, however, no appropriate way for this purpose. Although we admit Heisenberg's philosophy of investigating the structure of interactions through the multiple-production phenomena, his method of calculation is not more than a heuristic one.[3],[11] The sudden approximations due to Oppenheimer and others,[12] and Imamura, Kobayakawa[13] can not reproduce the angular distribution of secondaries. The statistical theories,[14] originally due to Fermi, calculate only the phase volumes and therefore these theories have yielded little information on the interaction of elementary particles. The hydrodynamical theories, due to Landau and others, can not but push the problem of interaction behind the assumption of the hydrodynamical picture (see, however, reference 4)). Investigations on the validity of hydrodynamical picture have not succeeded, so far, in deducing information on the interactions.[15]

We have been looking for some characteristic features of high energy phenomena which are useful as a guide for our investigations. Now, the method that we propose at this stage originates from the observation that a variety of events arises in the extremely high energy region. For example, the multiplicity of meson production has been found to widely fluctuate event by event even if we restrict our attention to a narrow range of the energy of colliding nucleons.[6] This variety is nothing but the indication that we have a liberty to select some events which are especially suitable for our purpose. Taking into account that the events with the smaller multiplicity should be handled rather easily as far as the theoretical consideration is concerned, we have decided to examine the single-pion production process as a first step. It must, however, be admitted that there has been no experimental finding about

the single-pion production at extremely high energy. Then, is it possible to compare our theory with experiment?

Yes, it is possible! There may be no direct evidence, yet we can still make guesses by the use of those experimental data on the multiple-pion production. It has been found, on the one hand, that the transverse momentum of the secondary pion is almost constant over a wide range of the multiplicity n and the energy E of the colliding nucleons.[6),16),17)] And, on the other hand, it has been pointed out by one of the authors (H.E.) that the angular distribution of pions varies rather systematically with n and E.[7)] These relation may be extrapolated to yield the characteristics of the single-pion-production process, thus making the comparison possible between the experiments and our theoretical calculations.

The idea can be summarized as follows: Of the variety of events at extremely high energy, i.e. the collisions with the smaller multiplicity and those with the larger multiplicity, we calculate the characteristics of the former and then compare the results with the extrapolations from the latter.

This constitutes our "extrapolation philosophy".

As the application of our philosophy to the analysis of the angular distribution, we have to add one more comment. In this paper, following Hoang[18)] and the Bristol group,[6)] we use index m to represent the sharpness of the angular distribution. It is defined for each event through

$$\langle \mathrm{cosec}\,\theta \rangle_{av} \equiv 1.25\sqrt{m} \tag{2·1}$$

where θ is the angle of emission in the C.M.S., and the average is taken over secondary particles event by event. It is noteworthy that this m has been found experimentally not to fluctuate so much if n and E are fixed. Along the line of our philosophy, we also define m in the case of single-pion production through

$$\int \mathrm{cosec}\,\theta \cdot \frac{d\sigma}{d\Omega}\, d\Omega \Big/ \int \frac{d\sigma}{d\Omega} d\Omega \equiv 1.25\sqrt{m}. \tag{2·2}$$

In this case, the theoretical calculation tells us the variation of m with the energy E of nucleons in the C.M.S., which might be approximated by

$$m = cE^{\alpha}, \tag{2·3}$$

where c and α are some constants. In the case of the multiple-production also, there must be a relation like (2·3) at least for fixed multiplicity n. In other words, c and α must be functions of the multiplicity, $c=c(n)$, $\alpha=\alpha(n)$.*

Now, if the changes of $c(n)$ and $\alpha(n)$ with the change of n are not so fast and if the theoretical value $\alpha(1)$ is correct, the plot of the experimental

* Note that n discriminates mutually orthogonal out-going channels in the sense of quantum mechanics.

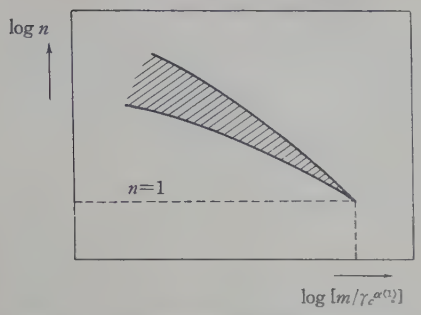


Fig. 1. An illustration of the extrapolation philosophy.

data, n against m/E^{α}, will look like Fig. 1.

From this experimental plot, we can infer the value of m for $n=1$, and compare this with the theoretical value.*

This example shows the way how our extrapolation philosophy works. Of course, there may be many other ways of extrapolation also. If we could use not only the cosec θ-moment but also the other moments of the angular distribution, the production process and therefore the interactions of particles would be investigated more thoroughly.

§ 3. Perturbation treatment of the single-pion-production process. I

——Pion-nucleon coupling of pseudo-scalar type——

Along the line described in the previous section, we investigate the nucleon-nucleon collision with single-pion production. The following reasoning will allow us to try the perturbation method. As a heuristic argument, one of the authors (H.E.) has once introduced the concept of impact parameter into the analysis of experimental data.[7] There has been shown the possibility of inferring the impact parameter of each nucleon-collision from the experimentally observable quantities, especially the angular distribution of mesons. In addition, there has been found a fact, quite an understandable fact, that, for the nucleon-nucleon collision with energy $E \gtrsim 30mc^2$ (m=nucleon mass**), the multiplicity of meson production is a decreasing function of the impact parameter. Then we may safely apply the perturbation method to those events with the smaller multiplicity, since it would be a reasonable expectation that the larger the impact parameter, the smaller the disturbance which the nucleons should suffer during the collision process.

Many papers have been published on the perturbation calculation of the single-pion-production process,[19),20)] but none of them treated the angular distribution at extremely high energy.

At first, we make a tentative calculation for the reaction $p+p \to p+p+\pi^0$; at extremely high energy the situations will be similar for other reactions.*** In this section, we take the meson-nucleon interaction to be of the pseudo-scalar type, $G\bar{\psi}\gamma_5\boldsymbol{\tau}\psi\boldsymbol{\phi}$, and obtain the matrix element for the process just mentioned (see Fig. 2):****

* The relation thus found in this paper among n, m and E is somewhat different from the relation in the previous work.[7]

** Not to confuse the mass of nucleon with the index of peakedness, Eq. (2·2).

*** This point will be discussed at the end of Appendix A.

**** We use the unit system $\hbar=c=1$, throughout.

$$M=-i\frac{1}{(2\pi)^{7/2}}G^3\sqrt{\frac{m^4}{2\omega E_1E_2E_3E_4}}\,T, \tag{3·1}$$

with
$$T=(1-\not{P}_{12}-\not{P}_{34}+\not{P}_{12}\not{P}_{34})T_0 \tag{3·2}$$

and

$$T_0=\left[\frac{1}{2(p_1\cdot k)-\mu^2}+\frac{1}{2(p_3\cdot k)+\mu^2}\right][\bar{u}(p_3)k_\mu\gamma^\mu u(p_1)][\bar{u}(p_4)\gamma_5 u(p_2)]\cdot\frac{1}{(p_2-p_4)^2-\mu^2} \tag{3·3}$$

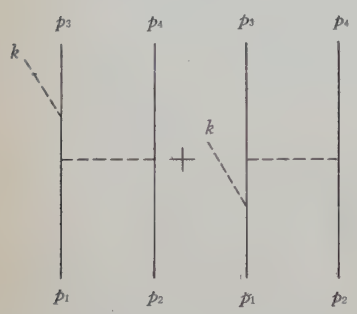


Fig. 2. The Feynman diagram corresponding to T_0, Eq. (3·3) and Eq. (4·2).

where p_1, p_2 are the four momenta of incident protons; p_3, p_4 and k are those of out-going protons and π^0 meson, respectively. m and μ are the masses of proton and π^0 respectively. $\not{P}$ is the Pauli exchange operator for the protons. Other notations will be self-explanatory.

In the C.M.S. the cross section is given by

$$d\sigma=(2\pi)^2\frac{1}{2v}\overline{SS}M^\dagger M \times\delta^{(4)}(p_1+p_2-p_3-p_4-k)\,d^3p_3d^3p_4d^3k \tag{3·4}$$

where v is the speed of the incident nucleons, $v\approx 1$.

Because of the identity:

$$T^\dagger T=(1+\not{P}_{12})(1+\not{P}_{34})[T_0^\dagger(1-\not{P}_{12}-\not{P}_{34}+\not{P}_{12}\not{P}_{34})T_0],$$

it is sufficient to calculate the contributions from the terms in square brackets. Taking into account that we are investigating phenomena at extremely high energy, we may safely neglect the masses in the numerators,* while in the denominators we may not, since such neglect sometimes makes the denominator zero. As will be seen in the following, the masses in the denominators play an important role in determining the angular distribution (see Eq. (3·6) and (3·7)). In this approximation, we have

$$\overline{SS}T_0^\dagger T_0=\frac{2}{(2m)^4}\left[\frac{(p_3\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_1\cdot k)}{2(p_3\cdot k)+\mu^2}+1\right]\cdot V(2,4),$$

$$\overline{SS}T_0^\dagger\not{P}_{12}T_0=(1+\not{P}_{12})\frac{1}{(2m)^4}\left[\frac{1}{2}\left\{\frac{(p_3\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_2\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_2\cdot k)}{2(p_3\cdot k)+\mu^2}+\frac{1}{2}\right\}\right.$$
$$-\frac{1}{2}\frac{(p_1\cdot p_2)(p_4\cdot k)}{(p_1+p_2)\cdot p_4-2m^2+\mu^2}\left\{\frac{1}{2(p_1\cdot k)-\mu^2}+\frac{1}{2(p_2\cdot k)-\mu^2}+\frac{1}{2(p_3\cdot k)+\mu^2}\right.$$
$$\left.\left.+\frac{1}{(p_1+p_2)\cdot k-\mu^2}\left(\frac{(p_3\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_3\cdot k)}{2(p_2\cdot k)-\mu^2}\right)\right\}\right]\cdot V(2,4), \tag{3·5}$$

* In Appendix A, we discuss the errors hereby introduced.

$$\sum_{f\,i} S\bar{S}T_0^{\dagger}\not{p}_{34}T_0=(1+\not{p}_{34})\frac{1}{(2m)^4}\Big[\frac{1}{2}\Big\{\frac{(p_4\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_1\cdot k)}{2(p_3\cdot k)+\mu^2}+\frac{(p_4\cdot k)}{2(p_3\cdot k)+\mu^2}+\frac{1}{2}\Big\}$$

$$-\frac{1}{2}\frac{(p_2\cdot k)(p_3\cdot p_4)}{p_2\cdot(p_3+p_4)-2m^2+\mu^2}\Big\{\frac{1}{2(p_1\cdot k)-\mu^2}+\frac{1}{2(p_3\cdot k)+\mu^2}+$$

$$+\frac{1}{2(p_4\cdot k)+\mu^2}+\frac{2(p_1\cdot k)}{[2(p_3\cdot k)+\mu^2][2(p_4\cdot k)+\mu^2]}\Big\}\Big]\cdot V(2,4),$$

$$\sum_{f\,i} S\bar{S}T_0^{\dagger}\not{p}_{12}\not{p}_{34}T_0=0,$$

with
$$V(2,4)=\frac{1}{(p_2-p_4)^2-\mu^2}.$$

The calculation to follow is straightforward (cf. Appendix A), and only the results will be summarized below. The incident energy of nucleons in the C.M.S., $E=E_1=E_2$ is taken to be

$$E=10^{10},\ 10^{11},\ 10^{12}\ \text{ev}.$$

(i) *The angular distribution of the pion* The angular distribution of the pion with its energy ω fixed is peaked forwards and backwards, and is given by

$$\frac{d^2\sigma}{d\omega d\Omega}=(1+\not{p}_{12})\frac{1}{8\pi}\Big(\frac{G^2}{4\pi}\Big)^3\frac{\omega}{E^4}\Big[\frac{E^2}{(p_1\cdot k)-(\mu^2/2)}a(E,\omega,\theta)+c(E,\omega,\theta)\Big]$$

$$\cong\frac{1}{8\pi}\Big(\frac{G^2}{4\pi}\Big)^3\frac{1}{E^3}\Big[\frac{a(E,\omega,\theta)}{(1-\cos\theta)+\frac{1}{2}\Big(\frac{m^2}{E^2}+\frac{\mu^2}{\omega^2}\Big)\cos\theta-\frac{1}{2}\frac{\mu^2}{E\omega}}$$

$$+\frac{\omega}{E}c(E,\omega,\theta)\Big] \qquad (3\cdot6)$$

$$+[\theta\to\pi-\theta],$$

where θ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{k}$. $a(E,\omega,\theta)$ and $c(E,\omega,\theta)$ are given in the Appendix [Eq. (A·7)]. As is exemplified in the Table I (the case of $E=10^{11}$ ev), they depend not strongly on θ, and so the dominating factor in the angular distribution (3·6) is $1/[(p_1\cdot k)-(\mu^2/2)]$ which stems from the nucleon propagator.

In these tables, we also include the data which we obtain by neglecting the Pauli exchange effect. The reason for this inclusion is twofold: to show the large cancellation due to the exchange effect on the one hand and to make the future reference easier on the other. As will be seen later, the angular distribution and the transverse momentum distribution are rather better fitted to the calculation without the exchange effect.

Now, the sharpness of the forward and backward peaks may conveniently be represented by the half-width [see Eq. (3·6)];

Table I. Illustration of $a(E, \omega, \theta)$ and $c(E, \omega, \theta)$. (The case of $E=10^{11}$ ev.)

(A) With the exchange effect taken into account (Eq. (3·2) in full).

$a(E, \omega, \theta)$

ω/E \ θ	0°	10°	45°	135°
0.1	0.083	0.090	0.20	0.41
0.4	0.50	0.54	1.14	1.89
0.9	6.8	8.9	9.8	13.2

$c(E, \omega, \theta)$

ω/E \ θ	0°	10°	45°	135°
0.1	0.83	1.17	1.34	1.43
0.4	2.9	4.1	4.9	6.9
0.9	5.1	5.7	7.5	23

(B) Without the exchange effect (the $\not{P}$'s in Eq. (3·2) neglected)

$a(E, \omega, \theta)$

ω/E \ θ	0°	10°	45°	135°
0.1	4.1	4.1	4.1	3.9
0.4	6.2	6.2	5.9	4.2
0.9	18.2	18.0	12.0	4.4

$c(E, \omega, \theta)$

ω/E \ θ	0°	10°	45°	135°
0.1	9.4	11.7	13.0	13.1
0.4	12.8	15.5	15.3	12.0
0.9	33	35	22	10.1

$$\theta_{1/2} \simeq \sqrt{\frac{m^2}{E^2} + \frac{\mu^2}{\omega^2} - \frac{\mu^2}{E\omega}}. \tag{3·7}$$

We see, from this formula, that the transverse momentum is of the order of $\omega\theta_{1/2} \simeq \mu \sim m$.

The angular distributions integrated over pion energy $d\sigma/d\Omega$, are shown in Fig. 3.

For the sake of easy reference, we present there the angular distribution without the exchange effect (at $E=10^{11}$ ev). Also, there is shown the angular distribution (at $E=10^{11}$ ev) which would be obtained if $a(E, \omega, \theta)=1$, $c(E, \omega, \theta)=0$. This fictitious angular distribution will illustrate the effect of a and c.

(ii) *The index of peakedness, m* Now, according to Eq. (2·2) we calculate

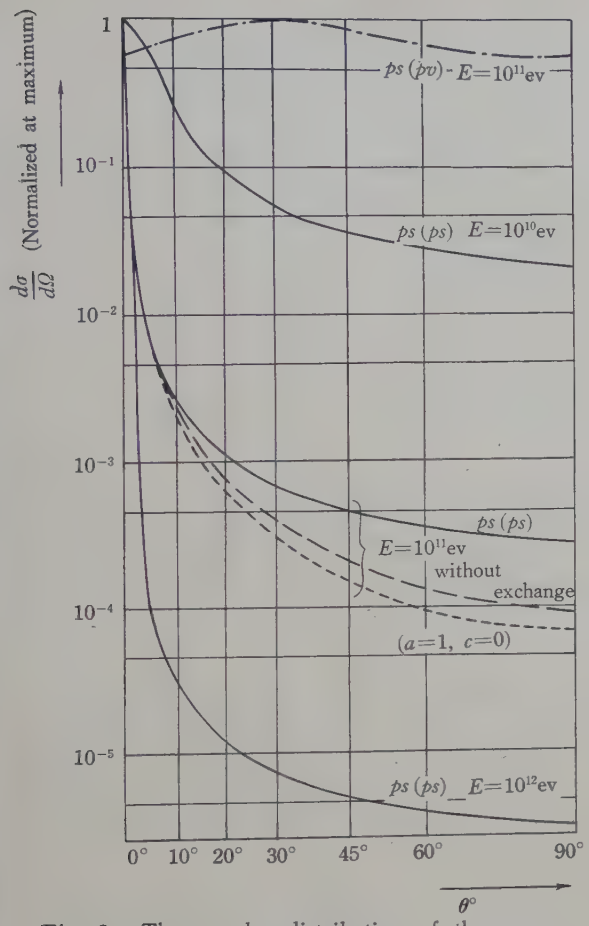


Fig. 3. The angular distribution of the meson calculated by the lowest order perturbation theory. Note that the ordinate is in the logarithmic scale.

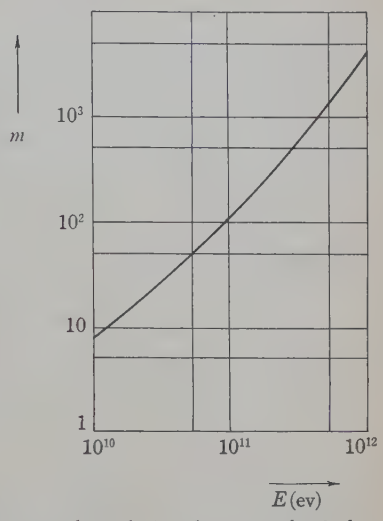


Fig. 4. The relation between the index of peakedness m and the energy E of the incident nucleon in C.M.S., which is obtained by the lowest order perturbation theory for the pion-nucleon coupling of pseudo-scalar type.

the index of peakedness of the angular distribution, Eq. (3·6), obtaining those values presented in Fig. 4 (see also Table II).

From this figures we see the approximate relation* corresponding to Eq. (2·3) :

$$m \propto E^{1.5}, \quad (E \gtrsim 10^{11} \text{ ev}).^{**} \qquad (3\cdot8)$$

In the case of the single-pion production, therefore, $m/\gamma_c^{1.5}$ is independent of γ_c (for $\gamma_c \gtrsim 10^2$), where γ_c is the Lorentz factor of the incident nucleons in the C.M.S., $E=mc^2\gamma_c$. Along the line of our extrapolation philosophy explained in § 2, we try a relation $n_s=g(m/\gamma_c^{1.5})$ by plotting the experimental data on Fig. 5. The first thing we observe on this figure is that the experimental points lie along a rather smooth curve, allowing the extrapolation to the case of single-pion production. This fact supports the result (3·8) of our calculation and

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* The fictitious angular distribution ($a=1$, $c=0$ in Eq. (3·6); illustrated in Fig. 3) gives $m \simeq (4/5)\pi E^2/\ln E^2$, where energy E is measured in unit of the nucleon mass. We obtain a similar relation to this when we neglect the exchange effect.

** This restriction may be regarded as corresponding to that in the impact parameter diagram,[7] $\gamma_c \gtrsim 30$.

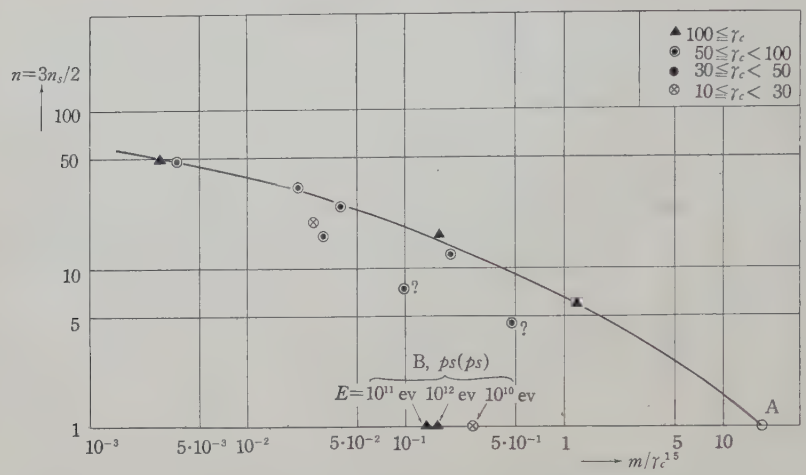


Fig. 5. The plot corresponding to Fig. 1. The experimental materials are due to the Bristol group,[6] A. Debenedetti et al.[22] and M. Schein et al.[23] The question mark beside the point tells us that the corresponding event has not forward-backward symmetry of the angular distribution and so the γ_c-determination is uncertain. The multiplicity n is taken to be three-seconds of the number of shower tracks n_s.

moreover justifies our method of approach, the extrapolation philosophy. The second is that the point B, the result of our calculation for single-pion production with *ps*-coupling, disagrees with the point A of extrapolation from the multiple-production data by a factor of $\sim 10^2$. This disagreement shows that the forward-, backward-peak should be higher than those illustrated in Fig. 3 by a factor of ~ 10 [cf. Eq. (2·2)].

As is suggested by the comparison between Table IA and IB, the angular distribution without the exchange effect yields a good fit with the extrapolation, and for this case, $m \propto \gamma_c^2$.*

(iii) *The transverse momentum of the pion* In Fig. 6, there are shown the distributions of the transverse momentum p_T which is normalized to 1 at $p_T = 600$ Mev/c. Although the peak position is independent of the energy E in accordance with experiment,[6),16)] the positions themselves are somewhat higher than the experimental finding, $p_T \sim 450$ Mev/c. In addition, the widths of the distributions are too broad to be consistent with experiment. As to these points, we may note that the neglect of the exchange effect yields better results, again.

(iv) *The inelasticity* The mean energy of the pion is calculated approximately to be $0.37\,E$ for all energies E under consideration. Therefore, the mean inelasticity $\langle \eta \rangle_{av}$, that is, the ratio of the mean energy of the pion to

* These facts were reported in our paper submitted to the 9th Conference on High Energy Nuclear Physics at Kiev, in 1959.

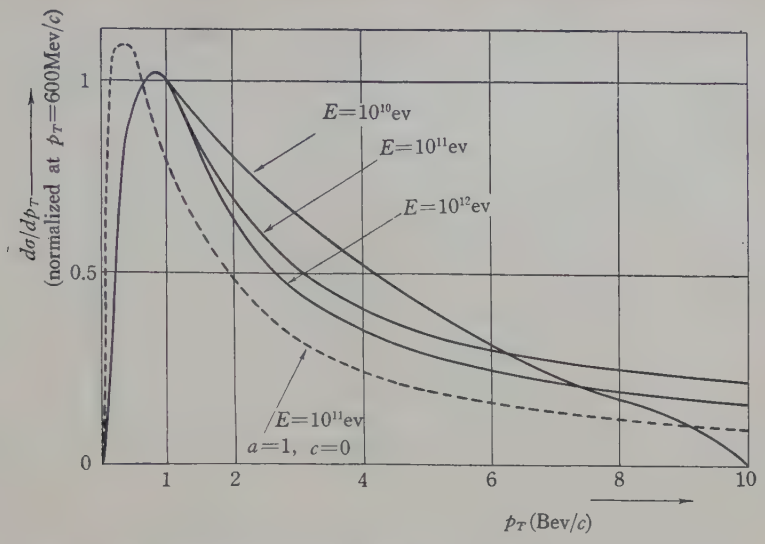


Fig. 6. The distribution of the transverse momentum of the meson calculated by the lowest order perturbation theory. The distribution corresponding to the hypothetical angular distribution is also shown, which is almost the same as that which would be obtained by neglecting the Pauli exchange effect.

the available energy of the nucleon collision, comes out to be 0.37, which is not inconsistent with values suggested by the multiple-production experiments.[6),17),21)] Note that the simple phase space calculation yields 1/3 for the inelasticity. This means that the interaction favours the high energy side of the pion spectrum.

(v) *The total cross section* The total cross section σ calculated for the process $p+p\to p+p+\pi^0$ is tabulated in Table II together with other characteristics mentioned above.

Table II.

E(ev,C.M.S.)	10^{10}	10^{11}	10^{12}
σ(mb), $\frac{G^2}{4\pi}=15$	156	3.0	0.049
m	8.7	142	4800
$\langle\eta\rangle_{av}$	0.36	0.37	0.37

The energy dependence of the cross section can be written as

$$\sigma(p+p\to p+p+\pi^0)\cong 12\left(\frac{G^2}{4\pi}\right)^3\left(\frac{\hbar}{mc}\right)^2\left(\frac{m}{E}\right)^2\ln\left(\frac{E}{m}\right),\quad (E\gg mc^2),\qquad (3\cdot 9)$$

and is almost in agreement with the approximate calculation due to C. Morette.[20)]

(vi) *Nucleon recoil* In connection with the impact-parameter argument presented at the beginning of this section, we have examined the nucleon-recoil

momentum transverse to its direction of incidence. The distribution of it has turned out to have a pronounced peak near μc, thus justifying our anticipation of the large impact parameter.

In conclusion, we hope that it should improve the fit of the theoretical calculation with the extrapolation to take into account some corrections, for example, the final state interaction. The calculation for the two-pion-production process would be helpful for the extrapolation.

§ 4. Perturbation treatment of the single-pion-production process. II

——Pion-nucleon coupling of pseudo-vector type——

In the case of the pion-nucleon coupling of pseudo-vector type, $i(F/\mu)\cdot\bar{\psi}\gamma_5\gamma^\mu\tau\psi\partial_\mu\boldsymbol{\phi}$, the matrix element corresponding to Eq. (3·1) becomes

$$M=-i\frac{1}{(2\pi)^{7/2}}\left(\frac{2m}{\mu}F\right)^3\sqrt{\frac{m^4}{2\omega E_1E_2E_3E_4}}\;T, \tag{4·1}$$

where T is defined by Eq. (3·2) together with

$$T_0=\left[\bar{u}(p_3)\left\{\frac{1}{m}-\frac{k_\mu\gamma^\mu}{2(p_1\cdot k)-\mu^2}-\frac{k_\mu\gamma^\mu}{2(p_3\cdot k)+\mu^2}\right\}u(p_1)\right]\cdot[\bar{u}(p_4)\gamma_5 u(p_2)]\cdot V(2,4). \tag{4·2}$$

The cross section is given by Eq. (3·4). In the approximation described in the previous section, only the term $(1/m\cdot)\bar{u}(p_3)u(p_1)$ survives in the first brackets of Eq. (4·2). Then, the equations corresponding to Eq. (3·5) becomes as follows:

$$\begin{aligned}
&\underset{f\,i}{S\bar{S}}T_0^\dagger T_0=\frac{2}{(2m)^4m^2}(p_1\cdot p_3)\cdot V(2,4),\\
&\underset{f\,i}{S\bar{S}}T_0^\dagger \not{p}_{12}T_0=(1+\not{p}_{12})\frac{1}{(2m)^4 2m^2}\left[(p_2\cdot p_3)-\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{(p_1+p_2)\cdot p_4-2m^2+\mu^2}\right]\cdot V(2,4),\\
&\underset{f\,i}{S\bar{S}}T_0^\dagger \not{p}_{34}T_0=(1+\not{p}_{34})\frac{1}{(2m)^4 2m^2}\left[(p_1\cdot p_4)-\frac{(p_1\cdot p_2)(p_3\cdot p_4)}{p_2\cdot(p_3+p_4)-2m^2+\mu^2}\right]\cdot V(2,4),\\
&\underset{f\,i}{S\bar{S}}T_0^\dagger \not{p}_{12}\not{p}_{34}T_0=0.
\end{aligned} \tag{4·3}$$

The nuclear force factor $V(2,4)$ is defined by Eq. (3·5). The calculation to follow is similar to that for the case of pseudo-scalar coupling (see the Appendix), and we obtain the cross section given by Eq. (A·9). The angular distribution $d\sigma/d\Omega$ at $E=10^{11}$ ev is shown in Fig. 3 together with the angular distribution in the case of pseudo-scalar coupling. As is seen at once from this figure, the pseudo-vector coupling cannot yield the reasonable value of m, Eq. (2·2), as far as the perturbation calculation of the lowest order is concerned. However, the examination on the nucleon recoil transverse to its

direction of incidence, which turns out to be comparable to E, suggests the invalidity of our perturbation calculation in this case.

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Appendix

The calculation of the cross section

A. The total cross section of the process $p+p\to p+p+\pi^0$ due to the pion-nucleon coupling of the pseudo-scalar type is given by [cf. Eq. (3·4) and Fig. A1]

$$\sigma=\frac{1}{16\pi E\sqrt{E^2-m^2}}\left(\frac{G^2}{4\pi}\right)^3\int_{-1}^{1}d\cos\theta\int_{\mu}^{\omega_{max}}d\omega\int_{E_4^-}^{E_4^+}dE_4\int_{0}^{2\pi}d\phi(2m)^4\langle\underset{f\,i}{S\bar{S}}T^\dagger T\rangle_{cons},\quad(\text{A}\cdot1)$$

where the matrix T is given by Eqs. (3·1)$\sim$(3·3). The bracket $\langle\ \rangle_{cons}$ means that the δ-function in Eq. (3·4) has already been integrated out, i.e. we should set $p_1+p_2-p_3-p_4-k=0$ in the matrix elements. And,

$$\begin{aligned}&\omega_{max}=[E^2-m^2+(\mu^2/4)]/E,\\&E_4^{\pm}=\frac{1}{2}\left[2E-\omega\pm\frac{\sqrt{\omega^2-\mu^2}\sqrt{(2E-\omega)^2-(\omega^2-\mu^2)-4m^2}}{\sqrt{(2E-\omega)^2-(\omega^2-\mu^2)}}\right].\end{aligned}\qquad(\text{A}\cdot2)$$

In the following we shall sketch the integration procedure for (A·1). Almost the same procedure applies to the case of the pseudo-vector coupling, too.

To explain the approximation involved, we begin by writing the exact expression, for example,

$$(2m)^4\underset{f\,i}{S\bar{S}}T_0^\dagger T_0$$

$$=4\frac{(p_2\cdot p_4)-m^2}{[(p_2-p_4)^2-\mu^2]^2}\left[\frac{1}{2(p_1\cdot k)-\mu^2}+\frac{1}{2(p_3\cdot k)+\mu^2}\right]^2\cdot[2(p_1\cdot k)(p_3\cdot k)-\mu^2(p_1\cdot p_3)+\mu^2m^2]. \tag{A·3}$$

Rewriting this into the form:

$$(2m)^4\underset{f\ i}{\boldsymbol{S}}\bar{S}T_0^{\dagger}T_0=\left\{2\left[\frac{(p_3\cdot k)}{2(p_1\cdot k)-\mu^2}+\frac{(p_1\cdot k)}{2(p_3\cdot k)+\mu^2}+1\right]\right.$$
$$-2\mu^2\left[\frac{1}{2(p_1\cdot k)-\mu^2}\right]^2\cdot[(p_1\cdot k)-(p_3\cdot k)]$$
$$\left.+\left\{\begin{array}{l}\text{terms with } O(\mu^2, m^2) \text{ as a}\\ \text{factor but without } \dfrac{1}{2(p_1\cdot k)-\mu^2}\end{array}\right\}\right\}V(2,4), \tag{A·4}$$

with $$V(2,4)\equiv\frac{1}{(p_2-p_4)^2-\mu^2}.$$

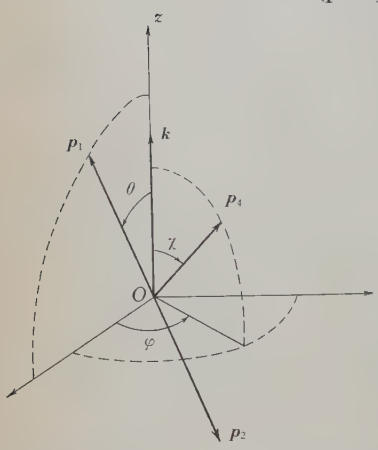


Fig. A1.

We take the first line only as the main contribution. The reason why we have singled out the term in the second line is that the factor $1/[2(p_1\cdot k)-\mu^2]$ makes its contribution, solely at $\theta=0$, similar in order of magnitude to the contribution from the term in the first line and we are interested in the index of peakedness, Eq. (2·2). We have calculated the contribution at $\theta=0$ of the terms such as that shown in the second line of Eq. (A·4), and found that it makes the forward, backward peaks due to the main terms lower by a factor $1-\varepsilon$ with $\varepsilon=(1-\eta)/12$ and $\eta=\omega/E$. Since the energy spectrum of the meson favours large η, we can neglect this ε-correction.

Now, we take the polar coordinates as shown in Fig. A1, and use the unit system $\hbar=c=E=1$.

We consider the integration over ϕ of the nuclear force factor $V(2,4)$. The scalar product is given by

$$(p_2\cdot p_4)=E_4+|\boldsymbol{p}_2|\cdot|\boldsymbol{p}_4|(\cos\theta\ \cos\chi+\sin\theta\ \sin\chi\ \cos\phi).$$

Since

$$\begin{aligned}(p_4\cdot k)&=(p_1+p_2-p_3-k)\cdot k\\&=2\omega-(p_3\cdot k)-\mu^2,\\(p_3\cdot k)&=[(p_3+k)^2-m^2-\mu^2]/2=[(p_1+p_2-p_4)^2-m^2-\mu^2]/2\\&=2(1-E_4)-(\mu^2/2),\end{aligned} \tag{A·5}$$

$|\boldsymbol{p}_4|\cos\chi$ can be written as

$$|\boldsymbol{p}_4|\cos\chi=\frac{E_4\omega-(p_4\cdot k)}{|\boldsymbol{k}|}=-\frac{(2-\omega)E_4+2(1-\omega)-(\mu^2/2)}{\sqrt{\omega^2-\mu^2}}.$$

Making use of these equations, we obtain

$$\int_{\mu}^{2\pi}\frac{1}{(p_2-p_4)^2-\mu^2}d\phi=\pi\frac{\sqrt{\omega^2-\mu^2}}{R},\quad R=\sqrt{AE_4^2+BE_4+C}\ ,$$

where A, B and C are given, in the form expanded in powers of masses, by Table AI.

Table AI. The coefficients of the series expansion of A, B and C ($x\equiv\cos\theta$)

	$m^0,\ \mu^0$	$m^2,\ \mu^0$	$\mu^2,\ m^0$
A	$(2-\omega-\omega x)^2$	$-4(1-\omega)+(2-\omega)\omega x-\omega^2x^2$	$((2-\omega)/\omega-x)x$
B	$-4(1-\omega)(2-\omega-\omega x)$	$2(4-6\omega+\omega^2+\omega x)$	$(\omega-1)(\omega+2)$ $+\{2-(2/\omega)-\omega(1-\omega)\}x$
C	$4(1-\omega)^2$	$-4(1-\omega)^2+\omega^2$ $-4\omega(1-\omega)x-\omega^2x^2$	$2(1-\omega)(1+\omega x)$

Then, calculating the scalar products in the same way as in Eq. (A·5) and integrating over E_4, we get

$$\sigma=\int_{-1}^{1}d\cos\theta\int_{\mu}^{\omega_{max}}d\omega\ 2\pi\frac{d^2\sigma}{d\omega d\Omega}. \tag{A·6}$$

The integrand is defined by Eq. (3·6) together with the following coefficients ($x\equiv\cos\theta$) :

$$a=-\frac{2\omega+1}{2}\frac{\omega-(2-\omega)x}{(2-\omega-\omega x)^2}+\left[\frac{1}{\omega}-\frac{1-\omega}{2-\omega-\omega x}\left\{\frac{1}{\omega}+\frac{\omega(1+x)}{2}\right\}\right]J_0(\omega,\ x)$$
$$+\frac{1}{2}\left(\omega-\frac{1}{\omega}\right)J_1(\omega,\ x)\ -\frac{1}{2}\frac{\omega+(2-\omega)x}{(2-\omega+\omega x)^2}$$
$$+\frac{1}{\omega}\left(1-\frac{1-\omega}{2-\omega+\omega x}\right)J_0(\omega,\ -x)+\frac{1}{2}\left(\omega-\frac{1}{\omega}\right)J_1(\omega,\ -x),$$
$$c=J_0(\omega,\ x)-\frac{1}{4}(1-\omega)J_1(\omega,\ x)+\frac{\omega}{8}\left[3(1-\omega)+(1-3\omega)x\right]_2\frac{1}{-\omega-\omega x}J_2(\omega,\ x)$$
$$+\frac{1-\omega}{8}(3-x)\frac{\omega}{2-\omega-\omega x}J_3(\omega,\ x), \tag{A·7}$$

where

$$J_0(\omega, x)\equiv\int_{E_4^-}^{E_4^+}\frac{dE_4}{R}=\frac{1}{2-\omega-\omega x}\ln\left|\frac{4(1-\omega)(2-\omega-\omega x)^2}{g}\right|,$$

$$J_1(\omega, x)\equiv\int_{E_4^-}^{E_4^+}\frac{2}{(p_1+p_2)\cdot p_4-2m^2+\mu^2}\frac{dE_4}{R}=\frac{1}{2(1-\omega)}\ln\left|\frac{16(1-\omega)^2}{g}\right|,$$

$$J_2(\omega, x)\equiv(1-x)\int_{E_4^-}^{E_4^+}\frac{2}{(p_3\cdot k)+(\mu^2/2)}\frac{dE_4}{R}=\frac{1}{\omega}\ln\left|\frac{8\omega(1-\omega)^2(1-x)(2-\omega-\omega x)}{sg}\right|,$$

$$J_3(\omega, x)\equiv(1+x)\int_{E_4^-}^{E_4^+}\frac{2}{(p_4\cdot k)+(\mu^2/2)}\frac{dE_4}{R}=\frac{1}{\omega(1-\omega)}\ln\left|\frac{16\omega^2(1-\omega)^4(1+x)^2}{sg}\right|,$$

and (A·8)

$$s\equiv\omega^2m^2+(1-\omega)\mu^2,$$

$$g\equiv(1-x)^2\omega^2m^2+2(2-\omega-\omega x)(1-\omega)\mu^2.$$

One remark is in order about the cancellation described just after Eq. (3·6). This cancellation due to the Pauli exchange effect amounts to the complete vanishing of the coefficient of $\ln[1/g]$ in $a(E, \omega, \theta)$ at $\theta=0$. As we have seen in the text, this cancellation makes the value m smaller and so the fit with experiments worse. Similar calculation shows that such cancellation as this occurs also in the case of $p+p\to n+p+\pi^+$, the only reaction that the p-p collision brings about producing a single pion other than that treated in the text.

B. For the case of the pion-nucleon coupling of the pseudo-vector type, Eq. (4·1) and Eq. (4·2), we can use Eq. (A·1) with a replacement $G\to(2m/\mu)F$ and the calculation similar to the above one gives the cross section Eq. (A·6) with

$$\frac{d^2\sigma}{d\omega d\Omega}=\frac{1}{16\pi}\frac{1}{m^2}\left[\frac{1}{4\pi}\left(\frac{2m}{\mu}F^2\right)\right]^3\times$$

$$\times\omega\left[4-\frac{6\omega(\omega+\omega x-2x)}{(2-\omega-\omega x)^2}+\left\{2-\omega+3\omega x-\frac{8(1-\omega)}{2-\omega-\omega x}\right\}J_0(\omega, x)\right.$$

$$\left.+4(1-\omega)J_1(\omega, x)\right]$$

$$+[\text{terms with } x \text{ replaced by } -x]. \qquad (A\cdot9)$$

The error introduced by our approximation neglecting masses is greatest at $\theta=0$, π and is estimated to be $\sim-20\%$ there.

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Strange Neutrino and a Composite Model for Elementary Particle. I*

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A composite model of the elementary particle is proposed by taking the stable particles, namely proton, electron, neutrino and strange neutrino (ν'), as the basic particles. The strange neutrino was introduced by mass reversal theory as the β-decay product of Λ-particle. It differs from the usual neutrino by mass parity and carries a unit of strangeness. Two neutrinos are both left-handed two-component Dirac spinors.

This model simplifies the mechanism of interactions very much because the basic particles are limited to the stable ones. It is noted that the charge independent strong interactions *uniquely* require "*vacuum-cloud*" which is always attached to the core of the elementary particle. The important role of the vacuum-cloud will be revealed throughout this series of papers I—III.

The interactions observed are empirically classified by the three independent reactions inside the core as follows: (a) The origin of the strong interactions is the reaction $P \leftrightarrow \pi^0 P$ where π^0 is just the symbol for the state $P\overline{P}$, (b) the reactions responsible for the leptonic decays of baryons are only $\mu\nu \leftrightarrow e\nu$ and $\mu\nu' \leftrightarrow e\nu'$, and (c) the non-leptonic decays of baryons take place through the strangeness non-conserving reaction, $\nu' \leftrightarrow \pi^0\nu$. The complete selection rules for the leptonic events are established. In our model the event $\Sigma^+ \to N + e^+ + \nu'$ is possible, while Sakata's model forbids this. The existence of a new neutral pion $\pi^{0\prime}$ belonging to $I=0$ is suggested.

§ 1. Introduction

We present in this paper a possible composite model for the elementary particle, using the information obtained in the previous paper[1] entitled "Mass Reversal and Weak Interactions". Among several mass reversal theories[2] Nishijima's theory[3] and ours[4] are consistent and have the explicit operator of mass reversal, while others such as chirality theory[5] seem to have some difficulties. The operator of mass reversal takes the simpler expression in the Heisenberg representation in Nishijima's theory and in the interaction representation in ours. Nevertheless, the individual mass reversal MR(1) of our theory seems to contain some obscure points. One of our purposes is to find out the correct interpretation of MR(1) and the way to modify the universal $V-A$ theory to produce the better fit with the experimental evidences from the model established. We take for granted the correctness of charge independence, the

* This series of papers is essentially the rearranged version of the reports (I)—(III) which appeared in the Soryushiron-Kenkyu (mimeographed circular in Japanese) **20**, 470; **21**, 500, 684 (1960).

conventional strangeness theory and the $V-A$ theory in general cases.

The stable leptons including the strange neutrino and the stable proton are chosen as the basic particles. There arises a fundamental but very difficult problem as to how the leptons can be bounded to proton. We do not tackle with this problem directly, but try empirically the classification of the interactions based on the composite model to see if there are some evidences which may concern with the solution of this problem. The classification is rather satisfactory. One of the important features of our composite model is the appearance of "vacuum-cloud" which comes from the fact that neutron and Λ-particle are not the elementary particles but the composite ones. The empirical discrimination between the characters of usual neutrino and strange neutrino is also attempted because there is no other way at present to distinguish between them.

For convenience sake, let us review two important results of our mass reversal theory in the following.

(1) The neutrino ν produced in β-decay of neutron is not completely the same as the neutrino ν' accompanied by β-decay of Λ-particle but they differ only in mass parity. We call the latter the strange neutrino ν'. Both neutrinos ν and ν' are left-handed two-component Dirac spinors. We are at present far from having a new Dirac equation for neutrino which will explicitly distinguish between ν and ν'. Therefore, we try in this paper to find out the difference of characters between ν and ν' empirically. The importance of the strange neutrino will be easily understood from the facts that Sachs' strangeness theory was derived from the definite relation between mass parity and strangeness[6] and that the interactions between hyperons and heavy mesons taken to derive Sachs' rule were experimentally approved.

(2) It was proved that the strange particle produced at least one neutrino in its leptonic decays. The process $K^+ \to \pi^+ + \nu' + \bar{\nu}$ was forbidden by the invariance under mass parity conjugation.

In § 2 we make a fundamental hypothesis that the basic particles should be the stable ones, namely proton, electron, neutrinos ν and ν', and the other particles are the composite of these. Examining the role of the basic particles, it is found that proton, electron, ν' and ν are the carriers of nucleon number, charge, strangeness and zero element of the additive group respectively and that in the hyperon-meson system, proton, Λ-particle and neutron, which are the basic particles of Sakata's composite model,[7] can be regarded as the carriers of charge, strangeness and zero element of the additive group respectively. Then the composite particles are constructed regardless of the type of interactions with the help of the charge conservation and the nucleon number conservation.

In § 3 the charge independent strong interactions are translated into the expression of the composite particle theory. This translation is *unique* and the results are stated as a theorem. The strong interactions occur through the reaction $P \leftrightarrow \pi^0 P$ ($\pi^0 \equiv P\bar{P}$) inside the "core" of the elementary particle and there

appears around the particle a new type of cloud having the character of both cloud and vacuum, which is, therefore, called "*vacuum-cloud*". This vacuum-cloud is apparently related to the hermiticity of the hamiltonian. The parity conservation in the strong interactions is quite natural in our standpoint, since the two-component neutrinos, which are the parity violating factors, have nothing to do with the strong interactions. It is important that the symbol $P\bar{P}$ represents both pion π^0 with isotopic spin $I=1$ and a new pion $\pi^{0\prime}$ with $I=0$ which is just the excited state of the usual pion π^0. That the mass of $\pi^{0\prime}$ is 301 Mev is suggested from the following type of the very accurate mass formula, originally due to Ohmae,[8]

$$Mass=R[1/(4n+m)^2-1/(4n+m+m'/2)^2] \qquad \text{(A)}$$

where $n=1,\ 2\,;\ m=0,\ 1/2\,;\ m'=1,\ 2,\ 3,\ \cdots$, and $R=38.3$ Bev, which is the new Rydberg-like constant. $4n+m$ and $4n+m+m'/2$ are called "*mass quantum number*", of which the details will be explained in the third paper of this series.

Section 4 is devoted to the leptonic weak interactions. In our model the leptonic decays happen when the constituents of the elementary particle just come out of "core", if necessary, through the reactions $\mu\nu(\nu')\leftrightarrow e\nu(\nu')$ only. And we can deduce from the experiments now available an empirical law which may reflect the difference of the couplings of ν and ν' in the core. The empirical law I states that the universal weak coupling constant f associated with ν production is a little larger than the universal coupling constant f' associated with ν' production when $\hbar=c=r_0=1$ is taken, r_0 being 10^{-13} cm. The possibility for explaining all decays only by the selection rule $\Delta I=1/2$ seems to be open to us if the doublet of universal coupling in the weak interactions and a new neutral pion $\pi^{0\prime}$ are taken into account. We also refer to the general features of vacuum-cloud.

In § 5 a new interpretation of MR(1) is suggested in connection with Heisenberg's unified field theory[9] and the non-leptonic decays of hyperons are analyzed in order to find out the empirical law characterizing the slow decays. The empirical law II states that the non-leptonic decays of baryons take place through the strangeness non-conserving reaction $\nu'\leftrightarrow\pi^0\nu$ and that the up-down asymmetry in the decays becomes maximum only when the hyperon decays into the ground state, which is proton. The independence of two reactions $\mu\nu(\nu')\leftrightarrow e\nu(\nu')$ and $\nu'\leftrightarrow\pi^0\nu$ are generally guaranteed by the mass reversal theory. A modification of universal $V-A$ theory is also indicated by means of MR(1). It is a remarkable fact that the two reactions responsible for the slow decays take exactly the form of $V-A$ couplings.

In § 6 it is shown that the leptonic decays accompany the leptons in a pair such as (e,ν), (μ,ν), (e,ν') and (μ,ν'). It is noted that the unwanted processes $\mu+p\rightarrow p+e$ and $\mu\rightarrow e+e^++e^-$ are forbidden because of the structures of ele-

mentary particles and the special situation of muon. We also refer to the possible decay $\Sigma^+ \to N + e^+ + \nu'$, which is allowed due to the special structure of Σ^+-particle, but forbidden in Sakata's model. Finally a comment on the mass parity conjugation is added.

§ 2. Fundamental hypothesis and structure of elementary particle

As is well known, a boson can be composed of the fermions, but the converse is not true. It will, therefore, be the simplest way to take the fermions only as the basic particles, although it is also possible to choose the fermions and several bosons as the basic ones. And if some of the basic particles are unstable, it is doubtful whether the stable particles can be composed of these or not. For these reasons we take the following assumption:

Fundamental Assumption *Only the stable particles can be taken as the basic particles and all other unstable "elementary" particles are the composites of the basic particles.*

Thus in our model the proton (P), electron (e), neutrino and the strange neutrino are taken as the basic particles. On the other hand, Sakata's model has the unstable Λ-particle and neutron in the basic particles. But his choice is not against our principle as long as the leptons are kept beyond the scope of the discussion, for the stable particle concerned is only the proton which is not the composite but the basic particle.

In addition to the above assumption we have another clue to making the composite model of the particle, namely, the fact that all the elementary par-

Table I. Fermions. The anti-particles are excluded from the table, for the situation is not changed by the inclusion of the anti-particles.

	ν	ν'	e^-	μ^-	P	N	Λ	Σ^+	Σ^0	Σ^-	Ξ^0	Ξ^-
N	0	0	0	0	1	1	1	1	1	1	1	1
Q	0	0	−1	−1	1	0	0	1	0	−1	0	−1
S	0	1	0	0	0	0	−1	−1	−1	−1	−2	−2
mass (Mev)			0.5	105.7			1115		1189			

Table II. Bosons

	γ	π^+	π^0	π^-	K^+	K^0
N	0	0	0	0	0	0
Q	0	1	0	−1	1	0
S	0	0	0	0	1	1
mass (Mev)	0		135			

ticles including leptons are very well distinguished by the eigenvalues of nucleon number (N), charge (Q) and strangeness (S) as shown in Tables I and II. But there are a few exceptional cases. The discrimination between μ^- and e^- or between Σ^0 and Λ must be done by a new quantum number like the mass quantum number because both μ^- and e^- or both Σ^0 and Λ have the same eigenvalues of (N, Q, S). The strangeness of Λ which is the carrier of S in Sakata's model can be transferred to ν', because the relation between strangeness and mass parity is applicable even to the leptons, and the leptonic decays always produce at least one neutrino. In this way the strange neutrino is regarded as the carrier of strangeness in our model,* proton and electron being, of course, the carrier of nucleon number and charge respectively. Then, what is the role of neutrino ν? If we remember that the eigenvalues of N, Q and S satisfy the additive law, the neutrino is easily found to be the zero element of the additive group.

Let us carefully examine the roles of P, e, ν and ν'. ν and e^-, both belonging to the eigenvalues $N=0$ and $S=0$, have $Q=0$ and -1 respectively. This means that they can be regarded as the elements for building up the particles with various eigenvalues of Q. And ν and ν', both belonging to $N=Q=0$, have $S=0$ and 1 respectively, which means that they can be regarded as the elements for making up the particles with various eigenvalues of S. Therefore, it is concluded that the stable leptons, namely e^-, ν and ν', are the minimum number of elements for setting up the particles with the various eigenvalues of Q and S but with the same eigenvalue of $N=0$. In this sense the lepton family can be said to have the simple character. If this statement has any sense, muon (μ^-) and e^- will be expected to be the minimum number of elements associated with a new quantum number.

In order to apply the same argument to the particles with $N=1$ we must introduce the auxiliary basic particles as many as we need. Because the proton with $N=1$ can be regarded as the carrier of charge, a neutral particle with $N=1$ and $S=0$, neutron, is needed to form together with proton the necessary elements of the additive group for Q. Further, another neutral element with $Q=0$ and $S=-1$, which is called Λ-particle, is required to form together with neutron (N) the elements of the additive group for S. The effectively fundamental particles for making the various particles with $N=1$ are, therefore, proton, neutron and Λ-particle which were the basic particles in Sakata's model. In this sense the information obtained by Sakata's model is available for the assignment of the specific structures to the elementary particles and the actual assignment is done by using Tables I and II along with the estimation of

* In the extended strangeness theory[1] in which the strangeness is assigned to the leptons too, the statement as "$\Delta S=1$ for the weak interaction" is not always correct as it depends on the decay interactions.

mass value. The structures obtained independently of the type of interactions are as follows:

$$\left.\begin{array}{l}\text{proton}\to P\ (\text{stable})\\ \text{neutron}\to Pe^-\bar{\nu}\\ \Lambda\to Pe^-\bar{\nu}'\end{array}\right\} \tag{1}$$

$$\left.\begin{array}{l}\Sigma^+\to(\Lambda P\bar{N})\equiv Pe^-\bar{\nu}'P\bar{P}e^+\nu\\ \Sigma^0\to(\Lambda P\bar{P})\equiv Pe^-\bar{\nu}'P\bar{P}\\ \quad\text{or}\ (\Lambda N\bar{N})\equiv Pe^-\bar{\nu}'P\bar{P}e^-\bar{\nu}e^+\nu\\ \quad\text{or}\ (\Lambda\Lambda\bar{\Lambda})\equiv Pe^-\bar{\nu}'P\bar{P}e^-\bar{\nu}'e^+\nu'\\ \Sigma^-\to(\Lambda\bar{P}N)\equiv Pe^-\bar{\nu}'P\bar{P}e^-\bar{\nu}\end{array}\right\}\to Pe^-\bar{\nu}'P\bar{P} \tag{2}$$

$$\left.\begin{array}{l}\pi^+\to(P\bar{N})\equiv P\bar{P}e^+\nu\\ \pi^0\to(P\bar{P})\equiv P\bar{P}\\ \quad\text{or}\ (N\bar{N})\equiv P\bar{P}e^-\bar{\nu}e^+\nu\\ \quad\text{or}\ (\Lambda\bar{\Lambda})\equiv P\bar{P}e^-\bar{\nu}'e^+\nu'\\ \pi^-\to(\bar{P}N)\equiv P\bar{P}e^-\bar{\nu}\\ K^+\to(\bar{\Lambda}P)\equiv P\bar{P}e^+\nu'\\ K^0\to(\bar{\Lambda}N)\equiv P\bar{P}e^+\nu'e^-\bar{\nu}\end{array}\right\}\to P\bar{P} \tag{3}$$

where the simplest structures with $Q=0$, $N=1$ and $S=0$ and with $Q=0$, $N=1$ and $S=-1$ were assigned to neutron and Λ-particle respectively and π^0 belonging to the eigenvalue $(0, 0, 0)$ of (N, Q, S) is shown in § 3 to be reduced to the simpler structure $P\bar{P}$ and accordingly Σ^0 is reduced to $Pe^-\bar{\nu}'P\bar{P}$.

There will be several arguments against the structures (1)−(3). One of the most serious problems will be as follows. Can the leptons having no strong couplings be bounded to proton core? We do not know the correct answer for this at present, but there are some possibilities.

(a) One easy way is to take into account the fact that the structures (1)−(3) have the ambiguity by the matters with $(N, Q, S)\equiv(0, 0, 0)$. If $P\bar{P}\equiv\pi^0$ is chosen as the matter with the value $(0, 0, 0)$ and added to the structure (1), the structures of P, N and Λ turn out, by means of (3), to be

$$P\to P,\quad N\to P\pi^-\quad\text{and}\quad\Lambda\to PK^-, \tag{4}$$

which resemble Goldhaber's model[10] in the point that K-particle carries a unit of strangeness. Corresponding to this change we make an appropriate change for the structure (2) so that all the baryons can be considered to be composed by the strong interactions. Thus the problem is reduced to find the way in

which the leptons are bounded to bosons. The relation (3) suggests that a boson will be formed if the leptons are sandwiched in between P and $\bar{P}$. This will be possible because the clouds of P and $\bar{P}$ will have the leptons as suggested from the β-decays of Λ and N.

(b) Another answer will be given if it is realized in nature that the weak interaction of leptons belonging to the second class with regard to the renormalization becomes stronger than the usual strong interaction at short distance. If this is the case, the high energy leptons will be bounded in the small domain called "core" of the elementary particle. In addition to this it may be possible in the core that the forces working are non-local and that some of the conventional conservation laws are broken. These situations will be favorable for the leptons to be bounded to proton.

However, we do not take the above easy way but keep the original structure without mod. (0, 0, 0) just assuming that the structure (1) can provide the observed mass values of N and Λ-particle. In order to explain the mass difference between N and Λ-particle from (1), the knowledge about the binding forces of neutrinos ν and ν' inside the core will be required. Unfortunately, we do not know any satisfactory theory which suggests what kind of force is effective in the core. In such a situation we attempt in this paper to find out empirically the laws which may give the information on the unknown binding forces concerning the neutrinos.

§ 3. Strong interaction

In this section it is aimed to express the content of the charge independence of the strong interactions in terms of the composite particle theory. Let us first assume that the interactions having $\Delta S=0$ are strong. Then the following important theorem is derived as a direct consequence of the structures (1)−(3) for the elementary particles.

Theorem *The origin of the strong interaction is the reaction* $P \leftrightarrow \pi^0 P$ *in the core of the elementary particle where* π^0 *is a state represented by the symbol* $P\bar{P}$. *And a new type of cloud named "vacuum-cloud" must always be attached to the core and it is symbolized by* $[(e^-\bar{\nu}\cdot e^+\nu),\ (e^-\bar{\nu}'\cdot e^+\nu')]$. *When the isotopic spin is taken into account, the state* $P\bar{P}$ *splits into the usual pion* π^0 *with* $I=1$ *and a new pion* $\pi^{0\prime}$ *with* $I=0$.

Hereafter the symbol π^0 is used in place of the state $P\bar{P}$, unless otherwise stated, and the vacuum-cloud will be symbolized simply by $(e^-\bar{\nu}\cdot e^+\nu)$ or $(e^-\bar{\nu}'\cdot e^+\nu')$ depending on the parts to be emphasized. In virtue of the vacuum-cloud the structure of Σ^0 is reduced to a simpler one $Pe^-\bar{\nu}'P\bar{P}$. From the theorem it is obvious that strong interactions conserve parity.

Proof Among the interactions obeying $\Delta S=0$, we take, for example,

$$N\to P+\pi^-,\quad \Lambda\to P+K^- \tag{5a}$$

$$P\to N+\pi^+,\quad P\to\Lambda+K^+ \tag{5b}$$

$$P\to P+\pi^0,\quad N\to N+\pi^0 \tag{5c}$$

which are, from the structures (1)−(3), expressed by means of the basic particles as

$$\left.\begin{aligned} Pe^-\bar{\nu}&\to P+P\bar{P}e^-\bar{\nu}\\ Pe^-\bar{\nu}'&\to P+P\bar{P}e^-\bar{\nu}'\end{aligned}\right\} \tag{5a}'$$

$$\left.\begin{aligned} P&\to Pe^-\bar{\nu}+P\bar{P}e^+\nu\\ P&\to Pe^-\bar{\nu}'+P\bar{P}e^+\nu'\end{aligned}\right\} \tag{5b}'$$

$$\left.\begin{aligned} P&\to P+P\bar{P}(\alpha+\beta e^-\bar{\nu}e^+\nu+\gamma e^-\bar{\nu}'e^+\nu')\\ Pe^-\bar{\nu}&\to Pe^-\bar{\nu}+P\bar{P}(\alpha+\beta e^-\bar{\nu}e^+\nu+\gamma e^-\bar{\nu}'e^+\nu')\end{aligned}\right\} \tag{5c}'$$

where π^0 is assumed to be a linear combination of $P\bar{P}$, $N\bar{N}$ and $\Lambda\bar{\Lambda}$, and α, β and γ are the arbitrary constants satisfying $\alpha^2+\beta^2+\gamma^2=1$.

Now we must consider the mechanism of the interaction in a general way. Suppose that every elementary particle has the several channels of interaction between the inside of elementary particle and the outside. Each channel will be specified by the way in which the available basic particles supplied by the elementary particle itself and the vacuum-cloud are separated into the several groups in the inside of the elementary particle. During the passage of interaction channel each group will continue growing into the actual composite particle by means of the unknown binding force. We further a speculation about the properties of this unknown force. Once the group is formed, the members of the group and some of symmetry properties which are the attribute concerning the groups will not be changed by this binding force until the group becomes the particle. And the basic particles in the group will behave inside the core as if they were the free particles.

From the speculation mentioned above Eqs. (5a)′−(5c)′ can be regarded as the reactions among the basic particles in the core corresponding to the strong interactions (5a)−(5c). Thus the solution of Eqs. (5a)′−(5c)′ provides the origin of the strong interaction. Because the basic particles are stable and independent of each other, the solution is *uniquely* obtained by comparing the number of each basic particle in both sides of the equations. The solution is

$$\left.\begin{aligned} &P\leftrightarrow P\bar{P}P=\pi^0P,\ \text{from (5a)}'\\ &e^-\bar{\nu}\cdot e^+\nu=e^-\bar{\nu}'\cdot e^+\nu'=(0,0,0)=\textit{vacuum},\ \text{from (5b)}'\end{aligned}\right\} \tag{6}$$

which automatically satisfies Eq. (5c)′, π^0 being reduced to $P\bar{P}$. Even when the charge independent interactions are taken instead of the strangeness conserving ones, the solution (6) is derived again. Therefore, we think that the essential point of the charge independence lies in the mathematical formulation

of the theory to produce the definite relations about the primary couplings.

Let us forget the isotopic spin of the baryons and perform the reaction $P \to \pi^0 P$ to Λ-particle, for example, then

$$\Lambda \equiv P e^- \bar{\nu}' \to (P\pi^0) e^- \bar{\nu}' \to P + K^- \\ \searrow \Lambda + \pi^0. \tag{7}$$

Here appears a new channel $\Lambda \to \Lambda + \pi^0$ which is not contained in the charge independent theory. When we remember the isotopic spin, the pion appearing through the new channel should be a new pion $\pi^{0\prime}$ belonging to $I=0$. In this way $\pi^{0\prime}$ can be put within the framework of the charge independent theory. But it is assumed for the present that $\pi^{0\prime}$ interacts with the strange particles only, for there are no strong experimental evidences to require this pion $\pi^{0\prime}$. It is easily seen that the same argument is valid to all the other interactions of the charge independent theory, completing the proof of the theorem.

Note that the general mechanism of interaction is much simplified in our model because the basic particles are limited to the stable ones, and that the hermitian conjugate interactions (5a) and (5b) have played an important role for the solution, which suggests the important relation between the vacuum-cloud and the hermiticity of the hamiltonian. In the next section we shall study the effect of the vacuum-cloud in a general way. In the third paper it will be suggested that the mass of $\pi^{0\prime}$ is 301 Mev. Here we just mention that $\pi^{0\prime}$ is an excited state of $P\bar{P}(\equiv\pi^0)$. The reaction $P \leftrightarrow \pi^0 P$ can be rewritten as $P\bar{P}P\bar{P} \leftrightarrow 0$, which means that $P\bar{P}P\bar{P}$ is also a state similar to the vacuum-cloud. In the second paper the vacuum-cloud will be shown to be a highly degenerate state containing many pions and kaons, etc., and to be symbolized by

$$[(P\bar{P}P\bar{P}),\ (e^-\bar{\nu}\cdot e^+\nu),\ (e^-\bar{\nu}'\cdot e^+\nu')].$$

If the reaction $P \to \pi^0 P$ is applied to pion and kaon, the boson-boson interactions come out through the following processes,

$$\pi^+ \equiv P\bar{P}e^+\nu \to P\pi^0\bar{P}\pi^0 e^+\nu \to \pi^0+\pi^0+\pi^+, \text{ etc.,} \tag{8a}$$

$$\left.\begin{array}{l} K^+ \equiv P\bar{P}e^+\nu' \to P\pi^0\bar{P}\pi^0 e^+\nu' \to \pi^0+\pi^0+K^+, \\ \text{(When vacuum-cloud is taken into account)} \\ P\pi^0\bar{P}\pi^0 e^+\nu'(e^-\bar{\nu}\cdot e^+\nu) \to K^0+\pi^0+\pi^+ \end{array}\right\} \tag{8b}$$

whose couplings may be taken to be of the same order as the strong nucleon-pion coupling. On the other hand it may be useful to find the difference of behaviors between two neutrinos ν and ν' inside the core in the fact that the pion-nucleon coupling g is a little stronger than the kaon-nucleon one g'. More exactly speaking, we assume that when the factor coming from the charge independence is discarded, the interactions between nucleons have the universal coupling constant g and the couplings between nucleon and strange hyperon

are universally g' and presumably that the couplings between the strange hyperons are g'' in such a way that $g/g'=g'/g''$. This assumption does not seem to meet with any evidences against the experiment, but to favor the description of the non-leptonic decay of baryon under the selection rule $\Delta I=1/2$ only. It is a remarkable fact in connection with the origin of the strong interaction, namely $P\leftrightarrow\pi^0 P$, that the decay events violating the selection rule $\Delta I=1/2$ always contain the neutral pion in their decay products.

In order to illustrate the usefulness of the above assumption let us consider the branching ratio $(K^+\to\pi^++\pi^0)/(K^0\to\pi^++\pi^-)$ under the selection rule $\Delta I=1/2$. The event $K^+\to\pi^++\pi^0$ is evidently forbidden by the $\Delta I=1/2$ rule when π^0 has $I=1$. Fortunately, we have another neutral pion $\pi^{0\prime}$ with $I=0$. And if $\pi^{0\prime}$ is really participating in the decay of K^+, the selection rule allows the event $K^+\to\pi^++\pi^{0\prime}$ where $\pi^{0\prime}$ comes through the interactions like $\Lambda\to\Lambda+\pi^{0\prime}$ in the intermediate state. Therefore, the branching ratio is proportional to $(fg'g'')^2/(fg'g)^2=(g''/g)^2$ where f is the coupling constant of the weak interaction responsible for the above decays. When $g''^2/g^2=g''^2/g'^2\approx 1/10$ is assumed from the spirit of the global symmetry, the observed branching ratio 1/500 will come out because the contribution to the ratio from the mass difference between $\pi^{0\prime}$ and pion and from the matrix elements concerned is estimated to be only a factor. And it remains to be examined whether or not the above explanation of the branching ratio is consistent with the other decay modes of kaon, of which the details will not be discussed here.

§ 4. Leptonic weak interaction

From the general mechanism of interaction it is almost evident that the leptonic event is just the separation of the constituents from the core through some reactions like the alpha particle emission from nucleus. In our standpoint there must be a great difference in the contribution to the masses of N and Λ-particle between ν and ν'.* Is there any experimental evidence which reflects such a difference of character of two neutrinos as the above great one? The answer for this is summarized in the following empirical law.

Empirical Law I *The leptonic decay events are observed when the basic particles come out from the core of elementary particle through the vacuum-cloud. The reactions in the core required for the leptonic decays are only* $e\nu\leftrightarrow\mu\nu$ *and* $e\nu'\leftrightarrow\mu\nu'$. *And for all the weak interactions there is the universality of strength of the coupling constants in the system of the unit* $\hbar=c=r_0(\sim 10^{-13}$ *cm*$)=1$ *in such a way that the coupling constant* f *is realized when* ν *takes part in the interaction and the coupling constant* f' *is observed when* ν' *is produced,* $(f'/f)^2$ *being the order of* 1/10.

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* To express this great difference we will assign the different mass quantum numbers to ν and ν' in the third paper.

The reactions $e\nu \leftrightarrow \mu\nu$ and $e\nu' \leftrightarrow \mu\nu'$ are required for guaranteeing the well-known invariance of the decay hamiltonian under the exchange of $\mu \leftrightarrow e$. Following Empirical Law I, let us write down the possible decay modes to know their coupling constants.

$$\begin{aligned}
&N \to P + e^- + \bar{\nu} && \text{(coupling constant : } f\text{)} \\
&\Lambda \to P + e^- + \bar{\nu}' && (\quad ,, \qquad f') \\
&\Lambda \to P + \mu^- + \bar{\nu}' && (\quad ,, \qquad f') \\
&\pi^\pm \to \pi^0 + e^\pm + \nu && (\quad ,, \qquad f) \\
&K^+ \to \pi^0 + e^+ + \nu' && (\quad ,, \qquad f') \\
&K^+ \to \pi^0 + \mu^+ + \nu' && (\quad ,, \qquad f')
\end{aligned} \tag{9}$$

where π^0 stands for the usual pion π^0 or $\pi^{0\prime}$. The mode $\pi \to \pi^0 + e + \nu$ was first introduced by Feynman-Gell-Mann[11] in order that the vector part of β-decay coupling requires no renormalization.

Next let us refer to the effect of the vacuum-cloud. When the pion in the three-body decays of bosons in (9) is absorbed into the vacuum-cloud $(P\bar{P}P\bar{P}) \equiv (\pi^0 \cdot \pi^0)$ by means of strong reaction $P \leftrightarrow \pi^0 P$, there come out the two-body decays such as

$$\begin{aligned}
&K^+ \to e^+ + \nu', \text{ and } K^+ \to \mu^+ + \nu' \quad \text{(coupling : } f'\text{)} \\
&\pi \to e + \nu \qquad \text{and } \pi \to \mu + \nu \qquad \text{(coupling : } f\text{)}.
\end{aligned} \tag{10}$$

Although stressed in § 3, the vacuum-cloud is in any case inevitable to derive the hermitian conjugate interactions. For example, the hermitian conjugate interaction $P \to N + e^+ + \nu$ of the slow β-decay is possible through the following processes. The proton surrounded with the vacuum-cloud is denoted by $P \cdot (e^-\bar{\nu} \cdot e^+\nu)$ and is separated into $Pe^-\bar{\nu} + e^+\nu \to N + e^+ + \nu$, which is just the wanted interaction. In the similar way the hermitian conjugate ones of the strong interaction and the non-leptonic interaction are obtained by taking into account in addition the reactions $P \leftrightarrow \pi^0 P$ and $\nu' \leftrightarrow \pi^0 \nu$ respectively. On the other hand, the hermitian conjugate interactions have the same coupling, which will allow us to propose a rule concerning the vacuum-cloud as follows :

Rule *The interaction appearing through the effect of the vacuum-cloud and the reactions characterizing the interaction itself has the same coupling as the original interaction.*

Obviously, this rule is correct for the hermitian conjugate interactions. If this rule is applied to the general case, the interaction (10) can take the couplings as indicated in (10) and also the interactions (8a) and (8b) take the couplings g and g' respectively. The interactions (10) differ from the corresponding ones in (9) by the contribution of strong interaction in the virtual state. Therefore, the contribution of the strong interaction to the matrix

element of the above decay processes must be almost a unity, which is just Ogawa's empirical law.[12] In this situation we expect that the above rule is valid not only for the weak interaction but also for the strong interaction.

On the other hand, Wakasa[13] pointed out that f' could be written as $f'=f\varepsilon$ where ε was considered to represent the non-local effect in the decay hamiltonian and that the better fit with the experiments was given by choosing $\varepsilon=1/4\sim1/10$. In our model the non-local effect due to the space-time extension of the elementary particle is always expected and the rule concerning the non-local effect ε is determined *a priori* from the structures (1)−(3) as summarized in Empirical Law I. But in Wakasa's approach we must choose the interactions showing the non-local effect in such a way that the best fit with the experimental evidences is obtained.

Finally, let us examine the consistency of Empirical Law I, taking some examples. Detailed calculation is not made in this paper because of the lack of the dynamical theory for the composite particle. The decay $\Lambda\to P+e^-+\bar{\nu}'$ is known to have the smaller decay rate than the prediction in the universal $V-A$ theory by, at least, one order of magnitude. The same is true for the $K\to\mu+\nu'$ decay whose rate in the $V-A$ theory is about 20 times shorter than the observed one. These are the reasons for taking the different couplings f or f' as indicated in (9) and (10). The consistency of the assignment of the couplings as in (9) and (10) will be easily recognized from the approach by Sakita[14] who calculated the partial lifetime of $K\to\mu+\nu'$ by dispersion technique on the basis of the $V-A$ type Fermi interactions with the couplings as in (9) and also from the several works by Oneda[15] who made the comprehensive discussions along the similar line to ours. Add that the ratio $W(\Lambda\to P+e^-+\bar{\nu}')/W_{(exp)}(\Lambda\to P+\pi^-)\approx1/1000$[15] where the decay probabilities $W(\Lambda\to P+e^-+\bar{\nu}')$ and $W(\Lambda\to P+\pi^-)$ were estimated by taking as the primary interactions $K_{e_3}\to\pi^0+e^-+\bar{\nu}'$ and $K^0\to\pi^+ +\pi^-$ with the coupling f' respectively and $W_{exp}(\Lambda\to P+\pi^-)$ is the experimental value.

§ 5. Non-leptonic decay interaction and MR(1)

When the non-leptonic decays of hyperons were examined in connection with the possible reactions inside the core, a distinct regularity about the reaction responsible for the non-leptonic decays was found. This regularity is stated as follows:

Empirical Law II *The strangeness non-conserving reaction $\nu'\leftrightarrow\pi^0\nu$ is the origin of the non-leptonic decays of baryons.* The asymmetry parameters*

* $K_{\pi 2}$ and $K_{\pi 3}$ take place through the reactions $\nu'\leftrightarrow\pi^0\nu$, $P\leftrightarrow\pi^0 P$ and the effect of vacuum-cloud.

for the decays of hyperon become maximum only when the hyperon decays into the ground state, which is the proton.

The latter statement of Empirical Law II seems to suggest the existence of mass levels for the elementary particle and such is the case as shown in the third paper. The reaction $\nu' \leftrightarrow \pi^0 \nu$ has in general nothing to do with the leptonic decays of baryons because in the extended strangeness theory the leptonic processes and the non-leptonic decays of hyperons statisfy the selection rules $\Delta S=0$ (mod. 2) and $\Delta S=1$ respectively. The deduction of Empirical Law II is presented in connection with Heisenberg's unified theory because our model apparently has several points in common with his theory. The discussions in this section are partly based on the speculation about the mechanism of interaction in § 3. In fact the symmetry property of the weak interaction is derived by treating all the particles concerned as free. And the results obtained on this assumption seems to be in accordance with the experiments.

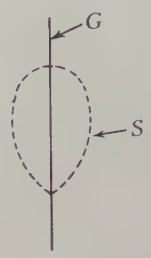


Fig. 1. The full line (G) indicates the Green function of the particle with zero mass and the dotted ones (S) represent those of the particles with non-zero mass.

In the Heisenberg theory[9] the nucleon is evidently represented by the graph in Fig. 1. Comparing the graph with the structure of nucleon in our model, we can say that the full line corresponds to the neutrino and the two dotted lines to proton and electron. This resemblance of graph and structure will give us an important suggestion about a new interpretation of MR(1). Moreover, in the Heisenberg theory the wave function $\varphi(x)$ for the one-particle state satisfies the equation in a certain approximation

$$(\gamma_\nu q_\nu) Q(-q^2)\varphi(x)=0, \tag{11}$$

which is invariant under the transformation $\varphi(x) \to \gamma_5 \varphi(x)$. In his theory this transformation is due to the neutrino-like property of the field operator, which suggests in our model the γ_5-transformation for the particles containing the basic neutrinos.

Let us permit the invariance of the interaction hamiltonian under the following transformation:

$$\left.\begin{aligned} &\psi_P \to \psi_P \\ &\psi_N \to \gamma_5 \psi_N, \quad \psi_\Lambda \to \gamma_5 \psi_\Lambda \quad \text{and} \quad \psi_\Sigma \to \gamma_5 \psi_\Sigma, \end{aligned}\right\} \tag{12}$$

where ψ_P is not changed because the proton does not contain the neutrino. Assuming the interactions responsible for the non-leptonic decays to be

$$f \bar{\psi}_\alpha \gamma_\mu O \psi_\beta \partial_\mu \phi_\pi, \tag{13}$$

the matrix O is determined by the requirement of the invariance of (13) under the transformation (12) as follows:

$$O \doteqdot 1+\gamma_5 \quad \text{for} \quad \Lambda \to P+\pi^-, \tag{14a}$$

$$O = a+b\gamma_5 \quad \text{for} \quad \Lambda \to N+\pi^0, \tag{14b}$$

$$O = 1+\gamma_5 \quad \text{for} \quad \Sigma^+ \to P+\pi^0, \tag{14c}$$

$$O = a'+b'\gamma_5 \quad \text{for other decays of } \Sigma, \tag{14d}$$

where a, a', b and b' are the arbitrary constants. By the same procedure the interactions of β-decay and μ-capture are shown to be the same as those predicted by the $V-A$ theory. The MR(1) theory also yields the same results as the above except the cases of (14b) and (14d). This is the reason why the invariance by MR(1) is considered to be correct for all the weak interactions. But we rather take the view that the operation of MR(1) will simply be the gauge transformation of the neutrino field.

Now we are ready to obtain some evidences for Empirical Law II. First of all let us demonstrate how the decay event $\Lambda \to P+\pi^-$ takes place. The strange neutrino $\bar{\nu}'$ parting from the main part of Λ-particle changes by the reaction $\nu' \to \pi^0\nu$ into $\pi^0\bar{\nu}$ which will come out straightforward from the core of Λ-particle. The $\pi^0\bar{\nu}$ must always accompany e^- leaving proton alone, otherwise $\pi^0\bar{\nu}$ cannot be the particle listed in (1)$-$(3). Thus the event $\Lambda \to P+\pi^-$ takes place through the following process:

$$\Lambda \equiv Pe^-\bar{\nu}' \to Pe^-(\pi^0\bar{\nu}) \to P+\pi^0e^-\bar{\nu} \to P+\pi^- \tag{14a}'$$

Similarly the event $\Lambda \to N+\pi^0$ occurs through the diagram as

$$\Lambda \equiv Pe^-\bar{\nu}' \to Pe^-(\pi^0\bar{\nu}) \to$$

and further the separated neutrino $\bar{\nu}$ makes the *recombination* into proton to finish the decay as

$$\to Pe^-\bar{\nu}+\pi^0 \to N+\pi^0. \tag{14b}'$$

There is apparently a difference of the reactions between the events (14a)$'$ and (14b)$'$. The same thing happens in the decays of Σ-particle. These are the reasons why Empirical Law II is presented. Let us carefully examine the processes (14a)$'$ and (14b)$'$. The wave function of Λ-particle, say ψ_Λ, seems to behave like the operation which is proportional to the creation operator $\psi_{\bar{\nu}'}$ of neutrino in case of (14a)$'$ and to the operator $\bar{\psi}_{\bar{\nu}}\psi_{\bar{\nu}'}$ in case of (14b)$'$. Corresponding to the gauge transformations $\psi_\nu \to \gamma_5\psi_\nu$ and $\psi_{\nu'} \to \gamma_5\psi_{\nu'}$ of the neutrinos, the event (14a)$'$ must be invariant under $\psi_\Lambda \to \gamma_5\psi_\Lambda$, while in the event (14b)$'$ ψ_Λ does not undergo such a transformation, hence produces the results (14a) and (14b) again.

By the same argument the decays of Σ-particle are distinguished. Under the reaction $\bar{\nu}' \to \pi^0\bar{\nu}$, Σ-particle changes as

$$\Sigma^+ \equiv Pe^-\bar{\nu}' \cdot P\bar{P}e^+\nu \to Pe^-(\pi^0\bar{\nu}) \cdot P\bar{P}e^+\nu \Rightarrow$$

and (a) when the neutrino $\bar{\nu}$ produced *runs away* from the core of Σ-particle, carrying e^- to the outside of the core

$$\Rightarrow (P\pi^0)\cdot P\bar{P}(e^-\bar{\nu}\cdot e^+\nu)$$
$$\rightarrow P+\bar{P}P+(\text{vacuum-cloud})\rightarrow P+\pi^0 \qquad (14c)'$$

and (b) when the neutrino $\bar{\nu}$ makes the *recombination* before going out of the core

$$\Rightarrow (P\pi^0)e^-\bar{\nu}+P\bar{P}e^+\nu\rightarrow Pe^-\bar{\nu}+P\bar{P}e^+\nu\rightarrow N+\pi^+ \qquad (14d)'$$

where for both events the strong reaction $P\pi^0\rightarrow P$ and the effect of vacuum-cloud are taken into account. Now it will be evident that (14c)′ is invariant under $\psi_\Sigma\rightarrow\gamma_5\psi_\Sigma$, while (14d)′ have not such a transformation property, thus leading to the results (14c) and (14d). It is important that the decay events (14) are sharply distinguished by the reactions such as "*run away*" and "*recombination*" and that the above procedure can be taken to be another interpretation of (12), suggesting more clearly that the transformation (12) is due to the gauge transformation of neutrinos.

The application of such a procedure to Fermi interactions should be done carefully. In Fermi interactions the neutrino leaving the fermion keeps the neutrino-like character, while in the non-leptonic interaction it loses its character to be transmuted into a boson. Thus in β-decay and μ-capture interactions wave functions of proton and neutron are not allowed to perform γ_5-transformation as (12), producing the following results:

$$\left.\begin{aligned}&\text{for } \beta\text{-decay,}\quad H=f(\bar{\psi}_P\gamma_\mu(a+b\gamma_5)\psi_N)(\bar{\psi}_e\gamma_\mu(1+\gamma_5)\psi_\nu)\\&\text{for } \mu\text{-capture,}\quad H=f(\bar{\psi}_P\gamma_\mu(a'+b'\gamma_5)\psi_N)(\bar{\psi}_\mu\gamma_\mu(1+\gamma_5)\psi_\nu)\end{aligned}\right\} \qquad (15)$$

where a, b, a' and b' are the arbitrary constants. These hamiltonians are more favorable for explaining the ratio C_V/C_A of couplings of β-decay than the universal $V-A$ theory in which the coupling ratio seems to be hardly explained even when the effects of the strong interactions are taken into account. But in our model this ratio is arbitrary.

It is known from the experiment that the asymmetry parameters for (14a) and (14c) are maximum. The results in (14) are, therefore, consistent with the experiment. The discussion about the decays of Σ-particle has several points in common with the orthodox method by Takeda and Kato.[16] These points are that for the explanation of the asymmetry parameter of decays of Σ-particle both the strong interaction and the structure of Σ-particle, which is dissociated like $\Lambda\cdot\pi$, have played an important role. Note that the similar discussions in this section will be applicable to the decays of the cascade particle. In the conventional theory having the $\Delta I=1/2$ rule only for the decays the asymmetry parameter of the decay $\Lambda\rightarrow N+\pi^0$ is shown to be the same as that of $\Lambda\rightarrow P+\pi^-$, while in our case it is not always the case. Therefore, it

can be a test of our model to see whether the event $\Lambda \to N + \pi^0$ has the maximum asymmetry parameter or not.

§ 6. Selection rule for decay events

It is easily seen that all the unstable particles produce the leptons in their leptonic decays in the pair of $(e\nu)$ and $(e\nu')$ or $(\mu\nu)$ and $(\mu\nu')$ because in our model they always keep the neutrinos in the form of $(e\nu)$ and $(e\nu')$ or $(\mu\nu)$ and $(\mu\nu')$. Therefore, the selection rule will become complete if the following three modes can be forbidden.

$$\mu^- + P \to P + e^-, \quad \mu \to e + e^+ + e^- \text{ and } \mu \to e + \gamma \tag{16}$$

This is the main problem in this section.

When the muon is captured by the proton core, there is no other route than the usual one to $N \cdot \nu$ because in our model the reactions in the core are limited to $\mu\nu \leftrightarrow e\nu$, $\mu\nu' \leftrightarrow e\nu'$, $P \leftrightarrow \pi^0 P$ and $\nu' \leftrightarrow \pi^0 \nu$; besides $P\nu\bar{\nu}$ is not the proton itself. In this situation $\mu^- + P \to P + e^-$ is not observed. If the necessary reaction $\mu\nu \leftrightarrow e\nu$ is considered to be responsible for the decay $\mu \to e + \nu + \bar{\nu}$, the reaction $\mu \cdot e \leftrightarrow e \cdot e$ would be required in order to make the process $\mu \to e + e^+ + e^-$ possible. Fortunately such a reaction was not needed at all. We hope that this fact may be the correct answer to the question why the three electron decay of muon is not observed. On the other hand, the study of the mass formula (A) suggests that the muon is *in a certain sense* the "*excited state*" of the electron. The unique decay mode of muon is, therefore, the transition to the ground state, which is just the event $\mu \to e + \nu + \bar{\nu}$. Thus the possible mode $\mu \to e + e^+ + e^-$ is forbidden.

The $F_{\mu\nu}$ type interaction for the process $\mu \to e + \gamma$ can be forbidden, provided the invariance of hamiltonian under mass reversal and the universality of the electromagnetic interaction are satisfied. Therefore, $\mu \to e + \gamma$ can rarely occur only through the two weak Fermi interactions because the $\mu \to e + e^+ + e^-$ has been forbidden. Thus the complete selection rules are established for the leptonic decays. For the same purpose the intermediate charged boson theory has been investigated by many authors. But such a theory has the defect that the $\mu \to e + \gamma$ is possibly observed as shown by Feinberg et al.,[17] whereas in our model the proton plays the similar role as the charged boson, so that the defect of the charged boson theory does not appear.

It should be mentioned that a special situation exists in the leptonic decays of Σ'^+-particle. In our model the decay $\Sigma'^+ \to N + e^+ + \nu'$ is possible through the reaction $\nu' \leftrightarrow \pi^0 \nu$, although in Sakata's model it is not allowed. This strange situation comes from the fact that Σ'^+-particle contains two kinds of neutrinos ν and ν' in the core. K^0 also will have the special situation in the leptonic decays. Look at the structure of K^0 in (3). Now let us show the diagram leading to the actual decay. For Σ'^+-particle, the reaction $\nu' \leftrightarrow \pi^0 \nu$ is just the

simultaneous reactions $\nu' \to \pi^0 \nu$ and $\nu \to \pi^0 \nu'$ and Σ^+-particle changes as follows:

$$\begin{aligned}\Sigma^+ &\equiv Pe^- \bar{\nu}' \cdot P\bar{P}e^+ \nu \to Pe^- (\bar{\nu}\pi^0) \cdot P\bar{P}e^+ (\pi^0 \nu') \\ &= Pe^- \bar{\nu} \cdot P\bar{P}e^+ \nu' \to (P \cdot P\bar{P}) e^- \bar{\nu} \cdot e^+ \nu' \to Pe^- \bar{\nu} \cdot e^+ \nu' \\ &\to N + e^+ + \nu'\end{aligned}$$

where $P\bar{P}$ in the final stage is absorbed into proton by means of the strong reaction $P\pi^0 \to P$. Note that this decay obeys the rule $\Delta S = 1$ in the conventional strangeness theory but satisfies the rule $\Delta S = 2$ in the extended one. This cannot be the reason for discarding that event, because the mass reversal theory demands the selection rule to be $\Delta S = 0$ (mod. 2) for the leptonic decays.

Finally a comment is added on the possible process $K \to \pi + \bar{\nu} + \nu'$, which was forbidden in the mass reversal theory by requiring a new symmetry named "mass parity conjugation". The mass parity conjugation, which implies the invariance by the interchange of ν and ν', is possible from the fact that the reactions in the core are invariant by the interchange of ν and ν'. And fortunately the structure of kaon itself does not allow the process $K \to \pi + \bar{\nu} + \nu'$.

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A Remark on the Strangeness and Isotopic Spin for Leptons*

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It is examined whether the strangeness and isotopic spin assignment for leptons proposed by Gamba, Marshak, Okubo and Klein fits the leptonic weak interactions to the same extent as in the case of the weak interactions between baryons and mesons. It is shown that there should exist two kinds of neutral massless lepton (ordinary and strange neutrino) in order to accept the selection rule $|\Delta S|=1$. Then the all observed weak interactions satisfy this rule, but the strangeness assignment allows some strangeness conserving leptonic processes such as $K^0\to\mu^+ + e^-$ and $K^+\to\mu^+ + e^- + \pi^+$. Further, there are some undesirable processes in which strangeness changes by one unit, for instance, $\mu^+\to 2e^+ + e^-$ and $\mu^- + p\to p + e^-$. At the present status of experiments, it is impossible to get any definite conclusion about the isotopic spin selection rule. It is however shown that, except for $\mu\to e+\nu+\tilde{\nu}$, $\mu^- + p\to n+\nu$ and probably $\pi\to\mu+\nu$, the selection rule $|\Delta I|=1/2$ may be regarded to hold in fairly good approximation, if we restrict ourselves to the interactions in which the strangeness change is ± 1. The significance of the existence of the aforementioned important exceptional cases is discussed.

§ 1. Introduction

As to baryons and mesons, baryon number, strangeness (or hypercharge), isotopic spin and its third component can be regarded as such a set of quantum numbers that descriminates each state of elementary particles. Namely, they are not only sufficient to specify the states of elementary particles, except for leptons, but are the most suitable quantities to describe the selection rules in the interactions between baryons and mesons. On the other hand, there are many attempts to assign isotopic spin to leptons, and by one of us (T. O.) it is pointed out that leptons also should have the internal degree of freedom of hypercharge,[1] and the existence of a similar relation for leptons between electric charge, hypercharge and the third component of isotopic spin to the one of baryons and mesons is suggested.[2] But any definite answer to these problems is not yet obtained. Among many attempts the recent assignment proposed by Gamba-Marshak-Okubo[3] and Klein[4] called our attention,** because it exhibits two kinds of analogy between

* The content of this article was read at the Meeting of Physical Society of Japan held on October 16, 1960, at Kyūshū University.

** Here it may be noticed that the assignment assumed in Heisenberg's unified theory of elementary particles[5] coincides with their assignment.

baryons and leptons, that is, the one in the expression of electric charge and the other in the mass spectra of (ν, e, μ) and of (p, n, Λ). Further a new selection rule is proposed on the basis of the analogy by them, but this selection rule has nothing to do with the isotopic spin and strangeness selection rule in ordinary sense.

Their assignment allows the following expression for electric charge,

$$Q=I_3+\frac{S-L}{2} \tag{1}$$

where S is the analogue of strangeness for baryons and mesons and L is the lepton number. Further, e^-, μ^- and ν are assumed to be leptons. Here we may ask why I_3 appearing in the above equation is not a simple parameter and must be regarded as the third component of a vector in a three-dimensional rotation space (isotopic spin space). For baryons and mesons, there are strong interctions which are invariant under the rotation of the isotopic spin space, and it is quite clear that the isotopic spin is a very convenient concept to understand the behaviour of elementary particles. Usually the various kinds of the isotopic spin assignment for leptons are based only on the mass spectrum and charge multiplicity. But the other important role of isotopic spin, the selection rule with respect to it in the leptonic process, is not taken into account. Unfortunately, leptons seem to have no strong interactions. Therefore, we should consider that as to leptonic processes the invariance under the rotation of isotopic spin space does not hold in any case, if we keep the parallelism between the isotopic spin of baryons and that of leptons. The only way to justify the isotopic spin assignment of leptons may be to ascertain whether it fits the selection rules for weak interactions, putting aside the problem why leptons have no charge independent interactions. The same arguments hold on the quantity S.

In non-leptonic weak interactions, the selection rule $|\Delta S|=1$ on strangeness S is well established, and the the selection rule $|\Delta I|=1/2$ on isotopic spin I holds in the fairly good approximation.[6] In this article, it is investigated to what extent the selection rules $|\Delta S|=1$ and $|\Delta I|=1/2$ and the assignment of S and I for leptons proposed by Gamba-Marshak-Okubo and Klein are adopted to the leptonic weak interactions.

§ 2. Strangeness selection rule and the existence of two kinds of neutrino

At first, we want to point out that there must be two kinds of neutral massless lepton (ordinary and strange neutrino) in order to accept the proposed strangeness assignment for (μ, e, ν), (Table 1), and the selection rule $|\Delta S|=1$. As easily seen, the neutral lepton emitted in the β-decay of neutron must have the non-zero value of strangeness. Therefore, this lepton must be considered different from the ordinary neutrino, ν, which is the member of the iso-doublet

(ν, e). Further, taking account of the relation (1), this must be the anti-particle of a neutral massless lepton ω with $S=+1$ and $I=0$ (strange neutrino), that is,

$$n\to p+e^-+\tilde{\omega}.$$

Table I

	μ^-	ν, e^-	ω
S	-1	0	$+1$
I	0	1/2	0
I_3	0	$+(1/2)$, $-(1/2)$	0

The other possibility* that ω is the neutral member of an iso-triplet is rejected if we exclude the existence of the doubly charged elementary particles. The assignment adopted here is listed in Table I.

In the following the main leptonic interactions satisfying the selection rule $|\Delta S|=1$ are shown, in order to specify the neutrino appearing in various processes:

$$\pi^+\to\begin{cases}\mu^++\nu,\\ e^++\omega,\end{cases}$$

$$\mu^+\to\begin{cases}e^++\nu+\tilde{\nu},\\ e^++\omega+\tilde{\omega},\end{cases}$$

$$\mu^-+p\to n+\nu,$$

$$K^+\to\begin{cases}\mu^++\omega,\\ \mu^++\omega+\pi^0,\\ e^++\nu+\pi^0\\ \qquad\vdots\\ e^++\nu,\end{cases}$$

$$\Sigma^+\to\begin{cases}\Lambda+e^++\omega,\\ n+e^++\nu,\end{cases}$$

and

$$\Lambda\to p+e^-+\tilde{\nu}.$$

The unobserved decay mode

$$K^\pm\to\mu^\mp+e^\pm+\pi^\pm$$

is forbidden, as there the strangeness change is two units. But, in the similar reaction,

$$K^\pm\to\mu^\pm+e^\mp+\pi^\pm,$$

the strangeness is conserved.

Concerning neutral K-mesons, the situation is somewhat complicated, due to

* There remains one more possibility in which ω and μ^- constitute an iso-doublet. But this scheme is of little interest, as no difference between (ν,e^-) and (ω,μ^-) appears there.

the particle mixture character of neutral K-mesons in their decay processes. The observed decay

$$K_2^0 \to e^{\pm} + \nu + \pi^{\mp}$$

is consistent with our selection rule. The same decay scheme for K_1^0 is also possible. The reaction

$$K_2^0 \to \begin{cases} \mu^+ + \omega + \pi^- \\ \mu^- + \tilde{\omega} + \pi^+ \end{cases}$$

can be regarded as allowed in the sense that the upper mode occurs through K^0-amplitude and the lower one through $\tilde{K}^0$-amplitude in K_2^0. We have the same situation about the decay mode

$$K^0_{1,2} \to \pi^{\pm} + \pi^0 + \mu^{\mp} + \omega(\tilde{\omega}).$$

Our scheme fails not only to reject the mode

$$K^0_{1,2} \to e^{\pm} + \mu^{\mp},$$

but imposes a strong interaction between K^0-($\tilde{K}^0$-)meson and $\mu^+ e^-(\mu^- e^+)$ pair, becuase in $K^0 \to \mu^+ + e^-$ and $\tilde{K}^0 \to \mu^- + e^+$ strangeness is conserved. But this decay mode is expected not to occur in nature,* and to be forbidden by some other selection rules or assumptions.

Finally it may be remarked that there are some undesirable leptonic processes, which have the property $|\Delta S| = 1$, for instance, $\mu^+ \to e^+ + e^+ + e^-$ and $\mu^- + p \to p + e^-$.

§ 3. Isotopic spin selection rule

The second problem is to examine to what extent the selection rule $|\Delta I| = 1/2$ fits the leptonic weak interactions. At the present stage, we can not give a definite answer to this problem, because the sufficient results about the concerned experiments are not obtained yet.

The only one observed process which satisfies the above selection rule exactly is $K^+ \to \mu^+ + \omega$, and this is due to the iso-singlet nature of μ as well as ω. But this process is the main decay mode of charged K-meson, and amounts to about 88 percent of the observed leptonic decay of K^+ meson.[6]

The main objection to the $|\Delta I| = 1/2$ rule is the decay $\mu \to e + \nu + \tilde{\nu}$ and the μ capture process $\mu^- + p \to n + \nu$. By the selection rule the interaction which discribes the former process is restricted to such a combination of two terms:

$$(\bar{E}\tau x)\mu(\bar{E}\tau E)$$

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* The reaction $K^+ \to \mu^+ + \nu$, which is the counter reaction to $K^0 \to \mu^+ + e^-$ from the point of view of isotopic spin is also a stangeness conserving process. The observed $K_{\mu 2}$ decay is interpreted as $K^+ \to \mu^+ + \omega$, as mentioned above.

$$=2(\bar{\nu}\,\mu)(\bar{e}\,\nu)-(\bar{e}\,\mu)[(\bar{\nu}\,\nu)-(\bar{e}\,e)]$$
$$=(\bar{\nu}\,\mu)(\bar{e}\,\nu)+(\bar{e}\,\mu)(\bar{e}\,e),$$

where the type of interaction is assumed to be vector and axial vector, the concerned γ-matrices being omitted, and the Fierz's relation[7] is used in the transition from the second expression to the last one. Further, the spurion technique[8] is used and

$$E=\begin{pmatrix}\nu\\ e^-\end{pmatrix},\quad x=\begin{pmatrix}0\\ 1\end{pmatrix},\quad y=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

The other expression $(\bar{E}x)\mu(\bar{E}E)$ gives the same combination. Therefore, the single term $(\bar{\nu}\mu)(\bar{e}\nu)$ does not satisfy the selection rule $|\Delta I|=1/2$, and by the rule the process $\mu^-\to 2e^-+e^+$ should be observed at the equal probability to the ordinary μ-e-decay. But, the anomalous μ-e-decay has not been observed. The only way to avoid this objection in this scheme is to forbid the process $\mu\to e+\nu+\tilde{\nu}$ entirely by some unknown selection rule, and to assume that the neutral massless leptons in μ-e-decay is ω and $\tilde{\omega}$. The process $\mu\to e+\omega+\tilde{\omega}$ exactly satisfies the $|\Delta I|=1/2$ rule, owing to the iso-singlet nature of μ and ω. The capture process is accompanied with the process $p+\mu\to p+e$ and/or $n+\mu\to n+e$ in the spurion interactions, because we have

$$(\bar{N}\boldsymbol{\tau}N)\bar{\mu}(x^*\boldsymbol{\tau}E)=2(\bar{p}n)(\bar{\mu}\nu)-\{(\bar{p}p)-(\bar{n}n)\}(\bar{\mu}e)$$

and we cannot isolate the term $(\bar{p}n)(\bar{\mu}\nu)$ from the others without violating the $|\Delta I|=1/2$ rule, even if we combine the above interaction with another one

$$(\bar{N}N)\bar{\mu}(x^*E)=\{(\bar{p}p)+(\bar{n}n)\}(\bar{\mu}e).$$

On the other hand, the recent experiments[9] report the reaction rate

$$\frac{W(\mu^-+N\to e^-+N)}{W(\mu^-+N\to\nu+N)}=(4^{+3}_{-2})\times 10^{-6},$$

in contradiction with the $|\Delta I|=1/2$ rule which predicts for this ratio to be nearly equal to $1/2$.

As to π-μ, π-e, β-decay processes of nucleon and of Λ-particle, fortunately the unwanted modes which appear in the concerned interactions are dropped out for some other reasons. In the first two problems, by our selection rule the combinations of the interactions are

$$\sqrt{2}\,(\bar{\mu}\nu)\pi^-(\bar{\mu}e)\pi^0$$

and

$$\sqrt{2}\,(\bar{\omega}e)\pi^+(\bar{\omega}\nu)\pi^0,$$

respectively. But the decay modes $\pi^0\to e+\tilde{\mu}$ and $\pi^0\to\nu+\tilde{\omega}$ are not observed in

both cases,* because the 2γ-decay of π^0 meson is much faster than them. The β-decay interaction for nucleon is coupled with the two undesirable processes in such a way that

$$(\bar{E}\tau y)\omega(\bar{N}\tau N)=2(\bar{e}\omega)(\bar{p}n)+(\bar{\nu}\omega)\{(\bar{p}p)-(\bar{n}n)\}.$$

But the probabilities of the modes $p\to p+\nu+\tilde{\omega}$ and $n\to n+\nu+\tilde{\omega}$ vanish because the concerned volume in phase space is zero. The β-decay of Λ-particle $\Lambda\to p+e^-+\tilde{\nu}$ is accompanied with the processes $\Lambda\to n+\nu+\tilde{\nu}$ and $\Lambda\to n+e^-+e^+$ in the spurion interaction

$$(\bar{N}\tau x)\Lambda(\bar{E}\tau E)=2(\bar{p}\Lambda)(\bar{e}\nu)-(\bar{n}\Lambda)\{(\bar{\nu}\nu)-(\bar{e}e)\}.$$

Among them only the first process is observed.[10] The second one cannot be observed, but the last one is observable even if it is very difficult to detect the process. Further, the interaction $(\bar{n}\Lambda)(\bar{e}e)$ induces the unobserved decay process $K^0\to\pi^0+e^++e^-$ through the chain $K^0\to\Lambda+\tilde{n}\to\Lambda+\tilde{n}+\pi^0\to e^++e^-+\pi^0$. But the concerned interaction term $(\bar{n}\Lambda)(\bar{e}e)$ can be eliminated without violating the $|\Delta I|=1/2$ rule, if we couple the above spurion interaction with another one,

$$(\bar{N}x)\Lambda(\bar{E}E)=(\bar{n}\Lambda)\{(\bar{e}e)+(\bar{\nu}\nu)\}.$$

Of course, under the present stage of experiments the branching ratio of the decay mode $K^0\to\pi^0+e^++e^-$ to other modes need not be zero and may be very small. Therefore, the interaction term $(\bar{n}\Lambda)(\bar{e}e)$ may remain, if it is taken into account that we have observed very few examples of the reaction $\Lambda\to p+e^-+\tilde{\nu}$.

About the other processes, at the present stage of experiments we can say neither yes nor no to our question about the $|\Delta I|=1/2$ rule.

The remaining observed decay modes of charged K-mesons are $K^+\to e^++\nu+\pi^0$ and $K^+\to\mu^++\omega+\pi^0$. The second process is coupled to $K^0\to\mu^-+\tilde{\omega}+\pi^+$, and the spurion interaction is

$$(y^*\tau K)\pi(\bar{\omega}\mu)=(\bar{\omega}\mu)(\pi^0K^++\sqrt{2}\pi^+K^0).$$

Therefore, the decay rate should be

$$\frac{W(K^+\to\mu^++\omega+\pi^0)}{W(K^0\to\mu^-+\tilde{\omega}+\pi^+)}=\frac{1}{2}$$

where the particle mixture problem is put aside.** This naturally coincides with the previously reported conjecture by Okubo et al.,[11] but we have no experimental value to this ratio. There are three kinds of spurion interactions for $K\to E+\widetilde{E}+\pi$;

* The appearance of the term $(\bar{\mu}e)\pi^0$ in the above combination may be regarded as the one of the objections against the selection rule $|\Delta I|=1/2$. Because the existence of this term brings about the reaction $N+\mu\to N+\pi^0+\mu\to N+e$, and the probability for this reaction must be very small compared with the μ-capture process, as mentioned above, even if it was not zero.

** If we consider that $K_2^0\to\mu^-+\omega+\pi^+$ can only occur through $\widetilde{K}^0$ amplitude in K_2^0 as was mentioned in § 3, this mode should not occur in nature in virtue of our selection rule.

$$H_1=(x^*\tau K)(\bar{E}E)\pi$$
$$=(\sqrt{2}\pi^-K^+-\pi^0K^0)((\bar{\nu}\,\nu)+(\bar{e}e)),$$
$$H_2=(x^*K)(\bar{E}\tau E)\pi$$
$$=K^0[(\bar{\nu}\,\nu)\pi^0+\sqrt{2}\,(\bar{\nu}e)\pi^+ +\sqrt{2}\,(\bar{e}\nu)\pi^- -(\bar{e}\,e)\pi^0]$$

and

$$H_3=\frac{1}{i\sqrt{2}}((x^*\tau K)\times\pi)(\bar{E}\tau E)$$
$$=-(\bar{\nu}\nu)K^+\pi^- +(\bar{\nu}e)(K^0\pi^+ +\sqrt{2}K^+\pi^0)$$
$$-(\bar{e}\nu)K^0\pi^- +(\bar{e}e)K^+\pi^-.$$

Similarly, for $K^c=\begin{pmatrix}\widetilde{K}^0\\ -K^-\end{pmatrix}$ we have

$$H_1{}^c=(y^*\tau K^c)(\bar{E}E)\pi$$
$$=(\widetilde{K}^0\pi^0-\sqrt{2}K^-\pi^+)((\bar{\nu}\,\nu)+(\bar{e}\,e)),$$
$$H_2{}^c=(y^*K^c)(\bar{E}\tau E)\pi$$
$$=\widetilde{K}^0[(\bar{\nu}\,\nu)\pi^0+\sqrt{2}\,(\bar{\nu}\,e)\pi^- +\sqrt{2}\,(\bar{e}\,\nu)\pi^- -(\bar{e}e)\pi^0],$$

$$H_3{}^c=\frac{1}{i\sqrt{2}}((y^*\tau K^c)\times\pi)(\bar{E}\tau E)$$
$$=-(\bar{\nu}\,\nu)K^-\pi^+ -(\bar{\nu}\,e)K^0\pi^+$$
$$+(\bar{e}\nu)(\widetilde{K}^0\pi^- +\sqrt{2}K^-\pi^0)+(\bar{e}e)K^-\pi^+.$$

Table II

	$H_1{}^{(1)}$	$H_2{}^{(1)}$	$H_3{}^{(1)}$	$H_1{}^{(2)}$	$H_2{}^{(2)}$	$H_3{}^{(2)}$
$(\bar{\nu}\,\nu)K^+\pi^-$	-1		$+1/\sqrt{2}$	-1		$-1/\sqrt{2}$
$(\bar{e}\,e)K^+\pi^-$	-1		$-1/\sqrt{2}$	-1		$+1/\sqrt{2}$
$(\bar{\nu}\,e)K^+\pi^0$			-1			$+1$
$(\bar{\nu}\,\nu)K_1{}^0\pi^0$	$+1$	$+1$				
$(\bar{e}\,e)K_1{}^0\pi^0$	$+1$	-1				
$(\bar{\nu}\,e)K_1{}^0\pi^+$		$+\sqrt{2}$	-1			
$(\bar{e}\,\nu)K_1{}^0\pi^-$		$+\sqrt{2}$	$+1$			
$(\bar{\nu}\,\nu)K_2{}^0\pi^0$				$+1$	$+1$	
$(\bar{e}\,e)K_2{}^0\pi^0$				$+1$	-1	
$(\bar{\nu}\,e)K_2{}^0\pi^+$					$+\sqrt{2}$	$+1$
$(\bar{e}\,\nu)K_2{}^0\pi^-$					$+\sqrt{2}$	-1
$(\bar{\nu}\,\nu)K^-\pi^+$	-1		$-1/\sqrt{2}$	$+1$		$-1/\sqrt{2}$
$(\bar{e}\,e)K^-\pi^+$	-1		$+1/\sqrt{2}$	$+1$		$+1/\sqrt{2}$
$(\bar{e}\,\nu)K^-\pi^0$			$+1$			1

In decay interaction the neutral K-mesons should appear in the modes $K_1^0 = 1/\sqrt{2}\cdot(K^0+\widetilde{K}^0)$ and $K_2^0 = 1/\sqrt{2}\cdot(K^0-\widetilde{K}^0)$, and we must combine H_i with $H_i^c (i=1, 2, 3)$ in suitable ways. Namely, for K_1^0 we must consider $H_1^{(1)} = 1/\sqrt{2}\cdot(H_1^c - H_1)$, $H_2^{(1)} = 1/\sqrt{2}\cdot(H_2^c + H_2)$ and $H_3^{(1)} = 1/\sqrt{2}\cdot(H_3^c - H_3)$,and for K_2^0 $H_1^{(2)} = -1/\sqrt{2}\cdot(H_1^c + H_1)$, $H_2^{(2)} = 1/\sqrt{2}\cdot(H_2 - H_2^c)$ and $H_3^{(2)} = 1/\sqrt{2}\cdot(H_3^c + H_3)$. The possible interaction terms and their coefficients are tabulated in Table II. For these decay modes we have very little experimental results. Among of them only $K^+ \to e^+ + \nu + \pi^0$ and $K_2^0 \to e^- + \tilde{\nu} + \pi^+$ are observed.

The missing decay mode $K^+ \to e^+ + \nu$ is coupled with $K^0 \to \nu + \tilde{\nu}$ and $K^0 \to e^+ + e^-$ in our scheme. But the former is forbidden in consideration of spin angular momentum. The latter is also unobserved yet, and can be separated from the others in such a way that the $|\Delta I| = 1/2$ rule is not violated.

$$H_a = H_1 - H_2$$

where

$$H_1 = (x^* \boldsymbol{\tau} K)(\bar{E}\boldsymbol{\tau}E) = 2K^+(\bar{\nu}e) - K^0\{(\bar{\nu}\nu) - (\bar{e}e)\}$$

and

$$H_2 = (xK)(\bar{E}E) = K^0\{(\bar{e}e) + (\bar{\nu}\nu)\}.$$

If β-decay of Σ-particle is to be considered,[12] there are two possibilities, that is, the daughter baryon may be either Λ or n. In the former case the spurion interaction is

$$(\bar{\Lambda}\boldsymbol{\Sigma})\bar{\omega}(y^*\boldsymbol{\tau}E) = (\bar{\Lambda}\Sigma^0)(\bar{\omega}\nu) + \sqrt{2}(\bar{\Lambda}\Sigma^+)(\bar{\omega}e),$$

and the observation of the decay described by the first term is very difficult. If the daughter is neutron, the branching ratio between various possible modes is complicated. Here we do not give the explicit forms of the spurion interactions, which are obtained by writing down $(\bar{N}\boldsymbol{\tau}x)\boldsymbol{\Sigma}(\bar{E}E)$, $(Nx)\boldsymbol{\Sigma}(\bar{E}\boldsymbol{\tau}E)$ and $((\bar{N}\boldsymbol{\tau}x)\times\boldsymbol{\Sigma})(\bar{E}\boldsymbol{\tau}E)$.

§ 4. Summary and discussions

In the preceding sections, we have examined how far the selection rules $|\Delta S| = 1$ and $|\Delta I| = 1/2$ fit the leptonic weak interactions, when we accept the strangeness and isotopic spin assignment proposed by Gamba-Marshak-Okubo and Klein. More definitely speaking, we have checked whether the assignment can have the same competence as the one for baryons and mesons. In § 3 we used the spurion technique to check the $|\Delta I| = 1/2$ selection rule, but the interactions may be regarded as the effective ones describing the concerned processes. Therefore, we may consider that the discussions hold as the kinematical analysis

between the initial and final state for the decay processes, and the results remain valid even if any interactions, which do not violate the $|\Delta I|=1/2$ selection rule, are taken into account. Further, as previously emphasised,[2] the spurion technique has the merit that it enables us to correlate the selection rule $|\Delta I|=1/2$ with the $|\Delta S|=1$ selection rule phenomenologically.

As to the strangeness selection rule, firstly it is remarkable that the two kinds of neutral massless lepton should exist: the ordinary neutrino ν and the strange neutrino ω. Some authors[13] also proposed the theory of two kinds of neutrino which are different from each other in the sign of chirality. But our discussions do not refer to the chirality character, and the ordinary and strange neutrino may have either the same chirality or the opposite chirality. Here the difference between them is in the isotopic spin and strangeness assignment. The existence of the two kinds of neutrino may be checked by the experiments to see whether the neutral massless lepton emitted in the decay $\pi^+\to\mu^++\nu$ gives rise to the inverse β-decay $\omega+n\to p+e^-$, which in our theory should not occur, or any combinations of such two leptonic processes listed in § 2, that is, the one refers to ν and the other to ω. All observed weak interactions satisfy the selection rule $|\Delta S|=1$, if we add the strange neutrino to the previously proposed scheme. But by the assignment there appear many strangeness conserving leptonic processes such as $K^0\to\mu^++e^-$ and $K^\pm\to\mu^\pm+e^\mp+\pi^\pm$. Further, there are many undesirable reactions which have the property $|\Delta S|=1$, for instance, $\mu^\pm\to 2e^\pm+e^\mp$ and $\mu^-+p\to p+e^-$.

Except for $\mu\to e+\nu+\tilde{\nu}$, $\mu^-+p\to n+\nu$ and probably $\pi\to\mu+\nu$, the selection rule $|\Delta I|=1/2$ might be considered to hold fairly well for the interactions with $|\Delta S|=1$, if we remember the extent of approximation of the rule for non-leptonic weak interactions. Unfortunately, we cannot say much whether the isotopic spin assignment is to be accepted or not, as the experiments of the branching ratio in the decay processes are incomplete at the present stage.

Now the important exceptional cases for the selection rules $|\Delta S|=1$ and $|\Delta I|=1/2$ have a remarkable character. Namely, they refer to the neutral lepton currents such as $(\bar{\mu}e)$, $(\bar{e}e)$ and probably $(\bar{\nu}\nu)$, if we use the terminology of the conserving current hypothesis.[14] Therefore, it seems sufficient only to prohibit the neutral lepton current, in order to remove the objections. But if we exclude the neutral lepton current, the interactions with $|\Delta I|=3/2$ are coupled with the spurion interactions, and strongly violate the selection rule $|\Delta I|=1/2$. On the other hand, the interaction with $|\Delta I|=3/2$ is limited to be smaller than ten percent for non-leptonic processes.[6]

Another possiblity to remove the objections might be to assume μ^+, instead of μ^-, as particle. It is well known[15] that this assumption and the conservation law of lepton number prohibit many undesirable processes such as $\mu\to e+e+e$ and $\mu+p\to p+e$. On the other hand, in order to explain the experimental results on μ-e-decay under this assumption, the neutral lepton coupled with μ in the

Fermi type interaction must be assumed to have the opposite chirality to the one of the neutral lepton coupled with e in the interaction.[16] If we identify these neutrinos with our strange and ordinary neutrino, respectively, the μ-e-decay is described as

$$\mu^+ \rightarrow e^+ + \nu + \omega.$$

Further, if we assume μ^+ as iso-singlet, according to Eq. (1), the strangeness of μ^+ becomes $+3$. Then the strangeness change in the process is two units. If we assume that μ^+ and ω constitute an iso-doublet, Eq. (1) requires that this doublet has $S=2$. Then the μ-decay process satisfies the selection rule $|\Delta S|=1$, but the π-decay and β-decay of nucleon cannot satisfy the rule. Therefore, these alternatives are not preferable either.

In this article we have discussed the selection rules for leptonic processes, under the assumption that the strangeness and isotopic spin of leptons assigned by Gamba-Marshak-Okubo and Klein play the same role as the ones for non-leptonic processes originally discussed by Nishijima and Gell-Mann. Here no discussion is given about the formal scheme to express the phenomenological selection rule. But we may say that the strangeness assignment is not promising. Further, there is no suitable physical parameter that may enable us to set up an additive selection rule for leptonic processes, as easily checked by the method explained in the appendix of the previous article.[1] In the light of the failure of strangeness selection rule discussed above, we may refer to the previous article by one of us (T. O.).[1] There it was proposed to reinterpret hypercharge* of baryons as the expectation value of a hypercharge operator, and then it was pointed out that in this theory leptons also should have this internal degree of freedom. By this procedure the new treatment of hypercharge gave rise to a new kind of selection rule to leptonic processes as well as to non-leptonic processes, and the undesirable leptonic processes were selected out.

Concerning the isotopic spin selection rule, if the undesirable leptonic processes discussed above are selected out by a suitable method which is independent of isotopic spin, the results of our analysis show that the selection rule $|\Delta I|=1/2$ becomes a fairly good approximation. But the isotopic spin assignment is intimately related to the strangeness assignment, as will be seen from Eq. (1), and the isotopic spin assignment would be changed when the treatment of strangeness is modified in any sense. Therefore, it is very questionable whether the success of the $|\Delta I|=1/2$ selection rule reached here will remain in the future stage. In the previous article by one of us (T. O.),[2] it is proposed that in a zeroth approximation the new isotopic spin defined in such a way that all baryons are iso-doublet seems to be more suitable. The reason was that by the assignment

* In the article the terminology "hyperonic charge" was used instead of "hypercharge" deliberately, as there the relation between electric charge and hypercharge was put aside.

isotopic spin is conserved in weak non-leptonic interactions as well as in strong interactions. Further, there it is also proposed to consider that all leptons are iso-singlet in that scheme. Some authors[11),17)] attempted to extrapolate the $|\Delta I|=1/2$ selection rule to the conserving baryon-meson-currents. The experimental check of their results will offer some clue to this problem, because in such theories it can be regarded that the isotopic spin of leptons are effectively assumed to be zero.

Some authors technically use two types of iso-doublet (ν, e) and (ν, μ) with the same ν, but it will be non-sense. However, it may be possible to set up a theory in which sometimes neutrino effectively plays the role of a counterpart of the doublet (ν, e) and in other times of that of the doublet (ν, μ). The scheme proposed by Schwinger[18)] is of such a character, and there leptons are combined in an iso-triplet.

Here we may raise a formal objection to regarding electron and neutrino as an iso-doublet. Neutrino can have definite chirality, but it is not the case for electron due to the non-vanishing mass. Therefore, it seems rather difficult to accept the idea of electron-neutrino iso-doublet. The only way to overcome this objection within the scheme may be to assume that the bare electron has no rest-mass and that the electromagnetic interaction such as the magnetic interaction or the one with cutoff, in which the constant with the dimension of length is included in some sense, gives rise to non-vanishing mass. Further, there is no leptonic interaction without neutrino, but electron is not the essential partner of leptonic weak interactions.

Concluding this article we may emphasize the importance of seeking some physical quantities, which not only specify the states of leptons but also describe the selection rules for leptonic processes well, in order to understand the problem of the mysterious difference between electron and μ-meson, both of which show very similar character except for their difference in mass, and the fact that leptons have no strong interaction.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica tipe) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables

Low Density Fermion System with Strong Attractions

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January 4, 1961

In the previous note[1] it has been shown that if the scattering length of two-particle system is negative low density system of Fermions must have a distribution which is drastically changed from that of degenerate Fermi gas. In the case of attractive interaction strong enough to create a bound state the scattering length takes a positive value. In this note the structure of such a system in the low density limit will be studied. The parameter ε_0 used in I refers to the magnitude of diffuseness of Fermi surface. As the distribution in the momentum space is then determined by the internal structure of two-particle bound state ε_0 will take a value independent of the density. Thus we will assume that $\beta^2=2m\varepsilon_0/k_F^2$ is inversely proportional to k_F^2 which will be verified later. m is the effective mass and k_F is the Fermi momentum. One can simplify Eq. (2) in I assuming that the two-body interaction $v(r)$ has a hard core at $r<c$ and a square well with depth V at $c<r<a=b+c$. Here distances are measured in the nondimensional unit, i.e. in the unit of the reciprocal of the Fermi momentum. In a low density system range parameters a, b and c are small as compared with unity, but well depth parameter $\alpha^2=mV/k_F^2$ becomes large. In this case the solution at $a<r$ can be obtained as follows,

$$u(r)=\frac{2C}{\pi}\int_1^\infty d\kappa\frac{\kappa}{\kappa^2+\beta^2}\sin\kappa r+\frac{2C}{\pi}\int_0^1 d\kappa\frac{\kappa}{\kappa^2-\beta^2}\sin\kappa r,$$

where C is an arbitrary constant. As the integral along $0<\kappa<1$ gives small contribution, it may be modified suitably and we get

$$u(r)=Ce^{-\beta r}.$$

The solution at $c<r<a$ can be similarly derived and is given by

$$u(r)=C'\sin\sqrt{\alpha^2-\beta^2}(r-c),$$

where C' is an arbitrary constant. The boundary condition at $r=a$ gives the condition for the existence of the non-trivial solution. It is the same as the condition for the existence of the bound state in the two-particle system. ε_0 is then found to be equal to half

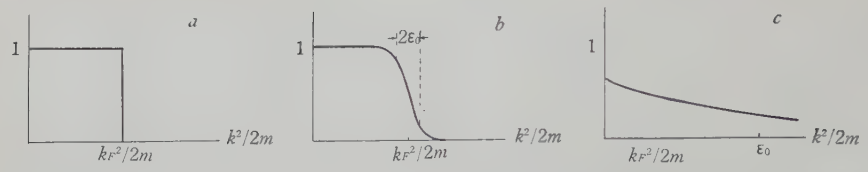


Fig. 1. Distributions of particles in momentum space. $k^2/2m$ means kinetic energy of a particle, k_F Fermi momentum and ε_0 is diffuseness parameter. Distribution in the case of repulsive interaction (a); weak attraction (b); and strong attraction (c).

the binding energy which is independent of the density. Our assumption is thus verified if there exists a bound state in the two-particle system.

The characteristics of the low density system can now be stated. In the case of repulsive interaction (positive scattering length and no bound states) ε_0 vanishes and Fermi distribution is stable. On the other hand, for weak attraction (negative scattering length) ε_0 is proportional to k_F^2 and diffused Fermi distribution is stable. Finally, for strong attraction (positive scattering length and bound states) ε_0 is independent of k_F and Fermi distribution is diffused completely (see Figs.) since the diffuseness parameter ε_0 is sufficiently larger than $k_F^2/2m$. This represents the fact that the system is an assembly of the composite particles.

The above results can also be derived shape-independently, that is, by means of the discussions which do not make any particular assumption regarding the shape of the two-body interaction. This will be treated in a subsequent paper. The author wishes to thank Drs. Iwamoto, Terasawa and Yasuno for discussions.

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The Cerenkov Self-Excitation of the Electron*

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An interesting variation of the Cerenkov effect has been discussed by Frank.[1] Consider a charged particle having internal excited states traversing a medium at super-light velocities ; under these circumstances not only will the particle emit Cerenkov radiation but it can also make simultaneous transitions *upward* into its excited states.

Quantitatively this situation is described by

$$\Delta U = -E_\nu(1-\beta n_\nu \cos\theta) \qquad (1)$$

where E_ν is the Cerenkov energy, at frequency ν, radiated into the cone of half angle θ ; ΔU is the increment of internal energy ; n_ν is the corresponding index of refraction ; and $\beta = v/c$. Clearly if $\beta n_\nu \cos\theta > 1$ then $\Delta U > 0$; and this is the condition for Cerenkov self-excitation.

* Supported by the National Science Foundation.

We should like to consider some of the possibilities which arise if this principle is applied to a hypothetical electron-muon interaction.

A semi-classical precedent for this already exists in the proposal of Bohm and Weinstein[2] that certain mesons may be regarded as excited states of the electron. We recall briefly that according to a number of classical extended-electron theories the electron should be capable of carrying out highly oscillatory motions even without the action of any external forces (classical Zitterbewegung). These oscillatory modes can persist only for certain characteristic frequencies ($\lesssim 10^{23}\,\mathrm{sec}^{-1}$) which have a natural interpretation as excited states of the electron.

The connection with mesons arises through a rudimentary quantization: If the smallest of these characteristic frequencies

$$\omega_1 \sim \frac{3}{2}\,\frac{mc^3}{e^2} \qquad (2)$$

is converted into its energy equivalent, one finds that the total energy of the lowest excited state is given by

$$mc^2(1+3/(2\alpha)) = 105.54\ \mathrm{Mev} \qquad (3)$$

($\alpha \equiv$ fine structure constant) a value close to the observed muon rest mass

$$M_\mu = 105.65 \pm .01\ \mathrm{Mev}.$$

Note that (3) is a slight modification of Nambu's formula.[3]

Whether one takes this interpretation literally or relies on a phenomenological muon-electron coupling,[4] it is now possible to consider the effect of a Cerenkov self-excitation: In the extreme relativistic case (1) simplifies to

$$\Delta U = E_\nu\, \Delta n_\nu \qquad (4)$$

where $\Delta n_\nu = n_\nu - 1$. If we now identify ΔU with the electron-muon energy gap $\sim 10^8$ ev and overestimate E_ν by E, the energy of the incident electron, (4) becomes

$$E > 10^8/\Delta n_\nu. \qquad (5)$$

Finally we require the observation that Δn_ν can be significantly different from zero, even at these enormous frequencies, in a "medium" provided by the vacuum polarization in the vicinity of an atomic nucleus. At a distance of approximately one (electron) Compton wavelength from the nucleus, the Euler-Kockel theory[5] should still be a qualitatively reliable guide; on this basis one can estimate that

$$\Delta n_\nu \sim \frac{7}{180} Z^2 \alpha^3 \qquad (6)$$

where Z is the nuclear charge. At $Z=10$, for example, $\Delta n_\nu \sim 6\times 10^{-6}$. Inserting this estimate back into (5), we finally obtain

$$E > 1.6\times 10^{13}\ \mathrm{ev}. \qquad (7)$$

The Cerenkov self-excitation could therefore drive a direct electron-muon conversion at very high energies.[6] If this process has a significant cross section, one could expect observable consequences in the structure of high-energy cosmic-ray cascades.

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Nucleon-Nucleus Scattering and the Choice between Sets of Nucleon-Nucleon Phase Shifts

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It has become fairly common practice to use the elastic scattering of nucleons from nuclei to differentiate between various sets of nucleon-nucleon phase shifts.[1,2] The usual procedure is to use the impulse approximation to calculate a nucleon-nucleus optical potential and then to obtain the scattering polarization from this potential in first Born approximation: In effect this gives the scattering resulting from the direct interaction between the incident nucleon and one nuclear nucleon, the nucleus then recoiling as a unit. If the scattering angle is sufficiently small, it is assumed that the effects of being off the energy shell can be neglected and the nucleon-nucleus scattering amplitude can then be expressed in terms of free nucleon-nucleon scattering phase shifts. Sakamoto[1] has carried out such a calculation at several energies in an attempt to distinguish between the phase shift sets given by meson theory,[3] by Gammel and Thaler,[4] and by Signell and Marshak.[5] The purpose of this note is to point out that multiple scattering effects can be expected to obscure the differences between polarizations calculated with different sets and hence make choices based upon the Born approximation unreliable.

At 150 Mev, the predicted polarization is the same for all three phase shift sets at laboratory scattering angles less than 5°, so that an experimental choice between the three calculations of Sakamoto[1] is only possible in the range 5° to 15° where the predicted polarizations differ by about 0.1 and the experimental points range from 0.4 to 0.6.[1] Over this angular interval, the meson polarization rises from 0.3 to 0.5, the Gammel-Thaler polarization goes from 0.4 to 0.6, and the Signell-Marshak curve goes from 0.5 to 0.7 so that the Gammel-Thaler phase shifts are preferred if the Born approximation polarization is to be believed.

However, over this same angular interval, absorption effects due to multiple scattering can decrease the Born approximation polarization by at least 0.1, bringing the Signell-Marshak curve down so as to give an excellent fit to the experimental data and dropping the Gammel-Thaler results below the data.[2] At 90 Mev, absorption effects

are expected to be even greater, the effect decreasing with increasing energy.[2] One could then assume that it is the Signell-Marshak phase shift set which is now to be preferred. However, it would be wiser at this time, before multiple scattering effects are more thoroughly understood, to make no such conclusions and to rely on additional free-nucleon scattering data which is becoming available for a differentiation among the different sets of scattering parameters.

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Elastic Scattering of Fast Electrons by the Helium Atom and the Interelectronic Correlation

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The cross section for elastic scattering of fast electrons by the helium atom given by the Born approximation is calculated by using some accurate wave functions describing the ground state of the atom. It is the purpose of this work to examine the dependence of the results of the Born approximation on the choice of the ground state function. This study is also an extension of that of Huzinaga,[1] who recently pointed out the importance of the problem.

The Born approximation gives the differential cross section $2\pi I(\theta)\sin\theta d\theta$ for elastic scattering into the small angle element $d\theta$ near θ of an electron with the incident energy E by the atomic potential $V(r)$ as

$$I(\theta)=4\left|\int_0^\infty \frac{[\sin(Kr)/Kr]}{Kr}\,V(r)\,r^2\,dr\right|^2, \tag{1}$$

where $K=2\sqrt{2E}\sin(\theta/2)$ and all the quantities are expressed in the atomic units. The potential $V(r)$ is given in terms of the normalized ground state wave function $u(\boldsymbol{r}_1, \boldsymbol{r}_2)$ as

$$V(r)=\iint\left[\sum_{s=1}^{2}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}-\frac{2}{r}\right]\times|u(\boldsymbol{r}_1,\boldsymbol{r}_2)|^2\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{2}$$

The following analytic ground state functions are used in our calculation:

1) Correlated closed shell[2]

$$u_1\sim\exp[-1.8497(r_1+r_2)]\times(1+0.3658\,r_{12}).$$

2) Correlated open shell[3]

$$u_2\sim(1+\mathrm{P}_{12})\exp(-1.436r_1-2.208\,r_2)\times(1+0.2924\,r_{12}).$$

The calculated differential cross

section strongly depends on the ground state function only at small scattering angles and so we give our results for $\theta=0$ in the Table. It should be noted that $I(0)$ is independent of the incident energy according to (1). For convenience, we adopt the differential cross section $I_0(0)$ for $\theta=0$ calculated from the simplest (uncorrelated closed shell) function

$$u_0 \sim \exp[-1.6875(r_1+r_2)]$$

as standard and give $I(0)/I_0(0)$ for u_1 and u_2 in the Table. The value of $I_0(0)$ is 0.49327 in atomic unit. Further Huzinaga's result obtained from the uncorrelated open shell wave function

$$u_3 \sim (1+P_{12})\exp(-1.19r_1-2.184r_2)$$

is also included in the Table for comparison. The symbol P_{12} in the expression for u_2 and u_3 means the operator interchanging the space coordinates $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ of the atomic electrons. To indicate the accuracy of various functions, the non-relativistic energy as well as its precision compared with the exact non-relativistic eigenvalue -2.90372[3] is also given in the Table.

Table I

wave function	energy (precision)	$I(0)/I_0(0)$
u_0	-2.84767 (98.0696%)	1
u_1	-2.89112 (99.2217%)	1.0447
u_2	-2.90142 (99.9208%)	1.2714
u_3	-2.8756 (99.0316%)	1.367

Now it is well known that the observed scattered intensity at small scattering angles is much greater than that calculated from the simplest wave function u_0 according to the Born approximation.[4] For instance, the observed value of $I(0)$ is $I(0)/I_0(0) \approx 1.5$ for the incident energies $350\sim700$ eV. The discrepancy has been ascribed to the neglect of polarization of the helium due to the incident electron.[4] However, Huzinaga suggested that considerable improvement of the theoretical value could be brought about by the use of the more accurate function u_3 within the Born approximation.

As shown in the Table, our calculation based on the functions u_1 and u_2, which are more accurate than u_3, gives smaller values of $I(0)$. In view of our result, Huzinaga's conclusion turns out to be too optimistic, though qualitatively correct. The reason for the large value of $I(0)$ obtained from u_3 is that u_3 has a too long tail at the outer parts of the atom, which is necessary for taking the interelectronic correlation into account within this restricted functional form. On the other hand, the functions u_1 and u_2 describing the correlation more precisely yield smaller values of $I(0)$. In particular, the value calculated from u_2 should be regarded to be the most accurate. Consequently, the improvement of the theoretical differential cross section by the use of accurate functions is not so remarkable as Huzinaga claimed, although the choice of the atomic wave function is important to evaluate accurate values of cross sections.

The author wishes to thank Mr. K. Katsuura for helpful comments and checking calculations.

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On the "Nagoya Model" of the Elementary Particles

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Gamba, Marshak and Okubo[1] have observed that, at least as far as the weak interactions are concerned, the three baryons Λ, N, P are equivalent to the three leptons μ^-, e^-, ν, in the sense that any reaction permitted or observed for one of the two groups is permitted for the second and conversely. It is remarkable that the three baryons entering the Gamba, Marshak and Okubo principle are the fundamental particles of the Sakata-Okun model. On the basis of this principles Z. Maki, M. Nakagawa, Y. Ohnuki and S. Sakata[2] have proposed a unified model of elementary particles (called "the Nagoya Model") in which the three basic baryons are supposed to be composed of a matter B^+ together with ν, e^- and μ^- respectively. So they denote

$$P \equiv \langle B^+ \nu \rangle,$$

$$N \equiv \langle B^+ e^- \rangle,$$

$$\Lambda \equiv \langle B^+ \mu^- \rangle.$$

It is also supposed that separation of a baryon into B^+ and lepton is very hard or may be impossible. So they write (p. 1178): "We have described a new model of elementary particles which, unifying all elementary particles into the three leptons (ν, e^- and μ^-) and a new kind of matter B^+, could account for the existence of three basic baryons (P, N and Λ) strongly interacting in charge independent ways".

In 1955 and 1956 I have proposed and investigated a composite baryon model in 3), reported 4) and 5), and in 6), reported in 7). In this model the nucleon is supposed to be composed of a very heavy boson (called "nuclonid") and a fermion with spin 1/2 (called "electronid"). The hyperons are excited states of the nucleon (of the system nuclonid-electronid) in this model. The stability of the nucleon against fission in nuclonid and electronid is guaranteed by a very great mass defect in the system nuclonid-electronid. The interaction between the nuclonid and the electronid in nucleon is due to the K and other mesons. To this model I have applied the Heitler-London theory of the hydrogen molecule and have found an explanation for the repulsion between nucleons at very short distances, and for the stability of the α-particles.

There are many differences between the "Nagoya Model" and mine which consists chiefly in the conformity of the "Nagoya Model" with the princi-

ples of Gamba, Marshak and Okubo on the one hand and Sakata-Okun on the other. (My model was given before the establishment of this principles). Nevertheless the basic idea consisting in composing the nucleon and the Λ-hyperon of a nuclonid (or B^+ matter) and a fermion-electronid (or ν, e^- and μ^-) is almost the same in the both models. Therefore it would be better to call the "Nagoya Model" the "Sofia-Nagoya Model".

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Double Dispersion Representation of Vertex Function

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We denote the three external lines of the vertex function by A, B and C, and the corresponding mass variables by z_1, z_2 and z_3, and the lowest thresholds of the three particles A, B and C by a, b and c, respectively.

In the present note, we shall prove that, if one of z_i's, say z_1, is real negative, the vertex function $F(z_1, z_2, z_3)$ satisfies the following double dispersion representation,

$$F(z_1, z_2, z_3) = \frac{1}{\pi^2}\int_{b^2}^{\infty} d\sigma_2^2 \int_{c^2}^{\infty} d\sigma_3^2 \frac{A_{23}(z_1, \sigma_2^2, \sigma_3^2)}{(\sigma_2^2 - z_2)(\sigma_3^2 - z_3)}. \tag{1}$$

The representation (1) makes use of the information derivable from the local commutativity and the mass spectral conditions.

As is well known, if two of z_i's, say z_1 and z_2, are real negative, for the vertex function the following dispersion relation holds,

$$F(z_1, z_2, z_3) = \frac{1}{\pi}\int_{c^2}^{\infty} d\sigma_3^2 \frac{A_3(z_1, z_2, \sigma_3^2)}{\sigma_3^2 - z_3}. \tag{2}$$

For fixed σ_3^2, the absorptive part $A_3(z_1, z_2 \sigma_3^2)$ defined by (2) is an analytic function of z_1 and z_2 and has the integral representation in the form[1]

$$A_3(z_1, z_2, \sigma_3^2) = \int dm_1 dm_2 dm_3 \times \varphi(\sigma_3, m_1, m_2, m_3) \times A^p(z_1, z_2, \sigma_3^2; m_1, m_2, m_3), \tag{3}$$

where A^p is the absorptive part of the lowest order perturbation theory. The non-vanishing region of the weight function φ is[1]

$\varphi = 0$ unless

$$\begin{cases} m_1 \geq 0, \ m_2 \geq 0, \ m_3 \geq 0, \ m_1 + m_2 \leq \sigma_3, \\ m_2 + m_3 \geq a, \ m_3 + m_1 \geq b. \end{cases} \tag{4}$$

The concrete expression of A^p is[1)]

$$A^p=[\lambda(z_1, z_2, \sigma_3^2)]^{-1/2}\log\frac{X+Y^{1/2}}{X-Y^{1/2}},$$

where

$$X=2m_3^2+\sigma_3^2-(z_1+z_2+m_1^2+m_2^2)$$
$$-(z_1-z_2)(m_1^2-m_2^2)/\sigma_3^2 \qquad (5)$$

and

$$Y=\lambda(z_1, z_2, \sigma_3^2)\lambda(m_1^2, m_2^2, \sigma_3^2)/\sigma_3^4$$

with

$$\lambda(x, y, z)=x^2+y^2+z^2$$
$$-2(xy+yz+zx).$$

From (5), we easily see that A^p can be rewritten as

$$A^p=\int_{\alpha-\beta}^{\alpha+\beta}d\sigma_2^2\frac{[\lambda(z_1, \sigma_2^2, \sigma_3^2)]^{-1/2}}{\sigma_2^2-z_2},$$

where

$$\alpha=m_3^2+m_1^2-(z_1-m_2^2-m_3^2)$$
$$\times(\sigma_3^2-m_1^2-m_2^2)/(2m_2^2)$$

and (6)

$$\beta=[\lambda(z_1, m_2^2, m_3^2)$$
$$\times\lambda(m_1^2, m_2^2, \sigma_3^2)]^{1/2}/(2m_2^2).$$

If z_1 is real negative, α and β are real positive, and $\alpha-\beta$ becomes minimum at

$$m_3(\sigma_3^2-m_1^2-m_2^2)$$
$$+m_1(z_1-m_2^2-m_3^2)=0, \qquad (7)$$

as a function of σ_3 defined in $\infty>\sigma_3^2\geq(m_1+m_2)^2$. Inserting (7) into $\alpha-\beta$, we see that the minimum value of $\alpha-\beta$ is $(m_3+m_1)^2$. The minimum value of $(m_3+m_1)^2$ in the region (4) is, of course, b^2. The above discussions are sufficient to complete the proof of the double dispersion representation (1).

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The Equivalence of Hamiltonians of Holstein-Primakoff and Dyson in Spin-Wave Theory in Ferromagnetism

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The Heisenberg model of a ferromagnet is described by the following Hamiltonian,

$$\mathcal{H}=g\mu H\sum_l S_l^z-2J\sum_{\langle l, m\rangle}\boldsymbol{S}_l\cdot\boldsymbol{S}_m. \qquad (1)$$

Here the notations are those that are commonly used. Following Holstein and Primakoff,[1)] we express (1) by the creation and annihilation operators of spin deviation as follows,

$$\mathcal{H}=g\mu H\sum_l(S-a_l^*a_l)$$
$$-2J\sum_{\langle l, m\rangle}[(S-a_l^*a_l)(S-a_m^*a_m)$$
$$+S(f_l a_l a_m^* f_m+a_l^* f_l f_m a_m)], \qquad (2)$$

where

$$f_l=(1-a_l^*a_l/2S)^{1/2}. \qquad (3)$$

We have calculated[2)] the spontaneous magnetization from (2) by the spin-wave method taking account of the spin-wave interactions as the perturbation and obtained the same result to the first order in 1/S as Dyson's[3)] which is regarded as rigorous at low

temperatures. Now we shall show that (2) can be reduced to Dyson's Hamiltonian $\mathcal{H}_D$ by a simple transformation so that (2) is completely equivalent to $\mathcal{H}_D$.

Let us transform $\mathcal{H}$ by a non-unitary matrix T as follows,

$$\mathcal{H}'=T^{-1}\mathcal{H}T, \tag{4}$$

where $\mathcal{H}'$ is expressed by a_l', $a_l'^*$, etc. instead of a_l, a_l^*, etc.; a_l' and $a_l'^*$ are defined by

$$a_l'=T^{-1}a_lT,\ a_l'^*=T^{-1}a_l^*T, \tag{5}$$

and they satisfy the following commutation rule,

$$[a_l',\ a_m'^*]=\delta_{l,m}. \tag{6}$$

We note that a_l' and $a_l'^*$ are not Hermitian conjugate to each other, therefore $\mathcal{H}'$ is not Hermitian. The eigenvalues of $\mathcal{H}'$, however, are the same as those of $\mathcal{H}$. We introduce the operators α_l and α_l^* in the forms

$$\begin{aligned} a_l'&=(1-\alpha_l^*\alpha_l/2S)^{1/2}\alpha_l,\\ a_l'^*&=\alpha_l^*(1-\alpha_l^*\alpha_l/2S)^{-1/2}, \end{aligned} \tag{7}$$

where α_l and α_l^* satisfy the following relations,

$$\alpha_l^*\alpha_l=a_l'^*a_l', \tag{8}$$

$$[\alpha_l,\ \alpha_m^*]=\delta_{l,m}. \tag{9}$$

Using (7) to (9), $\mathcal{H}'$ can be written as follows,

$$\begin{aligned} \mathcal{H}'=&Ng\mu SH-NzJS^2-g\mu H\sum_l\alpha_l^*\alpha_l\\ &+2JS\sum_{\langle l,m\rangle}(\alpha_l^*-\alpha_m^*)(\alpha_l-\alpha_m)\\ &+J\sum_{\langle l,m\rangle}\alpha_l^*\alpha_m^*(\alpha_l-\alpha_m)^2. \end{aligned} \tag{10}$$

This is perfectly in agreement with $\mathcal{H}_D$.

A similar method can be applied to antiferromagnetism. We use the operators α_l, α_l^* defined in (7) for + spin sublattice and β_m, β_m^* which are derived from the annihilation and creation operators b_m', $b_m'^*$ of the spin deviation for $-$spin sublattice as follows,

$$\begin{aligned} b_m'&=\left(1-\frac{\beta_m^*\beta_m}{2S}\right)^{-1/2}\beta_m,\\ b_m'^*&=\beta_m^*\left(1-\frac{\beta_m^*\beta_m}{2S}\right)^{1/2} \end{aligned} \tag{11}$$

$$[\beta_m,\ \beta_{m'}^*]=\delta_{m,m'}. \tag{12}$$

Thus the Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}'=&-z|J|NS^2-\kappa NS^2\\ &+2|J|S\sum_{\langle l,m\rangle}(\alpha_l^*\alpha_l+\beta_m^*\beta_m\\ &+\alpha_l\beta_m+\alpha_l^*\beta_m^*)\\ &+g\mu H(-\sum_l\alpha_l^*\alpha_l+\sum_m\beta_m^*\beta_m)\\ &+2\kappa s(1-1/2S)\\ &\times(\sum_l\alpha_l^*\alpha_l+\sum_m\beta_m^*\beta_m)\\ &-|J|\sum_{\langle l,m\rangle}\alpha_l^*(\alpha_l+\beta_m^*)^2\beta_m\\ &-\kappa(\sum_l\alpha_l^{*2}\alpha_l^2+\sum_m\beta_m^{*2}\beta_m^2), \end{aligned} \tag{13}$$

where H is directed along the antiferromagnetic axis, and κ is a constant proportional to uniaxial anisotropy energy. It is a very interesting problem to obtain a ground-state energy from (13).

The author thanks Dr. Yoshiharu Kitano for his helpful discussions.

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On the Critical Scattering of Neutrons

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Van Hove[1] proposed to describe the inelastic part of the critical scattering of neutrons by a ferromagnet by introducing spin diffusion, and argued that the diffusion constant of spin would be zero at the Curie point and would increase linearly with temperature in the vicinity of that point. De Gennes[2] gave a very concise theory of critical scattering and calculated the diffusion constant. However, his treatment, although very ingenious, is still phenomenological in nature and a microscopic basis of his theory remains to be elucidated.

In this short note, we shall give an expression for the diffusion constant in terms of the correlation functions of certain spin operators, and calculate it adopting the Heisenberg model of ferromagnet.

First define the Fourier transforms of spin operators,

$$S(\boldsymbol{k}) = \sum_m S_m \exp[i\boldsymbol{k}\cdot\boldsymbol{R}_m],$$

and consider a time-displaced correlation function of the z-components of two spin operators,

$$\langle\{S^z(-\boldsymbol{k}),\ S^z(\boldsymbol{k},\ t)\}\rangle,$$

where the curly bracket means the symmetrized product.

Above the Curie temperature and in the absence of external magnetic field, we have the following identity:

$$\langle\{S^z(-\boldsymbol{k}),\ S^z(\boldsymbol{k},\ t)\}\rangle - \langle\{S^z(-\boldsymbol{k}),\ S^z(\boldsymbol{k})\}\rangle = -\int_0^t d\tau\,(t-\tau)\langle\{\dot{S}^z(-\boldsymbol{k}),\ \dot{S}^z(\boldsymbol{k},\tau)\}\rangle, \tag{1}$$

where the dots imply the differentiation with respect to time. The integrand on the right-hand side is the correlation function of torques acting upon the respective spins and is something like the correlation of forces acting on a particle performing a Brownian motion. Our basic assumption here is that the correlation function of torques decays very much faster compared with the macroscopic relaxation caused by the diffusion process. Then, if the time t in Eq. (1) is comparable with the macroscopic relaxation time, the right-hand side of Eq. (1) reduces to

$$-t\int_0^\infty d\tau\langle\{\dot{S}^z(-\boldsymbol{k}),\ \dot{S}^z(\boldsymbol{k},\ \tau)\}\rangle. \tag{2}$$

On the other hand, the diffusion constant Λ is related to the left-hand side of Eq. (1) for small values of k as follows:[3]

$$-tk^2\Lambda\langle\{S^z(-\boldsymbol{k}),\ S^z(\boldsymbol{k})\}\rangle. \tag{3}$$

Thus, equating Eqs. (2) and (3) leads to the following expression for the diffusion constant:

$$\Lambda = \lim_{k\to 0}\frac{1}{k^2\langle[S^z(0)]^2\rangle} \times\int_0^\infty dt\langle\{\dot{S}^z(-\boldsymbol{k}),\ \dot{S}^z(\boldsymbol{k},\ t)\}\rangle.$$

The use of the equation of motion for spin operators further reduces this expression to the following:

$$\Lambda=\frac{1}{3N^3\phi_0(0)}\int_0^\infty dt\sum_p\sum_q\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\times J'(p)J'(q)\langle\{F(\boldsymbol{p}),F(\boldsymbol{q},t)\}\rangle, \tag{4}$$

where

$$\phi_0(0)\equiv\frac{1}{N}\langle[S^z(0)]^2\rangle=\frac{S(S+1)}{3}\frac{\chi}{\chi_0},$$

$$F(\boldsymbol{q})\equiv\{\overset{+}{S}(\boldsymbol{q}),\overset{-}{S}(-\boldsymbol{q})\}-\langle\{\overset{+}{S}(\boldsymbol{q}),\overset{-}{S}(-\boldsymbol{q})\}\rangle,$$

$$J(q)\equiv\sum_m J_{mf}\exp[i\boldsymbol{q}\cdot(\boldsymbol{R}_m-\boldsymbol{R}_f)],$$

where χ is the susceptibility of the system, χ_0 the paramagnetic susceptibility of an isolated ion, and S the magnitude of spin of an ion. The Fourier transform of the exchange integral $J(q)$ will be assumed to be the parabolic form with the cutoff at $q=q_m$ defined by $q_m^3/6\pi^2=N/V$. Thus the problem is reduced to the calculation of the correlation functions appearing in Eq. (4).

In the absence of any reliable method to calculate this correlation function, we have assumed the Gaussian decay:

$$\exp\left[-\frac{\pi}{4}\left(\frac{t}{\tau_c}\right)^2\right].$$

Then, Eq. (4) leads to

$$\Lambda\cong\frac{16}{3}\tau_c\left(\frac{J_0}{\hbar}\right)^2S(S+1)(\kappa_1 r_1)^2\times\left(\frac{a^2}{r_1^2q_m}\right)^2f\left(\frac{\kappa_1}{q_m}\right), \tag{5}$$

where

$$f(y)\equiv1+\frac{1}{2}\frac{y^2}{1+y^2}-\frac{3}{2}y\cot^{-1}y,$$

and κ_1, r_1 are the parameters characterizing the static correlation of spins, which have been calculated by Elliott and Marshall.[4] The magnitude of exchange integral between nearest neighbors, and the lattice constant have been denoted by J_0 and a. The decay constant τ_c is nearly constant with respect to temperature. Thus the temperature dependence of the diffusion constant in the vicinity of the Curie temperature is determined by the inverse of the susceptibility $\chi_0/\chi=(\kappa_1 r_1)^2$, which is in agreement with the phenomenological theories of van Hove and de Gennes.

Explicit calculation of Eq. (5) has been done for iron in the neighborhood of the Curie temperature, assuming $200^\circ K$ for J_0, and replacing κ_1 and r_1 by the observed ones obtained from the elastic part of the critical scattering of iron.[5] Thus, Eq. (5) yields

$$\Lambda\cong0.9\times10^{-5}(T-T_c)\ \mathrm{cm^2/sec},$$

which is about one half of the observed value,[5] but is reasonable considering the uncertainty of the quantities κ_1, r_1 and the exchange integral. As a matter of fact, a factor of two can be easily restored by using another set of values for these quantities still consistent with the experiments.[5,6]

Detailed account will appear shortly.

The authors wish to express their thanks to Professor T. Matsubara and to Professor K. Tomita for their enlightening discussions and encouragement.

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P-Wave Resonance in Pion-Pion Scattering

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Pion-pion scattering in the $I=l=1$ state is treated by the Tamm-Dancoff method. By making an approximation in which configurations containing up to three particles of meson and nucleon pair are taken into account in the 4th order pion-pion interaction kernel, the existence of resonance is concluded. By adjusting the coupling constant $g^2/4\pi$ between pion and nucleon so as to fit the pion-pion scattering length $a=0.1\sim0.3\mu^{-3}$ in the $I=l=1$ state, we have found the resonance level of 600~400 Mev and resonance width of 140~40 Mev, which is quite consistent with what adopted by Takeda et al. in their theory of the 2nd~3rd maximum in pion-nucleon scattering and also by Frazer-Fulco in the explanation of isotopic vector magnetic form factor of a nucleon. We adopt $g^2/4\pi=6.1\sim6.8$, which is somewhat smaller than the conventional one. This small coupling constant is interpreted as due to the inclusion of the effects of scattering and annihilation of a virtual nucleon pair.

§ 1. Introduction

The resonance formula for p-wave pion-pion scattering has been obtained theoretically by Miyamoto,[1] recently. This formula was derived by solving Chew-Mandelstam's equation, and predicts the location of the resonance level and its width as the function of pion-pion scattering length a. He has also shown that the same formula can be deduced from the chain approximation taking the Hamiltonian as $-2\pi a(\varphi_\alpha\partial_\mu\varphi_\beta-\varphi_\beta\partial_\mu\varphi_\alpha)^2$ for the p-wave interaction.

In the present meson theory, pions can be considered to interact mutually through the virtual nucleon-antinucleon pair formation and annihilation. In the above Hamiltonian, the nucleon loop is replaced by a point interaction. However, it seems to be worthwhile to investigate the pion-pion interaction taking into account the effects of virtual nucleon pair. As for the p-wave scattering, we can calculate the matrix element for the process going through nucleon loop, for example, for the process of Fig. 1, without ambiguity, because it does not involve any divergence. Igi and Kawarabayashi[2] calculated the matrix element for the process of Fig. 1, by using the perturbation and k/M-expansion (k is the pion momentum, M the nucleon mass), and obtained the attractive potential for the $I=l=1$ state. We try to calculate the matrix element for the pion-pion scattering in the better approximation than the lowest perturbation theory, and to see whether or not the existence of resonance level can be derived on the basis of the *ps* meson theory.

For this purpose it is desirable to solve the Bethe-Salpeter equation for

the process of Fig. 2. However, it is not easy to do so even approximately. If we performed it by the expansion to power series of k/M, there would appear a divergence, and the cutoff of a virtual momentum should be introduced to get a finite result. Then it has a quadratic dependence on the cutoff momentum. In order to avoid the difficulty of sensitive dependence on the cutoff momentum, we employ another more powerful method, such as the Tamm-Dancoff method, which was extensively used by Dyson-Bethe in the problem of pion-nucleon scattering. The advantage of our method is that we are free from any divergence. In this method, we make also the following approximation, namely for the 4th order pion-pion interaction kernel, we restrict the configurations of states to (2, 0), (1, 1) and (0, 1), where (m, n) represents the configuration containing m pions and n nucleon pairs. Then the potentials for pion-pion scattering which should be taken into account are those shown in Fig. 3. These interaction kernels involve the scattering in the $I=l=1$ state in the main, because they always contain as the intermediate state the state (0, 1) which can be in ${}^3S$ state. In our approximation of Fig. 3 there is only a small amount of $I=0$ state, since $I=0$ state involves (0, 1) states of ${}^3P_0$, ${}^3P_2$ which contribute less than ${}^3S$ by the centrifugal penetration factor. Therefore, this approximation is favourable for taking up only the $I=l=1$ state. If we want to treat the scattering in $I=0$, or 2, it is necessary to include the state (2, 1) at least.

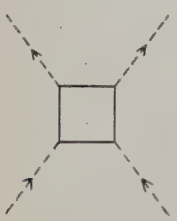


Fig. 1. Feynman diagram for the lowest order pion-pion scattering process.

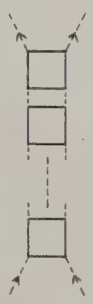


Fig. 2. Feynman diagram for higher order pion-pion scattering process.



Fig. 3. Diagrams for the 4th order pion-pion interaction kernels which are taken into account in our approximation.

§ 2. Equation for scattering amplitude and its solution

We start from a set of Schrödinger equations for $a_\lambda^{(m,n)}$:

$$(W-E_\lambda^{(m,n)})\,a_\lambda^{(m,n)}=\sum_{\lambda',m',n'}(\lambda, m, n|H'|\lambda', m', n')\,a_{\lambda'}^{(m',n')}, \qquad (1)$$

where $a_\lambda^{(m,n)}$ is the amplitude corresponding to the configuration (m, n) in the expansion of Schrödinger functional Ψ in terms of the eigenfunctions of number operator, i.e.

$$\Psi=\sum_{\lambda,m,n} a_\lambda^{(m,n)}|\lambda, m, n\rangle, \tag{2}$$

and $E_\lambda^{(m,n)}$ is the energy of the state $|\lambda, m, n\rangle$. λ represents the momenta, spin and isotopic spin of particles which are involved in that configuration. W is the actual total energy of the system. We adopt as the interaction Hamiltonian $H'=g\psi^{\dagger}\tau_\alpha\psi\phi_\alpha$. After we deduce the integral equation for the amplitude $a_\lambda^{(2,0)}$ which has the 4th order interaction kernel corresponding to the processes shown in Fig. 3, we separate the equation for the $I=l=1$ state, by using the expansion of $a_\lambda^{(2,0)}$ in terms of the partial wave amplitudes $a_l^I(k, k_0)$,

$$a_\lambda^{(2,0)}=\sum_{I,l}(2l+1)P_l(\cos\theta)\mathrm{P}^I a_l^I(k, k_0), \tag{3}$$

where P^I is the projection operator for the isotopic spin state I, and k, k_0 are the magnitudes of momenta in the c.m. system of the intermediate and incoming pions, respectively. The integral equation for $a_1^1(k, k_0)$ is

$$(\omega_0-\omega_k)a_1^1(k, k_0)=-G\frac{k}{\omega_k}\int\frac{dk'k'^3}{\omega_{k'}}I(k, k'; k_0)a_1^1(k', k_0), \tag{4}$$

where

$$I(k, k'; k_0)=\int_M \frac{dE_p\sqrt{E_p^2-M^2}(2E_p^2+M^2)}{E_p\sqrt{E_p^2+k^2}\sqrt{E_p^2+k'^2}(E_p-\omega_0)(E_p+\sqrt{E_p^2+k^2}+\omega_k-2\omega_0)*} \\ *(E_p+\sqrt{E_p^2+k'^2}+\omega_{k'}-2\omega_0) \\ \times A(k, k'; k_0), \tag{5}$$

$$A(k, k'; k_0)=1+\frac{3}{5}\frac{2E_p^2+3M^2}{2E_p^2+M^2}\left(\frac{pk}{E_p^2+k^2}\right)^2+(k\to k')+\cdots, \tag{6}$$

$$W=2\omega_0,$$

and

$$G=\left(\frac{g^2}{4\pi}\right)^2\frac{4}{9\pi^2}. \tag{7}$$

$I(k, k'; k_0)$ is the integration corresponding to nucleon-antinucleon loop. Its integrand is expanded in powers of $(pk/E_p^2+k^2)^2$ and $(pk'/E_p^2+k'^2)^2$, and $A(k, k'; k_0)$ represents the power series in these expansions. In the following calculations we take into account up to the second order terms in $A(k, k'; k_0)$, but the contributions from the second order terms can be shown to be the order of several percent of that from the first term.

For the non-singular amplitude $f(k, k_0)$ introduced by

$$a_1^1(k, k_0)=\delta(W-2\omega_k)+P\frac{1}{W-2\omega_k}f(k, k_0), \tag{8}$$

the equation is obtained as

$$f(k, k_0) = -G \frac{kk_0^2}{\omega_k} I(k, k_0; k_0) + G \frac{k}{\omega_k} \int \frac{dk' k'^3}{\omega_{k'}(\omega_{k'} - \omega_0)} I(k, k'; k_0) f(k', k_0), \tag{9}$$

and scattering phase shift δ is given by

$$f(k_0, k_0) = -\frac{1}{\pi} \tan \delta .$$

In order to solve Eq. (9) we rewrite it as

$$F(k, k_0) = 1 + G \int_{\mu} \frac{d\omega_{k'} k'^3}{\omega_{k'}(\omega_{k'} - \omega_0)} \frac{I(k, k'; k_0) I(k', k_0; k_0)}{I(k, k_0; k_0)} F(k', k_0), \tag{10}$$

$$F(k, k_0) \equiv -f(k, k_0) / G \frac{kk_0^2}{\omega_k} I(k, k_0; k_0) .$$

Here we notice that the dependence of $I(k, k'; k_0)/I(k, k_0; k_0)$ on k is very small as shown in Appendix I; so, we can neglect this dependence with good approximation. In this case, $F(k', k_0)$ becomes independent of k' and we can immediately write down the solution of Eq. (9) as follows,

$$f(k, k_0) = -G \frac{kk_0^2}{\omega_k} I(k, k_0) [1 - GJ(k, k_0)]^{-1}, \tag{11}$$

where

$$J(k, k_0) = \int_{\mu} \frac{d\omega_{k'} k'^3}{\omega_{k'}(\omega_{k'} - \omega_0)} \frac{I(k, k') I(k', k_0)}{I(k, k_0)}, \tag{12}$$

and $I(k, k')$ is the abbreviation of $I(k, k'; k_0)$. The solution has the same form as the solution obtained by the variational principle of Schwinger. The integration on E_p in $I(k, k_0)$ and the integration on k' in $J(k, k_0)$ are both divergence free, and we can evaluate them without making use of any cutoff procedure.

§ 3. Results and discussions

$I(k_0, k_0)$ and $J(k_0, k_0)$ are calculated numerically, and the results are shown in Figs. 4 and 5 respectively. It is noticed that the value of $J(k_0, k_0)$ is considerably large. If we adopt $g^2/4\pi = 15$ $(G \sim 10)$, $GJ(k_0, k_0)$ becomes always larger than 1, and the pion-pion force becomes repulsive, though it is attractive in the Born approximation. Then the resonance phenomenon does not take place.

These circumstances seem to depend on the treatment of nucleon loop. Because the nucleon pair can be in the ^3S state for p-wave pion-pion scattering, they would be in short distances and interact with considerably strong and unknown forces. These interactions, which correspond to the exchanges of many mesons between them and to the annihilation of them into four or more mesons, are not taken into account in our approximation. And yet it might

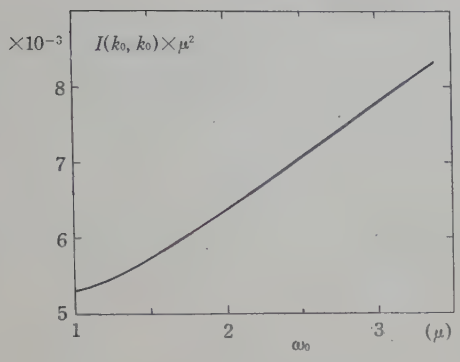


Fig. 4. $I(k_0, k_0)$ obtained by numerical integration on E_p.

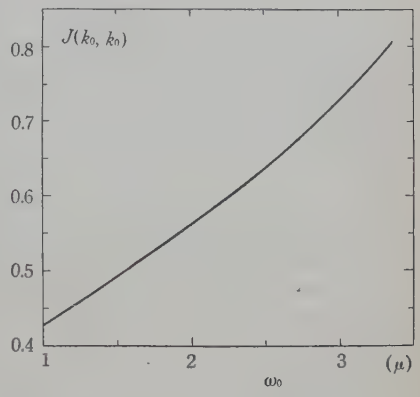


Fig. 5. $J(k_0, k_0)$ obtained by numerical calculation of the k'-integration in (12).

be impossible to treat these effects of virtual nucleon pairs correctly in the present field theory. In the treatment by Miyamoto, the effects of scattering and annihilation of virtual nucleon pair are included in the scattering length a. In our case, it is considered that these effects diminish the value of $g^2/4\pi$ effectively. This circumstance is similar to the case of pion decay, for example, where it was necessary to suppress the effects of virtual nucleon loop in order to obtain an adequate value for the lifetime. According to the idea of Miyamoto, we will determine the value of effective coupling constant so as to be given the experimental value to the scattering length. The scattering length is defined as a function of $g^2/4\pi$, as follows:

$$a=\lim_{k_0\to 0}\frac{\delta}{k_0^3}=\frac{\pi GI(0,0)}{\mu[1-GJ(0,0)]}. \tag{13}$$

When a value is given to a, the effective coupling constant is decided as the function of a. By analyzing the experimental data on one pion production in pion-nucleon collision, Rodberg[3] has given

$$a=0.2\,\mu^{-3}.$$

When we adopt $a=0.1\sim 0.3\mu^{-3}$, the magnitude of $g^2/4\pi$ is decided as shown in Table I. At this time, the resonance takes place. The resonance energy $2\omega_r$ and its total width Γ (against an abscissa ω_0) can be obtained and are also given in Table I.* The values in parenthesis are those that are obtained from the resonance formula by Miyamoto. $2\omega_r$ and Γ in Table I are consistent with the values assumed by Takeda[4] and Itabashi at el.[5] or Frazer and Fulco.[6] In the former, $2\omega_r\sim 600$ Mev

* After the calculation reported here was performed, we found in the reports of the Ninth Conference on High Energy Nuclear Physics in Kiev that the cross section of pion-pion scattering had also been calculated on the basis of the Tamm-Dancoff approximation by Silin and Tamm, and that the resonance in the $I=l=1$ state had been found at $\sim$500 Mev only with $g^2/4\pi=5$.

was assumed for the interpretation of high energy maximum in pion-nucleon scattering, and in the latter, $2\omega_r \sim 500$ Mev and $\Gamma \sim 60$ Mev were assumed in the analysis on the structure of nucleon. These results show the possibility that the pion-pion force can be explained on the basis of the conventional *ps* meson theory.

Table I

$a(\mu^{-3})$	$g^2/4\pi$	$2\omega_r$ (Mev)	Γ (Mev)
0.1	6.1	600	140
0.2	6.6	455	50
0.3	6.8	400	40
(0.24)		(560)	(130)

Table II

E_{max}	1.4 M			1.5 M		
ω_{max}	2 M	3 M	∞	2 M	3 M	∞
a (μ^{-3})	0.13	0.16	0.19	0.21	0.29	0.45
$2\omega_r$ (Mev)	630	560	510	510	440	390
Γ (Mev)	200	140	95	90	55	30

Instead of using the effective coupling constant, we can use the cutoff procedure in order to suppress the effect of virtual high energy nucleon pair, while $g^2/4\pi=15$ is retained. We introduce a cutoff E_{max} phenomenologically instead of taking into account the virtual annihilation of a virtual nucleon pair. We interpret $2E_{max}$ as a radius within which a nucleon and an anti-nucleon are annihilated. We select the values of E_{max} and ω_{max} so as to give the reasonable value for the scattering length a. For the cases of $E_{max}=1.4M$ and $1.5M$, and for some values of ω_{max}, a is calculated as shown in Table II. The resonance level $2\omega_r$ and the width Γ are also given in Table

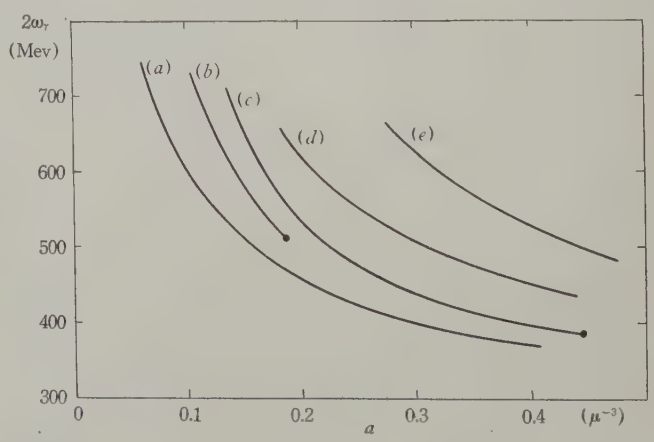


Fig. 6. Relations of $2\omega_r$ to a. Curve (a) corresponds to that given in Table I, (b) and (c) to those of $E_{max}=1.5M$ and $1.4M$, respectively, given in Table II. (d) and (e) are those obtained from the resonance formula by Miyamoto with the cutoff momenta $2M$ and M, respectively.

II. As it was anticipated, the results depend considerably on the value of E_{max}, while the apparent independence on ω_{max} indicates the dominance of low energy part in the pion-pion scattering. By making this analysis with cutoff, we find that the resonance level is almost uniquely determined, so long as we take the cutoff E_{max} so as to give an experimental scattering length. This analysis also confirms our philosophy of pology.

The relation of $2\omega_r$ or Γ to a in this case with cutoff is consistent with the case of Table I. Fig. 6 graphically shows the relation between a and $2\omega_r$ which are given in Tables I and II. Curve (a) corresponds to the case given in Table I, which is obtained by changing the magnitude of $g^2/4\pi$. Curves (b) and (c) correspond to the cases of $E_{max}=1.5M$ and $1.4M$, respectively, given in Table II. The starting points of these curves are those for $\omega_{max}=\infty$. Curves (d) and (e) are those that are obtained from the resonance formula by Miyamoto with the cutoff momenta $2M$ and M, respectively.

§ 4. Conclusion

As mentioned above the simple meson theoretic treatment on p-wave pion-pion scattering gives the plausible results for resonance when the effects of virtual high energy nucleon pair are included in $g^2/4\pi$ or scattering length a. The effects of virtual high energy nucleon-antinucleon interaction in the ${}^3S$ state should not be predicted by the present meson theory, but these effects which occur at relatively high threshold energy $2M$ can be included approximately in the scattering length a in accordance with the principle of pology. Then the remaining p-wave pion-pion scattering can be treated rather correctly in the present meson theory, if we limit ourselves to low energy scattering problem. As is shown in Fig. 6, the resonance level is almost uniquely determined by the value of the scattering length. Accordingly, it is desirable to decide the scattering length a accurately by the experiments of pion production by pion $\pi+N\to\pi+\pi+N$.

Acknowledgments

The author would like to express her gratitude to Professor N. Fukuda for his encouragement. Also the author would like to extend her sincere thanks to Dr. Y. Miyamoto for valuable discussions.

Appendix I

In solving the integral equation (9) we neglected the dependence of $I(k, k'; k_0)/I(k, k_0; k_0)$ on k. In this approxima-

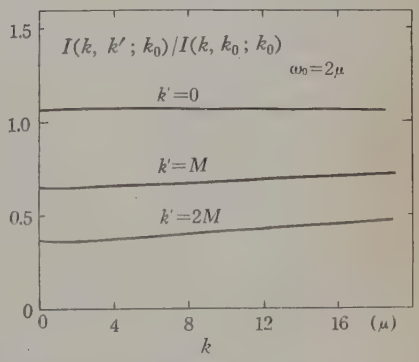


Fig. 7. k-dependence of $I(k, k'; k_0)/I(k, k_0; k_0)$ for the cases $\omega_0=2\mu$ and $k'=0$, M and $2M$.

tion $F(k', k_0)$ in Eq. (10) become independent of k'. Fig. 7 shows the k-dependence of $I(k, k'; k_0)/I(k, k_0; k_0)$ for $\omega_0=2\mu$ and $k'=0$, M, and $2M$. In the k'-integration, most contributions are given from the region up to $k'\sim 2M$, where the change of $I(k, k'; k_0)/I(k, k; k_0)$ with k is very slow; so, this dependence may be neglected in good approximation.

Appendix II

If we include one more configuration (2, 1) in the 4th order pion-pion interaction kernel, the kernel of type shown in Fig. 8 must be taken into account in addition to those shown in Fig. 3. In this case, $I(k, k')$ becomes to be replaced by $I(k, k')[1+(E_p-\omega_0)/(2E_p+\omega_k+\omega_{k'}-2\omega_0)]$. It is estimated as a result of this replacement that the magnitudes of $g^2/4\pi$ in Table I are decreased by about fifteen or smaller percent in the approximation treating the pion energy $\omega_{k'}$ as to be small compared with the nucleon energy E_p, and all above mentioned but the magnitude of $g^2/4\pi$ remain unchanged.



Fig. 8. Diagram of pion-pion 4th order interaction kernel which involves the configuration (2,1).

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Quantum Statistical Analogue of Ward's Identity

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Considering the quantum statistical electrodynamics, we prove an identity which corresponds to Ward's in quantum electrodynamics. In the course of the discussion, an emphasis is put on the periodicity properties of the functions which appear therein.

§ 1. Introduction

In the recent developments of the quantum statistics, the techniques of the quantum field theory have been applied extensively. These applications are made possible by the observation of the formal analogy between the partition function and the S-matrix. Indeed, Matsubara[1] was the first to show that the Feynman-diagram technique could be successfully applied to the perturbation expansion of the partition function. The point was to reduce the calculation of the trace to a kind of contraction procedure. In those works along this line of thought, attention has been concentrated mainly around the non-relativistic systems.

When we tackle the relativistic systems, we confront with a difficulty, the ultraviolet divergence which is familiar in the quantum field theory. Also in the case of quantum statistics, one may expect to bypass this difficulty through the renormalization procedure. The discussion of the renormalization presents some interesting problems, which we shall work out in a separate paper. As a preparatory step, we show in this note the parallelism between the quantum electrodynamics (Q.E.D.) and the statistical quantum electrodynamics (S.E.D.) with respect to the gauge invariance which, as is well known, plays an important role in the renormalization programme. In the course of the preparation for the publication of our work, we found Fradkin's paper on the same subject.[2] Because of the difference in the method of attack, we think it worthwhile to publish our work, too.

§ 2. Temperature-Heisenberg picture

Let us begin with the field operator $F_S(\boldsymbol{x})$ in the Schrödinger picture and hereby define the temperature-Heisenberg operator on the one hand:

$$\boldsymbol{F}(\boldsymbol{x}, \tau) \equiv \rho^{-1}(\tau) F_S(\boldsymbol{x}) \rho(\tau), \tag{2·1}$$

with density matrix defined by

$$\rho(\tau) \equiv \exp[-(H-\mu N)\tau], \tag{2·2}$$

and, on the other, the usual Heisenberg operator:

$$F(\boldsymbol{x}, t) \equiv e^{iHt} F_S(\boldsymbol{x})\, e^{-iHt}, \tag{2·3}$$

where τ, t, H, N and μ are the inverse temperature $1/kT$, time, the total Hamiltonian, the number operator (negaton number minus positon number), and the chemical potential, respectively. Then, the temperature-derivative of $\boldsymbol{F}(\boldsymbol{x}, \tau)$ is connected with the time-derivative of $F(\boldsymbol{x}, t)$:

$$\begin{aligned}\frac{\partial}{\partial\tau}\boldsymbol{F}(\boldsymbol{x}, \tau) &= \rho^{-1}(\tau)[H-\mu N,\ F_S(\boldsymbol{x})]\rho(\tau) \\ &= \rho^{-1}(\tau)\, e^{-iHt}\left\{-i\frac{\partial F(\boldsymbol{x}, t)}{\partial t} - \mu[N,\ F(\boldsymbol{x}, t)]\right\} e^{iHt}\rho(\tau).\end{aligned} \tag{2·4}$$

Keeping this equation in mind, we can interpret the relations among the usual Heisenberg operators into those among the temperature-Heisenberg operators by the correspondence:

$$\frac{\partial}{\partial\tau}\boldsymbol{F}(\boldsymbol{x}, \tau) \leftrightarrow -i\frac{\partial}{\partial t}F(\boldsymbol{x}, t), \tag{2·5}$$

when the operator F commutes with the number operator N.

Then, the equal-temperature commutators in S.E.D. can be written down easily, e.g.

$$\left.\begin{aligned}&\left[\boldsymbol{A}_\mu(\boldsymbol{x}, \tau),\ \frac{\partial}{\partial\tau}\boldsymbol{A}_\nu(\boldsymbol{x}', \tau)\right] = \delta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{x}'), \\ &[\boldsymbol{\psi}(\boldsymbol{x}, \tau),\ \boldsymbol{j}_0(\boldsymbol{x}', \tau)] = e\boldsymbol{\psi}(\boldsymbol{x}, \tau)\delta(\boldsymbol{x}-\boldsymbol{x}'), \\ &[\boldsymbol{A}_\mu(\boldsymbol{x}, \tau),\ \boldsymbol{j}_0(\boldsymbol{x}', \tau)] = 0,\end{aligned}\right\} \tag{2·6}$$

where $\boldsymbol{A}_\mu$ and $\boldsymbol{\psi}$ are the field operators for photons and electrons, respectively. The current operator $\boldsymbol{j}_\mu$ satisfies

with

$$\left.\begin{aligned}&\boldsymbol{j}_\mu = -\Box\,\boldsymbol{A}_\mu, \\ &\Box \equiv \frac{\partial^2}{\partial x_k \partial x_k} + \frac{\partial^2}{\partial\tau^2} = \frac{\partial^2}{\partial x_\mu \partial x_\mu}.\end{aligned}\right\} \tag{2·7}*$$

In the same way, the equation of continuity in Q.E.D.,

$$\frac{\partial}{\partial x_k} j_k + \frac{\partial}{\partial t}\, j_0 = 0 \tag{2·8}$$

---

* 1) $\boldsymbol{x} = (x_1, x_2, x_3)$, and $\tau = x_4$ we write simply $(\boldsymbol{x}, x_4) = x$. The Latin index appearing twice implies the summation over it from 1 to 3, the Greek index from 1 to 4.

(*continued on next page*)

implies

$$\frac{\partial}{\partial x_k}\boldsymbol{j}_k+i\frac{\partial}{\partial \tau}\boldsymbol{j}_0=0$$

or, with the notation $\tau=x_4$, $i\boldsymbol{j}_0=\boldsymbol{j}_4$,

$$\frac{\partial}{\partial x_\mu}\boldsymbol{j}_\mu(x)=0. \tag{2·9}$$

Now, to establish an identity in S.E.D. which corresponds to Ward's in Q.E.D., we here follow Nishijima's formalism developed for Q.E.D.[3] The basic equation is the following relation among the temperature T-products:

$$\begin{aligned}\Box_x \mathrm{T}[\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots]\\ =-\mathrm{T}[\boldsymbol{j}_\mu(x), \boldsymbol{A}_\nu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots]\\ -\frac{\delta}{\delta \boldsymbol{A}_\mu(x)}\mathrm{T}[\boldsymbol{A}_\mu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots],\end{aligned}$$

where

$$\frac{\delta}{\delta \boldsymbol{A}_\mu(x)}\boldsymbol{A}_\nu(x')=\delta_{\mu\nu}\delta(x-x'), \tag{2·10}$$

and

$$\delta(x-x')=\prod_{\mu=1}^{4}\delta(x_\mu-x_\mu').$$

Making use of Eqs. (2·6) and (2·9), we can eliminate $\boldsymbol{j}_\mu$ and arrive at

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\Box_x \mathrm{T}[\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots]\\ =ie[\delta(x-y)-\delta(x-y')]\mathrm{T}[\boldsymbol{A}_\nu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots]\\ -\frac{\partial}{\partial x_\mu}\frac{\delta}{\delta \boldsymbol{A}_\mu(x)}\mathrm{T}[\boldsymbol{A}_\nu(x'), \cdots, \boldsymbol{\psi}(y), \bar{\boldsymbol{\psi}}(y'), \cdots].\end{aligned} \tag{2·11}$$

As special cases, we get

$$\frac{\partial}{\partial x_\mu}\Box_x \mathrm{T}[\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x')]=-\frac{\partial}{\partial x_\nu}\delta(x-x'), \tag{2·12}$$

---

* [2] If $\boldsymbol{j}_\mu$ were defined to be the current operators solely of the electrons, this equation would sometimes lead to apparent contradiction: Take the statistical average [cf. Eq. (3·2)] of both sides of this equation, then the right-hand side vanishes identically since $\langle \boldsymbol{A}_\mu(x)\rangle$ is independent of x, while the left-hand side (the 0-th component) does not in the case that $\langle N\rangle \neq 0$. This is not so surprising. Because we are dealing with the infinite homogeneous system, the non-zero charge density makes the electromagnetic field mathematically indefinite. To avoid this difficulty, there is a quite natural way, i. e. to define the current operator as

$$\boldsymbol{j}_\mu=\boldsymbol{j}_\mu{}^{QED}-\langle \boldsymbol{j}_\mu{}^{QED}\rangle$$

where the first term, $\boldsymbol{j}_\mu{}^{QED}=(ie/2)[\bar{\psi}, \gamma_\mu\psi]$, is the contribution from the electrons and the second term is that from the back ground ions (the total system is assumed to be neutral).

$$\frac{\partial}{\partial y_\mu}\Box_y \mathrm{T}[\psi(x), \bar{\psi}(y), \boldsymbol{A}_\mu(y)] = ie[\delta(x-y)-\delta(y-x')]\mathrm{T}[\psi(y), \bar{\psi}(y')]. \tag{2·13}$$

§ 3. Green functions

Some remarks are in order about the periodicity properties of the Green functions in S.E.D.

The Green function for a photon in S.E.D. is defined by*

$$\boldsymbol{D}_{\mu\nu}'(x-x') \equiv \langle \mathrm{T}[\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x')]\rangle - \langle \boldsymbol{A}_\mu(x)\rangle\langle \boldsymbol{A}_\nu(x')\rangle, \tag{3·1}$$

$$(0 \leqq x_4,\ x_4' \leqq \beta),$$

where the statistical average of an operator $\boldsymbol{\Omega}$ is defined by

$$\langle \boldsymbol{\Omega}\rangle \equiv \mathrm{Tr}[\rho(\beta)\boldsymbol{\Omega}]/\mathrm{Tr}[\rho(\beta)], \tag{3·2}$$

with the temperature of the system T and $\beta \equiv 1/kT$. From these definitions, it follows immediately that

$$\boldsymbol{D}_{\mu\nu}'(\boldsymbol{x}, \beta-\tau) = \boldsymbol{D}_{\mu\nu}'(\boldsymbol{x}, -\tau), \quad \text{for } 0 \leqq \tau \leqq \beta, \tag{3·3}$$

and especially,

$$\boldsymbol{D}_{\mu\nu}'(\boldsymbol{x}, \pm\beta) = \boldsymbol{D}_{\mu\nu}'(\boldsymbol{x}, 0).$$

Taking these properties into account, we can extend the Green function all over the real x_4-axis in such a way that it becomes a periodic function with the period β. This situation is shown schematically in Fig. 1.

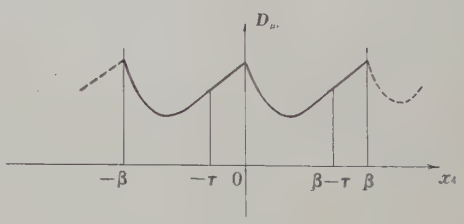


Fig. 1.

Now, the statistical average of Eq. (2·12) gives

$$\Box_x \frac{\partial}{\partial x_\mu}\boldsymbol{D}_{\mu\nu}'(x-x') = -\frac{\partial}{\partial x_\nu}\delta(x-x')$$

for the open interval $-\beta < x_4 - x_4' < \beta$. The derivatives of the left-hand side of this equation at $x_4 - x_4' = \pm\beta$ are determined by the periodic extension of $\boldsymbol{D}_{\mu\nu}'$, and consequently

$$\left.\begin{aligned} &\Box_x \frac{\partial}{\partial x_\mu}\boldsymbol{D}_{\mu\nu}'(x-x') = -\frac{\partial}{\partial x_\nu}\delta_2(x-x') \\ &\text{where} \\ &\delta_2(x) \equiv \delta(\boldsymbol{x})\cdot\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\exp\left[-i\frac{2n\pi}{\beta}x_4\right] \end{aligned}\right\} \tag{3·4}$$

* Because of the chemical potential, Furry's theorem does not necessarily hold. So, we keep the second term, which, however, does not depend on x and x'.

is the periodic extension of $\delta(x_4)$ with the period β.*

In the same manner, we define the Green function for an electron:

$$S'(x-x')=\langle \mathrm{T}[\psi(x), \bar{\psi}(x')]\rangle,$$
$$(0\leqq x_4, x_4'\leqq\beta). \qquad (3\cdot5)$$

Due to the relation which stems from the definition itself,

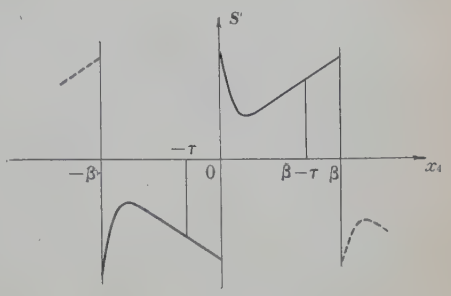


Fig. 2.

$$S'(x, \beta-\tau)=-S'(x, -\tau), \quad \text{for } 0\leqq\tau\leqq\beta, \qquad (3\cdot6)$$

we can extend this function "anti"-periodically all over the real x_4-axis. This situation is shown schematically in Fig. 2.

§ 4. Identity corresponding to Ward's

Let

$$f_\mu(u, v)\equiv\langle \mathrm{T}[\psi(x), \bar{\psi}(x'), A_\mu(y)]\rangle, \qquad (4\cdot1)$$

where the space-coordinates are suppressed in the left-hand side and

$$u\equiv x_4-y_4, \quad v\equiv y_4-x_4',$$

then, we can establish the following property from this definition (Fig. 3):

$$\left.\begin{aligned} &f_\mu(u+\beta, v)=-f_\mu(u, v)\\ &\quad \text{for } (u, v)\in \mathrm{I}, \text{ i.e. } \begin{cases} -\beta<u, v, u+\beta<\beta,\\ -\beta<u+v, u+v+\beta<\beta,\end{cases}\\ \text{and}\quad &f_\mu(u, v+\beta)=-f_\mu(u, v)\\ &\quad \text{for } (u, v)\in \mathrm{II}, \text{ i.e. } \begin{cases} -\beta<u, v, v+\beta<\beta\\ -\beta<u+v, u+v+\beta<\beta.\end{cases}\end{aligned}\right\} \qquad (4\cdot2)$$

As before, we can extend the function Eq. (4·1), all over the (u, v)-plane with the aid of these "anti"-periodicity properties.

Now, we take the statistical average of Eq. (2·13). The periodicity properties of the left-hand side described by Eq. (4·2) tell us that the δ-functions in the right-hand should be replaced by its "anti"-periodic extension:**

---

* It must be taken for granted here that $\int_a^b \delta(x_4)f(x_4)dx_4=1/2\cdot f(0)$ for $b>a=0$ or $0=b>a$. This δ_2-function acts on the periodic functions with period β (the functions with "even" frequencies, see Eq. (4·5)) as if it were the usual δ-function.

** See the footnote above. This δ_1-function acts on the "anti"-periodic functions [the functions with "odd" frequencies, see Eq. (4·5)] as if it were the usual δ-function.

$$\delta_1(x)\equiv\delta(x)\frac{1}{\beta}\sum_{n=-\infty}^{+\infty}\exp\left[-i\frac{(2n+1)\pi}{\beta}x_4\right]. \tag{4·3}$$

The vertex function $\boldsymbol{\Gamma}_\nu$ is connected with the function defined by Eq. (4·1):

$$\langle \mathrm{T}[\boldsymbol{\psi}(x), \bar{\boldsymbol{\psi}}(x'), \boldsymbol{A}_\mu(y)]\rangle$$
$$=ie\int \boldsymbol{S}'(x-\xi)\boldsymbol{\Gamma}_\nu'(\xi-\eta, \eta-\zeta)\boldsymbol{S}'(\zeta-x')$$
$$\times \boldsymbol{D}_{\mu\nu}'(\eta-y)d\xi d\eta d\zeta-\boldsymbol{S}'(x-x')\langle \boldsymbol{A}_\mu(y)\rangle, \tag{4·4}$$

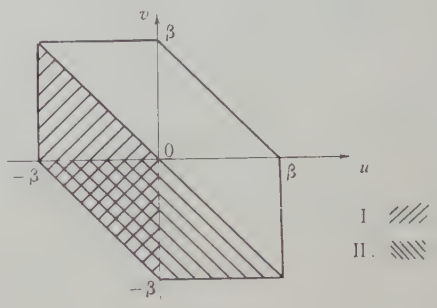


Fig. 3.

where, for example,

$$\int d\xi=\int_0^\beta d\xi_4\int_{-\infty}^{+\infty}d\xi_1\int_{-\infty}^{+\infty}d\xi_2\int_{-\infty}^{+\infty}d\xi_3.$$

We may note here that the periodicity properties of the left-hand side, Eq. (4·4), are consistent with those of the right-hand side. As is shown in the Appendix, the vertex function has also the "anti"-periodicity properties and can therefore be expanded into the Fourier Series,

$$\boldsymbol{\Gamma}_\mu'(x, x')=\left[\frac{1}{\beta(2\pi)^3}\right]^2\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\int d\boldsymbol{p}\int d\boldsymbol{p}'\,\boldsymbol{\Gamma}_\mu(p, p')$$
$$\times\exp[-i\boldsymbol{p}\boldsymbol{x}-ip_{4n}x_4]\exp[-i\boldsymbol{p}'\boldsymbol{x}'-ip_{4m}x_4'] \tag{4·5}$$

with the "odd" frequencies:

$$p_{4k}=(2n+1)\frac{\pi}{\beta},\quad (k=n, m).$$

Making use of Eq. (3·4) to eliminate $\boldsymbol{D}_{\mu\nu}'$ in Eq. (4·4), we obtain, since the second term in Eq. (4·4) does not depend on y,

$$-\int \boldsymbol{S}'(x-\xi)\frac{\partial}{\partial y_\nu}\boldsymbol{\Gamma}_\nu'(\xi-y, y-\zeta)\boldsymbol{S}'(\zeta-x')d\xi\,d\eta$$
$$=\boldsymbol{S}'(y-x')\delta_1(x-y)-\boldsymbol{S}'(x-y)\delta_1(y-x'), \tag{4·6}$$

where we have made an integration by parts with the aid of the fact that $\boldsymbol{\Gamma}_\nu(\xi-\eta, \eta-\zeta)$ is a periodic function of η_4 with the period β. We define the Fourier coefficients of $\boldsymbol{S}$ in a manner similar to Eq. (4·5), then Eq. (4·6) yields

$$i\boldsymbol{S}'(p)(p-p')_\nu\boldsymbol{\Gamma}_\nu(p, p')\boldsymbol{S}'(p')=\boldsymbol{S}'(p')-\boldsymbol{S}'(p),$$
or
$$i(p-p')_\nu\boldsymbol{\Gamma}_\nu(p, p')=[\boldsymbol{S}'(p)]^{-1}-[\boldsymbol{S}'(p')]^{-1} \tag{4·7}$$

where, as is seen from Eq. (3·6),

$$p_4=(2n+1)\frac{\pi}{\beta},\quad p_4'=(2n'+1)\frac{\pi}{\beta},\quad (n,\ n'=0,\ \pm1,\ \cdots).$$

This is just the quantum statistical analogue of the generalized Ward's identity.[3),4)] Of course, our result, Eq. (4·7), coincides with Fradkin's.[2)]

Appendix

In this Appendix, we show that the vertex operator $\boldsymbol{\Gamma}_\mu(p,\ p')$ has odd frequencies, p_4 and p_4'. We begin the proof by adding to the Hamiltonian a term due to the external source[5)] $J_\mu(x)$:

$$H_{\text{ext}}(\tau)=-e\int J_\mu(x)\boldsymbol{A}_\mu(x)d\boldsymbol{x}. \tag{A·1}$$

We assume that the electron Green function $\boldsymbol{S}[J|x,\ \tilde{\xi}]$, which includes the effect of the source, has its inverse such that

$$\int \boldsymbol{S}[J|x,\ \tilde{\xi}]\boldsymbol{S}^{-1}[J|\tilde{\xi},\ y]d\tilde{\xi}=\delta(x-y),\quad (0\leqq x_4,\ y_4\leqq\beta). \tag{A·2}$$

Then, the vertex operator is given by

$$\boldsymbol{\Gamma}_\mu(x-z,\ z-y)\equiv\lim_{J\to0}\frac{\delta}{\delta\langle ie\boldsymbol{A}_\mu(z)\rangle}\boldsymbol{S}^{-1}[J|x,\ y]. \tag{A·3}$$

This equation can be rewritten as

$$\boldsymbol{\Gamma}_\mu(x-z,\ z-y)=-\int\boldsymbol{S}'^{-1}(x-\tilde{\xi})\left\{\lim_{J\to0}\frac{\delta}{\delta\langle ie\boldsymbol{A}_\mu(z)\rangle}\boldsymbol{S}[J|\tilde{\xi}\eta]\right\}\boldsymbol{S}'^{-1}(\eta-y)d\tilde{\xi}\,d\eta, \tag{A·4}$$

where the electron Green function is given by

$$\boldsymbol{S}'(x-\tilde{\xi})=\lim_{J\to0}\boldsymbol{S}[J|x,\ \tilde{\xi}],$$

$$\boldsymbol{S}'^{-1}(x-\tilde{\xi})=\lim_{J\to0}\boldsymbol{S}^{-1}[J|x,\ \tilde{\xi}].$$

According to Eq. (A·4), the periodicity properties of $\boldsymbol{\Gamma}_\mu(u,\ v)$ with respect to $u=x_4-z_4$ and $v=z_4-y_4$ are governed by $\boldsymbol{S}'^{-1}(x-\xi)$ and $\boldsymbol{S}'^{-1}(\eta-y)$ in the right-hand side, respectively.

Now, as is shown in the text, $\boldsymbol{S}'$ is "anti"-periodic with period β. Then, we can see from Eq. (A·2) that $\boldsymbol{S}'^{-1}(\tilde{\xi}-y)$ is also "anti"-periodic, because this function has the variable y in common with the δ-function in the right-hand side which must be replaced by the "anti"-periodic δ_1 in the limit $J\to0$, $\boldsymbol{S}[J|x,\ \tilde{\xi}]\to\boldsymbol{S}'(x-\tilde{\xi})$. This completes the proof.

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On an Extension of the Mathematical Framework of the Quantum Theory

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Extending the usual framework of the quantum theory, we have succeeded in getting a new mathematical framework. The most essential character of the usual theory is preserved in the new theory, that is, the set of physical quantities can be represented by an operator algebra in a separable Hilbert space. In the new theory, on the other hand, we can rigorously treat the interaction Hamiltonians, which cannot be treated in the usual framework. In our new framework there are always solutions of the Schrödinger equation for any Hamiltonian. However, the most serious defect of our theory is in the fact that we have no guarantee of the uniqueness of solutions.

§ 1. Introduction

The outline of the mathematical framework of the usual quantum theory is as follows. For any physical system there is a separable Hilbert space $\mathfrak{H}$, and every physical quantity is represented by an operator a in $\mathfrak{H}$ and every state by a vector $\Psi \in \mathfrak{H}$ of norm 1. The expectation value of a in a state Ψ is given by the inner product $(\Psi, a\Psi)$. The change of the system with time can be described by a one-parameter group $U(t)$ of unitary operators, and in the Schrödinger representation $\Psi(t)$ is given by

$$\Psi(t) = U(t)\Psi(0). \tag{1·1}$$

Stone's theorem indicates that there is a self-adjoint operator H in $\mathfrak{H}$ such that

$$i(d/dt)U(t) = HU(t), \quad U(t) = \exp(-iHt), \tag{1·2}$$

and H is called the Hamiltonian of this system [ref. 1), p. 598]. Since $U(t)$ are unitary operators in $\mathfrak{H}$, it is possible to use the Heisenberg representation and consider that physical quantities change with time in accordance with the equation

$$a(t) = U^{-1}(t)\,a(0)\,U(t), \tag{1·3}$$

and that states are unchanged.

On the one hand the usual mathematical framework of the quantum theory is in harmony with the physical contents of the theory, and at first sight it seems to be almost impossible to separate these two aspects of the theory.

But, on the other hand, there is little room for doubt that the improvement of the framework is inevitable in order to deal rigorously with the quantum theory of interacting fields. This is due to the fact that, because the usual interaction Hamiltonian H cannot be an operator in $\mathfrak{H}$, $U(t)$ cannot be a one-parameter group of unitary operators in $\mathfrak{H}$. In fact, if $U(t)$ is to be physically meaningful, it should be continuous in some sense. But, since $\mathfrak{H}$ is separable, even the weak measurability of $U(t)$ implies the strong continuity of it. Namely, if $(\Psi_1, U(t)\Psi_2)$ is measurable for any two vectors $\Psi_1, \Psi_2 \in \mathfrak{H}$, $U(t)\Psi$ is strongly continuous in $\mathfrak{H}$ for any $\Psi \in \mathfrak{H}$ [ref. 1), pp. 74, 598], and there is a self-adjoint operator H in $\mathfrak{H}$ which satisfies Eq. (1·2). This means that, when we deal with the interacting fields, $U(t)$ in general cannot be weakly measurable, and the incompetence of the usual framework is obvious.

Now we must seek for an extended framework which, on the one hand, preserves most of the essential character of the usual theory and, on the other hand, enables us to overcome the above difficulties. In our opinion, that the set of physical quantities at a time t forms a *-algebra is the most essential part of the usual theory, and this almost inevitably implies that the *-algebra is represented by an operator algebra $\mathfrak{A}$ in a separable Hilbert space $\mathfrak{H}$.[2] Therefore, in this paper, we accept this assumption from the outset, and furthermore we consider the set $\mathfrak{B}$ of bounded operators instead of $\mathfrak{A}$ for brevity's sake. It is well-known that $\mathfrak{B}$ forms a B^*-algebra [ref. 1), pp. 22, 160; ref. 3)].

As to the definition of "state," however, there is much room for improvement. The most essential role of a state is that it is a linear functional on $\mathfrak{B}$,[2] and it is rather secondary in importance that a state f can be represented as

$$f(a) = (\Psi, a\Psi), \quad a \in \mathfrak{B}, \quad \Psi \in \mathfrak{H}, \quad \|\Psi\| = 1. \tag{1·4}$$

With due attention to this point, we assume that a state f is a positive linear functional on $\mathfrak{B}$ but not necessarily written as in Eq. (1·4). Further we assume that a state f is of norm 1, namely

$$\sup_{\|a\| \leq 1} |f(a)| = 1.$$

This is nothing but to assume that f is continuous. Let the set of all these functionals be $\bar{S}$. Then it is clear that functionals in $\bar{S}$ are not necessarily written as in Eq. (1·4), but it is known that the set S of linear functionals of the form

$$f(a) = \sum_{i=1}^{\infty} (\Psi_i, a\Psi_i), \quad \sum_i \|\Psi_i\|^2 = 1, \tag{1·4'}$$

is everywhere dense in $\bar{S}$ in some topology [ref. 4), p. 300]. Therefore it is very natural to assume that a state can be represented as a limit of a sequence of functionals $f_n(a) = (\Psi_n, a\Psi_n)$, $n = 1, 2, \cdots$, in some sense.

In order to treat the extended notion of states on the line parallel to the usual one, we introduce a non-separable Hilbert space $\mathfrak{H}$, in which $\mathfrak{H}$ is contained. Then we can show that $\mathfrak{B}$ is isomorphic to a B^*-algebra $\mathfrak{B}$ of bounded operators in $\mathfrak{H}$,† and that every state f can be represented by a vector $\Psi\in\mathfrak{H}$ as

$$f(a)=(\Psi, a\Psi),\ a\in\mathfrak{B},\ \Psi\in\mathfrak{H},\ \|\Psi\|=1. \tag{1·4''}$$

Consequently it is possible to say that our new framework is a natural extension of the usual one.

All the above arguments refer to a system at a time t. Now it is necessary to introduce a one-parameter group $\boldsymbol{U}(t)$ of unitary operators which satisfies Eq. (1·2) in some appropriate sense. As stated above, it will be impossible to define such $U(t)$ in $\mathfrak{H}$, but the obstacle disappears in $\mathfrak{H}$. In fact, even if $\boldsymbol{U}(t)$ is weakly measurable, namely, even if $(\Psi_1, \boldsymbol{U}(t)\Psi_2)$ is measurable, $\boldsymbol{U}(t)$ is not necessarily strongly measurable, and Stone's theorem does not hold. Hence, though $\boldsymbol{U}(t)$ is an operator in $\mathfrak{H}$, $\boldsymbol{H}$ in Eq. (1·2) is not so in general.

The fact that $\boldsymbol{U}(t)$ is an operator in $\mathfrak{H}$ and not in $\mathfrak{H}$ is most significant in our new theory. This implies that the unitary operator $\boldsymbol{U}(t)$, which describes the change of the system with time, does not belong to $\mathfrak{B}$, that is, $\boldsymbol{U}(t)$ is not a physical quantity and not "observable" in the original sense. Therefore, in our theory, the Schrödinger representation is more adequate than the Heisenberg one. In fact, since

$$\boldsymbol{a}(t)=\boldsymbol{U}^{-1}(t)\boldsymbol{a}(0)\boldsymbol{U}(t) \tag{1·3'}$$

is an operator in $\mathfrak{H}$ but cannot belong to $\mathfrak{B}$, the problem becomes troublesome in the Heisenberg representation.†† On the other hand, the equation

$$\Psi(t)=\boldsymbol{U}(t)\Psi(0) \tag{1·1'}$$

has a clear-cut meaning in our theory and causes no trouble.

§ 2. Non-separable Hilbert space $\mathfrak{H}$

As a preliminary we begin with a definition of a μ-Limit of any bounded sequence. Let ξ be a bounded sequence of complex numbers, that is,

$$\xi=\{\alpha_n;\ n=1, 2, \cdots\},\ \sup_n|\alpha_n|<+\infty,$$

and $\mathfrak{m}$ be the set of all these sequences. By defining addition and scalar multiplication by

$$\left.\begin{aligned}&\xi+\eta=\{\alpha_n\}+\{\beta_n\}=\{\alpha_n+\beta_n\},\\&\alpha\xi=\alpha\{\alpha_n\}=\{\alpha\alpha_n\},\end{aligned}\right\} \tag{2·1}$$

† In this paper we use symbols of bold face letters in order to indicate that they refer to $\mathfrak{H}$ and not to $\mathfrak{H}$.

†† When we consider the Lorentz invariance of the theory, it becomes clear that the interaction representation is most adequate, but, in this paper, we do not enter into this problem.

and the norm of ξ by

$$\|\xi\| = \sup_n |\alpha_n|, \tag{2·2}$$

$\mathfrak{m}$ becomes a Banach space [ref. 5), pp. 11, 53]. Let $\mathfrak{c}$ be the set of all elements of $\mathfrak{m}$ each of which has a finite limit $\lim_{n\to\infty} \alpha_n$. Then $\mathfrak{c}$ is also a Banach space and is a closed linear subspace of $\mathfrak{m}$ [ref. 5), p. 11].

In this paper we use the following conventional notation:

$$\mathbf{0} = \{0, 0, \cdots\}, \quad \mathbf{1} = \{1, 1, \cdots\},$$

$$|\xi| = \{|\alpha_n|\}, \quad \xi^2 = \{(\alpha_n)^2\}, \quad \bar{\xi} = \{\bar{\alpha}_n\},$$

where $\bar{\alpha}$ is the conjugate complex number to α. Further we write

$$\bar{M} = \{\bar{\xi}\,;\, \xi \in M\}$$

for any subset M of $\mathfrak{m}$. $\xi = \{\alpha_n\}$ is said to be real if all α_n are real. It is said to be positive and we write $\xi \geq \mathbf{0}$, if $\alpha_n \geq 0$, $n = 1, 2, \cdots$. If a linear functional μ on $\mathfrak{m}$ or $\mathfrak{c}$ satisfies the condition that $\xi \geq \mathbf{0}$ implies $\mu(\xi) \geq 0$, then μ is said to be positive.

Put

$$\mu_0(\xi) = \lim_{n\to\infty} \alpha_n, \quad \xi = \{\alpha_n\} \in \mathfrak{c}. \tag{2·3}$$

Then μ_0 is a positive linear functional on $\mathfrak{c}$ of norm 1, and, by extension, we can get a linear functional μ on $\mathfrak{m}$ of norm 1 by virtue of Hahn-Banach's theorem [ref. 6), p. 111]. Now we shall show that μ can be taken to be positive.

Theorem 1. *There is a positive linear functional μ on $\mathfrak{m}$ which is an extension of μ_0 and of norm 1.*

Proof. Putting

$$B_1 = B + \mathbf{1}, \quad B = \{\xi\,;\, \|\xi\| < 1, \xi \in \mathfrak{m}\},$$

and

$$M_0 = \{\xi\,;\, \mu_0(\xi) = 0, \xi \in \mathfrak{c}\},$$

we can show that B_1 is an open convex set, and that $B_1 \cap M_0 = \phi$. Let $\mathfrak{m}^R$, B_1^R and M_0^R be the subsets of all real elements of $\mathfrak{m}$, B_1 and M_0, respectively. It is easy to show that $\mathfrak{m}^R$ becomes a Banach space over the real number field, and that M_0^R is a linear subspace of it. Furthermore, B_1^R is open and convex in $\mathfrak{m}^R$ and $B_1^R \cap M_0^R = \phi$. Hahn-Banach's theorem indicates that there is a closed hyperplane M^R which contains M_0^R and does not intersect with B_1^R [ref. 6), p. 69], that is,

$$M^R \supset M_0^R, \quad B_1^R \cap M^R = \phi. \tag{2·4}$$

Let M be the linear subspace in $\mathfrak{m}$ which is generated by M^R. Then M is a closed hyperplane in $\mathfrak{m}$, and $\bar{M} = M$. Further, Eq. (2·4) implies that

$$M \supset M_0, \quad B_1 \cap M = \phi. \tag{2·4′}$$

In other words, there is a continuous linear functional μ on $\mathfrak{m}$ which is an extension of μ_0, and M is defined by the equation $\mu(\hat{\xi})=0$. It is obvious that, if $\hat{\xi}$ is real, $\mu(\hat{\xi})$ is real, too, and that

$$\mu(\hat{\xi})>0 \quad \text{on } B_1^R. \tag{2·5}$$

Put

$$\beta_n(\hat{\xi})=\mathrm{Min}\{\mathrm{Max}((\alpha_n/\|\hat{\xi}\|),\varepsilon),(1-\varepsilon)\}$$

for some $\hat{\xi}\geq\mathbf{0}$, and $1/2>\varepsilon>0$. Then

$$\eta(\varepsilon)=\{\beta_n(\varepsilon)\}\in B_1^R,$$

and $\eta(\epsilon)\to\hat{\xi}/\|\hat{\xi}\|$ as $\varepsilon\to 0$. From Eq. (2·5) and the continuity of μ, it follows that $\hat{\xi}\geq\mathbf{0}$ implies $\mu(\hat{\xi})\geq 0$, that is, μ is positive.

Lastly we shall prove that $\|\mu\|=1$. As $\mu(\mathbf{1})=1$, we have $\|\mu\|\geq 1$. Conversely, if $\hat{\xi}\in B$, and $|\alpha|=1$, then $\mathbf{1}-\alpha\hat{\xi}\in B_1$, and

$$\mu(\mathbf{1}-\alpha\hat{\xi})|=|1-\alpha\mu(\hat{\xi})|>0,$$

namely $|\mu(\hat{\xi})|\neq 1$ on B. This means that $|\mu(\hat{\xi})|<1$ on B, that is, $\|\mu\|\leq 1$. Q.E.D.

Now we can define a μ-Limit of any bounded sequence by the equation

$$\mu\text{-}\lim_{n\to\infty}\alpha_n=\mu\text{-Lim}(\hat{\xi})=\mu(\hat{\xi}),\quad \hat{\xi}=\{\alpha_n\}, \tag{2·6}$$

and this μ-Limit plays an essential role in the following.

Lemma 1. *If $\hat{\xi}\geq\mathbf{0}$, then the two equations μ-Lim$(\hat{\xi})=0$ and μ-Lim$(\hat{\xi}^2)=0$ are equivalent to each other.*

Proof. We can assume that $\|\hat{\xi}\|\leq 1$ without loss of generality. Since $1\geq\alpha_n\geq 0$, $\alpha_n\geq(\alpha_n)^2$ and $\mu\text{-Lim}(\hat{\xi})\geq\mu\text{-Lim}(\hat{\xi}^2)$. Thus, μ-Lim$(\hat{\xi})=0$ implies μ-Lim$(\hat{\xi}^2)=0$. Conversely, if μ-Lim$(\hat{\xi}^2)=0$, then for any real α

$$0\leq(1-\alpha\alpha_n)^2=1-2\alpha\alpha_n+\alpha^2(\alpha_n)^2,$$

and

$$0\leq\mu\text{-Lim}(1-\alpha\alpha_n)^2=1-2\alpha\mu\text{-Lim}\,\alpha_n+\alpha^2\mu\text{-Lim}(\alpha_n)^2=1-2\alpha\mu\text{-Lim}\,\alpha_n.$$

Thus μ-Lim $\alpha_n=0$. Q. E. D.

Lemma 2.

$$\mu\text{-Lim}(|\hat{\xi}|)\geq|\mu\text{-Lim}(\hat{\xi})|.$$

Proof. Let the closure of B_1 be $(B_1)^c$. If $\hat{\xi}\in(B_1)^c$, then the real part of μ-Lim$(\hat{\xi})$ is not negative. If $\|\hat{\xi}\|<1$, we can show that $|\hat{\xi}|-\hat{\xi}\in(B_1)^c$. Put

$$\mu\text{-Lim}(\hat{\xi})=|\mu\text{-Lim}(\hat{\xi})|\exp(i\theta).$$

Then $|\hat{\xi}|-\hat{\xi}\exp(-i\theta)$ belongs to $(B_1)^c$, and

$$\mu\text{-Lim}(|\hat{\xi}|-\hat{\xi}\exp(-i\theta))=\mu\text{-Lim}|\hat{\xi}|-|\mu\text{-Lim}(\hat{\xi})|$$

is real. Therefore,

$$\mu\text{-Lim}(|\hat{\xi}|-\hat{\xi}\exp(-i\theta))\geq 0,$$

that is,

$$\mu\text{-Lim}(|\hat{\xi}|)\geq|\mu\text{-Lim}(\hat{\xi})|. \qquad \text{Q. E. D.}$$

Now we shall show that we can define a non-separable Hilbert space $\mathfrak{H}$, which can be regarded as an extension of the separable Hilbert space $\mathfrak{H}$. Let ω be a bounded sequence of vectors in $\mathfrak{H}$, namely

$$\omega=\{\Psi_n;\ n=1, 2, \cdots\},\ \sup_n\|\Psi_n\|<+\infty,$$

and Ω be the set of all these sequences. By defining

$$\left.\begin{aligned}\omega^1+\omega^2&=\{\Psi_n^{1}\}+\{\Psi_n^{2}\}=\{\Psi_n^{1}+\Psi_n^{2}\},\\ \alpha\omega&=\alpha\{\Psi_n\}=\{\alpha\Psi_n\},\end{aligned}\right\} \qquad (2\cdot 7)$$

Ω becomes a linear space.

Lemma 3. *Put*

$$\Omega_0=\{\omega;\ \|\omega\|=0\},\quad \|\omega\|^2=\mu\text{-Lim}(\Psi_n, \Psi_n). \qquad (2\cdot 8)$$

Then Ω_0 is a linear subspace of Ω, and Ω/Ω_0 is also a linear space.

Proof. Lemma 1 shows that, if $\omega\in\Omega_0$, then $\mu\text{-Lim}\|\Psi_n\|=0$. Further, from Lemma 2 and the positiveness of μ, it follows that

$$\begin{aligned}|\mu\text{-Lim}(\Psi_n^{1}, \Psi_n^{2})|&\leq\mu\text{-Lim}|(\Psi_n^{1}, \Psi_n^{2})|\\ &\leq\mu\text{-Lim}(\|\Psi_n^{1}\|\cdot\|\Psi_n^{2}\|)\leq(\mu\text{-Lim}\|\Psi_n^{1}\|)\cdot\sup\|\Psi_n^{2}\|.\end{aligned}$$

Thus, for every pair of $\{\Psi_n^{1}\},\ \{\Psi_n^{2}\}\in\Omega_0$,

$$\mu\text{-Lim}(\Psi_n^{1}, \Psi_n^{2})=0,$$

and

$$\mu\text{-Lim}(\Psi_n^{1}+\Psi_n^{2}, \Psi_n^{1}+\Psi_n^{2})=0. \qquad \text{Q. E. D.}$$

Theorem 2. *Let $\tilde{\omega}\in\Omega/\Omega_0$ be the class containing ω. Then we can define an inner product $(\tilde{\omega}^1, \tilde{\omega}^2)$, and Ω/Ω_0 becomes a non-separable pre-Hilbert space.*

Proof. The proof of Lemma 3 shows that, if $\{\Psi_n^{1}\}\in\Omega_0$, then

$$\mu\text{-Lim}(\Psi_n^{1}, \Psi_n^{2})=0$$

for every $\{\Psi_n^{2}\}\in\Omega$. Therefore, we can uniquely define an inner product by the equation

$$(\tilde{\omega}^1, \tilde{\omega}^2)=\mu\text{-Lim}(\Psi_n^{1}, \Psi_n^{2}),\quad \{\Psi_n^{1}\}\in\tilde{\omega}^1,\ \{\Psi_n^{2}\}\in\tilde{\omega}^2, \qquad (2\cdot 9)$$

and we can show that $(\tilde{\omega}^1, \tilde{\omega}^2)$ is an inner product in a proper sense [ref. 1), p. 17], that is, Ω/Ω_0 is a pre-Hilbert space. The non-separability of Ω/Ω_0 can be easily shown. Q.E.D.

Let the completion of Ω/Ω_0 be $\mathfrak{H}$. Then $\mathfrak{H}$ is a non-separable Hilbert space, and we write $\mathfrak{H}_\mu$ with a subscript μ in order to express explicitly that

$\mathfrak{H}$ is defined by using the functional μ. In the following we write $\boldsymbol{\Psi}=\{\Psi_n\}$ when $\boldsymbol{\Psi}\in\mathfrak{H}$ is a class containing the sequence $\{\Psi_n\}$ for the sake of simplicity. Since the set of sequences $\{\Psi, \Psi, \cdots\}$, $\Psi\in\mathfrak{H}$, is isomorphic to $\mathfrak{H}$, $\mathfrak{H}$ can be regarded as a subspace of $\mathfrak{H}$. For any element a in $\mathfrak{B}$, we can define an operator $\boldsymbol{a}$ in $\mathfrak{H}$ by the equation

$$\boldsymbol{a}\boldsymbol{\Psi}=\{a\Psi_n\}, \tag{2·10}$$

and it is easy to show that $\boldsymbol{a}$ is also a bounded operator and that $\|\boldsymbol{a}\|=\|a\|$. We denote the set of these operators by $\mathfrak{B}$, and it follows that $\mathfrak{B}$ is isomorphic to $\mathfrak{B}$. It is obvious that any state f, which is the limit of a sequence of functionals $\{(\Psi_n, a\Psi_n)\}$ in a sense, can be written as in Eq. (1·4″). Thus we have succeeded in getting a new framework as stated in § 1.

§ 3. Physical consideration

In this section we shall consider the main features of the physical contents of our new theory. A state f which is given by Eq. (1·4″) differs essentially from that in Eq. (1·4) in the following sense. Since f is a continuous linear functional on $\mathfrak{B}$, it can be split uniquely into two parts f_1 and f_2, where f_1 is a linear functional on a subset $\mathfrak{J}\subset\mathfrak{B}$ of all completely continuous operators, and f_2 is orthogonal to $\mathfrak{J}$ [ref. 3), p. 49; ref. 7), p. 396], namely

$$f=f_1+f_2,\quad f_1\in\mathfrak{J}',\quad f_2\in\mathfrak{J}^{\perp}. \tag{3·1}$$

Since f_1 can be represented by Eq. (1·4′), it belongs to the same category as the states in Eq. (1·4), that is, f_1 is the same state as those in the usual theory. On the other hand, f_2 is of very different nature, and the following example shows the difference clearly. Let e be the unit element of $\mathfrak{B}$, $\{\Psi_i; i=1, 2, \cdots\}$ be a complete orthonormal system of $\mathfrak{H}$, and P_i be a projection on each one-dimensional subspace generated by Ψ_i. Then

$$f_2(e)=\|f_2\|,\quad f_2(P_i)=0,\quad i=1, 2, \cdots.$$

Therefore, though $e=P_1+P_2+\cdots$,

$$f_2(e)\neq\sum_{i=1}^{\infty}f_2(P_i).$$

At first sight this seems to be very curious. Nevertheless it is possible to consider that this feature is rather appropriate for the representation of the state of a particle accompanied with the cloud of the so-called proper field. For example, there is a state $\boldsymbol{\Psi}$, in which we may say that all states Ψ_i, $i=1, 2, \cdots$, are equally probable.

When a Hamiltonian H is not an operator in $\mathfrak{H}$, a one-parameter group $\boldsymbol{U}(t)$ of unitary operators can be defined as follows. In general, H can be represented by the equation

$$H=\sum_{i=1}^{\infty} h_i, \tag{3·2}$$

where h_i is a self-adjoint operators in $\mathfrak{H}$. Put

$$U_n(t)=\exp(-iH_nt), \quad H_n=\sum_{i=1}^{n} h_i. \tag{3·3}$$

Then we can define $\boldsymbol{U}(t)$ by the equation

$$\boldsymbol{U}(t)\boldsymbol{\Psi}=\{U_n(t)\Psi_n\}, \tag{3·4}$$

and it is easy to show that $\boldsymbol{U}(t)$ is a one-parameter group of unitary operators in $\mathfrak{H}$. By formal differentiation, we get

$$i(d/dt)\,(\boldsymbol{U}(t)\boldsymbol{\Psi}, \boldsymbol{a}\boldsymbol{U}(t)\boldsymbol{\Psi})=(\boldsymbol{U}(t)\boldsymbol{\Psi}, [\boldsymbol{a}, \boldsymbol{H}]\boldsymbol{U}(t)\boldsymbol{\Psi}), \tag{3·5}$$

where

$$\boldsymbol{H}\boldsymbol{\Psi}=\{H_1\Psi_1, H_2\Psi_2, \cdots\}. \tag{3·6}$$

Thus we may say that $\boldsymbol{U}(t)$ satisfies Eq. (1·2) in a sense and that it is the required one. It is, however, necessary to note the following two points. First, in order that Eq. (3·5) may hold, the two operations $i(d/dt)$ and μ-Lim must be commutative. Secondly, $\boldsymbol{H}\boldsymbol{\Psi}$ does not belong to $\mathfrak{H}$ in general, and the right-hand side of Eq. (3·5) is merely a formal inner product.

When the renormalization is necessary, the situation becomes more complicated, but the essential character of the problem remains unchanged. Let $H_{free}(m_0)$ and $H_{int}(e_0)$ be the free Hamiltonian and the interaction one respectively. Putting

$$H_n=H_{free}(m_0)+H_{int}^n(e_0),$$

and

$$\lim_{n\to\infty} H_{int}^n(e_0)=H_{int}(e_0),$$

we get the sequence $\{U_n(t)\}=\{\exp(iH_nt)\}$. Now we must renormalize the mass and the charge in each U_n, and each U_n should be written by the observed mass m and the observed charge e, instead of m_0 and e_0. Then we must take the μ-Limit of this renormalized $\{U_n\}$, and we get the desired one-parameter group $\boldsymbol{U}(t)$ of unitary operators in $\mathfrak{H}$. As m and e are definite, m_0 and e_0 should have definite values m_0^n and e_0^n corresponding to each H_n. But it is not necessary that the sequences $\{m_0^n\}$ and $\{e_0^n\}$ have finite limits which are not equal to zero. In fact, it is very plausible that $e_0^n\to 0$ as $n\to\infty$, that is, $H_{int}^n(e_0^n)\to 0$. If this is the case, we cannot write the Hamiltonian in an explicit form. But this is no matter to us, because $\boldsymbol{U}(t)$ is defined completely without using such a Hamiltonian.

Moreover, our theory has several formal advantages. For example, in Dirac's positron theory, two kinds of vacua Ψ_0 and $\boldsymbol{\Psi}_0$ coexist in $\mathfrak{H}$. As a matter of fact, if $\Psi_0\in\mathfrak{H}$ is the vacuum in which no electron exists in any posi-

tive and negative energy state, then the vacuum $\Psi_0 \in \mathfrak{H}$ in which all the negative energy states are occupied can be represented by

$$\Psi_0 = \{\Psi_0, \alpha_1 a_1 \Psi_0, \alpha_2 a_2 a_1 \Psi_0, \cdots\},$$

where a_i, $i=1, 2, \cdots$, is a creation operator of an electron in a negative energy state and α_i is a normalizing factor. There is another example. As the Hamiltonian H is not necessarily an operator in $\mathfrak{H}$, the so-called zero-point energy causes no difficulty. In fact, even if $H=\alpha_1+\alpha_2+\cdots=+\infty$,

$$U(t) = \{\exp(-iH_n t)\}, \quad H_n = \alpha_1+\alpha_2+\cdots+\alpha_n,$$

properly defines a one-parameter group of unitary operators in $\mathfrak{H}$.

All the above arguments show that our new framework is wide enough to contain the present quantum theory of fields. Unfortunately, it is too wide, and it contains many unphysical contents. Before we can apply our theory to a concrete physical problem, we must solve several problems: Which of the infinitely many $\mathfrak{H}_\mu$ has a physical meaning [see the Appendix]?; do the different representations of H in Eq. (3·2) give the same result to a physical problem?; and so on. All these problems will be studied on another occasion.

Appendix

In this Appendix we shall show an example of reasonable extensions of μ_0, which is uniquely defined in a subspace $\mathfrak{n} \supset \mathfrak{c}$ of $\mathfrak{m}$. We believe that every physically meaningful functional should coincides with this extension on $\mathfrak{n}$.

Let τ be an operator on $\mathfrak{m}$ such that

$$\tau\hat{\xi} = \tau\{\alpha_1, \alpha_2, \alpha_3, \cdots\} = \{\alpha_2, \alpha_3, \cdots\}. \tag{A·1}$$

Then τ is a linear operator and $\|\tau\|=1$. Put

$$\hat{\xi}_k = (1/k)\sum_{l=1}^{k} \tau^l \hat{\xi} \tag{A·2}$$

for any $\hat{\xi} \in \mathfrak{m}$. If $\{\hat{\xi}_k\}$ has a limit in $\mathfrak{m}$, we denote it by $\hat{\xi}_\infty$, namely $\hat{\xi}_k \to \hat{\xi}_\infty$ in $\mathfrak{m}$ as $k\to\infty$. Let $\mathfrak{n}$ be the set of all elements each of which has a corresponding $\hat{\xi}_\infty$. Then we can show that $\mathfrak{n}$ is a closed linear subspace of $\mathfrak{m}$, and that $\mathfrak{n} \supset \mathfrak{c}$. We can also show that, if $\hat{\xi}$ is "periodic" or at least "almost periodic," it belongs to $\mathfrak{n}$. Since $\|\tau\|=1$, $\hat{\xi}_k \to \hat{\xi}_\infty$ implies $\tau\hat{\xi}_k \to \tau\hat{\xi}_\infty$, that is, $\hat{\xi} \in \mathfrak{n}$ implies $\tau\hat{\xi} \in \mathfrak{n}$, and $\tau\hat{\xi}_\infty = \hat{\xi}_\infty$, because

$$|\tau\hat{\xi}_k - \hat{\xi}_k\| = (1/k)\,\|\sum_{l=1}^{k} \tau^{l+1}\hat{\xi} - \sum_{l=1}^{k} \tau^l \hat{\xi}\| = (1/k)\,\|\tau^{k+1}\hat{\xi} - \tau\hat{\xi}\| \to 0.$$

Therefore, $\hat{\xi}_\infty$ is constant, and we write

$$\hat{\xi}_\infty = \{\xi_\infty, \xi_\infty, \cdots\} = \xi_\infty \cdot \mathbf{1}. \tag{A·3}$$

Putting

$$\mu_1(\hat{\xi}) = \xi_\infty, \quad \hat{\xi} \in \mathfrak{n}, \tag{A·4}$$

we can define a functional μ_1 on $\mathfrak{n}$, and it is easy to show that μ_1 is an extension of μ_0. μ_1 is a positive linear functional of norm 1 on $\mathfrak{n}$, and we can extend it on $\mathfrak{m}$, getting such a functional on $\mathfrak{m}$. It is obvious that μ_1 is a reasonable extension of μ_0, and we may rather say that μ_1 is a "mean" of ξ, instead of a "limit." In fact, μ_1 corresponds to the so-called Hölder's summation $(H, 1)$. We shall be able to utilize any other method of summation, but we do not know which of these methods is most suitable for our purpose.

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Dispersion Relations for Kaon-Nucleon Scattering and $KK\pi\pi$ Interaction*

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A general formalism is given in order to investigate the type of the K-Λ-nucleon and K-Σ-nucleon couplings together with the $KK\pi\pi$ interaction by applying the Cini-Fubini approximation to the Mandelstam representation for the K-nucleon scattering. By taking suitable combinations of invariant amplitudes for $K^{\pm}$-p and $K^{\pm}$-n, we are able to isolate the effects due to the isospin independent and dependent type of the $KK\pi\pi$ interaction. Necessary experimental data which make the independent determination of these coupling types possible are suggested.

We discuss in particular a simple method which allows us to determine the isospin dependent $KK\pi\pi$ interaction and then propose a suitable experiment for it.

§ 1. Introduction

The dispersion relations for the scattering of K-mesons by nucleons, as important clues for interpretation of K-meson data, were first written down by Sakurai,[1] by Amati and Vitale[2] and by the present author[3] in order to determine the type of the K-coupling and the magnitude of the coupling constant. Preliminary attempts to apply the dispersion relations to the fixing of the K-meson parity, by making use of a few data available at that time, were made by Matthews and Salam,[4] by Goebel[5] and by the present author.[6]

The dispersion theoretical approach indicates that a repulsive K^--p potential unambiguously leads to scalar K-mesons while an attractive K^--p potential favours a pseudoscalar type of the K-baryon coupling, if we assume the relative parity of the Λ and Σ hyperons to be positive and take the K^--p potential to be repulsive. The special difficulties caused by the presence of the unphysical region due to the absorption of K^--mesons by protons below the threshold were then investigated in detail by the analytic continuation of the physical amplitudes into the unphysical region with extensive use of the perturbation theory for the various possible parity scheme by Tuan.[7]

Now there is some evidence which seems to indicate the existence of the $KK\pi\pi$ interaction. For instance, pion production was reported[8] in K^- colli-

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sions by protons at 220 Mev above the threshold for elastic scattering. Some calculations have been successfully carried out[9] using the mechanism of $KK\pi\pi$ interaction which can also be expected to occur theoretically from the Feynman diagram through the virtual nucleon-antihyperon closed loop. Such an interaction, if it does exist, will play an important role in various strange particle reactions, such as K-nucleon scattering, K-hyperon production, and so on.

In a previous paper,[10] we investigated the $KK\pi\pi$ interaction from the viewpoint of the Hamiltonian formalism in which the K-meson was considered to be scattered via the exchange of two pions between the nucleon and the K-meson.

By applying the above theoretical scheme to the K-nucleon scattering, we have arrived at the conclusion that the $KK\pi\pi$ interaction of both isospin independent type and isospin dependent type are amalgamated into the direct K-hyperon-nucleon interaction to play an important role in the three K-nucleon scatterings.

The above-mentioned dispersion theoretical approach, however, is not altered even when the $KK\pi\pi$ interaction exists, since the forward scattering amplitude arising from the $KK\pi\pi$ interaction* is isospin independent and so was dropped from the dispersion relations used before; hence the above dispersion theoretical approach is not suitable for investigating the $KK\pi\pi$ interaction.

In this paper, we shall derive a formalism for investigating the $KK\pi\pi$ interaction simultaneously with the K-nucleon-hyperon interaction by applying the approximation introduced by Cini and Fubini.[11] They considered that only the contributions from the singularities of the scattering amplitude near the physically interesting range of variables need to be taken correctly into account, while the contributions from distant singularities can be expressed in the form of polynominal expansions. Such an approximation applied to the K-nucleon scattering amplitude postulated by Mandelstam leads to the fixed momentum dispersion relations in the energy variables which are valid at low energies, and allow us to estimate the $KK\pi\pi$ interaction via $K\bar{K}\rightarrow$ resonating pions$\rightarrow N\bar{N}$.

The effect of pion-pion resonance on low energy pion-nucleon scattering was successfully investigated by applying the same method by Bowcock, Cottingham and Lurie.[12] By confining our considerations to the suitable combinations of invariant amplitudes for $K^{\pm}$-p and $K^{\pm}$-n, we can separate the effect due to the isospin independent and dependent type of the $KK\pi\pi$ interaction.

In order to investigate the isospin dependent $KK\pi\pi$ interaction the S-wave threshold behavior is powerful while the $P(3/2)$ state threshold behavior is useful to get information about the isospin independent $KK\pi\pi$ interaction.

* We take the standpoint that the isospin independent $K^*K\pi_j\pi_j$ form can exist as a basic interaction from the renormalization requirement and that the isospin dependent $(K^*\tau\partial_\mu K-\partial_\mu K^*\tau K)\times(\pi T\partial_\mu\pi)$ type of interaction comes from the effective interaction of the closed loop.

§ 2 is devoted to the kinematical considerations, § 3 to the fixed momentum transfer dispersion relation for the K-nucleon scattering by applying the Cini-Fubini approximation to the Mandelstam representation, and § 4 to the calculation of the weight functions by making use of unitarity. In § 5 the general discussion is given for determining independently the direct interaction and $KK\pi\pi$ interaction and necessary experiment for the purpose is suggested. In § 6, a simple method which allows us to investigate the isospin dependent $KK\pi\pi$ interaction in the near future is suggested and an appropriate experiment is proposed. In § 7 some final remarks are given.

§ 2. Definition of variables and kinematical considerations

We will consider a K-nucleon scattering as given in Fig. 1. Let the four-vector momenta of the incident and outgoing K-mesons be k_1 and k_2, respectively, while those of the initial and final nucleons are p_1 and p_2. Let us define the Lorentz invariant variables

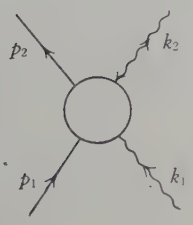


Fig. 1.

$$s=-(p_1+k_1)^2, \qquad (2\cdot1a)$$
$$u=-(p_1-k_2)^2, \qquad (2\cdot1b)$$
$$t=-(k_2-k_1)^2, \qquad (2\cdot1c)$$

which are just the square of energies in the center-of-mass system of the following three reactions:

I) $p_1+k_1\to p_2+k_2$ $(K+N\to K+N)$, $(2\cdot2a)$

II) $p_1-k_2\to p_2-k_1$ $(\bar{K}+N\to\bar{K}+N)$, $(2\cdot2b)$

III) $k_1-k_2\to p_2-p_1$ $(K+\bar{K}\to N+\bar{N})$. $(2\cdot2c)$

In (2·1a, b, c) there are of course only two independent variables since we have the usual relation

$$s+u+t=2N^2+2K^2, \qquad (2\cdot3)$$

which comes from the energy-momentum conservation. The masses of the pion, K-meson, nucleon, Λ hyperon and Σ hyperon are denoted by μ, K, N, Λ and Σ, respectively. For the kinematical analysis of the K-nucleon scattering we shall employ the same notation as what Chew, Goldberger, Low and Nambu[13] (hereafter CGLN) adopted in the pion-nucleon case.

The S matrix can be written as

$$S=\delta_{fi}-(2\pi)^4 i\delta^4(p_2+k_2-p_1-k_1)\left(\frac{N^2}{4E_1E_2\omega_1\omega_2}\right)^{1/2}\bar{u}_2Tu_1, \qquad (2\cdot4)$$

where E_1 and E_2 are the initial and final nucleon energies and, ω_1 and ω_2 the initial and final K-meson energies. Let $T_p^{(+)}$, $T_n^{(+)}$, $T_p^{(-)}$, $T_n^{(-)}$ denote the T matrix for K^+-p, K^+-n, K^--p and K^--n scattering, respectively. If T is to be invariant under Lorentz transformation, it must be of the form

$$T_{p,n}^{(\pm)}=-A_{p,n}^{(\pm)}+\frac{i\tilde{\gamma}}{2}(k_1+k_2)B_{p,n}^{(\pm)}, \tag{2·5}$$

where A and B are functions of s, u and t. Here we will introduce for the sake of convenience the following combinations of the invariant functions:

$$A_{p,n}^{(1)}(s,u,t)=\frac{A_{p,n}^{(-)}(s,u,t)+A_{p,n}^{(+)}(s,u,t)}{2}, \tag{2·6a}$$

$$A_{p,n}^{(2)}(s,u,t)=\frac{A_{p,n}^{(-)}(s,u,t)-A_{p,n}^{(+)}(s,u,t)}{2}, \tag{2·6b}$$

$$B_{p,n}^{(1)}(s,u,t)=\frac{B_{p,n}^{(-)}(s,u,t)+B_{p,n}^{(+)}(s,u,t)}{2}, \tag{2·6c}$$

$$B_{p,n}^{(2)}(s,u,t)=\frac{B_{p,n}^{(-)}(s,u,t)-B_{p,n}^{(+)}(s,u,t)}{2}, \tag{2·6d}$$

which satisfy the convenient crossing relations

$$A_{p,n}^{(1)}(s,u)=A_{p,n}^{(1)}(u,s), \tag{2·7a}$$

$$A_{p,n}^{(2)}(s,u)=-A_{p,n}^{(2)}(u,s), \tag{2·7b}$$

$$B_{p,n}^{(1)}(s,u)=-B_{p,n}^{(2)}(u,s), \tag{2·7c}$$

$$B_{p,n}^{(2)}(s,u)=B_{p,n}^{(2)}(u,s). \tag{2·7d}$$

The conventional scattering amplitudes f_1, f_2, which are simply related to the scattering cross sections in the center-of-mass system as

$$\frac{d\sigma}{d\Omega}=\sum|\langle f|f_1-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}_2\boldsymbol{\sigma}\cdot\boldsymbol{k}_1}{k_2k_1}f_2|i\rangle|^2, \tag{2·8}$$

may be written as

$$f_1=\frac{E+N}{2W}\cdot\frac{A+(W-N)B}{4\pi}, \tag{2·9a}$$

$$f_2=\frac{E-N}{2W}\cdot\frac{-A+(W+N)B}{4\pi}, \tag{2·9b}$$

where

$$W=\text{total energy}, \tag{2·10}$$

$$E=\text{total nucleon energy}, \tag{2·11}$$

and we have suppressed the suffices referring to charge states. The decomposition of f_1 and f_2 into states of definite angular momentum is given by

$$f_1=\sum_{l=0}^{\infty} f_{l+} P_{l+1}(\cos\theta)-\sum_{l=2}^{\infty} f_{l-} P_{l-1}'(\cos\theta), \qquad (2\cdot 12a)$$

$$f_2=\sum_{l=1}^{\infty}(f_{l-}-f_{l+})\, P_l'(\cos\theta), \qquad (2\cdot 12b)$$

where $f_{l\pm}$ is the scattering amplitude in the state with parity $-(-1)^l$ and total angular momentum $J=l\pm(2/1)$ and θ denotes the scattering angle in the center-of-mass system.

§ 3. Mandelstam representation for the scattering amplitude

In this section we shall assume that the invariant functions A and B satisfy the Mandelstam representation which correctly shows the location of singularities for elastic processes, and then apply the approximation introduced by Cini and Fubini.[11] By this approximation the two-dimensional representation is reduced to the one-dimensional one, which turns out to show the correct analytic properties in all variables, fulfilling the crossing symmetry and approximate unitarity. We shall use this one-dimensional representation in order to obtain an expression for the partial wave amplitudes by means of Eqs. (2·9) and (2·12).

Now for the K-nucleon forward scattering the dispersion relations have been proved in every order of perturbation theory,[14] but any rigorous proof has not yet been given based on field theoretical formalism even for this case,[15] let alone for the Mandelstam representation.

As a tool for the phenomenological analysis of experimental data, however, we shall optimistically assume the Mandelstam representation for the K-nucleon scattering to be valid. Let us consider this representation for one of the invariants, e.g. $A_p^{(-)}$ for K^--p scattering

$$\begin{aligned} A_p^{(-)}(s,u,t)=&-\frac{\pm\Lambda-N}{\Lambda^2-s}g_\Lambda^2-\frac{\pm\Sigma-N}{\Sigma^2-s}g_\Sigma^2 \\ &+\frac{1}{\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}du'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{p(13)}^{(-)}(s',t')}{(s'-s)(t'-t)} \\ &+\frac{1}{\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}du'\int_{4\mu^2}^{\infty}dt'\,\frac{A_{p(23)}^{(-)}(u',t')}{(u'-u)(t'-t)} \\ &+\frac{1}{\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}ds'\int_{(\Lambda+\mu)^2}^{\infty}du'\,\frac{A_{p(12)}^{(-)}(s',u')}{(s'-s)(u'-u)}\,. \end{aligned} \qquad (3\cdot 1)$$

Here the upper (lower) sign is to be taken in the case of pseudoscalar (scalar) coupling and g_Λ^2, g_Σ^2 are the rationalized, renormalized coupling constants. For

simplicity we will hereafter omit the one-dimensional term which does not alter any of the essential results obtained below.

We shall apply the Cini-Fubini method for the location of the singularities in order to obtain a simple one-dimensional representation. With t fixed, Eq. (3·1) leads to

$$A_p^{(-)}(s,u,t)=-\frac{\pm\Lambda-N}{\Lambda^2-s}g_\Lambda^2-\frac{\pm\Sigma-N}{\Sigma^2-s}g_\Sigma^2$$
$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds\frac{I_p^{(-)}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}du'\frac{\bar{I}_p^{(-)}(u',t)}{u'-u}. \qquad (3\cdot2)$$

The unitarity of the S matrix gives $I_p^{(-)}(s,t)$ and $\bar{I}_p^{(-)}(u,t)$ in the form of the sum over intermediate states involving K^-p reactions and K^+p reactions, respectively. Hence we can split $I_p^{(-)}(s,t)$ in the following form,

$$I_p^{(-)}(s,t)=I_{\bar{K}p}^{(-)}(s,t)+I_{\pi\Lambda}^{(-)}(s,t)+I_{\pi\Sigma}^{(-)}(s,t)+I_{mp}^{(-)}(s,t), \qquad (3\cdot3)$$

where $I_{mp}^{(-)}(s,t)=0$ for $s<(\Lambda+2\mu)^2$, since all possible states with baryon number one and strangeness minus one are to be taken into account. $I_{\bar{K}p}^{(-)}(s,t)$, $I_{\pi\Lambda}^{(-)}(s,t)$ and $I_{\pi\Sigma}^{(-)}(s,t)$ represent the contributions coming from the intermediate states with one K-meson and one nucleon, with one pion and one Λ particle, and with one pion and Σ particle respectively; $I_{mp}^{(-)}(s,t)$ is the contribution from all many-particle states with higher mass. Quite similarly $\bar{I}_p^{(-)}(u,t)$ can be separated into two parts:

$$\bar{I}_p^{(-)}(u,t)=I_p^{(+)}(u,t)$$
$$=I_{Kp}^{(+)}(u,t)+I_{mp}^{(+)}(u,t), \qquad (3\cdot4)$$

where $I_{mp}^{(+)}(u,t)=0$ for $u<(N+K+\mu)^2$, since in this case all admissible intermediate states have baryon number one and strangeness plus one. Substitution of (3·3) and (3·4) in (3·2) leads to

$$A_p^{(-)}(s,u,t)=-\frac{\pm\Lambda-N}{\Lambda^2-s}g_\Lambda^2-\frac{\pm\Sigma-N}{\Sigma^2-s}g_\Sigma^2$$
$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty}ds'\frac{I_{\bar{K}p}^{(-)}(s,t)}{s'-s}+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds'\frac{I_{\pi\Lambda}^{(-)}(s',t)}{(s'-s)}+\frac{1}{\pi}\int_{(\Sigma+\mu)^2}^{\infty}ds'\frac{I_{\pi\Sigma}^{(-)}(s',t)}{s'-s}$$
$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty}du'\frac{I_{\bar{K}p}^{(+)}(u',t)}{u'-u}$$
$$+\frac{1}{\pi}\int_{(\Lambda+2\mu)^2}^{\infty}ds'\frac{I_{mp}^{(-)}(s',t)}{s'-s}+\frac{1}{\pi}\int_{(N+K+\mu)^2}^{\infty}du'\frac{I_{mp}^{(+)}(u',t)}{u'-u}. \qquad (3\cdot5)$$

Let us now investigate the dependence on the variables of each term in the above expression separately by analyzing the corresponding perturbation graph.

$$\text{i)} \int_{(N+K)^2}^{\infty} ds' \frac{I_{\bar{K}p}^{(-)}(s', t)}{s'-s}. \tag{3·6i}$$

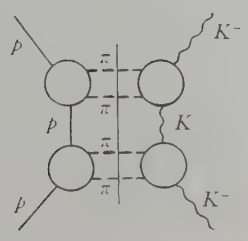


Fig. 2. Determination of the lowest singularity in t of $I_{\bar{K}p}^{(-)}(s, t)$.

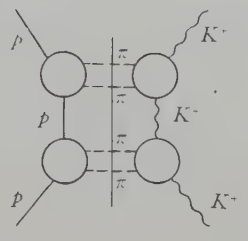


Fig. 3. Determination of the lowest singularity in t of $I_{Kp}^{(+)}(u, t)$.

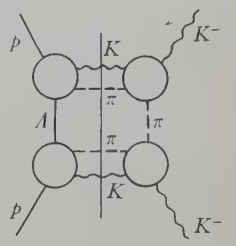


Fig.4. Determination of the lowest singularity in t of $I_{\pi\Lambda}^{(-)}(s, t)$.

In the above integral the lowest cut in the t variable begins at $16\mu^2$, while the cut in s begins at the elastic threshold $(N+K)^2$. This is graphically shown in Fig. 2.

From the location of the singularities we see that the integral (3·6i) has a strong s dependence and a weak t dependence and we are allowed to expand (3·6i) in powers of t/t' when t is small. Hence

$$I_{\bar{K}p}^{(-)}(s', t) \simeq a_{\bar{K}p}^{(-)}(s') + a'_{\bar{K}p}^{(-)}(s')t + \cdots. \tag{3·7}$$

For simplicity we shall keep here only the first two terms, which means that we take into account only the S and P waves in the K^--p scattering in the low energy and low momentum transfer region.

$$\text{ii)} \int_{(N+K)^2}^{\infty} du' \frac{I_{Kp}^{(+)}(u', t)}{u'-u}. \tag{3·6ii}$$

A quite similar discussion can be applied to Fig. 3 and yields

$$I_{Kp}^{(+)}(u', t) \simeq a_{Kp}^{(+)}(u') + a'_{Kp}^{(+)}(u')t. \tag{3·8}$$

$$\text{iii)} \int_{(\Lambda+\mu)^2}^{\infty} ds' \frac{I_{\pi\Lambda}^{(-)}(s,' t)}{s'-s}. \tag{3·6iii}$$

In this integral the lowest cut in t starts at $4(K+\mu)^2$ whereas the cut in s begins at $(\Lambda+\mu)^2$ below the elastic threshold (Fig. 4). Here also the expansion of $I_{\pi\Lambda}^{(-)}(s', t)$ about $t=0$ can be justified:

$$I^{(-)}_{\pi\Lambda}(s', t) \simeq a^{(-)}_{\pi\Lambda}(s') + a'^{(-)}_{\pi\Lambda}(s')t. \tag{3·9}$$

$$\text{iv)} \quad \int_{(\Sigma+\mu)^2}^{\infty} ds' \frac{I^{(-)}_{\pi\Sigma}(s', t)}{s'-s}. \tag{3·6iv}$$

Similarly from Fig. 5, $I^{(-)}_{\pi\Sigma}(s', t) \simeq a^{(-)}_{\pi\Sigma}(s') + a'^{(-)}_{\pi\Sigma}(s')t.$ (3·10)

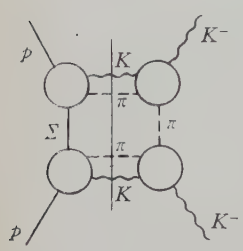


Fig. 5. Determination of the lowest singularity in t of $I^{(-)}_{\pi\Sigma}(s, t)$.

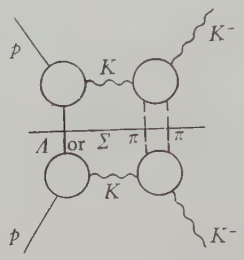


Fig. 6. Determination of the lowest singularity in s of $I^{(-)}_{\Lambda\pi\pi}$, $I^{(-)}_{\Sigma\pi\pi}$.

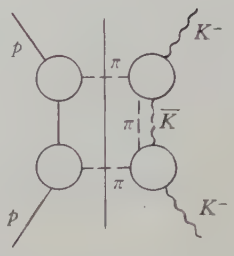


Fig. 7. Determination of the lowest singularity in t of $I^{(-)}_{NK\pi}$.

$$\text{v)} \quad \int_{(\Lambda+2\mu)^2}^{\infty} ds' \frac{I^{(-)}_{mp}(s', t)}{s'-s}. \tag{3·6v}$$

We shall divide $I^{(-)}_{mp}(s', t)$ as follows:

$$I^{(-)}_{mp}(s', t) = I^{(-)}_{\Lambda\pi\pi}(s', t) + I^{(-)}_{\Sigma\pi\pi}(s', t) + I^{(-)}_{NK\pi}(s', t) + R^{(-)}_{mp}(s', t). \tag{3·11}$$

For the first and second integrals, $\int_{(\Lambda+2\mu)^2}^{\infty} ds' \frac{I^{(-)}_{\Lambda\pi\pi}(s', t)}{s'-s}$ and $\int_{(\Sigma+2\mu)^2}^{\infty} ds' \frac{I^{(-)}_{\Sigma\pi\pi}(s', t)}{s'-s}$, the lowest limit of s is $(\Lambda+2\mu)^2$ and $(\Sigma+2\mu)^2$, respectively, while the cut in t begins at $4K^2$ (Fig. 6). Therefore we cannot disregard its s dependence but can expand $I^{(-)}_{\Lambda\pi\pi}(s', t)$ and $I^{(-)}_{\Sigma\pi\pi}(s', t)$ about $t=0$.

$$I^{(-)}_{\Lambda\pi\pi}(s', t) \simeq a^{(-)}_{\Lambda\pi\pi}(s') + a'^{(-)}_{\Lambda\pi\pi}(s')t. \tag{3·6va}$$

$$I^{(-)}_{\Sigma\pi\pi}(s', t) \simeq a^{(-)}_{\Sigma\pi\pi}(s') + a'^{(-)}_{\Sigma\pi\pi}(s')t. \tag{3·6vb}$$

For the third integral $\int_{(N+K+\mu)^2}^{\infty} ds' \frac{I^{(-)}_{NK\pi}(s', t)}{s'-s}$, however, the lowest cut in t begins at $4\mu^2$ whereas the cut in s begins at $(N+K+\mu)^2$, which is larger than $(N+K)^2$: $(N+K+\mu)^2-(N+K)^2 \approx 20\mu^2$. Accordingly, we cannot disregard the t dependence but can make an expansion in powers of $s-(N+K)^2/s'-(N+K)^2$.

By making use of the Mandelstam representation (3·1), we have

$$\int_{(N+K+\mu)^2}^{\infty} ds' \frac{I_{NK\pi}^{(-)}(s', t)}{s'-s} \simeq \int_{4\mu^2}^{\infty} dt' \frac{\alpha_p^{(-)}(s, t')}{t'-t}, \qquad (3\cdot 6\text{vc})$$

where

$$\alpha_p{}^{(-)}(s, t') \simeq \alpha_p{}^{(-)}((N+K)^2, t') + \alpha_p{}'^{(-)}((N+K)^2, t')\{s-(N+K)^2\}.$$

Here also we shall keep only the first two terms. Most of the contribution coming from $R_{mp}^{(-)}(s', t)$ can be absorbed into the integral (3·6vc) through the following diagrams (Fig. 8a, b). The weight function $\alpha(s, t')$ can be calculated

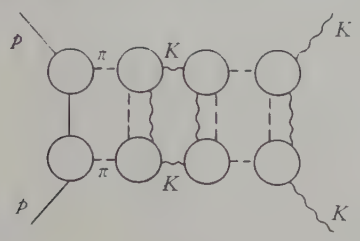


Fig. 8a.

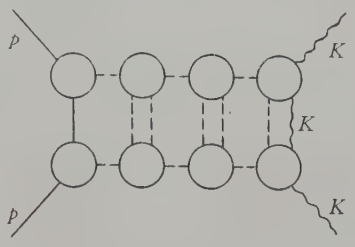


Fig. 8b.

by making use of the $KK\pi\pi$ interaction through $K\bar{K}\rightarrow$ resonating pions$\rightarrow N\bar{N}$.

$$\text{vi)} \int_{(N+K+\mu)^2}^{\infty} du' \frac{I_{mp}^{(+)}(u', t)}{u'-u}. \qquad (3\cdot 6\text{vi})$$

$I_{mp}^{(+)}(u', t)$ can also be split as

$$I_{mp}^{(+)}(u', t) = I_{NK\pi}^{(+)}(u', t) + R_{mp}^{(+)}(u', t) \qquad (3\cdot 12)$$

and similar consideration leads to the (3·vc) type of expression.

Collecting all our results together obtained above from i) through vi), we have for $A_p^{(-)}(s, u, t)$ the simple one-dimensional representation:

$$A_p^{(-)}(s, u, t) = -\frac{\pm\Lambda - N}{\Lambda^2 - s} g_\Lambda{}^2 - \frac{\pm\Sigma - N}{\Sigma^2 - s} g_\Sigma{}^2$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds' \frac{a_{Kp}^{(-)}(s') + a'^{(-)}_{Kp}(s')t}{s'-s} + \frac{1}{\pi}\int_{(N+K)^2}^{\infty} du' \frac{a_{Kp}^{(+)}(u') + a'^{(+)}_{Kp}(u')t}{u'-u}$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds' \frac{a_{\pi\Lambda}^{(-)}(s') + a_{\pi\pi\Lambda}^{(-)}(s') + a'^{(-)}_{\pi\Lambda}(s')t + a'^{(-)}_{\pi\pi\Lambda}(s')t}{s'-s}$$

$$+(\Lambda\rightarrow\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\alpha_p^{(-)}\{(N+K)^2, t'\}}{t'-t} + \frac{s-(N+K)^2}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\alpha'^{(-)}_p\{(N+K)^2, t'\}}{t'-t}$$

$$+ \text{ const. term.} \tag{3·13}$$

The last constant comes from a term which has only a weak dependence since the cuts are all distant.

Quite a similar discussion can be made for the invariant amplitude $A_p^{(+)}$ for K^+-p scattering; it is more convenient for later use, however, to write down the one-dimensional dispersion relations with respect to $A_{p,n}^{(1),(2)}$, $B_{p,n}^{(1),(2)}$ rather than $A_{p,n}^{(+),(-)}$, $B_{p,n}^{(+),(-)}$ because of the simple crossing relations (2·7a, b, c, d).

Thus for $A_p^{(1)}(s, u, t)$,

$$\begin{aligned} A_p^{(1)}(s,u,t) = &-\frac{g_\Lambda^2}{2}(\pm\Lambda - N)\left[\frac{1}{\Lambda^2-s}+\frac{1}{\Lambda^2-u}\right] - \frac{g_\Sigma^2}{2}(\pm\Sigma - N)\left[\frac{1}{\Sigma^2-s}+\frac{1}{\Sigma^2-u}\right] \\ &+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'\,[a_p^{(1)}(s')+a'^{(1)}_p(s')t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right] \\ &+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds'\,\frac{1}{2}[a_{\pi\Lambda}^{(-)}(s')+a_{\pi\pi\Lambda}^{(-)}(s')+a'^{(-)}_{\pi\Lambda}(s')t+a'^{(-)}_{\pi\pi\Lambda}(s')t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right] \\ &+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{\alpha_p^{(1)}(s,u,t')}{t'-t} \\ &+C_{\Lambda p}^{(1)}, \end{aligned} \tag{3·14}$$

where the real constant $C_{\Lambda p}^{(1)}$ stands for a contribution which has only a weak dependence on all three of the variables and

$$a_p^{(1)}(s') \equiv \frac{a_{Kp}^{(-)}(s')+a_{Kp}^{(+)}(s')}{2}, \tag{3·15}$$

$$a'^{(1)}_p(s') \equiv \frac{a'^{(-)}_{Kp}(s')+a'^{(+)}_{Kp}(s')}{2}, \tag{3·16}$$

$$\alpha_p^{(1)}(s,u,t') \equiv \frac{\alpha_{pK^-}+\alpha_{pK^+}}{2}, \tag{3·17}$$

$$\alpha'^{(1)}_p(s,u,t') \equiv \frac{\alpha'_{pK^-}+\alpha'_{pK^+}}{2}. \tag{3·18}$$

$A_p^{(2)}$ has a similar representation except that the constant must be zero, since it must be odd under interchange of s and u from crossing symmetry. So we obtain

$$\begin{aligned} A_p^{(2)}(s,u,t) = &-\frac{g_\Lambda^2}{2}(\pm\Lambda - N)\left[\frac{1}{\Lambda^2-s}-\frac{1}{\Lambda^2-u}\right] - \frac{g_\Sigma^2}{2}(\pm\Sigma - N)\left[\frac{1}{\Sigma^2-s}-\frac{1}{\Sigma^2-u}\right] \\ &+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'[a_p^{(2)}(s')+a'^{(2)}_p(s')t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right] \end{aligned}$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds' \frac{1}{2}[a_{\pi\Lambda}^{(-)}(s')+a_{\pi\pi\Lambda}^{(-)}(s')+a'^{(-)}_{\pi\Lambda}(s')t+a'^{(-)}_{\pi\pi\Lambda}(s')t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\alpha_p^{(2)}(s,u,t')}{t'-t}dt', \tag{3·19}$$

where
$$a_p^{(2)}(s') \equiv \frac{a_{\bar{K}p}^{(-)}(s)-a_{Kp}^{(+)}(s')}{2}, \quad \text{etc.,} \tag{3·20}$$

and
$$\alpha_p^{(2)}(s,u,t') \equiv \frac{\alpha_{pk^-}-\alpha_{pk^+}}{2}, \quad \text{etc.,} \tag{3·21}$$

which is odd under interchange of s and u.

Similar forms of the Mandelstam representation hold for both $B_p^{(1)}$ and $B_p^{(2)}$ except for a difference in the Born term.

$$B_p{}^{(1)}(s,u,t)=\frac{g_\Lambda{}^2}{2}\left[\frac{1}{M^2-s}-\frac{1}{M^2-u}\right]+\frac{g_\Sigma{}^2}{2}\left[\frac{1}{\Sigma^2-s}-\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'[b_p^{(1)}(s')+b'^{(1)}_p(s')t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds' \frac{1}{2}[b_{\pi\Lambda}^{(-)}(s')+b_{\pi\pi\Lambda}^{(-)}(s')+b'^{(-)}_{\pi\Lambda}(s')t+b'^{(-)}_{\pi\pi\Lambda}(s')t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\beta_p^{(1)}(s,u,t')}{t'-t}dt'. \tag{3·22}$$

$B_p^{(1)}$ also must be odd under interchange of s and u, so that the constant must be zero.

As for $B_p^{(2)}$, we have

$$B_p^{(2)}(s,u,t)=\frac{g_\Lambda{}^2}{2}\left[\frac{1}{M^2-s}+\frac{1}{M^2-u}\right]+\frac{g_\Sigma{}^2}{2}\left[\frac{1}{\Sigma^2-s}+\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'\,[b_p^{(2)}(s')+b_p'^{(2)}(s')t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds' \frac{1}{2}[b_{\pi\Lambda}^{(-)}(s')+b_{\pi\pi\Lambda}^{(-)}(s')+b'^{(-)}_{\pi\Lambda}(s')t+b'^{(-)}_{\pi\pi\Lambda}(s')t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{\beta_p^{(2)}(s,u,t')}{t'-t}\, dt'$$

$$+C_{Bp}^{(2)}. \tag{3·23}$$

Here

$$b_p^{(1)}(s') \equiv \frac{b_{\bar{K}p}^{(-)}(s')+b_{Kp}^{(+)}(s')}{2}, \quad \text{etc.,} \tag{3·24}$$

$$\beta_p^{(1)}(s,u,t') \equiv \frac{\beta_{pK^-}+\beta_{pK^+}}{2}, \quad \text{etc.,} \tag{3·25}$$

$$b_p^{(2)}(s') \equiv \frac{b_{\bar{K}p}^{(-)}(s')-b_{Kp}^{(+)}(s')}{2}, \quad \text{etc.,} \tag{3·26}$$

$$\beta_p^{(2)}(s,u,t') \equiv \frac{\beta_{pK^-}-\beta_{pK^+}}{2}, \quad \text{etc.} \tag{3·27}$$

There are four other relations for the K-neutron scattering which are exactly the same except that the protons are replaced by neutrons throughout, the term in g_Λ^2 being put equal to zero, and that g_Σ^2 is replaced by $2g_\Sigma^2$. Therefore we get the following representations for the four invariant amplitudes $A_n^{(1),(2)}$, $B_n^{(1),(2)}$:

$$A_n^{(1)}(s,u,t) = -g_\Sigma^2(\pm\Sigma-N)\left[\frac{1}{\Sigma^2-s}+\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'\left[a_n^{(1)}(s')+a_n'^{(1)}(s')t\right]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty} ds'\,\frac{1}{2}\left[a_{\pi\Lambda}^{(-)}(s')_n+a_{\pi\pi\Lambda}^{(-)}(s')_n+a'^{(-)}_{\pi\Lambda}(s')_n t+a'^{(-)}_{\pi\pi\Lambda}(s')_n t\right]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{\alpha_n^{(1)}(s,u,t')}{t'-t}\, dt'$$

$$+C_{An}^{(1)}, \tag{3·28}$$

where $a_{\pi\Lambda}^{(-)}(s')_n$'s are the weight functions in the case of K^--neutron absorption.

$$A_n^{(2)}(s,u,t) = -g_\Sigma^2(\pm\Sigma-N)\left[\frac{1}{\Sigma^2-s}-\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty} ds'\left[a_n^{(2)}(s')+a_n'^{(2)}(s')t\right]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds'\,\frac{1}{2}[a_{\pi\Lambda}^{(-)}(s')_n+a_{\pi\pi\Lambda}^{(-)}(s')_n+a'^{(-)}_{\pi\Lambda}(s')_n t+a'^{(-)}_{\pi\pi\Lambda}(s')_n t]\left[\frac{1}{s-s'}-\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{\alpha_n^{(2)}(s,u,t')}{t'-t}\,dt'. \tag{3·29}$$

$$B_n^{(1)}(s,u,t)=g_\Sigma{}^2\left[\frac{1}{\Sigma^2-s}-\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty}ds'[b_n^{(1)}(s')+b_n{}'^{(1)}(s')t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds'\,\frac{1}{2}[b_{\pi\Lambda}^{(-)}(s')_n+b_{\pi\pi\Lambda}^{(-)}(s')_n+b'^{(-)}_{\pi\Lambda}(s')_n t+b'^{(-)}_{\pi\pi\Lambda}(s')_n t]\left[\frac{1}{s'-s}-\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{\beta_n^{(1)}(s,u,t')}{t'-t}dt'. \tag{3·30}$$

$$B_n^{(2)}(s,u,t)=g_\Sigma{}^2\left[\frac{1}{\Sigma^2-s}+\frac{1}{\Sigma^2-u}\right]$$

$$+\frac{1}{\pi}\int_{(N+K)^2}^{\infty}ds'[b_n^{(2)}(s')+b_n{}'^{(2)}(s')t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+\frac{1}{\pi}\int_{(\Lambda+\mu)^2}^{\infty}ds'\,\frac{1}{2}[b_{\pi\Lambda}^{(-)}(s')_n+b_{\pi\pi\Lambda}^{(-)}(s')_n+b'^{(-)}_{\pi\Lambda}(s')_n t+b'^{(-)}_{\pi\pi\Lambda}(s)_n t]\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]$$

$$+(\Lambda\to\Sigma)$$

$$+\frac{1}{\pi}\int_{4\mu^2}^{\infty}dt'\,\frac{\beta_n^{(2)}(s,u,t')}{t'-t}dt'$$

$$+C_{Bn}^{(2)}.$$

The above mentioned approximate representations are valid for $|s-N^2|\lesssim 2N\times(K+\mu)$, $|u-N^2|\lesssim 2N(K+\mu)$ and $|t|\lesssim 16\mu^2$. The range of validity includes the physical regions of reactions I and II in § 2 up to the threshold for pion production.

§ 4. Calculation of the weight functions

In § 3, the eight independent invariant amplitudes $A_{p,n}^{(1),(2)}$ and $B_{p,n}^{(1),(2)}$ were expressed as simple one-dimensional dispersion integrals over some weight functions $a_{p,n}^{(1),(2)}$, $b_{p,n}^{(1),(2)}$, $\alpha_{p,n}^{(1),(2)}$ and $\beta_{p,n}^{(1),(2)}$, etc. These weight functions can be calculated by making use of unitarity.

i) The weight functions $I_{\bar{K}N}^{(-)}(s,t)$, etc., are equal to $\mathrm{Im}\, A_{\bar{K}N}^{(-)}(s,t)$ at least for s and t in the physical region of reaction I, and $\mathrm{Im}\, A_{\bar{K}N}^{(-)}(s',t)$ can be expanded as a sum over partial wave K-nucleon scattering amplitudes according to Eqs. (2·9) and (2·12). Keeping only the S and P waves in the polynomial expansions, we have

$$\mathrm{Im}\, A_p^{(1),(2)}(s',t) \simeq k'\left[\sigma_s{}^{(1)(2)} + 3\sigma_{p3/2}^{(1),(2)}\left(1-\frac{t}{2k'^2}\right)\right] - \frac{2N\omega'}{k'}\left[\sigma_{p1/2}^{(1),(2)} - \sigma_{p3/2}^{(1),(3)}\right] \quad (4\cdot1)$$

$$\simeq a_p{}^{(1),(2)}(s') + a'_p{}^{(1),(2)}(s')\,t. \quad (4\cdot1)'$$

Therefore we obtain

$$a_p{}^{(1),(2)}(s') \simeq k'\left[\sigma_s{}^{(1),(2)} + 3\sigma_{p3/2}^{(1),(2)}\right] - \frac{2N\omega'}{k'}\left[\sigma_{p1/2}^{(1),(2)} - \sigma_{p3/2}^{(1),(2)}\right], \quad (4\cdot2)$$

$$a_p{}'^{(1),(2)}(s') \simeq -\frac{3\sigma_{p3/2}^{(1),(2)}}{2k'}, \quad (4\cdot3)$$

where $$\sigma^{(1)} \equiv \frac{\sigma_{pK^-} + \sigma_{pK^+}}{2} \quad (4\cdot4)$$

and $$\sigma^{(2)} \equiv \frac{\sigma_{pK^-} - \sigma_{pK^+}}{2}. \quad (4\cdot5)$$

Similarly $\mathrm{Im}\, B_p^{(1),(2)}(s',t)$ can be expanded as follows:

$$\mathrm{Im}\, B_p^{(1),(2)}(s',t) \simeq \frac{2N}{k'}\left[\sigma_{p1/2}^{(1),(2)} - \sigma_{p3/2}^{(1),(2)}\right] \quad (4\cdot6)$$

$$\simeq b_p^{(1),(2)}(s') + b'_p{}^{(1),(2)}(s')\,t. \quad (4\cdot6)'$$

Eqs. (4·6) and (4·6)′ lead to

$$b_{p1}^{(1),(2)}(s') \simeq \frac{2N}{k'}\left[\sigma_{p1/2}^{(1),(2)} - \sigma_{p3/2}^{(1),(2)}\right], \quad (4\cdot7)$$

$$b_p{}'^{(1),(2)}(s') \simeq 0. \quad (4\cdot8)$$

The weight functions $I_{\pi\Lambda}^{(-)}(s,t)$, etc., are also equal to $\mathrm{Im}\, A_{\pi\Lambda}^{(-)}(s,t)$ for s and t in the physical region, and if necessary, it should be continued analytically into the unphysical region.[3),4),7)] Thus,

$$a_{\pi\Lambda}^{(-)}(s') \simeq k'\left[\{\sigma_{\pi\Lambda}\}_s + 3\{\sigma_{\pi\Lambda}\}_{p3/2}\right] - \frac{2N\omega'}{k'}\left[\{\sigma_{\pi\Lambda}\}_{p1/2} - \{\sigma_{\pi\Lambda}\}_{p3/2}\right], \quad (4\cdot9)$$

$$a'^{(-)}_{\pi\Lambda}(s') \simeq -\frac{3\{\sigma_{\pi\Lambda}\}_{p3/2}}{2k'}, \tag{4·10}$$

$$b^{(-)}_{\pi\Lambda}(s') \simeq \frac{2N}{k'}\left[\{\sigma_{\pi\Lambda}\}_{p1/2} - \{\sigma_{\pi\Lambda}\}_{p3/2}\right], \tag{4·11}$$

$$b'^{(-)}_{\pi\Lambda}(s') \simeq 0, \tag{4·12}$$

and so on.

Quite similar expressions can be obtained also for $a_n^{(1),(2)}$ and $b_n^{(1),(2)}$ except that the protons are replaced by the neutrons.

ii) *Evaluation of the two-pion contributions*

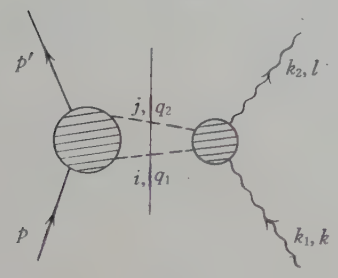


Fig. 9.

We now turn to the problem of determining the weight functions $\alpha_{p,n}^{(1),(2)}$ and $\beta_{p,n}^{(1),(2)}$. According to the discussion in § 3, these functions can be determined by making use of the unitarity condition:

$$\bar{u}(p')[-\alpha_p + i\gamma Q\beta_p]u(p)$$
$$= -\sqrt{2k_{10}}\,(2\pi)^{-2}\frac{1}{4}\sum_{i,j}\int d^3q_1\int d^3q_2\,\delta(k_1-k_2-q_1-q_2)$$
$$\times\langle p'|\bar{\eta}|q_1^{\,i}, q_2^{\,j}\rangle u(p)\langle q_1^{\,i}, q_2^{\,j}|J_l||k_1^{\,k}\rangle, \tag{4·13}$$

where the notation introduced by CGLN[13] (Fig. 9) is adopted. Here the mesons of the intermediate state have four-momenta q_1 and q_2 with isospin indices i and j, while the incoming K-meson of four-momentum k_1 with isospin index k is scattered into the outgoing K-meson of k_2 with index l. $\langle p'|\bar{\eta}|q_1^{\,i}, q_2^{\,j}\rangle$ and $\langle q_1^{\,i}, q_2^{\,j}|J_l|k_1^{\,k}\rangle$ are the T-matrix elements for the processes $N\bar{N}\to\pi\pi$ and $\pi\pi\to K\bar{K}$, respectively. Although the $KK\pi\pi$ interactions are extended, we shall make here a point interaction approximation by requiring Lorentz invariance and charge independence with the restriction that no derivative of order higher than second will be admitted. These requirements will also exhibit the characteristic features of the $KK\pi\pi$ interactions, which now take the following types:

$$H_1 = \lambda_1 K_i^* K_i \varphi_j \varphi_j, \tag{4·14}$$

$$H_2 = i\frac{\lambda_2}{K^2}(K^*\tau^{(\alpha)}\partial_\mu K - \partial_\mu K^*\tau^{(\alpha)}K)(\varphi T^{(\alpha)}\partial_\mu\varphi), \tag{4·15}$$

$$H_3 = \frac{\lambda_3}{K^2}(K_i^*\partial_\mu K_i)(\varphi_j\partial_\mu\varphi_j). \tag{4·16}$$

We choose hereafter H_1 as the isospin independent and H_2 as the isospin dependent $KK\pi\pi$ interaction since the other isospin independent interaction H_3 is not essentially different from H_1. In our approximation, as discussed in detail in § 3, it will be sufficient to take into account only the S and P wave

two-pion states. H_1 is responsible for the S wave and H_2 for the P wave state, respectively. It should also be noted that if we use for $\langle p', \bar{\eta}|q_1^i, q_2^j\rangle$ the physical $N\bar{N}\to\pi\pi$ matrix element including the $\pi\pi$ interaction, the effect of the resonating pions, and so on, (see Figs. 8. a, b) can automatically be included in the weight functions through (4·13).

As was defined in (3·17), (3·21), (3·25) and (3·27), we have

$$\alpha_p^{(1)}=\frac{\alpha_{pK^-}+\alpha_{pK^+}}{2}, \qquad \alpha_p^{(2)}=\frac{\alpha_{pK^-}-\alpha_{pK^+}}{2},$$

$$\beta_p^{(1)}=\frac{\beta_{pK^-}+\beta_{pK^+}}{2}, \qquad \beta_p^{(2)}=\frac{\beta_{pK^-}-\beta_{pK^+}}{2},$$

and so we shall first calculate, for example, α_{pK^+}, β_{pK^-} and α_{pK^-}, β_{pK^-} by making use of H_2.

Calculation of α_{pK^+} and β_{pK^+} arising from H_2

We can now get the current J_l from the effective Hamiltonian of the isospin dependent type H_2 according to

$$J_l=\frac{\delta H_2}{\delta K_l^*}=2i\frac{\lambda_2}{K^2}\tau_{lm}^{(\alpha)}\partial_\mu K_m\cdot\varphi_i T_{ij}{}^{(\alpha)}\partial_\mu\varphi_j, \tag{4·17}$$

which leads to

$$\begin{aligned}
&\langle p'|\bar{\eta}|q_1^i, q_2^j\rangle u(p)\langle q_1^i, q_2^j|J_l|k_1^k\rangle\\
&=\frac{-1}{\sqrt{2\omega_1 2\omega_2}}\bar{u}(p')\left\{[-A^{(+)}+i\gamma\bar{Q}B^{(+)}]\delta_{ij}+[-A^{(-)}+i\gamma\bar{Q}B^{(-)}]\frac{1}{2}[\tau_j,\tau_i]\right\}u(p)\\
&\quad\times\frac{4\lambda_2}{K^2}\frac{1}{\sqrt{2k_{10}2\omega_1 2\omega_2}}\left\{\tau^{(\alpha)}\cdot\tau^{(\alpha)}k_1\cdot(q_1-q_2)\frac{1}{2}[\tau_j,\tau_i]\right\}\\
&=-\frac{4\lambda_2}{K^2}\frac{1}{\sqrt{2k_{10}}}\frac{1}{2\omega_1 2\omega_2}\bar{u}(p')[\tau^{(\alpha)}\cdot\tau^{(\alpha)}k_1\cdot(q_1-q_2)][-A^{(-)}+i\gamma\bar{Q}B^{(-)}]u(p),
\end{aligned} \tag{4·18}$$

where

$$\bar{Q}\equiv\frac{1}{2}(q_1-q_2). \tag{4·19}$$

Substituting this equation into (4·13) and noticing $\tau^{(\alpha)}\cdot\tau^{(\alpha)}=1$ since the pK^+ system has the total isospin one, we have

$$\begin{aligned}
&-\alpha_{pK^+}+i\gamma Q_l\beta_{pK^+}\\
&=\frac{1}{4\pi^2}\frac{\lambda_2}{K^2}\int\frac{d^3q_1}{2\omega_1}\int\frac{d^3q_1}{2\omega_2}\delta(k_1-k_2-q_1-q_2)[k_1\cdot(q_1-q_2)][-A^{(-)}+i\gamma\bar{Q}B^{(-)}]\\
&=\frac{1}{4\pi^2}\frac{\lambda_2}{K^2}\int\frac{d^3q_1}{2\omega_1}\int\frac{d^3q_2}{2\omega_2}\delta(k_1-k_2-q_1-q_2)(k_1+k_2)_\mu\bar{Q}_\mu\\
&\qquad\times[-A^{(-)}+i\gamma\bar{Q}B^{(-)}],
\end{aligned} \tag{4·20}$$

where use has been made of the property

$$k_1(q_1-q_2)=\bar{Q}(k_1+k_2). \tag{4·21}$$

It should be noted that the expression (4·20) has the same form as the isotopic vector part of the nucleon electromagnetic structure,[16] and so may be written as

$$(4\cdot20)=\frac{1}{4\pi^2}\frac{\lambda_2}{K^2}\frac{4\pi^2}{e}[i\gamma_\mu(k_1+k_2)_\mu g^V{}_{1(2\pi)}(t) \tag{4·22}$$

$$+(k_1+k_2)_\mu\sigma_{\mu\nu}t_\nu g^V{}_{2(2\pi)}(t)].$$

Redefining

$$-q_1-q_2=k_2-k_1\equiv-2\kappa \tag{4·23}$$

together with $Q=1/2\cdot(k_1+k_2)$, we obtain

$$i\gamma_\mu(k_1+k_2)_\mu=2i\gamma_\mu Q_\mu, \tag{4·24}$$

$$(k_1+k_2)_\mu\sigma_{\mu\nu}t_\nu=-4(\kappa^2-N\nu_L-iN\gamma_\mu Q_\mu). \tag{4·25}$$

By using Eqs. (4·24) and (4·25), Eq. (4·22) becomes

$$(4\cdot22)=(4\cdot20)=+\frac{1}{4\pi^2}\frac{\lambda_2}{K^2}\frac{8\pi^2}{e}\Big\{-2(\kappa^2-N\nu_L)g^V_{2(2\pi)}(t)$$

$$+i\gamma_\mu Q_\mu[g^V_{1(2\pi)}(t)+2Ng^V_{2(2\pi)}(t)]\Big\}, \tag{4·26}$$

where ν_L is the incident K-meson energy in the laboratory system. Therefore we get

$$\alpha_{pK^+}=-\frac{4\lambda_2}{eK^2}(N\nu_L+(t/4))g^V_{2(2\pi)}(t), \tag{4·27}$$

$$\beta_{pK^+}=\frac{2\lambda_2}{eK^2}[g^V_{1(2\pi)}(t)+2Ng^V_{2(2\pi)}(t)]. \tag{4·28}$$

Calculation of α_{pK^-} and β_{pK^-} arising from H_2

We shall first rewrite the effective Hamiltonian H_2 in the case of pK^- scattering.

Define

$$K\equiv\begin{pmatrix}K^+\\K^0\end{pmatrix}, \tag{4·29}$$

$$\tilde{K}\equiv(\bar{K}^0, K^-). \tag{4·30}$$

Then

$$K^*=(K^{+*}, K^{0*}). \tag{4·31}$$

In order to define the total isotopic spin of the $\bar{K}$-nucleon system, we must make the following substitution,

$$K^*=-\tilde{K}i\tau_1, \tag{4·32}$$

or

$$K=i\tau_2\tilde{K}^*, \tag{4·32}'$$

in the expression (4·15) to obtain

$$H_2=i\frac{\lambda_2}{K^2}(-\widetilde{K}\tau^{(\alpha)}\partial_\mu\widetilde{K}^*+\partial_\mu\widetilde{K}\tau^{(\alpha)}\widetilde{K}^*)(\varphi T^{(\alpha)}\partial_\mu\varphi). \tag{4·33}$$

The current J_l in this case becomes

$$J_l=\frac{\delta H_2}{\delta\widetilde{K}_l^*}=2i\frac{\lambda_2}{K^2}\partial_\mu\widetilde{K}_m\tau_{ml}^{(\alpha)}\cdot\varphi_i T_{ij}^{(\alpha)}\partial_\mu\varphi_j. \tag{4·34}$$

By a calculation similar to that of the previous case we obtain

$$\alpha_{pK^-}=-\frac{4\lambda_2}{eK^2}\left(N\nu_L+\frac{t}{4}\right)g^V_{2(2\pi)}(t)\ , \tag{4·35}$$

$$\beta_{pK^-}=-\frac{2\lambda_2}{eK^2}[g^V_{1(2\pi)}(t)+2Ng^V_{2(2\pi)}(t)]. \tag{4·36}$$

Next we shall calculate the weight functions arising from H_l along the same lines.

Calculation of α_{pK^+}, α_{pK^-}, β_{pK^+} and β_{pK^-} arising from H_1

The unitarity condition (4·13) together with the effective Hamiltonian of the isospin independent type H_1 leads to

$$\alpha_{pK^+}=\alpha_{pK^-}=\frac{3\lambda_1}{8\pi^2}f(t), \tag{4·37}$$

where

$$\begin{aligned}f(t)&=\int d^4\bar{Q}\delta\left(\bar{Q}^2+\frac{q^2}{4}+\mu^2\right)\delta(2q\bar{Q})\ [-A^{(+)}+i\gamma QB^{(+)}]\\&=-\frac{8}{9}g_r^2\frac{1}{\sqrt{t}\sqrt{N^2-(t/4)}}\left[\frac{3\pi}{2N}k_r^2\left(1+\frac{t}{2k_r^2}\right)+\pi\omega_r\right]\tan^{-1}y_r(t)\\&+\frac{g_r^2\pi N\sqrt{(t/4)-\mu^2}}{\sqrt{t}\,(N^2-(t/4))}\left[1-\frac{1}{y_0(t)}\tan^{-1}y_0(t)\right]\\&-\frac{8}{9}\frac{g_r^2\pi N\sqrt{(t/4)-\mu^2}}{\sqrt{t}\,(N^2-(t/4))}\left[1-\frac{1}{y_r(t)}\tan^{-1}y_r(t)\right],\end{aligned} \tag{4·38}$$

where the notation of CKGZ[17] has been used.

$$\beta_{pK^+}=\beta_{pK^-}=0. \tag{4·39}$$

Here we have set all partial cross sections, except that for the (3/2, 3/2) state, equal to zero for the pion-nucleon scattering and approximated the latter by a delta function. In Fig. 10, we have plotted $f(t)$ for various values of t. The weight functions for the K-neutron scattering can also be calculated by making use of the unitarity relation (4·13) together with the relations:

$$T_n^{(-)}=T_1^{(-)}, \tag{4·40}$$

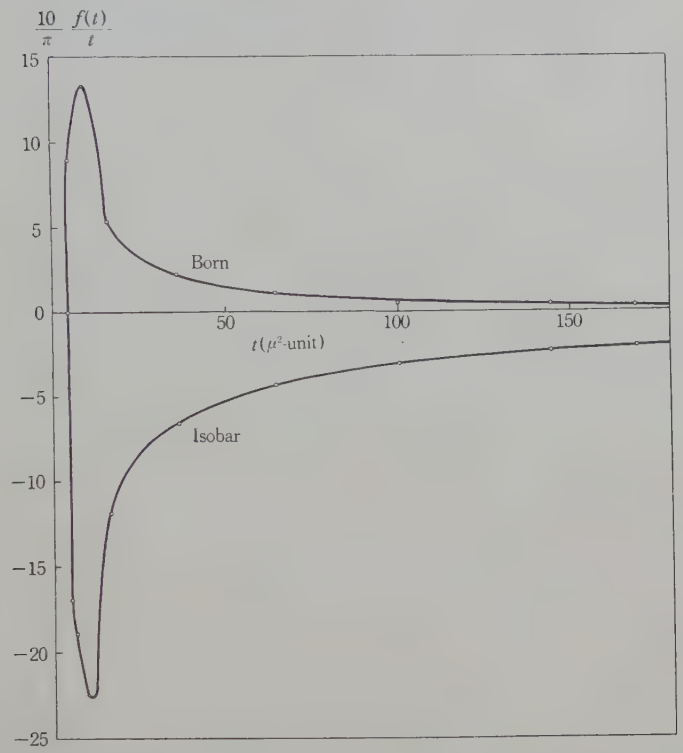


Fig. 10. Spectral function $f(t)$, arising from H_1, devided by t; it is the contribution from both the Born approximation term and the 3-3 isobar term.

$$T_n^{(+)}=(1/2)(T_1^{(+)}-T_0^{(+)}), \tag{4·41}$$

where, on the right-hand side, the upper and lower indices denote strangeness and isospin, respectively.

Calcutation of α_{nK^+} *and* β_{nK^+} *arising from* H_2

We now get

$$\alpha_{nK^+}=-\frac{8\lambda_2}{eK^2}(N\nu_L+t)g^V_{2(2\pi)}(t), \tag{4·42}$$

$$\beta_{nK^+}=\frac{4\lambda_2}{eK^2}[g^V_{1(2\pi)}(t)+2Ng^V_{2(2\pi)}(t)]. \tag{4·43}$$

Calculation of α_{nK^-} *and* β_{nK^-} *arising from* H_2

Quite similarly we have

$$\alpha_{nK^-}=\frac{8\lambda_2}{eK^2}(N\nu_L+t)g^V_{2(2\pi)}(t), \tag{4·44}$$

$$\beta_{nK^-} = -\frac{4\lambda_2}{eK^2}\left[g^V_{1(2\pi)}(t) + 2Ng^V_{2(2\pi)}(t)\right]. \tag{4·45}$$

α_{nK^+}, α_{nK^-}, β_{nK^+} and β_{nK^-} arisig from H_1

$$\alpha_{nK^+} = \alpha_{nK^-} = \frac{3\lambda_1}{8\pi^2} f(t), \tag{4·46}$$

where $f(t)$ is given as (4·32).

$$\beta_{nK^+} = \beta_{nK^-} = 0. \tag{4·47}$$

Collecting all our results concerning $\alpha_{pK^\pm}$, $\beta_{pK^\pm}$, $\alpha_{nK^\pm}$ and $\beta_{nK^\pm}$, we obtain the weight functions $\alpha_p^{(1)}$, $\alpha_p^{(2)}$, $\beta_p^{(1)}$ and $\beta_p^{(2)}$ which were introduced in (3·14), (3·19), (3·22) and (3·23) and those $\alpha_n^{(1)}$, $\alpha_n^{(2)}$, $\beta_n^{(1)}$ and $\beta_n^{(2)}$ introduced in (3·28), (3·29), (3·30) and (3·31), respectively. Namely,

$$\begin{aligned}
\alpha_p^{(1)} &= \frac{3\lambda_1}{8\pi^2} f(t), \\
\alpha_p^{(2)} &= \frac{4\lambda_2}{eK^2}\left(N\nu_L + \frac{t}{4}\right) g^V_{2(2\pi)}(t), \\
\beta_p^{(1)} &= 0, \\
\beta_p^{(2)} &= -\frac{2\lambda_2}{eK^2}\left[g^V_{1(2\pi)}(t) + 2Ng^V_{2(2\pi)}(t)\right],
\end{aligned} \tag{4·48}$$

and

$$\begin{aligned}
\alpha_p^{(1)} &= \frac{3\lambda_2}{8\pi^2} f(t), \\
\alpha_p^{(2)} &= \frac{8\lambda_2}{eK^2}(N\nu_L + t) g^V_{2(2\pi)}(t), \\
\beta_p^{(1)} &= 0, \\
\beta_p^{(2)} &= -\frac{4\lambda_2}{eK^2}\left[g^V_{1(2\pi)}(t) + 2Ng^V_{2(2\pi)}(t)\right].
\end{aligned} \tag{4·49}$$

§ 5. Type of the direct interactions and $KK\pi\pi$ interactions

Because of the lack of enough experimental data on the K-nucleon scattering and absorption processes and also of the ambiguity caused by the presence of unphysical regions due to the absorption of K^- by protons below the threshold, the independent determination of the type and magnitude of the K-N-Λ and K-N-Σ and $KK\pi\pi$ interactions is very difficult at the present stage. Although no definite conclusions can be drawn because of the lack of the experimental data, the analysis has been made on each of the interaction types

K-N-Λ and K-N-Σ by making use of the dispersion relations for the charge exchange scattering $K^+n\to K^0p$ and for the elastic scattering $K^+p\to K^+p$ by Sugano and Komatsuzawa.[17]

In order to investigate both the direct and $KK\pi\pi$ interaction, we shall employ a method which allows us to make maximum use of the limited experimental data available at present. In what follows we shall optimistically assume that the dispersion relations for the eight invariant amplitudes $A_{p,n}^{(1),(2)}$, $B_{p,n}^{(1),(2)}$ are all valid without subtraction and connect these amplitudes to the S-wave scattering length by making use of the relations (2·9a, b), (2·12a, b) and the approximate expressions for the A's and B's (3·14), (3·19), (3·22,) (3·23) evaluated in the Cini-Fubini approximation to the Mandelstam representation. Then the four relations can be derived as follows. First of all, we have

$$
\begin{aligned}
\mathrm{Re}\{f_p^{(1)}\}_s \equiv & \mathrm{Re}\frac{a_{pK^-}+a_{pK^+}}{2} \\
= & \frac{1}{4\pi}\frac{N}{N+K}[A_p^{(1)}(s_0,u_0,0)+KB_p^{(1)}(s_0,u_0,0)] \\
= & \frac{N}{N+K}\Bigg[-\left(\frac{g_\Lambda^2}{4\pi}\right)\frac{\pm\Lambda+N}{(\pm\Lambda+N)^2-K^2}-\left(\frac{g_\Sigma^2}{4\pi}\right)\frac{\pm\Sigma+N}{(\pm\Sigma+N)^2-K^2} \\
& +\frac{1}{4\pi^2}\int_{(N+K)^2}^{\infty}ds'\,k'\Bigg[\{\sigma_p^{(1)}\}_s+3\{\sigma_p^{(1)}\}_{p3/2}-\frac{2N(\omega'-K)}{k'}[\{\sigma_p^{(1)}\}_{p1/2}-\{\sigma_p^{(1)}\}_{p3/2}]\Bigg] \\
& \qquad\times\left[\frac{1}{s'-(N+K)^2}+\frac{1}{s'-(N-K)^2}\right] \\
& +\frac{P}{4\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}ds'\frac{k'}{2}\Bigg[\{\sigma_{\pi\Lambda}\}_s+3\{\sigma_{\pi\Lambda}\}_{p3/2}-\frac{2N(\omega'-K)}{k'}[\{\sigma_{\pi\Lambda}\}_{p1/2}-\{\sigma_{\pi\Lambda}\}_{p3/2}]\Bigg] \\
& \times\left[\frac{1}{s'-(N+K)^2}+\frac{1}{s'-(N-K)^2}\right]+(\pi\Lambda\to\pi\Sigma)+(\pi\Lambda\to\pi\pi\Lambda)+(\pi\Lambda\to\pi\pi\Sigma) \\
& +\frac{3}{8\pi^2}\left(\frac{\lambda_1}{4\pi}\right)\frac{1}{\pi}\int_{4\mu^2}dt'\frac{f(t')}{t'} \\
& +\frac{C_{Ap}^{(1)}}{4\pi}\Bigg], \qquad (5\cdot1)
\end{aligned}
$$

where the upper (lower) sign is hereafter to be taken in the case of pseudoscalar (scalar) coupling, and a_{pK^-} and a_{pK^+} are S-wave scattering lengths for the K^--p and K^+-p scattering, respectively. $s_0=(N+K)^2$, $u_0=(N-K)^2$ and $f(t')$ is given in (4·32). Next we have

$$
\mathrm{Re}\{f_p^{(2)}\}_s \equiv \mathrm{Re}\ \frac{a_{pK^-}-a_{pK^+}}{2}
$$

$$= \frac{1}{4\pi}\frac{N}{N+K}[A_p^{(2)}(s_0, u_0, 0)+KB_p^{(2)}(s_0, u_0, 0)]$$

$$= \frac{N}{N+K}\Big[\Big(\frac{g_\Lambda^2}{4\pi}\Big)\frac{K}{(\pm\Lambda+N)^2-K^2}+\Big(\frac{g_\Sigma^2}{4\pi}\Big)\frac{K}{(\pm\Sigma+N)^2-K^2}$$

$$+\frac{1}{4\pi^2}\int_{(N+K)^2}^{\infty}ds'k'\Big[\{\sigma_p^{(2)}\}_s+3\{\sigma_p^{(2)}\}_{p3/2}-\frac{2N(\omega'-K)}{k'}\big[\{\sigma_p^{(2)}\}_{p1/2}-\{\sigma_p^{(2)}\}_{p3/2}\big]\Big]$$

$$\times\Big[\frac{1}{s'-(N+K)^2}-\frac{1}{s'-(N-K)^2}\Big]$$

$$+\frac{P}{4\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}ds'\frac{k'}{2}\Big[\{\sigma_{\pi\Lambda}\}_s+3\{\sigma_{\pi\Lambda}\}_{p3/2}-\frac{2N(\omega'-K)}{k'}\big[\{\sigma_{\pi\Lambda}\}_{p1/2}-\{\sigma_{\pi\Lambda}\}_{p3/2}\big]\Big]$$

$$\times\Big[\frac{1}{s'-(N+K^2)}-\frac{1}{s'-(N-K)^2}\Big]$$

$$+(\pi\Lambda\to\pi\Sigma)+(\pi\Lambda\to\pi\pi\Lambda)+(\pi\Lambda\to\pi\pi\Sigma)$$

$$-\Big(\frac{\lambda_2}{4\pi}\Big)\frac{2}{eK}\frac{1}{\pi}\int_{4\mu^2}dt'\frac{g_{1(2\pi)}^V(t')}{t'}+\frac{KC_{Bp}^{(2)}}{4\pi}\Big]. \qquad (5\cdot2)$$

$$\mathrm{Re}\{f_n^{(1)}\}_s \equiv \mathrm{Re}\frac{a_{nK^-}+a_{nK^+}}{2}$$

$$= \frac{1}{4\pi}\frac{N}{N+K}[A_n^{(1)}(s_0, u_0, 0)+KB_n^{(1)}(s_0, u_0, 0)]$$

$$= \frac{N}{N+K}\Big[-\Big(\frac{g_\Sigma^2}{4\pi}\Big)\frac{2(\pm\Sigma+N)}{(\pm\Sigma+N)^2-K^2}$$

$$+\{p\to n \text{ in the corresponding term of } (5\cdot1)\}$$

$$+\{p\to n \text{ in the corresponding term of } (5\cdot1)\}$$

$$+(\pi\Lambda\to\pi\Sigma)+(\pi\Lambda\to\pi\pi\Lambda)+(\pi\Lambda\to\pi\pi\Sigma)$$

$$+\frac{3}{8\pi^2}\Big(\frac{\lambda_1}{4\pi}\Big)\frac{1}{\pi}\int_{4\mu^2}dt'\frac{f(t')}{t'}$$

$$+\frac{C_{An}^{(1)}}{4\pi}\Big], \qquad (5\cdot3)$$

$$\mathrm{Re}\{f_n^{(2)}\}_s \equiv \mathrm{Re}\frac{a_{nK^-}-a_{nK^+}}{2}$$

$$= \frac{1}{4\pi}\frac{N}{N+K}[A_n^{(2)}(s_0, u_0, 0)+KB_n^{(2)}(s_0, u_0, 0)]$$

$$= \frac{N}{N+K}\Bigg[\left(\frac{g_{\Sigma}^{2}}{4\pi}\right)\frac{2K}{(\pm\Sigma+N)^{2}-K^{2}}$$

$$+\{p\to n \text{ in the corresponding term of } (5\cdot2)\}$$

$$+\{p\to n \text{ in the corresponding term of } (5\cdot2)\}$$

$$+(\pi\Lambda\to\pi\Sigma)+(\pi\Lambda\to\pi\pi\Lambda)+(\pi\Lambda\to\pi\pi\Sigma)$$

$$-\left(\frac{\lambda_2}{4\pi}\right)\frac{4}{eK}\frac{1}{\pi}\int_{4\mu^2}dt'\,\frac{g^{V}_{1(2\pi)}(t')}{t'}+\frac{KC^{(2)}_{Bn}}{4\pi}\Bigg]. \tag{5·4}$$

It should be remarked that $\mathrm{Re}\,(a_{pK^-}+a_{pK^+})/2$ and $\mathrm{Re}\,(a_{nK^-}+a_{nK^+})/2$ are dependent on the $KK\pi\pi$ interaction of isospin independent type (4·14) which corresponds to an intermediate π-π resonance of $I=0$, while $\mathrm{Re}\,(a_{pK^-}-a_{pK^+})/2$ and $\mathrm{Re}\,(a_{nK^-}-a_{nK^+})/2$ are related to the $KK\pi\pi$ interaction of isospin dependent type (4·15) which corresponds to an $I=1$ π-π resonance state. Therefore, Eqs. (5·1) and (5·3) are suitable for determining λ_1 and Eqs. (5·2) and (5·4) for λ_2, $C^{(1)}_{Ap}$ $C^{(2)}_{Bp}$ etc., being, however, left as arbitrary parameters which come from high energy contributions to our dispersion integrals in the π-meson case.[12]

Now we shall make the following conjecture. When we introduce the experimental S-wave scattering length in Eqs. (5·2) and (5·4) to get λ_2, we can set $C^{(2)}_{Bp}$, $C^{(2)}_{Bn}$ equal to zero because the high energy contribution to the dispersion integrals seems small because of the Pomeranchuk theorem,[18] while $C^{(1)}_{Ap}$, $C^{(1)}_{An}$ are not expected to be small because of the bad convergence of the dispersion integral. Hence in order to investigate the isospin independent $KK\pi\pi$ interaction it is reliable to use, rather than Eqs. (5·1) and (5·3), a relation in which the P-wave scattering length is expressed as the sum of the dispersion integral and the contribution coming from the λ_1-type $KK\pi\pi$ interaction.

We think it preferable to analyze the $p(3/2)$ state rather than the $p(1/2)$ state because the scattering length of the $p(3/2)$ state $f^{(-)}_{p(3/2)}$ is expressed as

$$-\frac{3}{k^2}\mathrm{Re}\,f^{(1)}_{p3/2}\,|_{k^2=0}=\frac{1}{4\pi}\frac{2N}{N+K}\left[\frac{dA^{(1)}(s,t)}{dt}\Big|_{\substack{t=0\\s=s_0}}+K\,\frac{dB^{(1)}(s,t)}{dt}\,\Big|_{\substack{t=0\\s=s_0}}\right], \tag{5·5}$$

where the relations (3·14), (3·22) are to be introduced in the case of K-p scattering and Eqs. (3·28), (3·30) in the case of K-n scattering. In this way good convergence is assured and moreover free parameters completely vanish. On the other hand, difference of $f^{(1)}_{p(1/2)}$ and $f^{(1)}_{p(3/2)}$ is expressed as

$$\frac{1}{k^2}\{f^{(1)}_{p1/2}-f^{(1)}_{p3/2}\}\,|_{k^2=0}=\frac{1}{4\pi}\,\frac{1}{2N(N+K)}\{-A^{(1)}(s,t)|_{\substack{t=0\\s=s_0}}+(2N+K)B^{(1)}(s,t)|_{\substack{t=0\\s=s_0}}\}, \tag{5·6}$$

where the dispersion integrals have bad convergence and moreover, the arbitrary parameter does not disappear.

Summarizing the above discussion, we see that the S-wave scattering lengths $\mathrm{Re}\{f_p^{(2)}\}_s$ and $\mathrm{Re}\{f_n^{(2)}\}_s$ are connected as

$$\mathrm{Re}\{f_p^{(2)}\}_s \propto A_p^{(2)} + KB_p^{(2)} = \begin{Bmatrix} \alpha_s & g_{\Lambda,s}^2 \\ \text{or } \alpha_{ps} & g_{\Lambda,ps}^2 \end{Bmatrix} + \begin{Bmatrix} \beta_s & g_{\Sigma,s}^2 \\ \text{or} \beta_{ps} & g_{\Sigma,ps}^2 \end{Bmatrix} + \gamma\lambda_2 + \text{dispersion integral}, \tag{5·7}$$

$$\mathrm{Re}\{f_n^{(2)}\}_s \propto A_n^{(2)} + KB_n^{(2)} = \begin{Bmatrix} \beta_s' & g_{\Sigma,s}^2 \\ \text{or} \beta'_{ps} & g^2_{\Sigma,ps} \end{Bmatrix} + \gamma'\lambda_2 + \text{dispersion integral}, \tag{5·8}$$

by Eqs. (5·2) and (5·4). Here $\alpha_{s,ps}$, $\beta_{s,ps}$, $\beta'_{s,ps}$, γ and γ', etc., are quantities to be calculated and the dispersion integrals are expressed in terms of the partial wave cross sections in Eqs. (5·2) and (5·4). On the other hand, we get

$$\frac{1}{k^2}\mathrm{Re}\{f_p^{(1)}\}_{p3/2} \propto \begin{Bmatrix} \alpha_s'' & g_{\Lambda,s}^2 \\ \text{or } \alpha_{ps}'' & g_{\Lambda,ps}^2 \end{Bmatrix} + \begin{Bmatrix} \beta_s'' & g_{\Sigma,s}^2 \\ \text{or } \beta_{ps}'' & g_{\Sigma,p}^2 \end{Bmatrix} + \gamma''\lambda_1, \tag{5·9}$$

$$\frac{1}{k^2}\mathrm{Re}\{f_n^{(1)}\}_{p3/2} \propto \begin{Bmatrix} \beta_s''' & g_{\Sigma,s}^2 \\ \text{or } \beta'''_{ps} & g_{\Sigma,ps}^2 \end{Bmatrix} + \gamma'''\lambda_1. \tag{5·10}$$

Hence from Eqs. (5·7), (5·8), (5·9) and (5·10) we can determine independently the coupling type of K-N-Λ, K-N-Σ and the λ_1, λ_2 of the $KK\pi\pi$ interaction if appropriate experimental data can be obtained. For this determination we would like to propose the following three investigations:

i) To choose a suitable solution from the Dalitz solutions.[19]
ii) To investigate the threshold behavior for the P-wave phase shifts.
iii) To get the S- and P-wave cross sections separately for the K-nucleon, $\bar{K}$-nucleon and absorption processes.

§ 6. One proposal to get λ_2

In the last section we discussed that if appropriate experimental data are given, independent determination of the direct interaction and the $KK\pi\pi$ interaction is possible. In this section we shall discuss a simple method which allows us to determine the isospin dependent $KK\pi\pi$ interaction, and then propose a suitable experiment.

The dispersion relation for the K-nucleon forward scattering amplitude which was obtained in the previous analyses[3),4),5)] is,

$$\mathrm{Re}\,\frac{a_{pK^-} - a_{pK^+}}{2} = \frac{N}{N+K}\left[\left(\frac{g_\Lambda^2}{4\pi}\right)\begin{Bmatrix} (1/16.3)\ (1/K);\, ps \\ -1.15(1/K);\ s \end{Bmatrix} + \left(\frac{g_\Sigma^2}{4\pi}\right)\begin{Bmatrix} (1/17.6)(1/K);\, ps \\ -1.35(1/K);\ s \end{Bmatrix}\right.$$
$$\left. + \frac{1}{4\pi^2}\int_{(N+K)^2}^{\infty} ds' \frac{k'[\sigma_{K^-p}(s') - \sigma_{K^+p}(s')]}{2}\left\{\frac{1}{s'-(N+K)^2} - \frac{1}{s'-(N-K)^2}\right\}\right]$$

$$+\frac{1}{4\pi^2}\int_{(\Lambda+\mu)^2}^{\infty}ds'\,\frac{k'[\sigma_{\pi\Lambda}(s')+\sigma_{\pi\Sigma}(s')]}{2}\left\{\frac{1}{s'-(N+K)^2}-\frac{1}{s'-(N-K)^2}\right\}\Bigg], \quad (6\cdot1)$$

where $$\frac{\sigma_{K^-p}(s')-\sigma_{K^+p}(s')}{2}\simeq\{\sigma_p^{(2)}\}_s+3\{\sigma_p^{(2)}\}_{p(3/2)},$$

$$\sigma_{\pi\Lambda}(s')\simeq\{\sigma_{\pi\Lambda}\}_s+3\{\sigma_{\pi\Lambda}\}_{p(3/2)}\text{, etc.,} \quad (6\cdot2)$$

by making the partial wave expansion up to P waves. Now we shall make the following assumptions concerning (5·2):

i) The contribution from higher mass configurations is small because of the large denominator and of the Pomeranchuk theorem[18] as indicated in the last section; hence $C_{Bp}^{(3)}$ can be neglected.

ii) $\sigma(K^-p\to\pi\pi\Lambda)/\sigma(K^-p\to\pi\Lambda)$ is small in the low energy region as is shown from experiment.

Then comparison between Eq. (6·1) and Eq. (5·2) under the above two assumptions leads to

$$-\frac{N}{4\pi^2}\int_{(N+K)^2}^{\infty}ds'(\omega'-K)[\{\sigma_p^{(2)}\}_{p1/2}-\{\sigma_p^{(2)}\}_{p3/2}]\left[\frac{1}{s'-(N+K)^2}-\frac{1}{s'-(N-K)^2}\right]$$

$$-\frac{N}{4\pi^2}\,\text{Principal P.}\int_{(\Lambda+\mu)^2}^{\infty}ds'(\omega'-K)[\{\sigma_{\pi\Lambda}\}_{p1/2}-\{\sigma_{\pi\Lambda}\}_{p3/2}+(\Lambda\to\Sigma)]$$

$$\times\left[\frac{1}{s'-(N+K)^2}-\frac{1}{s'-(N-K)^2}\right]$$

$$=\left(\frac{\lambda_2}{4\pi}\right)\frac{1}{eK}\,\frac{1}{\pi}\int_{4\mu^2}dt'\,\frac{g_{1(2\pi)}^V(t')}{t'}. \quad (6\cdot3)$$

If the left-hand side is positive, we can get positive λ_2 which corresponds to a repulsive interaction, while if it is negative we obtain negative λ_2 which corresponds to an attractive interaction. Hence it is to be hoped that the partial wave cross sections $\sigma_{p1/2}$ and $\sigma_{p3/2}$ for the low energy K-p and $\bar{K}$-p scattering $\{\sigma_{\pi\Lambda}\}_{p1/2}$ and $\{\sigma_{\pi\Lambda}\}_{p3/2}$, etc., for the absorption processes will be investigated as soon as possible.

§ 7. Concluding remarks

General discusions have been given in order to investigate the direct K-coupling and $KK\pi\pi$ interaction. It turns out to be convenient to use Eq. (5·2) and (5·4) for obtaining information about λ_2. It should be noted here that we have to calculate the contribution coming from the two-pion state in the isovector part of the electric form factor,

$$G_{1(2\pi)}^V(0)=\frac{1}{\pi}\int_{4\mu^2}dt'\,\frac{g_{1(2\pi)}^V(t')}{t'}. \quad (7\cdot1)$$

This value gives $0.67\,e$ in the Born approximation (see Fig. 11). But a reliable value coming from the (3, 3) isobar state has not yet been obtained because of its unreliability. The function $f(t)/t$ in Eq. (4·32) is also plotted in Fig. 10. When we investigate the P and D wave threshold behavior, we need, moreover, the mean square radius of the anomalous magnetic moment and charge distributions due to the two-pion intermediate state, which was calculated by Y. Fujii.[20]

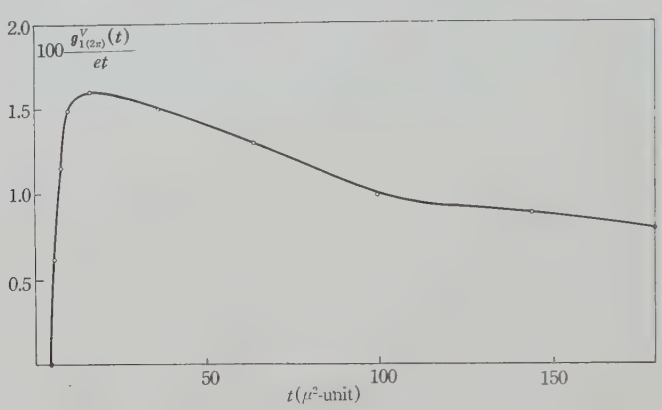


Fig. 11. Spectral function $g^V_{1(2\pi)}(t)$, arising from H_2, devided by t: the contribution from the Born approximation term.

If we get the K-N-Λ, K-N-Σ coupling constants, λ_1 and λ_2 from the threshold behavior for the S and P wave phase shifts, we can predict the energy dependence of the S, P and D phase shifts by making use of Eqs. (3·14) (3·19), (3·22) and (3·23).

It should be remarked that in order to determine the coupling types independently the $K^{\pm}$-p, $K^{\pm}$-n elastic scattering amplitudes have been used here, but a similar discussion can be made by making use of the K-p elastic and charge exchange amplitudes.[17]

We hope that experiments will be carried out to get the S- and P-wave cross sections separately for the K-nucleon, $\bar{K}$-nucleon and absorption processes, to get the threshold behavior of the S- and P-wave phase shifts, and to choose a suitable solution from the Dalitz solutions as soon as possible.

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Low-Energy Limit Theorems and Dispersion Relations

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Low-energy limit behaviour of various collision processes is re-investigated from the dispersion theoretic point of view, which turns out to be a powerful approach for obtaining low-energy limit theorems. Some low-energy limit theorems for the processes involving collisions of a boson with a fermion are obtained, using this method.

It is shown that for pion-baryon scattering, certain isotopic spin combinations of the P-wave spin-flip amplitudes satisfy simple low-energy limit theorems. In particular, for pion-nucleon scatering, isotopic even part of the P-wave spin-flip amplitude is shown to be simply related to the renormalized pion-nucleon coupling constant, from which the magnitude of the coupling constant can be determined:

$$f^2/4\pi = 0.09 \pm 0.02.$$

It is also shown that this determination is insensitive to the presence of (3, 3) resonance and a possible $T=J=1$ pion-pion resonance.

§ 1. Introduction

Thirring[1] was the first to treat in full generality the low-energy limit behaviour——the zero frequency limit behaviour——of the Compton scattering amplitude. His result was later generalized by Low[2] and Gell-Mann and Goldberger[3] who proved that the amplitude for Compton scattering of the nucleon is completely expressed up to terms of first order in the frequency of the photon by the Born approximation computed for a nucleon with renormalized charge and Pauli anomalous moment, interacting with arbitrary local and renormalizable fields. An analogous theorem for S-wave photo-pion production was proved by Kroll and Ruderman,[4] which states that "the matrix element for charged photomeson production at threshold, correct to all orders in the meson coupling constant, approaches the weak-coupling result in the limit of vanishing meson mass, provided that the meson coupling constant and nucleon mass are replaced by their renormalized values as conventionally defined".

This theorem was applied to the determination of the pion-nucleon coupling constant.

On the other hand, for pion-nucleon scattering, it is known that there is some ambiguity in the renormalization procedure of the coupling constant[5] and the low-energy limit theorem for this case does not hold if we take the Watson-Lepore renormalized coupling constant.

The general procedure for obtaining low-energy limit theorems, including the above-mentioned cases, is investigated in detail by Klein[6] and Schwinger[7] using the Green function method and it is shown that two additional para-

meters are needed there to determine the low-energy behaviour of the amplitude for pion-nucleon scattering. It seems very natural, at first sight, that the scattering amplitude is expressed correctly, in the low-energy limit, by the Born approximation with renormalized (observable) quantities. So it is necessary to investigate the condition under which the low-energy limit theorem holds or not. In § 2, we shall study the derivation of this theorem based on dispersion theoretic approach which turns out to give us a simple picture about the contents of the low-energy limit theorem. In § 3, low-energy limit behaviour of the amplitude for pion-baryon scattering is studied and it is shown that isotopic even part of the P-wave spin-flip amplitude of the pion-nucleon scattering satisfies a simple low-energy limit theorem, from which the magnitude of the pion-nucleon coupling constant can be determined, using low-energy P-wave phase-shifts data.[12] The result well agrees with the values so far obtained.

For pion-Λ-hyperon scattering, it is necessary to introduce one parameter to establish the theorem for S- and P-wave scattering, respectively. It can be shown that P-wave spin-flip amplitude is suppressed compared with the corresponding pion-nucleon P-wave scattering amplitude on account of the mass difference between m_Λ and m_Σ and can be expressed, in the low-energy limit, in terms of the S-wave scattering length.

It is also shown that we require, for pion-Σ-hyperon scattering, three additional parameters to determine the low-energy limit amplitudes but P-wave spin-flip amplitude can be determined by the renormalized coupling constant $g_{\Sigma\Sigma}/4\pi$ alone.

In § 4, low-energy behaviour of the amplitude for photo-pion production is investigated based on the analysis made in § 2 and we have exactly the same result with those given by Kroll and Ruderman and Klein.

The final section is devoted to the discussion of the derivation and the contents of the low-energy limit theorems and related problems.

§ 2. General consideration

We shall consider first the scattering of a boson, mass μ with a fermion, mass m. Let the four-momenta of the incident and the outgoing boson be q and q', respectively, while those of the initial and the final fermion are p and p', respectively (Fig. 1). Moreover, let $M(s,t)$ be one of the invariant scattering amplitudes and be assumed to satisfy the following dispersion relation in the neighborhood of zero-momentum transfer $t=0$.[8]

$$M(s,t)=\sum_{i=1}^{n}\frac{c_i}{m_i^2-s}+\int_{(m+\mu)^2}^{\infty}\frac{\rho(s',t)}{s'-s}ds', \qquad (2\cdot1)$$

where $s=-(p+q)^2$, total energy in the centre-of-mass system

and $t=+(q-q')$, momentum transfer.

The first term of the right-hand side of Eq. (2·1) represents the contributions from discrete states and $c_i(i=1,\cdots,n)$ are some constants independent of s and t. Their positions m_i^2, which are the square of the intermediate fermion mass and may in general be different from the square of the initial fermion mass m^2, naturally lie in the unphysical energy region on account of the stability condition. The second term is the contribution from continuous spectrum, physical regions in the case of the forward scattering.

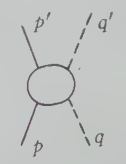


Fig. 1. Scattering of a boson with a fermion.

Now, we turn our attention to the limit of the vanishing external boson mass $\eta(\equiv\mu^2/m^2)\to 0$. For this purpose, it is convenient to rewrite Eq. (2·1) as follows, changing the variable s into ξ,

$$M'(\xi,t)=\sum_{i=1}^{n}\frac{c_i}{\xi_i-\xi}+\int_1^\infty\frac{\rho'(\xi',t)}{\xi'-\xi}d\xi', \qquad (2\cdot2)$$

where $s\equiv(m+\mu)^2\xi$ and $m_i^2\equiv(m+\mu)^2\xi_i$.

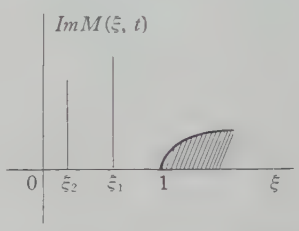


Fig. 2. Structure of the absorptive part of the scattering amplitude. The threshold corresponds to $\xi=1$.

The structure of the imaginary part of $M'(\xi,t)$, as is easily seen from Eq. (2·2), is graphically shown in Fig. 2. In the limit of $\eta=0$, positions of poles ξ_i moves as follows,

$$\lim_{\eta\to 0}\xi_i\to\xi_i^0\ (=m_i^2/m^2). \qquad (2\cdot3)$$

From (2·3), we see that in the limit of $\eta=0$ the particular pole, position of which is the same as the initial fermion mass $m_i^2=m^2$ and lies in the unphysical energy region, moves to attach the continuous spectrum in the physical region for the case of the forward scattering. These situations are illustrated in Figs. 2 and 3.

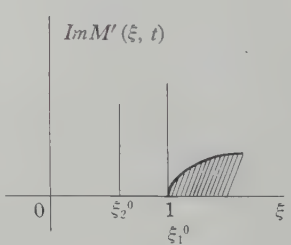


Fig. 3. Change of the structure of the absorptive part in the limit of the vanishing boson mass. The pole ξ_1, moves to attach to the physical region, while other pole remains in the unphysical region.

So, in this limiting case, the low-energy scattering amplitude $M'(\xi\approx 1, t\approx 0)$ is completely dominated by the contribution from the pole attached to the physical region and can be determined by the residue of its pole. This is the content of the so-called low-energy limit theorem when seen from the dispersion relation approach, which is nothing but an application of the idea of "pology". From this point of view, it is clear that the Compton scattering is a special case in which we do not need the limiting procedure

in order to get the low-energy theorem.

In practice, however, we must expand the scattering amplitude with respect to a properly chosen complete set which satisfies general invariance requirements and assume for each of its coefficients——invariant amplitudes——the dispersion relation like Eq. (2·1). Then, the scattering amplitude is determined, in the low-energy limit, by the contribution from a particular pole which attaches to the physical region, only if contribution from the pole appears in all invariant amplitudes, and in this case low-energy limit theorem holds. If, on the other hand, the contribution from the pole does not appear in some invariant amplitudes, the low-energy limit theorem does not in general hold and we must leave those amplitudes as free parameters to determine low-energy forms of the scattering amplitude. The former is the case of Compton scattering and photo-production of pion, while the latter corresponds to the case of the pion-baryon scattering.

We have so far assumed non-subtracted dispersion relation in Eq. (2·1) but this assumption is not essential for the derivation of the low-energy limit theorem, as is clarified in the final section.

§ 3. Pion-baryon scattering

3. 1 Pion-nucleon scattering

Dispersion theoretic approach to the low-energy pion-nucleon scattering has already been made by Chew et al.[9]

We follow their analyses, devoting our attention to the low-energy limit behaviour of the scattering amplitude as considered in § 2. We shall begin with reproducing their kinematical analysis for convenience. Let q, $i(i=1, 2, 3)$ be the state of the incident pion and q', j that of the outgoing one and p and p' are the initial and final nucleon momenta, respectively (Fig. 4). Then, the S-matrix element can be written as*

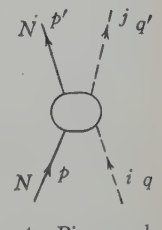


Fig. 4. Pion-nucleon scattering. The isotopic spin of the incident and outgoing pions are denoted by i and j, respectively.

$$S=\delta_{fi}-(2\pi)^4\delta(p'+q'-p-q)\left(\frac{m^2}{4E'E\omega'\omega}\right)^{1/2}\bar{u}(p')Tu(p). \tag{3·1}$$

The scattering amplitude T is decomposed, explicitly writing the isotopic spin dependence, as follows:

$$T=(-A^+ +i\gamma\cdot QB^+)\delta_{fi}+(-A^- +i\gamma\cdot QB^-)\frac{1}{2}[\tau_f, \tau_i] \tag{3·2}$$

where Q is given by

* In this paper four-dimensional inner product is defined according to $p\cdot q=\boldsymbol{p}\cdot\boldsymbol{q}-p_0q_0$, and spinor normalization is $\bar{u}(p)u(p)=1$.

$$Q=\frac{1}{2}(q+q').$$

$A^{\pm}$ and $B^{\pm}$,* called invariant amplitudes in § 2, are functions of s and t and are proved to satisfy the following dispersion relation in the neighborhood of $t=0$.[9]

$$\begin{aligned}
A^{\pm}(s,t)&=\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\rho_{1N}^{A\pm}(s',t)}{s'-s}ds'+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\rho_{2N}^{A\pm}(s_c',t)}{s_c'-s_c}ds_c',\\
B^{\pm}(s,t)&=g^2\left(\frac{1}{m^2-s}\mp\frac{1}{m^2-s_c}\right)\\
&+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\rho_{1N}^{B\pm}(s',t)}{s'-s}ds'+\frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{\rho_{2N}^{B\pm}(s_c',t)}{s_c'-s_c}ds_c',
\end{aligned}\tag{3·3}$$

where $s=-(p+q)^2$, $s_c=-(p-q')^2$.

If we define physical scattering amplitudes $f_1^{\pm}$ and $f_2^{\pm}$ according to

$$\begin{aligned}
f_1^{\pm}&=\left(\frac{E+m}{2W}\right)\left(\frac{A^{\pm}+(W-m)B^{\pm}}{4\pi}\right),\\
f_2^{\pm}&=\left(\frac{E-m}{2W}\right)\left(\frac{-A^{\pm}+(W+m)B^{\pm}}{4\pi}\right),
\end{aligned}\tag{3·4}$$

where E is the total nucleon energy and $W=E+\omega$, in the centre-of-mass systems.

Then, $f_1^{\pm}$ and $f_2^{\pm}$ are simply related to the phase-shift of the scattering:

$$\begin{aligned}
f_1^{\pm}&=\sum_{l=0}^{\infty}f_{l+}^{\pm}P_{l+1}'(\cos\theta)-\sum_{l=2}^{\infty}f_{l-}^{\pm}P_{l-1}'(\cos\theta),\\
f_2^{\pm}&=\sum_{l=1}^{\infty}(f_{l-}^{\pm}-f_{l+}^{\pm})P_l'(\cos\theta),
\end{aligned}\tag{3·5}$$

where $f_{l\pm}$ is the scattering amplitude in the state of parity $-(-1)^l$ and total angular momentum $j=l\pm 1/2$.

It is first to be noticed that the contribution from the pole does not appear in the amplitudes $A^{\pm}$, the fact essentially dependent on the spin and parity of the pion. In the limit of the vanishing external pion mass we have from Eq. (3·3) low-energy forms of $A(s,0)$ and $B(s,0)$, expanding them in power series of the pion energy, as follows:**

---

* $A^{\pm}$ and $B^{\pm}$ can be expressed in terms of the total isotopic spin as follows:

$$A^{+}=(1/3)(A^{1/2}+2A^{3/2}),\quad A^{-}=(1/3)(A^{1/2}-A^{3/2}),\text{ etc.}$$

** Note that from crossing symmetry, $A^{+}(\omega)$ and $B^{-}(\omega)$ are shown to be even functions of ω, while $A^{-}(\omega)$ and $B^{+}(\omega)$ are odd functions of ω.

$$
\begin{aligned}
A^+(\omega) &= \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} -\frac{\rho_{1N}^{A+}}{s'-m^2}\,ds' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\rho_{2N}^{A+}}{s_c'-m^2}\,ds_c' + O(\omega^2),\\
A^-(\omega) &= 0 + O(\omega),\\
B^+(\omega) &= -\frac{g^2}{m}\frac{1}{\omega} + O(\omega),\\
B^-(\omega) &= \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\rho_{1N}^{B-}}{s'-m^2}\,ds' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{\rho_{2N}^{B-}}{s_c'-m^2}\,ds_c' + O(\omega^2).
\end{aligned}
\tag{3·6}
$$

On the other hand, we have from (3·4), in the same low-energy region,

$$
\begin{aligned}
4\pi f_1^{\pm} &\cong \left(1-\frac{\omega}{m}\right)\{A^{\pm}(\omega)+\omega B^{\pm}(\omega)\},\\
4\pi f_2^{\pm} &\cong \frac{q^2}{4m^2}\left(1-\frac{\omega}{m}\right)\left\{-A^{\pm}(\omega)+2m\left(1+\frac{\omega}{2m}\right)B^{\pm}(\omega)\right\}.
\end{aligned}
\tag{3·7}
$$

Inserting (3·6) into (3·7), we obtain

$$
\begin{aligned}
4\pi f_1^+ &\cong \left(1-\frac{\omega}{m}\right)\left\{A^+(0)-\frac{g^2}{m}+O(\omega^2)\right\},\\
4\pi f_1^- &\cong \left(1-\frac{\omega}{m}\right)\{A^-(\omega)+\omega B^-(0)+O(\omega^3)\},\\
4\pi f_2^+ &\cong -\frac{g^2}{2m^2}\frac{q^2}{\omega}\left(1-\frac{\omega}{m}\right)\left(1+\frac{\omega}{2g^2}A^+(0)+\frac{\omega}{2m}\right)+O(\omega q^2),\\
4\pi f_2^- &\cong 0+O(\omega^2).
\end{aligned}
\tag{3·8}
$$

Eq. (3·8) shows that we require two independent amplitudes $A^+(0)$ and $A^-(\omega)+\omega B^-(0)$ to determine the low-energy behaviour of the pion-nucleon scattering amplitude, which correspond to two parameters introduced by Klein. We know that in order to have an agreement with experiment on S-wave scattering, $A^+(0)$ must be positive and almost cancel out the second negative (repulsive) term $-g^2/m$ in the right-hand side of f_1^+. Numerically we require

$$
\frac{1}{4\pi}A^+(0) \approx \left(\frac{g^2}{4\pi}\frac{1}{m}+\frac{a^+}{4\pi}\right) \approx 2.2\,\frac{1}{\mu}\,,
\tag{3·9}
$$

where a^+ is the S-wave scattering length.*

---

* Numerical values of S-wave scattering lengths

$$a^+ = (1/3)(a_1+2a_3),$$
$$a^- = (1/3)(a_1-a_3),$$

are given, according to the recent analysis,[10] as follows,

$$
\begin{aligned}
a^+ &= 0.001 \pm 0.003,\\
a^- &= 0.088 \pm 0.003.
\end{aligned}
\qquad (\text{in unit of } \mu^{-1})
$$

This value is very large and one of the difficulties in handling the S-wave problem seems to lie on this point. We also see that the amplitude f_2^+, the spin-flip amplitude as is seen from (3·5), can be written in terms of the eigenstate phase-shifts:

$$\delta_{11}+2\delta_{31}-\delta_{13}-2\delta_{33}\cong-\frac{3}{2}\left(\frac{g^2}{4\pi}\right)\frac{1}{m^2}\frac{q^3}{\omega}\left(1-\frac{\omega}{m}\right)\left(1+\frac{\omega}{2q^2}A^+(0)+\frac{\omega}{2m}\right), \tag{3·10}$$

where we have neglected the contribution from higher wave ($l\geq 2$) amplitudes and set approximately the amplitudes f_{TJ} as follows:

$$f_{TJ}=e^{i\delta_{TJ}}\sin\delta_{TJ}/q\approx\frac{1}{q}\delta_{TJ}. \tag{3·11}$$

Inserting (3·9) into (3·10) and neglecting the contribution of a^+, we obtain

$$\delta_{11}+2\delta_{31}-\delta_{13}-2\delta_{33}\cong-6\left(\frac{f^2}{4\pi}\right)\frac{q^3}{\omega}(1+O(\omega^2)). \tag{3·12}$$

Now, we can estimate from (3·12) the magnitude of the pion-nucleon coupling constant, using low-energy P-wave phase-shift data which are summarized in Table I.

Table I. Low-energy P-wave phase-shifts data

Phase-shifts	1958 CERN Conf.[11]	1960 Rochester Conf.[12]
δ_{33}/η^3	0.234±0.019	0.234±0.019
δ_{31}/η^3	−0.048±0.068	−0.032±0.022
δ_{13}/η^3	−0.055±0.062	−0.044+0.005
δ_{11}/η^3	−0.016±0.11	0.038±0.038
$\delta_{11}+2\delta_{31}-\delta_{13}-2\delta_{33}$	−0.527	−0.526

From Table I and Eq. (3·12), we obtain the following result,*

$$f^2/4\pi=0.09\pm0.02.$$

This result agrees well with values estimated from many other methods.[13] In this connection, it will be worthwhile to touch upon the effective-range approach. According to the effective range formula,[9] P-wave phase-shifts are given:

$$\tan\delta_\alpha=\lambda_\alpha\frac{q^3}{\omega}(1+\omega r_\alpha), \tag{3·13}$$

* We use the data from the Rochester Conference (1960).

where

$$\lambda_{11}=-\frac{8}{3}\frac{f^2}{4\pi},\quad \lambda_{33}=\frac{4}{3}\frac{f^2}{4\pi},\quad \lambda_{13}=\lambda_{31}=-\frac{2}{3}\frac{f^2}{4\pi},$$

$$r_{33}\cong\frac{1}{\omega_r},\quad r_{11}\cong r_{13}=r_{31}\cong -r_{33}, \tag{3·14}$$

ω_r: resonance energy.

It is to be noticed that resonance energy which appears in the effective range r_α (3·14) is completely cancelled in accordance with (3·12) when we make the combination such as (3·12), that is,

$$\lambda_{11}r_{11}+2\lambda_{31}r_{31}-\lambda_{13}r_{13}-2\lambda_{33}r_{33}\approx 0 \quad \text{(independent of } \omega_r\text{)}. \tag{3·15}$$

Thus, our method for the determination of the coupling constant is almost unaffected by the presence of the (3, 3) resonance. The same is true also for the effect of $T=J=1$ pion-pion resonance which is easily shown to contribute only to the odd amplitude of the isotopic spin.

It should also be noticed that good agreement can be achieved only when the isotopic spin even part of the S-wave scattering length a^+ is negligible. Hence, we see that the smallness of the S-wave scattering length a^+ is consistent with the low-energy P-wave phase-shifts data.

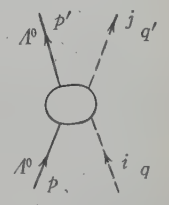


Fig. 5. Pion-Λ-hyperon scattering.

3. 2 Pion-Λ-hyperon scattering

Kinematics is the same as the case of the pion-nucleon scattering except for the isotopic spin dependence, so we shall not repeat it here.* The result is summarized in the form

$$T=(-A+i\vec{\tau}QB)\delta_{fi}, \tag{3·16}$$

$$A(s,t)=(m_\Lambda-m_\Sigma)g_{\Sigma\Lambda}^2\left\{\frac{1}{m_\Sigma^2-s}+\frac{1}{m_\Sigma^2-s_c}\right\}$$

$$+\frac{1}{\pi}\int_{(m_\Lambda+\mu)^2}^{\infty}\frac{\rho_{1\Lambda}^A(s',t)}{s'-s}ds'+\frac{1}{\pi}\int_{(m_\Lambda+\mu)^2}^{\infty}\frac{\rho_{2\Lambda}^A(s_c',t)}{s_c'-s_c}ds_c',$$

$$B(s,t)=g_{\Sigma\Lambda}^2\left\{\frac{1}{m_\Sigma^2-s}-\frac{1}{m_\Sigma^2-s_c}\right\} \tag{3·17}$$

$$+\frac{1}{\pi}\int_{(m_\Lambda+\mu)^2}^{\infty}\frac{\rho_{1\Lambda}^B(s',t)}{s'-s}ds'+\frac{1}{\pi}\int_{(m_\Lambda+\mu)^2}^{\infty}\frac{\rho_{2\Lambda}^B(s_c',t)}{s_c'-s_c}ds_c',$$

* Notations in this and following subsections are the same as the treatment of the pion-nucleon scattering as far as no confusion occurs.

where $g_{\Sigma\Lambda}^2$ is the renormalized pion-hyperon coupling constant and m_Λ and m_Σ represent masses of Λ and Σ hyperons, respectively.

The first difference from the case for pion-nucleon scattering appears in the contribution of the Born term to the amplitude A. Owing to the mass difference between m_Λ and m_Σ, this contribution does not disappear. The second difference is that we nevertheless require the whole amplitude $A(0)$ to determine the low-energy form since the Born term and the remaining term both contribute in the same order to the scattering amplitude even in the low-energy limit. This is again due to the mass difference m_Λ and m_Σ and the above analysis suggests us that mass difference plays a rather important role in the treatment of the low-energy pion-hyperon scattering.

Following exactly the same procedures as in the pion-nucleon scattering, we find

$$\begin{aligned} 4\pi f_1 &= A(0)+O(\omega), \\ 4\pi f_2 &= \frac{q^2}{4m_\Lambda^2}\{-A(0)+O(\omega)\}. \end{aligned} \tag{3·18}$$

From (3·18) the S-wave scattering length can be determined as

$$a=\frac{1}{4\pi}A(0) \tag{3·19}$$

and using this scattering length a, P-wave spin-flip amplitude f_2 can be determined:

$$\delta_{P1/2}-\delta_{P3/2}\simeq -a\frac{q^3}{4m_\Lambda^2}(1+O(\omega)). \tag{3·20}$$

It should be noticed that the P-wave spin-flip amplitude is suppressed compared with the corresponding pion-nucleon scattering amplitude. (See Eq. (3·12).)

3.3 Pion-Σ-hyperon scattering

Kinematical considerations are made in Appendix A. As is seen from (A·6), there are two kinds of pole contributions ——one from a Λ-hyperon intermediate state and the other from a Σ-hyperon intermediate state. In the low-energy limit we have from (A·6)

$$\begin{aligned} A^0 &= -\frac{2g_{\Sigma\Lambda}^2}{m_\Lambda+m_\Sigma}+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{A0}}{s'-m_\Sigma^2}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{A0}}{s_c'-m_\Sigma^2}ds_c'+O(\omega^2), \\ A^+ &= +\frac{2g_{\Sigma\Lambda}^2}{m_\Lambda+m_\Sigma}+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{A+}}{s'-m_\Sigma^2}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{A+}}{s_c'-m_\Sigma^2}ds_c'+O(\omega^2), \\ A^- &= 0+O(\omega), \\ B^0 &= 0+O(\omega), \end{aligned} \tag{3·21}$$

$$B^+=-\frac{g_{\Sigma\Sigma}^2}{m_\Sigma}\frac{1}{\omega}+O(\omega),$$

$$B^-=-\frac{2g_{\Sigma\Lambda}^2}{m_\Sigma^2-m_\Lambda^2}+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{B-}}{s'-m_\Sigma^2}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{B-}}{s_c'-s_c}ds_c'+O(\omega^2).$$

Inserting (3·21) into(A·5), we obtain

$$\begin{aligned}
&4\pi f_1^0=A^0(0)+O(\omega),\\
&4\pi f_1^+=A^+(0)-\frac{1}{m_\Sigma}g_{\Sigma\Sigma}^2+O(\omega),\\
&4\pi f_1=0+O(\omega),\\
&4\pi f_2^0=\frac{q^2}{4m_\Sigma^2}\{-A^0(0)+O(\omega)\},\\
&4\pi f_2^+=\frac{q^2}{4m_\Sigma^2}\left\{-A^+(0)-\frac{2}{\omega}g_{\Sigma\Sigma}^2+O(\omega)\right\},\\
&4\pi f_2^-=\frac{q^2}{4m_\Sigma^2}\{+2m_\Sigma B^-(0)+O(\omega)\}.
\end{aligned}\qquad(3\cdot22)$$

From (3·22), we see that there two independent amplitudes A^+ and A^0 are necessary to determine the S-wave scattering amplitude* and for P-wave spin-flip amplitude we have the following low-energy limit theorem,

$$\delta_{P1/2}^+-\delta_{P3/2}^+\cong-\frac{1}{2m_\Sigma^2}\left(\frac{g_{\Sigma\Sigma}^2}{4\pi}\right)\frac{q^2}{\omega}(1+O(\omega)).\quad(3\cdot23)$$

Eq. (3·23) can be applied to the determination of the coupling constant $g_{\Sigma\Sigma}^2/4\pi$ if low-energy P-wave phase-shifts are known. On the other hand, we have no analogous theorem which determines another coupling constant $g_{\Sigma\Lambda}^2/4\pi$. This is because even in the limit of the vanishing pion mass, the pole from a Λ-hyperon intermediate state cannot attach to the physical region on account of the mass difference between m_Λ and m_Σ. (Fig. 6)

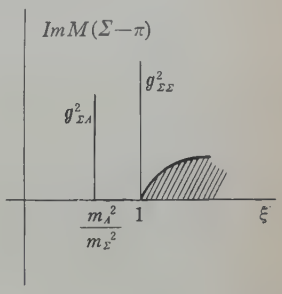


Fig. 6. Change of the structure of the absorptive part of the pion-Σ-hyperon scattering amplitude in the limit of the vanishing pion mass. The pole corresponding to a Σ-hyperon intermediate state attaches to the physical region.

§ 4. Photo-pion production

We shall consider the following three reactions,

* This is equivalent to the introduction of the following Hamiltonian for the S-wave scattering,[14]

$$H=\lambda_1(\bar{\psi}_\Sigma\cdot\psi_\Sigma)(\varphi\cdot\varphi)+\lambda_2(\bar{\psi}_\Sigma\cdot\varphi)(\psi_\Sigma\cdot\varphi).$$

$$\begin{aligned} \gamma+p &\to n+\pi^+ \quad (+), \\ &\to p+\pi^0 \quad (0), \\ \gamma+n &\to p+\pi^- \quad (-). \end{aligned} \tag{4·1}$$

For S-wave pion production, which case is well known as the Kroll-Ruderman theorem, we know that amplitudes of reactions (4·1) can be expressed in the low-energy limit,[4]

$$\begin{aligned} T_+^s &= \sqrt{2}\,\frac{eg}{2m} i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}, \\ T_0^s &= 0, \\ T_-^s &= \sqrt{2}\,\frac{eg}{2m} i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}. \end{aligned} \tag{4·2}$$

For P-wave pion production,[6] we have

$$\begin{aligned} T_+^p &= \sqrt{2}\,\frac{g}{2m}\left\{\mu_n \frac{\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\boldsymbol{\sigma}\cdot\boldsymbol{q}}{\omega} - \left(\frac{e}{2m}+\mu_p\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})}{k}\right\}, \\ T_0^p &= \frac{g}{2m}\left(\frac{e}{2m}+\mu_p\right)\left\{\frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})}{k} - \frac{\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\boldsymbol{\sigma}\cdot\boldsymbol{q}}{\omega}\right\}, \\ T_-^p &= \sqrt{2}\,\frac{g}{2m}\left\{\mu_n \frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})}{k} - \left(\frac{e}{2m}+\mu_p\right)\frac{\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})\boldsymbol{\sigma}\cdot\boldsymbol{q}}{\omega}\right\}, \end{aligned} \tag{4·3}$$

where $\boldsymbol{k}$ and $\boldsymbol{q}$ are momenta of the incoming photon and outgoing pion, respectively, $\boldsymbol{\varepsilon}$ is the polarization of the photon and $\mu_{p,n}$ represent anomalous magnetic moments of the proton and neutron, respectively.

In the following we derive the above results based on the analysis in § 2. For this purpose we first expand the total amplitude into invariant amplitudes with a properly chosen invariant set, which has already been given by Chew et al.[15] Making use of their results, the complete photo-pion production matrix element may be written,

$$M = M_A A + M_B B + M_C C + M_D D, \tag{4·4}$$

where quantities A, B, C and D are invariant amplitudes and functions of s and t as well as the nucleon isotopic spin. Furthermore M_A, M_B, M_C and M_D represent four independent basic matrices, which satisfy covariance and gauge invariance, and are given as follows,

$$\begin{aligned} M_A &= i\gamma_5\gamma\cdot\varepsilon\gamma k, \\ M_B &= 2i\gamma_5(P\varepsilon\cdot qk - Pk\cdot q\varepsilon), \\ M_C &= \gamma_5(\gamma\varepsilon\cdot qk - \gamma\cdot k q\varepsilon), \\ M_D &= 2\gamma_5(\gamma\varepsilon\cdot Pk - \gamma k\cdot P\varepsilon - im\gamma\cdot\varepsilon\cdot\gamma k), \end{aligned} \tag{4·5}$$

where $2P=p+p'$ (Fig. 7).

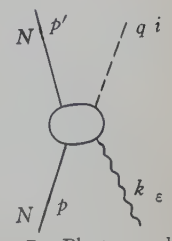


Fig. 7. Photo-production of pion.

Invariant amplitudes A, B, C and D are further decomposed, explicitly writing isotopic spin dependence, as follows,*

$$A=A^{+}I^{+}+A^{-}I^{-}+A^{0}I^{0},$$

etc.

where

$$I^{+}=\delta_{i3},$$

$$I^{-}=\frac{1}{2}[\tau_i,\tau_3], \tag{4·6}$$

$$I^{0}=\tau_i.$$

The 12 invariant amplitudes $A^{\pm 0}$, $B^{\pm 0}$, $C^{\pm 0}$ and $D^{\pm 0}$ are proved to satisfy dispersion relations for low momentum transfer.[16]

Investigating their pole contributions, it is easily seen that each amplitude has the pole contribution, the residue of which can be expressed in terms of the renormalized charge, anomalous magnetic moment of the nucleon and pion-nucleon coupling constant. So we find in this case that the low-energy theorem holds. In fact, in the low-energy limit, the whole amplitude up to linear of the pion momentum can be expressed:

$$\begin{aligned}T=&\begin{Bmatrix}0\\0\\1\end{Bmatrix}\frac{eg}{2m}i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+\begin{Bmatrix}1\\1\\-1\end{Bmatrix}\frac{g}{2m}\left(\frac{e}{2m}+\mu_r\right)\frac{i(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})(\boldsymbol{k}\cdot\boldsymbol{q})}{k}\\&+\begin{Bmatrix}0\\0\\-1\end{Bmatrix}\frac{g}{2m}\left(\frac{e}{2m}+\mu_r\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{q}\boldsymbol{\sigma}(\boldsymbol{k}\times\boldsymbol{\varepsilon})}{k}+\begin{Bmatrix}0\\0\\1\end{Bmatrix}\frac{eg}{2m}\frac{1}{k\omega-\boldsymbol{k}\cdot\boldsymbol{q}}\{\boldsymbol{\sigma}(\boldsymbol{k}-\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{\varepsilon}\}\\&+\begin{Bmatrix}-1\\-1\\1\end{Bmatrix}\frac{g}{2m}\left(\frac{e}{2m}+\mu_r\right)\frac{i(\boldsymbol{\sigma}\cdot\boldsymbol{k})(\boldsymbol{q}\cdot\boldsymbol{\varepsilon})}{k},\end{aligned} \tag{4·7}$$

where parentheses indicate the cases (+), (0) and (−) of (4·6), respectively and μ_r stands for $\mu_p-\mu_n$ in the case of (±) and $\mu_p+\mu_n$ in the case of (0). The first term of (4·7) represents the S-wave production and the second, the third and the last terms contribute to the P-wave production, while the fourth term which is called the retardation term, represents the contribution from pion current and contains higher wave contribution. It should be noticed that the appearance of the retardation term in the pole contribution is due to the gauge invariance. The results given in (4·2) and (4·3) immediately follow from (4·7).

* Note that the superscript (±0) bears no simple relation to the charge of the pion being produced, in contrast to (4·2) and (4·3).

§ 5. Discussions and concluding remarks

In the previous sections we have explored the contents of the so-called low-energy limit theorems of various collision processes based on dispersion relations, which makes it clear that the low-energy limit is nothing but the limit in which a pole attaches to the physical region and the pole contribution dominates the whole amplitude. In the course of the discussion, however, we have assumed the validity of dispersion relations but in order to establish the theorem, it is enough to consider them in the neighborhood of the momentum transfer $t=0$, in which case it is possible to prove them based on general principles of the quantum field theory.[8]

In order to justify the limiting procedure of $\eta\to 0$, we must know explicitly the dependence of the scattering amplitude. This is accomplished by the method of the integral representation of the scattering amplitude.[18]

The amplitude of the boson-fermion scattering, according to Lehmann, can be written as follws,

$$M(s,t,\eta)=\frac{1}{8\pi^2K(\eta)}\int du_0u_i\,du_i\,d\kappa_i^2\int_0^{2\pi}d\alpha\int_0^{\pi}d\beta_1\int_0^{\pi}d\beta_2\int_0^{2\pi}d\chi$$
$$\times\frac{\Phi(u_0,\,u_i^2,\,\kappa_i^2,\,\cos\alpha\sin\beta_1\sin\beta_2+\cos\beta_1\cos\beta_2,\,s)}{[x_1(\eta)-\cos(\theta-\chi)][x_2(\eta)-\cos(\chi-\alpha)]}, \qquad (5\cdot1)$$

where

$$x_i(\eta)=\frac{K^2(\eta)+u_i^2+\kappa_i^2-\{(1-\eta)m^2/2\sqrt{s}+u_{i0}\}^2}{2K(\eta)u_i\sin\beta_i},$$

$$K^2(\eta)=\frac{\{s+m^2(1-\eta)\}^2-4m^2s}{4s},$$

θ: scattering angle in the centre-of-mass system.

From (5·1), the analytic property of the scattering amplitude in η can be derived, i.e., the scattering amplitude $M(s,t,\eta)$ is an analytic function of η regular for $\mathrm{Re}\,\eta\leq\mu^2/m$ in the neighborhood of the real axis, provided $t\approx 0$. The limiting procedure of $\eta\to 0$ in the case of the forward scattering can, therefore, be justified.

Moreover, in the limit of $\eta=t=0$, the scattering amplitude is always proved to satisfy the dispersion relation even if Eq. (2·1) cannot be proved for $\eta\neq 0$. This shows that the derivation of the low-energy limit theorem in the previous sections may be applied also to the case in which the validity of the dispersion relation in general cannot be proved.

Next, we investigate how the type of the interaction affects the low-energy limit theorem. This problem is closely related to the subtraction problem in dispersion theory. Let us consider the case of pion-nucleon scattering. In this

case it is easily shown that use of subtracted dispersion relations does not alter the conclusion in § 3, that is, low-energy theorems depend only on the residue of the pole. Hence the type of interaction does not affect the validity of the low-energy limit theorems; for example, in the case of *ps-pv* coupling, we have the following expressions of the Born term instead of those in § 3.1:

$$\begin{aligned} A^{+}_{Born} &= \frac{g^2}{m}, \\ A^{-}_{Born} &= 0, \\ B^{+}_{Born} &= g^2\left\{\frac{1}{m^2-s}-\frac{1}{m^2-s_c}\right\}, \\ B^{-}_{Born} &= -2f^2/\mu^2+g^2\left\{\frac{1}{m^2-s}+\frac{1}{m^2-s_c}\right\}, \end{aligned} \tag{5·2}$$

where $f^2=\frac{\mu^2}{4m^2}g^2$.

The only difference from (3·3) is additional constant terms to the amplitudes A^{+} and B^{-} but this difference is completely absorbed in free parameters.* In this sense, low-energy limit theorem is insensitive to the type of the interaction.

The derivation of the theorem mainly depends on the kinematical analysis, as is seen in § 3 and § 4, that is, the content of the low-energy theorem is almost determined by the known masses, spins and parities of the particles which participate in the scattering processes under consideration.

Though, in this paper, we have confined ourselves to the two-body scattering and reaction processes, the present method can be applied to any other inelastic processes, if dispersion relations for those processes can be justified. For example, for double photo-production of *S*-wave pions, it is conjectured based on this analysis that the amplitude can be determined in terms of the renormalized charge, anomalous magnetic moment of the nucleon and the pion-nucleon coupling constant as well as the pion-nucleon *S*-wave scattering lengths.[17]

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Appendix A

Dispersion Relation of Pion-Σ-hyperon Scattering

Scattering amplitude T, which is defined in the same way as in the pion-

* The same argument can be applied to the effect of pion-pion interaction to the scattering amplitude.

nucleon scattering, can be decomposed in the Σ-hyperon isotopic spin space as follows:

$$T=-A+i\gamma QB, \qquad (A\cdot 1)$$

where

$$A=A^0\delta_{ji}+\frac{1}{2}\{t_j, t_i\}A^+ +\frac{1}{2}[t_j, t_i]A^-,$$

etc. $\qquad (A\cdot 2)$

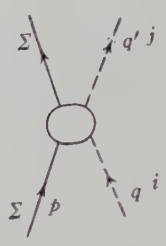


Fig. 8. Pion-Σ-hyperon scattering.

Here $t_i(i=1, 2, 3)$ is the vector component in the Σ-hyperon isotopic spin space and is represented:

$$t_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix},\quad t_2=\begin{pmatrix}0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix},\quad t_3=\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \qquad (A\cdot 3)$$

Invariant amplitudes $A^{\pm 0}$ and $B^{\pm 0}$ defined in (A·1) and (A·2) can be expressed in terms of the eigenamplitudes of the total isotopic spin as follows:

$$\begin{aligned}A^+&=-\frac{1}{6}A^2+\frac{1}{2}A^1-\frac{1}{3}A^0,\\ A^0&=\frac{2}{3}A^2 \qquad\quad +\frac{1}{3}A^0,\\ A^-&=\frac{1}{6}A^2+\frac{1}{2}A^1+\frac{1}{3}A^0,\end{aligned} \qquad (A\cdot 4)$$

etc.

Physical amplitudes $f_1^{\pm 0}$ and $f_2^{\pm 0}$ which are directly related to phase-shifts of the scattering can be expressed in terms of these invariant amplitudes in exactly the same way as the case for pion-nucleon scattering:

$$\begin{aligned}f_1^{\pm 0}&=\left(\frac{E+m_\Sigma}{2W}\right)\left(\frac{A^{\pm 0}+(W-m_\Sigma)B^{\pm 0}}{4\pi}\right),\\ f_2^{\pm 0}&=\left(\frac{E-m_\Sigma}{2W}\right)\left(\frac{-A^{\pm 0}+(W+m_\Sigma)B^{\pm 0}}{4\pi}\right).\end{aligned} \qquad (A\cdot 5)$$

Dispersion relations of pion-Σ-hyperon scattering, after these kinematical considerations, can be written down as follows:

$$A^0(s, t)=(m_\Sigma-m_\Lambda)g_{\Sigma\Lambda}^2\left\{\frac{1}{m_\Lambda^2-s}+\frac{1}{m_\Lambda^2-s_c}\right\}$$

$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{A0}(s', t)}{s'-s}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{A0}(s_c', t)}{s_c'-s_c}ds_c',$$

$$A^+(s,t)=(m_A-m_\Sigma)g_{\Sigma A}^2\left\{\frac{1}{m_A^2-s}+\frac{1}{m_A^2-s_c}\right\}$$
$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{A+}(s',t)}{s'-s}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{A+}(s_c',t)}{s_c'-s_c}ds_c',$$

$$A^-(s,t)=(m_\Sigma-m_A)g_{\Sigma A}^2\left\{\frac{1}{m_A^2-s}-\frac{1}{m_A^2-s_c}\right\}$$
$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{A-}(s',t)}{s'-s}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{A-}(s_c',t)}{s_c'-s_c}ds_c', \qquad (A\cdot6)$$

$$B^0(s,t)=g_{\Sigma A}^2\left\{\frac{1}{m_A^2-s}-\frac{1}{m_A^2-s_c}\right\}$$
$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{B0}(s',t)ds'}{s'-s}+\int_{(m_\Sigma+\mu)}^{\infty}\frac{\rho_{2\Sigma}^{B0}(s_c',t)}{s_c'-s_c}ds_c',$$

$$B^+(s,t)=-g_{\Sigma A}^2\left\{\frac{1}{m_A^2-s}-\frac{1}{m_A^2-s_c}\right\}+g_{\Sigma\Sigma}^2\left\{\frac{1}{m_\Sigma^2-s}-\frac{1}{m_\Sigma^2-s_c}\right\}$$
$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{B+}(s',t)}{s'-s}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{B+}(s_c',t)}{s_c'-s_c}ds_c',$$

$$B^-(s,t)=-g_{\Sigma A}^2\left\{\frac{1}{m_A^2-s}+\frac{1}{m_A^2-s_c}\right\}+g_{\Sigma\Sigma}^2\left\{\frac{1}{m_\Sigma^2-s}+\frac{1}{m_\Sigma^2-s_c}\right\}$$
$$+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{1\Sigma}^{B-}(s',t)}{s'-s}ds'+\int_{(m_\Sigma+\mu)^2}^{\infty}\frac{\rho_{2\Sigma}^{B-}(s_c',t)}{s_c'-s_c}ds_c',$$

where $g_{\Sigma A}^2$ and $g_{\Sigma\Sigma}^2$ are renormalized pion-hyperon coupling constants.

We have, from crossing relations, the following properties of spectral functions:

$$\begin{aligned}
&\rho_{1\Sigma}^{A0+}(s,t)\leftrightarrow\rho_{2\Sigma}^{A0+}(s_c,t),\\
&\rho_{1\Sigma}^{A-}(s,t)\leftrightarrow-\rho_{2\Sigma}^{A-}(s_c,t),\\
&\rho_{1\Sigma}^{B0+}(s,t)\leftrightarrow-\rho_{2\Sigma}^{B0+}(s_c,t),\\
&\rho_{2\Sigma}^{B-}(s,t)\leftrightarrow\rho_{2\Sigma}^{B-}(s_c,t).
\end{aligned}\qquad \text{for } s\leftrightarrow s_c \qquad (A\cdot7)$$

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The Effect of the Pion-Pion Interaction on the Pion-Nucleon Scattering

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It is suggested that the low-energy pion-nucleon p-wave phase shifts will be well understood by taking account of the effects of not only the pion-pion P-wave interaction but also the pion-pion S-wave interaction into the equations for the p-wave scattering amplitudes derived by Chew, Goldberger, Low and Nambu.

§ 1. Introduction

Chew, Goldberger, Low and Nambu (CGLN) have derived the equations for the partial wave amplitudes of the pion-nucleon scattering and have shown that they nearly reproduce the experimental p-wave phase shifts at the low energies, except that the theoretical $|\alpha_{11}|$ is about $3\sim4$ times as large as the experimental one.[1] According to the investigations by many authors,[2] however, the effect of the pion-pion interaction** ($\pi+\pi\rightarrow N+\bar{N}$) has been neglected there. Recently, Bowcock, Cottingham and Lurie[3] have shown that the pion-pion P-wave interaction has the effects to make α_{11} and α_{31} close to each other at the low energies and to make α_{11} larger than α_{31} at the intermediate energies, which are consistent with the experiments.

In this paper, we will show that the introductions of not only the pion-pion P-wave interaction but also the pion-pion S-wave interaction are rather favorable for understanding the pion-nucleon p-wave phase shifts at the low energies. The reason is as follows: The pion-pion S-wave interaction produces an isospin-independent pion-nucleon interaction, then if one assumes it to be attractive, one can make the rather smaller p-wave phase shifts predicted by CGLN plus the correction from the pion-pion P-wave interaction close to the experiments. In § 2 we will reproduce the p-wave phase shifts by the method introduced by CGLN and describe the corrections caused from the possible pion-pion interactions. In § 3 we will discuss how the p-wave phase shifts predicted by CGLN can be made close to the experimental ones by the assumed effects of the pion-pion interactions.

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** This has been naturally introduced in order to explain the electromagnetic structure of nucleon and also has been shown to be very suitable to understand the second and third resonance-like behaviors of the pion-nucleon scattering. See references 4) and 5).

§ 2. Theory of CGLN and the corrections from the pion-pion interactions

We will first reproduce the pion-nucleon p-wave scattering amplitudes, following the procedure given by CGLN. They can be derived from the following equations:

$$f^{(\pm)}_{p3/2}/q^3=\frac{2}{3}\frac{E+M}{2W}\{(A^{(\pm)\prime}(W^2,0)-2q^2A^{(\pm)\prime\prime}(W^2,0))$$
$$+(W-M)(B^{(\pm)\prime}(W^2,0)-2q^2B^{(\pm)\prime\prime}(W^2,0))\}/4\pi, \qquad (2\cdot1)$$

$$(f^{(\pm)}_{p1/2}-f^{(\pm)}_{p3/2})/q^2=\frac{1}{E+M}(B^{(\pm)}(W^2,0)-2q^2B^{(\pm)\prime}(W^2,0))/4\pi, \qquad (2\cdot2)$$

where the superscript $'$ represents the derivative with respect to $t(\equiv-4\kappa^2)$, at $t=0$.*

In (2·2), very small contributions from the pion-nucleon s- and d-waves to the left-hand side have been neglected. Here, we will examine to use the different equation (2·2) from the corresponding one of CGLN, for, if one used this instead of (2·2), it would be necessary to use the dispersion relation for $A^{(+)}(W^2,0)$, which, as to the convergence of the related dispersion integral, is the worst of the $A^{(\pm)}(W^2,0)$, $B^{(\pm)}(W^2,0)$ and these derivatives, and further, extra information about the effect of the pion-pion interaction on the $A^{(\pm)}(W^2,0)$ would later become necessary. By the use of the dispersion relations for $A^{(\pm)}$ and $B^{(\pm)}$ given by CGLN, from (2·1) and (2·2) we have the p-wave scattering amplitudes as follows:

$$\mathrm{Re}f_{33}/q^3=\frac{4}{3}\frac{f^2}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\left(\frac{1+\omega'/M}{1+\omega/M}\right)\left(\frac{E+M}{E'+M}\right)\frac{\mathrm{Im}f_{33}(\omega')}{q'^2(\omega'-\omega)}$$
$$+\frac{1}{9\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{q'^2(\omega'+\omega)},$$

$$\mathrm{Re}f_{13}/q^3=-\frac{2}{3}\frac{f^2}{\omega}+\frac{4}{9\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{q'^2(\omega'+\omega)},$$

$$\mathrm{Re}f_{31}/q^3=-\frac{2}{3}\frac{f^2}{\omega}-\frac{f^2}{M}+\frac{1}{\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{q'^2}\left(\frac{4}{9}\frac{1}{\omega'+\omega}+\frac{2}{3M}\right),$$

$$\mathrm{Re}f_{11}/q^3=-\frac{8}{3}\frac{f^2}{\omega}+\frac{2f^2}{M}+\frac{1}{\pi}\int_1^\infty d\omega'\frac{\mathrm{Im}f_{33}(\omega')}{q'^2}\left(\frac{16}{9}\frac{1}{\omega'+\omega}+\frac{2}{3M}\right). \qquad (2\cdot3)$$

* In this paper, the notations of CGLN will be used, unless specially mentioned, and the conventional units $\hbar=c=1$ and pion mass $\mu=1$ will be used.

These equations are derived by expanding the various terms which appear in the original equations with respect to ω/M and ω'/M and by neglecting these second orders. The last two equations in (2·3) are different from the corresponding ones of CGLN in the order of f^2/M. But, if one remembers the approximate relation $f^2 \simeq 2/3\pi \int_1^\infty d\omega' \mathrm{Im} f_{33}(\omega')/q'^2$, these differences are found to be not so numerically distinctive.*

Next, we will describe the contributions from the pion-pion interactions. The S- and P-wave pion-pion interactions contribute to $A^{(+)}(W^2, t)$ and to $A^{(-)}(W^2, t)$ and $B^{(-)}(W^2, t)$, respectively. It will be found that there appear no contributions to the $B^{(+)}(W^2, t)$ as long as one omits the higher partial waves other than S- and P-waves.[3),6)] Then, from (2·1) and (2·2), the pion-pion contributions will be written as

$$\begin{aligned} &\delta(f_{p3/2}^{(+)}/q^2) = a(q), \\ &\delta(f_{p3/2}^{(-)}/q^2) = \epsilon(q), \\ &\delta(f_{p1/2}^{(-)} - f_{p3/2}^{(-)})/q^2 = b(q), \\ &\delta(f_{p1/2}^{(+)} - f_{p3/2}^{(+)})/q^2 = 0, \end{aligned} \tag{2·4}$$

where $a(q)$ represents the contribution from the S-wave pion-pion interaction and $\epsilon(q)$ and $b(q)$ represent the contributions from the P-wave pion-pion interaction. Then, using the relations

$$f^{(+)} = \frac{1}{3}(f^{(1)} + 2f^{(3)}), \quad f^{(-)} = \frac{1}{3}(f^{(1)} - f^{(3)}), \tag{2·5}$$

the contribution to each p-wave amplitude becomes as follows:

$$\delta(f_{33}/q^2) = a(q) - \epsilon(q), \quad \delta(f_{13}/q^2) = a(q) + 2\epsilon(q), \tag{2·6}$$

$$\delta(f_{31}/q^2) = a(q) - b^*(q), \quad \delta(f_{11}/q^2) = a(q) + 2b^*(q), \tag{2·7}$$

$$b^*(q) \equiv b(q) + \epsilon(q), \tag{2·8}$$

which may be summed up as

$$\delta f = 3(\boldsymbol{p}\cdot\boldsymbol{q})a(q)\delta_{\alpha\beta} + [3(\boldsymbol{p}\cdot\boldsymbol{q})\epsilon(q) + (\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{q})b(q)][\tau_\alpha, \tau_\beta]/2, \tag{2·9}$$

where $\boldsymbol{q}$ and $\boldsymbol{p}$ are the initial and final pion momenta and β and α are the initial and final pion isospin indices. In the next section, we will try to estimate these parameters $a(q)$, $b^*(q)$ and $\epsilon(q)$ by comparing with the experiments at the low energies.

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* This will be very natural from the following consideration. The difference between (2·2) and that of CGLN is essentially in whether one uses the experimental or the theoretical s-wave scattering amplitudes. (And in (2·2) the contributions from the s-waves have been neglected, as they are estimated to be very small by using the experimental s-wave amplitudes.) However, the theoretical s-wave amplitudes predicted by CGLN have been shown to nearly reproduce the experiments without the pion-pion effect.[1)] Then, the results should be nearly equal to each other.

§ 3. Numerical estimation of the pion-pion effects. Discussion

We shall here restrict our attention to the pion-nucleon scattering within the low energy limit. The p-wave phase shifts predicted by (2·3) are numerically calculated as is shown in Table I, one taking the coupling constant $f^2 \simeq 0.08 \pm 0.01$. The experiment reported by Puppi in the 1958 Annual International Conference on High Energy Physics at CERN is also listed in Table I. Owing to the large experimental inaccuracy, in addition to the limitations on the theory of Cini and Fubini applied here, one might not be able to derive so decisive conclusions on the pion-pion effects, but we wish to derive as much information on these effects as possible.

Table I. The pion-nucleon p-wave phase shifts at the low energies are calculated by using (2·3). The experiment is taken from the report by Puppi in the 1958 Annual Conference at CERN.

f^2	0.07	0.08	0.09	experiment
α_{33}/q^3	0.196	0.210	0.223	0.234±0.019
α_{13}/q^3	−0.035	−0.041	−0.048	−0.039±0.022
α_{31}/q^3	−0.037	−0.045	−0.054	−0.044±0.005
α_{11}/q^3	−0.110	−0.132	−0.157	−0.038±0.038

From Table I it can be seen that the α_{13} and α_{31} are consistent with experiment, but (i) the $|\alpha_{11}|$ is about 3∼4 times as large as the experimental one, (ii) the α_{33} has a tendency to be smaller than the experimental one at least within $f^2 \simeq 0.08 \pm 0.01$, though this is not so remarkable as (i). In this paper, we will assume these discrepancies to be due to the neglect of the effects of the pion-pion interactions. These discrepancies might be considered to be due to the neglect of the effects of the higher partial waves of the pion-nucleon scattering on the dispersion integrals, but these effects will be found to be approximately included in the assumed pion-pion effects at least if one assumes that the two-pion exchange interaction will be dominant at the low energies.[2)]

Table II. The possible pion-pion effects $a(q)$, $\epsilon(q)$ and $b^*(q)$ in (2·6) and (2·7) are estimated.

f^2		0.07	0.08	0.09
a	from (2·6)	0.024±0.015	0.017±0.015	0.010±0.015
	from (2·7)	0.019±0.013	0.032±0.013	0.046±0.013
b^*		0.026±0.013	0.031±0.013	0.036±0.013
ϵ		−0.014±0.010	−0.007±0.010	−0.001±0.010

Comparing the p-wave phase shifts predicted by (2·3) with the experiment, we can then estimate $a(q)$ and $\epsilon(q)$ from (2·6) and can estimate $a(q)$ and $b^*(q)$ from (2·7). The two ways to estimate $a(q)$ are, of course, due to the last equation of (2·4) which comes from the assumption of the S- and P-wave interaction dominance. The results are shown in Table II.

The necessity of the positive b^* can be found from (2·7), as the positive b^* can make the α_{31} and α_{11} (which are predicted as $\alpha_{31} \geqslant \alpha_{11}$ in (2·3)) close to each other. But from this effect the α_{31} becomes rather smaller than the experimental one, then the positive $a(\approx b)$ becomes necessary, which is reflected in the third line of Table II. On the other hand, the positive a from (2·6) comes from the fact that, without this, the α_{33} is rather smaller than the experimental one. As to the $J=3/2$ states only (or $J=1/2$ states), if one properly took the coupling constant f^2 as the order of 0.10 (or 0.06), one could nearly reproduce the experiments without the pion-pion S-wave effect $a(q)$, but then one could not reproduce the experiments of the other $J=1/2$ states (or $J=3/2$ states). This circumstance is illustrated in Fig. 1. From Fig. 1 we can estimate the coupling constant f^2 and the pion-pion S-wave effect $a(q)$ so as to coincide the $a(q)$ from (2·6) with the $a(q)$ from (2·7). Then the results become as follows:

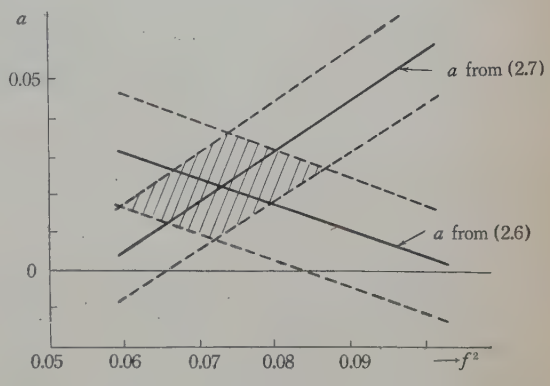


Fig. 1.

$$f^2=0.073\pm0.013, \quad a=0.022\pm0.013. \tag{3·1}$$

Thus, the attractive isospin-independent interaction $a(q)>0$ is found to be very favorable for understanding the low-energy p-wave phase shifts. Further, the above estimated f^2 is found to be close to the usual 0.08 ± 0.01, which is considered to support our assumption of the pion-pion S- and P-wave interactions dominance. We can then calculate $b^*(q)$ and $\epsilon(q)$ tentatively by taking $f^2=0.073$ as

$$b^*=0.028\pm0.013, \quad \epsilon=-0.012\pm0.010, \tag{3·2}$$

where the error in b^* arises almost from the large inaccuracy in α_{11}, but the $a-b^*$ will be fairly accurately determined from the data of α_{31} within ±0.005. These b^* and ϵ are consistent with the values calculated by Bowcock et al. in reference 3), where the pion-pion P-wave resonant interaction is assumed. The pion-pion S-wave effect $a(q)$ in this paper is about 1/2 of the one previously estimated from the different method.[6] Hence, the pion-pion S-wave scattering

length a_S will be revised as the order of $1/2\mu$ which is one half of the one previously estimated. However, these values of a_S might not be so reliable by the following two reasons: (i) In the treatment of the integral equation for $\pi+\pi\to N+N$ in reference 6), the imaginary part of the pion-nucleon scattering amplitude has been approximated by the partial waves for $t \geq 4\mu^2$ as has been made in the treatment of the electromagnetic structure of the nucleon by Federbush et al.,[8] which may, however, be a drastic assumption in view of the treatment of the electromagnetic structure of the nucleon by Frazer and Fulco,[4] and (ii) in our "phenomenological treatments", the assumed attractive force will essentially represent all the contributions other than the one from the dominant (3-3) state to the dispersion relation for $A^{(+)}$.

We will finally try to estimate the pion-pion corrections at the intermediate energies, for instance, at $q=1$ so as to reproduce the behaviors of the small phase shifts given by Chiu and Lomon,[7] taking $f^2=0.08$. It then follows that

$$a(q) \simeq 0.028, \quad b^*(q) \simeq 0.015, \quad \epsilon(q) \simeq +0.007,$$
$$\alpha_{13} \simeq +0.015\, q^3, \quad \alpha_{11} \simeq \alpha_{31} \simeq -0.018\, q^3 \quad \text{at } q=1. \qquad (3\cdot3)$$

From these, $a(q)$ and $b^*(q)$ may be slowly varying functions with the energies. However, in order to reproduce the positive α_{13} at the intermediate energies, it will be convenient if $\epsilon(q)$ rapidly increases from negative to positive as q goes to the intermediate energies. At any rate, from our analysis, it can be found that the assumed effects of the pion-pion S- and P-wave interactions are very favorable for understanding at least the low-energy p-wave pion-nucleon scattering. In order to see the pion-pion effects on the partial waves other than the p-waves, however, one has to know the explicit structures of these effects on the invariant scattering amplitudes $A^{(\pm)}$ and $B^{(\pm)}$. Here, we have not referred to this problem.

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Exact Treatment of the Bound State Problems in the Non-Relativistic Quantum Field Theory

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An exact equation for the bound state of nucleons in the static neutral scalar model is obtained by using the field equation for Heisenberg operator and the method of canonical transformation. The formulation of the eigenvalue problem is completed by the analyticity consideration. In particular, a boundary condition at $\tau=0$ ($-i\tau$ being the "negative imaginary" relative time) turns out that the derivative of the "wave function" with respect to τ must be zero at $\tau=0$. The abnormal solutions exist in this case as well as in the ladder approximation, but there is difference in nature between them for the zero meson mass case. The appearance of abnormal solutions is mainly due to the structure of the fundamental equation itself. This is demonstrated by handling a first order equation for the static neutral scalar model and the Bethe-Salpeter equation for the Lee model, in both cases of which the abnormal solutions do not appear.

§ 1. Introduction and summary

The so-called abnormal or redundant solutions of the Bethe-Salpeter (B-S) equation in the relativistic two-body problem were first found by Wick[1] and Cutkosky,[2] and discussed by many people. Some authors[3] tried to give a positive physical meaning to such solutions and the others,[4] on the contrary, to exclude them by some physical reasonings. These solutions represent the states which have a quantum number pertaining to the relative time variable, and all except the ground state (normal state) have not their non-relativistic correspondence. Recently Ohnuki, Takao and Umezawa[5] have shown that even in the non-relativistic case the B-S equation has abnormal solutions which do not correspond to the eigenstate of the total Hamiltonian.

All studies published so far concern with the lowest ladder approximation, and it is not yet clear whether the existence of abnormal states is inherent in the conventional field-theoretic treatment of the bound state problem or it is only due to the approximation employed. In this respect, the investigation of the complete B-S equation without any approximation is very desirable, but we do not know any example except for some trivial one as the Lee model in which the corrected ladder approximation is indeed exact.

Usually, the B-S equation is solved by transforming into an equivalent eigenvalue problem in the relative coordinate system of two particles. This formulation however is not always mathematically satisfactory, because some-

times the boundary condition at the origin of relative coordinate system is incomplete, and about it we could have no definite indication from general field-theoretic considerations. This is also one of the important points to be clarified in investigating the bound state problem.

We are now going to discuss mainly about the bound state of nucleons in the scalar neutral meson theory with fixed source,* starting with the derivation of an exact B-S equation. Owing to the technical complication of usual field-theoretic method, we are obliged to be content with the direct derivation of the equation by using the equations of motion of the Heisenberg operators and the method of canonical transformation. Of course, such a derivation is quite unsatisfactory from the general field-theoretic view-point, but as our aim is not the derivation but the close analysis of the complete B-S equation and, hoping that the same result would be obtained also by the standard Feynman-graphical technique, we are no more nervous about this point. The equation thus obtained is a differential equation in the relative time variable, and contains three terms: a term proportional to the square of the coupling constant which just corresponds to the ladder approximation and two terms proportional to the fourth power of the coupling constant. Hence, the limitation to the ladder approximation can be easily obtained.

The problem of solving this equation can be transformed into an eigenvalue problem in the imaginary relative time variable, following Wick,[1] by a method of analytic continuation. However, as stated above, we meet with the difficulty of the boundary condition at the origin of the relative time. The second method of handling the equation is then described, which utilizes the analytical structure of the B-S amplitude and the B-S equation, and transforms the B-S equation into a Fredholm's integral equation of second kind for the spectral function of the B-S amplitude and an eigenvalue equation.** The latter equations themselves give a complete description of the bound state problem, but practically it is difficult to solve them. However, it turns out that this eigenvalue equation is equivalent to the boundary condition at the origin of the relative time in the first method, and thus we can get complete information on the eigenvalue problem in the relative time variable from the general field-theoretic standpoint and, at the same time, make clear a close relation between two methods.

Now, returning to the first method, we have to solve the eigenvalue problem in the imaginary time variable under the boundary conditions that the "wave function" vanishes at infinity and its first derivative at the origin becomes zero. The equation to be solved under these boundary conditions is a second order ordinary differential equation of the type of one-dimensional eigenvalue equation in the quantum mechanics. In comparison with the case of

* Hereafter referred to as the neutral scalar model.

** A similar method has been discussed also by Nakanishi[6] in connection with Goldstein's eigenvalue problem.

ladder approximation treated by Ohnuki et al.,[5] the "potential" is regular here too but deeper everywhere. Therefore we have abnormal states even in this exact treatment, the nature of which is however somewhat different from the ladder approximation for the zero meson mass case because the potential tends to zero at infinity as the inverse of the variable whereas it does as the inverse square of the variable in the ladder approximation.

It is further a question to be decided whether the existence of abnormal states should be appreciated positively or denied by some physical or mathematical reason. The abnormal solutions have certainly an intimate relation to the freedom of relative time, the physical meaning of which has not yet been clarified, but the whole responsibility of its appearance is not always on the relative time. In order to understand this circumstance, we solve the same problem by considering a first order equation of the B-S amplitude and the B-S equation for the Lee model, in both cases of which we have no trouble, showing that the abnormal solutions originate rather from the structure of the fundamental equation itself.

§ 2. Complete Bethe-Salpeter equation

The system of two nucleons, fixed at $\boldsymbol{r}/2$ and $-\boldsymbol{r}/2$ respectively, interacting with the meson field is described by a Hamiltonian in the unrenormalized quantities:

$$H=m_0\sum_i N_i+\int\omega a^*(\boldsymbol{k})\,a(\boldsymbol{k})\,d\boldsymbol{k}$$
$$+\frac{g}{(2\pi)^{3/2}}\sum_i\int\frac{d\boldsymbol{k}}{(2\omega)^{1/2}}\left(a(\boldsymbol{k})\exp\left(\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right)+h.c.\right)N_i, \qquad (2\cdot1)^*$$

where the subscript i takes 1 and 2 and designates two nucleons, $N_i=\alpha_{i0}^*\alpha_{i0}$ is the nucleon number, $a^*(\boldsymbol{k})$ and $a(\boldsymbol{k})$ the meson operators, $\omega=\sqrt{\mu^2+\boldsymbol{k}^2}$ ($\mu=$ meson mass), and

$$\eta_i=\begin{cases}1 & \text{for } i=1\\ -1 & \text{for } i=2.\end{cases}$$

The commutation relations are

$$\{\alpha_{i0},\,\alpha_{j0}^*\}=\delta_{ij},\quad [a(\boldsymbol{k}),\,a^*(\boldsymbol{k}')]=\delta(\boldsymbol{k}-\boldsymbol{k}'). \qquad (2\cdot2)$$

For comparison and later references, we shall briefly sketch the main results by the canonical transformation. The transformation function is defined by

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* For simplicity of writing, we do not introduce a cutoff factor into the interaction from the beginning.

$$U=\exp(g\sum_i N_i(A_i-A_i^*)), \tag{2·3}$$

in which

$$A_i=\frac{1}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{k}}{\sqrt{2}\,\omega^{3/2}}a(\boldsymbol{k})\exp\left(\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right).$$

The field operators are transformed into their corresponding clothed operators:

$$\beta_{i0}\equiv U\alpha_{i0}U^{-1}=\alpha_{i0}\exp\{-g(A_i-A_i^*)\}=\alpha_i\exp(-gA_i)\exp(gA_i^*), \tag{2·4a}*$$

$$b(\boldsymbol{k})\equiv Ua(\boldsymbol{k})U^{-1}=a(\boldsymbol{k})+\frac{g}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}\,\omega^{3/2}}\sum_i N_i\exp\left(-\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right), \tag{2·4b}$$

where α_i is the renormalized operator of the nucleon defined by

$$\alpha_i\equiv Z^{-1/2}\alpha_{i0}=\exp\left(\frac{1}{2}\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^3}\right)\alpha_{i0}. \tag{2·5}$$

It is seen at once that

$$N_i=\alpha_{i0}^*\alpha_{i0}=\beta_{i0}^*\beta_{i0}. \tag{2·6}$$

The Hamiltonian can be written in a diagonal form in terms of the clothed operators as

$$H=m\sum_i N_i+\int\omega b^*(\boldsymbol{k})b(\boldsymbol{k})\,d\boldsymbol{k}-g^2V(r)N_1N_2, \tag{2·7}$$

where m is the renormalized mass of the nucleon defined by

$$m\equiv m_0-\delta m=m_0-\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^2}, \tag{2·8}$$

$$V(r)=e^{-\mu r}/4\pi r. \tag{2·9}$$

It is then very easy to see that for the bound state of two nucleons represented by

$$|\mathrm{B}\rangle=C(r)\beta_{10}^*\beta_{20}^*|0\rangle, \tag{2·10}$$

$|0\rangle$ being the vacuum,** the eigenvalue equation

$$H|\mathrm{B}\rangle=E|\mathrm{B}\rangle \tag{2·11}$$

is satisfied by the total energy of the value

$$E=2m-g^2V(r). \tag{2·12}$$

The field equations for nucleon in the Heisenberg representation derived

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* In the last equality we have used a special form of Hausdorff's formula for two non-commuting operators X and Y for which $[X, Y]$=c-number,

$$e^{X+Y}=e^Xe^Ye^{-[X,Y]/2}=e^Ye^Xe^{[X,Y]/2}.$$

** As is well known, all the vacuums (free, true and clothed) are identical.

from the Hamiltonian (2·1) or (2·7) are, in a form convenient for the present purpose,

$$(id/dt-m)\boldsymbol{\alpha}_{i0}(t)=\frac{g}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{k}}{(2\omega)^{1/2}}\left\{\boldsymbol{\alpha}_{i0}(t)\boldsymbol{b}(\boldsymbol{k},t)\exp\left(-\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right)\right.$$

$$\left.+\boldsymbol{b}^*(\boldsymbol{k},t)\boldsymbol{\alpha}_{i0}(t)\exp\left(-\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right)\right\}-g^2V(r)N_j\boldsymbol{\alpha}_{i0}(t),\ (i\neq j). \quad (2\cdot13)$$

The B-S amplitude is defined by

$$\varphi(1,2)\equiv\langle 0|T(\boldsymbol{\alpha}_{10}(t_1)\boldsymbol{\alpha}_{20}(t_2))|\mathrm{B}\rangle. \quad (2\cdot14)$$

Operating $(i\partial/\partial t_1-m)$ on this amplitude, and using the equation (2·13) for $\boldsymbol{\alpha}_{10}(t_1)$, the expression (2·10) of $|\mathrm{B}\rangle$ and the commutation relations, we have

$$(i\partial/\partial t_1-m)\varphi(1,2)=-g^2V(r)\theta(-t)\varphi(1,2)$$

$$-\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^3}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\{e^{-i\omega t}\theta(t)-e^{i\omega t}\theta(-t)\}\varphi(1,2). \quad (2\cdot15a)$$

Similarly,

$$(i\partial/\partial t_2-m)\varphi(1,2)=-g^2V(r)\theta(t)\varphi(1,2)$$

$$+\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^3}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\{e^{-i\omega t}\theta(t)-e^{i\omega t}\theta(-t)\}\varphi(1,2). \quad (2\cdot15b)$$

In these equations, t denotes the relative time t_1-t_2 of two nucleons. We shall leave the details of the derivation to the Appendix.

Iterating the operation, we get the following complete B-S equation:

$$(i\partial/\partial t_1-m)(i\partial/\partial t_2-m)\varphi(1,2)$$

$$=\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\{e^{-i\omega t}\theta(t)+e^{i\omega t}\theta(-t)\}\varphi(1,2)$$

$$+\frac{g^4}{(2\pi)^3}V(r)\int\frac{d\boldsymbol{k}}{2\omega^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\{e^{-i\omega t}\theta(t)+e^{i\omega t}\theta(-t)\}\varphi(1,2)$$

$$-\frac{g^4}{(2\pi)^6}\left[\int\frac{d\boldsymbol{k}}{2\omega^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\{e^{-i\omega t}\theta(t)+e^{i\omega t}\theta(-t)\}\right]^2\varphi(1,2). \quad (2\cdot16)$$

At first sight we find that on the right-hand side only the first term appears in the ladder approximation. The termination of the correction terms at the finite (fourth) power of coupling constant is of course due to the speciality of the model and can be properly understood remembering, for example, the expression of mass renormalization (2·8) where the correction terminates at the square of coupling constant. Taking this feature into account, it is not so unreasonable to expect that the same equation as (2·16) would come out also from the Feynman-graphical consideration. We therefore adopt the equation as the starting point of our discussion.

§ 3. Wick's method of analytic continuation

Defining

$$\varphi_+(1,2)=\theta(t)\langle 0|\boldsymbol{\alpha}_{10}(t_1)\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle, \tag{3·1a}$$

and

$$\varphi_-(1,2)=-\theta(-t)\langle 0|\boldsymbol{\alpha}_{20}(t_2)\boldsymbol{\alpha}_{10}(t_1)|\mathrm{B}\rangle, \tag{3·1b}$$

we see at once from their spectral decomposition with respect to the relative time that $\varphi_+(1,2)$ contains only positive frequencies whereas $\varphi_-(1,2)$ contains only negative frequencies, that is, we have the spectral representations,

$$\varphi_\pm(1,2)=e^{-iET}u_\pm(t)=e^{-iET}\theta(\pm t)\int_0^\infty \rho(\omega)\,e^{\mp i(B/2+\omega)t}\,d\omega, \tag{3·2}$$

where $T=(t_1+t_2)/2$, $B=2m-E$ is the binding energy of the bound state, and the spectral function may be written, separating out the one-nucleon contribution, as

$$\rho(\omega)=\sigma\cdot\delta(\omega)+\nu(\omega)\theta(\omega-\mu). \tag{3·3}$$

The same spectral function for both $\varphi_+(1,2)$ and $\varphi_-(1,2)$ is due to the symmetry property of the B-S amplitude. $u_\pm(t)$ thus can be analytically continuated to the lower (upper) half of complex t-plane. The same holds of the functions on the right-hand side of the equation (2·16). For the complete symmetry, it is sufficient to consider the equation in the lower half plane only and we shall hereafter omit the subscript $+$.

After all, with slight changes of variable and parameters, we have on the negative imaginary axis of the complex t-plane an equation of the form

$$(-d^2/dx^2+U(x;\lambda,\mu r))u=-\alpha^2 u, \tag{3·4}$$

where $x=\tau/r=it/r$ is real, $\lambda=(g/2\pi)^2$, and $\alpha=Br/2$. This equation is of the type of eigenvalue problem in the one-dimensional semi-infinite space with a "potential"

$$U(x;\lambda,\mu r)=-\lambda q''(x)+\lambda^2\pi e^{-\mu r}q'(x)+\lambda^2[q'(x)]^2, \tag{3·5}$$

in which

$$q(x)=\frac{1}{4\pi}\int\frac{d\boldsymbol{k}}{\omega^3}\cos(\boldsymbol{k}\cdot\boldsymbol{r})e^{-\omega r x}. \tag{3·6}$$

The "wave function" $u(x)$ must satisfy a boundary condition at infinity that $u(\infty)=0$, but at the origin $x=0$ it is not clear what boundary condition should be imposed on $u(x)$.* Therefore this eigenvalue problem is incomplete.

* Ohnuki et al.[5] considered an eigenvalue problem in the both-sided infinite space with the boundary conditions at $x=\pm\infty$, in favour of the symmetry of the potential $(U(-x)=U(x))$. However, in general, two analytic regions of $u(t)$ are separated completely and there is no analytic continuation from one half-plane to the other; the two regions touch one another at one point only $t=0$. It is therefore rigorous to treat the eigenvalue problem in each region under the boundary conditions at infinity and at the origin.

§ 4. Analytic properties of the Bethe-Salpeter equation

The Fourier transform of the B-S amplitude with respect to the relative energy p is defined by

$$\chi(p)=(-i)\,e^{iET}\int\varphi(1,2)e^{ipt}\,dt, \tag{4·1}$$

and calculated using (3·2) and (3·3) as

$$\chi(p)=\frac{\sigma}{p-B/2+i\varepsilon}+\int_{\mu}^{\infty}\frac{\nu(\omega)\,d\omega}{p-B/2-\omega+i\varepsilon}-\frac{\sigma}{p+B/2-i\varepsilon}-\int_{\mu}^{\infty}\frac{\nu(\omega)\,d\omega}{p+B/2+\omega-i\varepsilon}. \tag{4·2}$$

$\chi(p)$ thus has two single poles and two cuts as shown in Fig. 1.

The B-S equation (2·16) is transformed into an integral equation,

$$(p^2-(B/2)^2)\chi(p)=\int(K_+(p-q)+K_-(p-q))\chi(q)\,dq, \tag{4·3}$$

with singular kernels given by

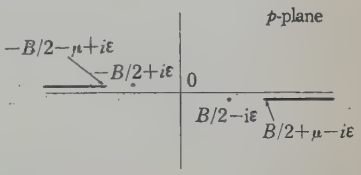


Fig. 1. Analyticity of $\chi(p)$ of the neutral scalar model in the complex p-plane.

$$K_{\pm}(p)=\frac{1}{2\pi i}\int_{\mu}^{\infty}\frac{k(\omega)\,d\omega}{\pm p-\omega+i\varepsilon}, \tag{4·4}$$

where

$$\begin{aligned}k(\omega)=&\theta(\omega-\mu)\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}'}{2\omega'}\cos(\boldsymbol{k}'\cdot\boldsymbol{r})\,\delta(\omega-\omega')\\&+\theta(\omega-\mu)\frac{g^4}{(2\pi)^3}V(r)\int\frac{d\boldsymbol{k}'}{2\omega'^2}\cos(\boldsymbol{k}'\cdot\boldsymbol{r})\,\delta(\omega-\omega')\\&-\theta(\omega-2\mu)\frac{g^4}{(2\pi)^6}\int\frac{d\boldsymbol{k}'}{2\omega'^2}\cos(\boldsymbol{k}'\cdot\boldsymbol{r})\int\frac{d\boldsymbol{k}''}{2\omega''^2}\cos(\boldsymbol{k}''\cdot\boldsymbol{r})\,\delta(\omega-\omega'-\omega'').\end{aligned} \tag{4·5}$$

This singular integral equation can be solved by a similar method of analytic continuation as in the preceding section, but we treat it in an alternative way.

The convolution integrals on the right-hand side of (4·3) can be performed by means of the formulae,

$$\int\frac{\chi(q)\,dq}{p-q-\omega+i\varepsilon}=-2\pi i\left\{\frac{\sigma}{p-B/2-\omega+i\varepsilon}+\int_{\mu}^{\infty}\frac{\nu(\omega')\,d\omega'}{p-B/2-\omega-\omega'+i\varepsilon}\right\}, \tag{4·6a}$$

and

$$\int \frac{\chi(q)\,dq}{p-q+\omega-i\varepsilon} = -2\pi i \left\{ \frac{\sigma}{p+B/2+\omega-i\varepsilon} + \int_{\mu}^{\infty} \frac{\nu(\omega')\,d\omega'}{p+B/2+\omega+\omega'-i\varepsilon} \right\}, \quad (4\cdot 6b)$$

which are easily verified by carrying out a contour integral including a large semicircle in the upper (lower) half of complex q-plane. Eq. (4·3) then becomes

$$(p^2-(B/2)^2)\chi(p) = -2\pi i \left\{ \sigma\cdot(K_+(p-B/2)+K_-(p+B/2)) + \int_{\mu}^{\infty} [K_+(p-B/2-\omega)+K_-(p+B/2+\omega)]\nu(\omega)\,d\omega \right\}. \quad (4\cdot 7)$$

Both sides of this equation must have the same analyticity. Putting $p=B/2$, we have*

$$B\cdot\sigma = -2\pi i \left\{ \sigma[K_+(0)+K_-(B)] + \int_{\mu}^{\infty} [K_+(-\omega)+K_-(B+\omega)]\nu(\omega)\,d\omega \right\}, \quad (4\cdot 8)$$

and the jump of both sides at the cut from $B/2+\mu$ to ∞ gives*

$$\omega(B+\omega)\nu(\omega) = -2\pi i \left\{ \sigma k(\omega) + \int_{\mu}^{\infty} k(\omega-\omega')\nu(\omega')\,d\omega' \right\}. \quad (4\cdot 9)$$

For the last equation, we consider the integrals,

$$\int_{\mu}^{\infty} (B+\omega)\nu(\omega)\,d\omega = 2\pi i \left\{ \sigma K_+(0) + \int_{\mu}^{\infty} K_+(-\omega)\nu(\omega)\,d\omega \right\}, \quad (4\cdot 10)$$

and

$$\int_{\mu}^{\infty} \omega\nu(\omega)\,d\omega = 2\pi i \left\{ \sigma K_-(B) + \int_{\mu}^{\infty} K_-(B+\omega)\nu(\omega)\,d\omega \right\}. \quad (4\cdot 11)$$

Combining (4·8), (4·10) and (4·11), finally, we get

$$B\left(\sigma + \int_{\mu}^{\infty} \nu(\omega)\,d\omega\right) + 2\int_{\mu}^{\infty} \omega\nu(\omega)\,d\omega = 0. \quad (4\cdot 12)$$

In this formulation, (4·9) and (4·12) are the fundamental equations. Eq. (4·9) for the spectral function $\nu(\omega)$ of the B-S amplitude is a Fredholm's integral equation of second kind and, contrary to the original integral equation (4·3), non-singular. Moreover, its kernel and free term are of the same form. Since $k(\omega)$ contains the factors $\theta(\omega-\mu)$ and $\theta(\omega-2\mu)$, for every finite values

* The same results are obtained for the other pole and cut, since $K_-(0)=K_+(0)$, $K_+(-B)=K_-(B)$, etc.

of ω, we can calculate $\nu(\omega)$ by finite times of iteration so far as the meson mass μ is different from zero. Eq. (4·12), with an insertion of the solution $\nu(\omega)$ of (4·9), is an eigenvalue equation, which is in general a very complicated transcendental equation to find B because $\nu(\omega)$ itself contains B as a parameter. It is interesting to notice that (4·12) gives a relation between the spectral function $\nu(\omega)$ and the binding energy B in no intervention of kernels $K_{\pm}$ of the B-S equation which may sometimes depend on the approximation of the theory. Therefore we may give a strict physical meaning to the spectral function $\nu(\omega)$ as, say, a probability function to find exchanging mesons of total energy ω.

Solving (4·9) in a power series of $g^2V(r)-B$, we obtain as the zeroth order solution

$$\nu_n(\omega)=\theta(\omega-\mu)\sigma\sum_{l=1}^{\infty}\frac{1}{l!}\left(-\frac{g^2}{(2\pi)^3}\right)^l\int\delta(\omega-\omega_1-\cdots-\omega_l)\prod_{i=1}^{l}\frac{d\boldsymbol{k}_i}{2\omega_i^3}\cos(\boldsymbol{k}_i\cdot\boldsymbol{r}), \tag{4·13}$$

and writing

$$\nu(\omega)=\nu_n(\omega)+(g^2V(r)-B)\nu_1(\omega), \tag{4·14}$$

$\nu_1(\omega)$ satisfies an integral equation,

$$\nu_1(\omega)=\frac{\nu_n(\omega)}{B+\omega}-\frac{2\pi i}{\omega(B+\omega)}\int_{\mu}^{\infty}k(\omega-\omega')\nu_1(\omega')\,d\omega', \tag{4·15}$$

and the eigenvalue equation (4·12) becomes

$$(g^2V(r)-B)\left(1+\int_{\mu}^{\infty}\nu_n(\omega)\,d\omega-\int_{\mu}^{\infty}(B+2\omega)\nu_1(\omega)\,d\omega\right)=0. \tag{4·16}$$

Thus $\nu_n(\omega)$ represents the spectral function of the normal state and the roots of the second factor of Eq. (4·16) would give abnormal states, if they exist. However, it is very troublesome to perform this plan to find abnormal roots of the eigenvalue equation, and we have not succeeded to get satisfactory conclusions. Nevertheless, it will soon become clear that the arguments of this section play an important role in the settlement of our problem.

§ 5. Eigenvalue problem

Let us now return to the eigenvalue problem in the imaginary relative time variable. First, we shall show that the eigenvalue equation in the last section brings a boundary condition at the origin and accordingly makes the eigenvalue problem complete. In fact, we have from (4·2)

$$\lim_{t\to+0}u'(t)=(-i/2)\left\{B\left(\sigma+\int_{\mu}^{\infty}\nu(\omega)\,d\omega\right)+2\int_{\mu}^{\infty}\omega\nu(\omega)\,d\omega\right\}=0, \tag{5·1}$$

because of (4·12). Analyticity argument concludes that we must take the solution of (3·4) whose derivative vanishes at the origin.*

Summing up, we have the following eigenvalue problem: *to find such solutions of the equation,*

$$(-d^2/dx^2+U(x;\lambda,\mu r))u=-\alpha^2 u \quad \textit{for} \quad x\geq 0, \tag{5·2}$$

that satisfy the boundary conditions,

$$u'(0)=0 \quad \textit{and} \quad u(\infty)=0. \tag{5·3}$$

The correspondence between two methods of solving the B-S equation, that is, the method formulated just above and that presented in the last section, is quite clear; the differential equation (5·2) and the boundary conditions (5·3) correspond to the integral equation (4·9) and the eigenvalue equation (4·12), respectively.

i) *The case* $\mu=0$

$q'(x)$ is given, from (3·6), by

$$q'(x)=-\frac{r}{4\pi}\int\frac{d\boldsymbol{k}}{k^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\,e^{-krx}=-\operatorname{arccot}x. \tag{5·4}$$

We have thus an equation,

$$(-d^2/dx^2+U(x;\lambda))u=-\alpha^2 u, \tag{5·5}$$

with an "attractive potential"

$$U(x;\lambda)=-\frac{\lambda}{1+x^2}-\lambda^2\pi\operatorname{arccot}x+\lambda^2(\operatorname{arccot}x)^2, \tag{5·6}$$

which is regular everywhere and an even function of x.** When $\alpha=\lambda\pi/2$, this equation can be easily integrated,[7] giving, except for integration constants,

$$u_n(x)=\exp\left[-\lambda\int^x\arctan x'\cdot dx'\right], \tag{5·7}$$

which indeed satisfies boundary conditions (5·3) and is the normal solution in accordance with (4·13). For other values of α, we cannot find a solution in a closed form, but qualitative arguments will be sufficient to lead us to final conclusions.

In comparison with the ladder approximation, where we have only first one term of the potential (5·6), our potential tends to zero at infinity more

* It is worthy to point out that this boundary condition should be imposed on also in the ladder approximation since, as stated above, the relation (4·12) is valid independently of the approximation.

** This is at once seen by rewriting it as

$$U(x;\lambda)=-\frac{\lambda}{1+x^2}+\lambda^2(\arctan x)^2-(\lambda\pi/2)^2.$$

slowly because of the term $-\lambda^2\pi$ arccot x which behaves as $-\lambda^2\pi/x$ for large x. This will bring an essentially different aspect of "levels" between our exact treatment and the ladder approximation.

Before entering into the discussion of levels, we shall point out a somewhat unfamiliar feature of the exact equation. Usually, the equations describing the one-dimensional eigenvalue problem in the quantum mechanics are of hypergeometric type or can be transformed into it. Our equation, in general, does not fit in either case. Therefore, we must at first ascertain the existence of the solution having a correct behavior at infinity. Unfortunately, due to the presence of arccot x, it is difficult to say that a formal development of the solution at the irregular singular point ($x=\infty$) of the equation by the standard method provides indeed an asymptotic expansion. However, we have found in the mathematical literature[8] a theorem stating, when applied to our equation, that *if*

$$\int^{\infty} |U(x;\lambda)|dx=\infty, \tag{5·8}$$

but

$$\int^{\infty} |U(x;\lambda)|^2 dx<\infty, \tag{5·9}$$

then there is a solution (5·5) *of possessing the asymptotic form*

$$u=\exp\left[-\alpha x-(1/2\alpha)\int_0^x U(x';\lambda)dx'+O(1)\right], \tag{5·10}$$

which fits the boundary condition at infinity. In our case the condition (5·9) is certainly satisfied.

No further questions arise because Eq. (5·5) is regular for every finite value of x. It is therefore straightforward to reach at a conclusion that *our problem has many solutions other than the normal one.* As is seen from (5·7), the normal solution has no node and accordingly corresponds to the ground state. The sole problem to be settled is whether these solutions are of the well type or the Coulomb type or some others. Here the *well* type means that there are a finite number of levels for every finite value of λ, and the *Coulomb* type means that there are an infinite number of levels even for any small but non-zero value of λ. In the ladder approximation it has been shown[5] that in the case $\mu=0$ we have a rather strange feature, so to speak, the *mixed* type, namely, there are no abnormal solutions for $\lambda<1/4$ and occurs a degeneration of an infinite numbers of solutions at $\lambda=1/4$, and that the solutions belong to the well type for $\mu\neq 0$. These results are essentially based on the behavior of the potential as $x\to\infty$.

For a close qualitative analysis of the levels, the use of the following

comparison theorem is quite convenient: *If we consider two attractive potentials* $U_1(x;\lambda)$ *and* $U_2(x;\lambda)$ *and suppose that, for a given* λ,

$$|U_1(x;\lambda)| \geq |U_2(x;\lambda)| \quad \textit{for all } x, \tag{5·11}$$

then, under the same boundary conditions, the number of levels for $U_1(x;\lambda)$ *is not less than that for* $U_2(x;\lambda)$. From this theorem, we can get at once another conclusion that *the exact equation possesses not less abnormal solutions than the ladder-approximate equation does,* since

$$-\lambda^2\pi \operatorname{arccot} x + \lambda^2 (\operatorname{arccot} x)^2 < 0 \quad \text{for } 0 \leq x < \infty. \tag{5·12}$$

A further use of the comparison theorem is made with well potentials completely embedded in the exact potential. As is well known, the number of levels of the well potential* depends on a characteristic parameter proportional to a product of the depth and the square of the width and increases more and more as the parameter becomes larger and larger. Now, owing to the behavior like $-\lambda^2\pi/x$ of the exact potential as $x\to\infty$, we can take for comparison a well potential with arbitrarily large characteristic parameter for any small but non-zero λ. Therefore *the levels in the case* $\mu=0$ *are not of the mixed type but of the Coulomb type.*

ii) *The case* $\mu\neq 0$

The qualitative nature of levels in this case is, unlike the zero mass case, rather akin to that in the ladder approximation, because here too the potential is of short range; *the levels belong to the well type.* All other aspects for $\mu=0$ can be generalized straightforward to this case. For example, the normal state is the ground state and it remains true that the exact equation possesses not less abnormal solutions than the ladder-approximate equation does, since $q'(x)<0$, $q''(x)>0$ and $q'(0)=-(\pi/2)e^{-\mu r}$, so that

$$\lambda^2\pi e^{-\mu r} q'(x) + \lambda^2 [q'(x)]^2 < 0 \quad \text{for } 0 \leq x < \infty. \tag{5·13}$$

Summarizing the results, the exact B-S equation for the bound state in the neutral scalar model has many solutions which are of the Coulomb type for $\mu=0$ and of the well type for $\mu\neq 0$.

§ 6. First order equation

It has been seen above that so long as one starts from the B-S equation, no matter whether it is exact or approximate, and solves it by transforming into an eigenvalue problem in the imaginary relative time variable, there are in general many solutions other than the normal one. People may interpret this fact in their own way. Some may think that as the abnormal

* For the present purpose, it is necessary and sufficient to consider only even levels.

solutions are connected intimately with the freedom of relative time, they would be inherent in the B-S formalism and should be interpreted properly. The other may suppose that this is not so but is due to the insufficiency of physical conditions to solve the problem. And it may also be conjectured that the method of solving the B-S equation is not formulated strictly, or, to be concrete, the B-S equation (2·11) and the eigenvalue problem (5·2 and 3) is not mathematically equivalent.

Here we shall present an alternative try to solve the bound state problem in terms of the same B-S amplitude $\varphi(1, 2)$. It is based on the first order equations for $\varphi(1, 2)$, (2·15). Although these equations, of course, have a decisive disadvantage that we do not know the general way to derive them field-theoretically, we consider them as an example of fundamental equation for the B-S amplitude having no abnormal solutions and want to show that no trouble can occur if one treats the problem properly.

To this end, there are many ways. The simplest one is to use both equations, (2·15a) and (2·15b), and to add them. We then have

$$(i\partial/\partial T-2m)\varphi(1, 2)=-g^2 V(r)\varphi(1, 2), \tag{6·1}$$

which becomes, on account of (3·2),

$$(B-g^2 V(r))\varphi(1, 2)=0. \tag{6·2}$$

Certainly, there is *only* the normal solution. Employing any other method, the result would be unaltered. For example, when we apply a method of analytic continuation to (2·15a), for instance, a first order differential equation is obtained, for which we have, except for the normalization factor, a unique (normal) solution satisfying a boundary condition at infinity. In this case, it is not only unnecessary but also impossible to impose the boundary condition at the origin arbitrarily, in no contradiction with the condition at infinity. In the method of § 4, when applied to (2·15a), the eigenvalue equation reduces simply to

$$(B-g^2 V(r))\sigma=0, \tag{6·3}$$

giving the normal root only, and the integral equation can be solved by iteration and provides just the same expression for $\nu(\omega)$ as (4·13).

§ 7. Bethe-Salpeter equation in the Lee model

Another exactly soluble example of the bound state problem is the V-N potential in the Lee model. Weinberg[9] has already discussed exhaustively by solving the eigenvalue problem of the total Hamiltonian. We shall treat the same problem by setting up the B-S equation for the V-N bound state and show that this equation has no abnormal solutions.

V-particle and N-particle (mass m_N) are fixed at $\boldsymbol{r}/2$ and $-\boldsymbol{r}/2$ in

interaction with θ-particle (mass μ). The Hamiltonian of such a system is, in the unrenormalized quantities,

$$H=m_0\sum_i \alpha_{i0}^*\alpha_{i0}+m_N\sum_i \beta_i^*\beta_i+\int \omega a^*(\boldsymbol{k})\,a(\boldsymbol{k})\,d\boldsymbol{k}$$
$$+g_0\sum_i\int v(\omega)\left(a(\boldsymbol{k})\,\alpha_{i0}^*\beta_i\exp\left(\frac{1}{2}i\eta_i\boldsymbol{k}\cdot\boldsymbol{r}\right)+h.c.\right)d\boldsymbol{k}, \qquad (7\cdot1)$$

in which the notation is almost self-evident except for

$$v(\omega)=(2\pi)^{-3/2}\frac{f(\omega)}{(2\omega)^{1/2}},$$

with $f(\omega)$ a cutoff factor, and the subscript i and η_i are the same as in (2·1).

Some results concerning the one-body problem will become necessary, but, referring them to other papers,[10] we here confine ourselves to write an expression for the renormalized propagator of V. It is defined by

$$S_{Vc}(t_1-t_2)=-i\langle 0|T(\boldsymbol{\alpha}_i(t_1)\boldsymbol{\alpha}_i^*(t_2))|0\rangle, \qquad (7\cdot2)$$

$\boldsymbol{\alpha}_i(t_1)$ being the renormalized Heisenberg operator of V, and its Fourier transform is

$$S_{Vc}(p)=1/h(p-m_N+i\varepsilon), \qquad (7\cdot3)$$

where the function $h(z)$ denotes

$$h(z)=(z+m_N-m)\left\{1-g^2(z+m_N-m)\int\frac{v^2(\omega)\,d\boldsymbol{k}}{(z-m)\,(m_N-m+\omega)^2}\right\}, \qquad (7\cdot4)$$

in which m is the renormalized mass of V,* g the renormalized coupling constant. $h(z)$ has the following analyticity:

$$h(m-m_N)=0 \qquad (7\cdot5a)$$

and

$$h(z)\ \textit{has a cut from}\ \mu\ \textit{to}\ \infty. \qquad (7\cdot5b)$$

The B-S amplitude of V-N system is defined by

$$\varphi^{(\pm)}(1,2)=\langle 0|T(\boldsymbol{\alpha}_1(t_1)\boldsymbol{\beta}_2(t_2)\mp\boldsymbol{\beta}_1(t_2)\boldsymbol{\alpha}_2(t_1))|\mathrm{B}\rangle, \qquad (7\cdot6)$$

according as the state of even or odd parity is taken. It is easily seen, from the graphical consideration, that in the Lee model the ladder approximation is indeed exact, if only the propagator of V is modified. The usual field-theoretical procedure then gives the following integral equation for the B-S amplitude,

---

* For the stability of V, we assume $m<m_N+\mu$.

$$\varphi^{(\pm)}(1,2) = \mp g^2 \int dt_3 dt_4 \theta(t_3-t_4) S_{Ve}(t_1-t_3) S_N(t_2-t_4)$$

$$\times \int d\boldsymbol{k} v^2(\omega) \exp\{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega(t_3-t_4)\} \cdot \varphi^{(\pm)}(4,3), \qquad (7\cdot7)$$

where $S_N(t_2-t_4)$ is the propagator of N and its Fourier transform is represented by $S_N(p) = (p-m_N+i\varepsilon)^{-1}$.

Let us solve this equation by the same method as described in § 4. Transforming the B-S amplitude as

$$\varphi^{(\pm)}(1,2) = \frac{1}{2\pi} e^{-iET} \int \chi^{(\pm)}(p)\, e^{-ipt}\, dp, \qquad (7\cdot8)$$

in which $t=t_1-t_2$ is the relative time, $a=m/(m+m_N)$, $b=m_N/(m+m_N)$, $T=at_1+bt_2$ and E, of course, the total energy of the V-N system, we have from (7·7), with a little manipulation,

$$(p-bE+m_N)h(p+aE-m_N+i\varepsilon)\chi^{(\pm)}(p)$$

$$= \mp \frac{1}{2\pi i} \int j(p+q+(a-b)E+i\varepsilon)\chi^{(\pm)}(q)\, dq, \qquad (7\cdot9)$$

where

$$j(z) = g^2 \int d\boldsymbol{k} v^2(\omega) \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r})}{z-\omega}. \qquad (7\cdot10)$$

Now, in a way similar to that in the neutral scalar model, $\chi^{(\pm)}(p)$ can be written in its spectral representation,

$$\chi^{(\pm)}(p) = \frac{\sigma_1^{(\pm)}}{p+aE-m+i\varepsilon} + \int_{\mu}^{\infty} \frac{\nu^{(\pm)}(\omega)\, d\omega}{p+aE-m_N-\omega+i\varepsilon} + \frac{\sigma_2^{(\pm)}}{p-bE+m_N-i\varepsilon}, \qquad (7\cdot11)$$

namely, $\chi^{(\pm)}(p)$ has two single poles and a cut as shown in Fig. 2. In comparison with Fig. 1, the absence of a cut on the left is due to that the N-state cannot be modified. Then the integration on the right-hand side of (7·9) can be easily carried out by means of a formula like (4·6b), giving

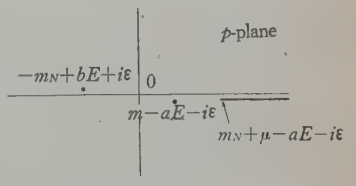


Fig. 2. Analyticity of $\chi^{(\pm)}(p)$ of the Lee model in the complex p-plane.

$$(p-bE+m_N)h(p+aE-m_N+i\varepsilon)\chi^{(\pm)}(p)$$

$$= \mp j(p+aE-m_N+i\varepsilon)\sigma_2^{(\pm)}. \qquad (7\cdot12)$$

This equation corresponds to (4·7) in the neutral scalar model, but is essentially simpler than the latter; putting $p=-m_N+bE$, we have at once an eigenvalue equation,

$$[h(E-2m_N) \pm j(E-2m_N)]\sigma_2^{(\pm)} = 0, \qquad (7\cdot13)$$

in contrary to (4·8) in the neutral scalar model which still contains the spectral function $\nu(\omega)$. This eigenvalue equation is identical with Weinberg's Eq. (16±), so that the B-S equation in the Lee model turns out to give the correct values of total energy of the V-N system. We note in addition that $\sigma_1^{(\pm)}$ and $\nu^{(\pm)}(\omega)$ can be found from (7·12), respectively, by putting $p=m-aE$ and evaluating the jump at the cut from $m_N+\mu-aE$ to ∞.

§ 8. Concluding remarks

In this paper, we have studied, without any approximation, a central problem of the bound state in the quantum field theory about the neutral scalar model and the Lee model. The main analysis was devoted to the exact B-S equation of the neutral scalar model and its consequences. Following Wick, a method of analytical continuation has been used to transform the B-S equation into the eigenvalue equation in the imaginary relative time variable. Some results thus obtained seem to be generally applicable, whereas the others would be dependent on the model.

It is specially noteworthy that we have found a boundary condition at the origin of relative time from the analyticity of B-S amplitude and B-S equation. Though such information might be obtained by the analyticity consideration of B-S amplitude only, taking no account of the B-S equation, but it is sure that the aid of the B-S equation would make the analysis easy. The generalization of this idea to the relativistic case is very desirable and will probably cast a new light on the bound state problms in the relativistic quantum field theory. In this respect, the recent works of some authors on the analyticity of B-S amplitude and B-S equation are remarkable.[6,11,12]

The exact B-S equation in the neutral scalar model has been shown to possess many solutions among which the ground state is normal. An essentially different aspect from the ladder approximation is that, in the case $\mu=0$, there are an infinite number of levels for any small but non-zero value of the coupling constant. Therefore a rather curious appearance of abnormal solutions found in the ladder approximation[1,2,5] seems entirely dependent on the approximation. A fact that the normal state is always a ground state might be a conclusive factor to exclude abnormal states in connection with the analyticity of the B-S amplitude.

The above-stated conclusions will have a general adequacy. Moreover, we have obtained such a result that the exact B-S equation contains not less levels than the ladder-approximative equation does, but of course this and others cannot hold independently of the model.

We have employed all hands, so far known to us, to attack the bound state problem in the neutral scalar model and reached at the above conclusions. We have found, on the other hand, that the same problem gives a unique

solution if one solves the first order equation instead of the B-S equation and that in the Lee model the B-S equation gives rise to no troubles.

There are many ways to understand these facts, but they may be divided into three broad classes: to affirm the abnormal solutions, to deny them physically, and to deny them mathematically. Since the first two standpoints have already been discussed by several authors, we should like to mention only the last one and to emphasize that there would be places for the possibilities to exclude abnormal solutions mathematically. Here the word "mathematically" should be understood in the following sense. We have at the beginning a Hamiltonian describing two-particle system, and define a B-S amplitude, set up an equation, and then solve it by a certain method. At each stage, we must pay attention to the mathematical equivalence of the formulation. This is rather trivial in the first order equation for the B-S amplitude in the neutral scalar model, and it is comparatively straightforward to see this equivalence in the B-S equation of the Lee model because of the simple structure of B-S amplitude.

The situation seems more complicated in the B-S formalism of the neutral scalar model. In this case, the fundamental equation is singular. Two methods have been developed to avoid the singularity. In the first method described in § 5, the equation is reduced to a second order differential equation. In comparison with the case of first order equation, more restrictions on the solution are of course necessary and in fact, besides the boundary condition at infinity, the condition at the origin is supplied by the analyticity consideration. Formally, these two conditions suffice to solve the eigenvalue problem. However, there are other various properties of normal solution which, though automatically fulfilled in the first order equation, must be taken into account to select the correct solution in the second order equation. One example may be the absence of nodes in the normal solution. Anyway, the behaviors of solution at the finite value of variable are very important in connection with the singularity of the fundamental equation. These behaviors should be informed by the analytical structure of the B-S amplitude and the B-S equation. The analysis along this line will probably give a clue to settle the problem of abnormal solutions, at least, in the neutral scalar model.

On the other hand, the second method described in § 4 provides the integral equation and the eigenvalue equation simultaneously, the latter being equivalent to the above-mentioned boundary condition at the origin. The B-S equation of the Lee model was solved by this method and brought no troubles. These facts might allow us to expect that all information about the B-S amplitude and the B-S equation is entirely contained and thus only the normal solution is automatically selected in this method. With much regret, we have not succeeded to solve the eigenvalue equation, but there may remain a trial to ascertain whether the abnormal roots exist or not by considering some limiting values of parameters, for which the abnormal solutions appear in the first

method, and thus reducing the equation to a simple form.

In conclusion, we should like to stress again the importance of studying the analyticity in the bound state problem in quantum field theory.

The author wishes to express his cordial thanks to Prof. Y. Tanikawa for his valuable criticism and warm encouragement throughout this work. He is also grateful to Mr. Y. Kato for many helpful discussions.

Appendix

Derivation of Eq. (2·15a)

For the preliminaries, we remark that the unitary operator U, (2·3), may be regarded as expressed in terms of either α_{i0} and $a(\boldsymbol{k})$ or β_{i0} and $b(\boldsymbol{k})$. In the following calculations, the latter view-point will be convenient. Another remark concerns with the clothed operator of the meson in the Heisenberg representation. Since $b(\boldsymbol{k})$ commutes with N_i, $\boldsymbol{b}(\boldsymbol{k}, t)$ can be easily written as

$$\boldsymbol{b}(\boldsymbol{k}, t)=e^{iHt}b(\boldsymbol{k})\,e^{-iHt}=b(\boldsymbol{k})\,e^{-i\omega t}. \tag{A·1}$$

First, operating $(i\partial/\partial t_1-m)$ on $\varphi_+(1, 2)$, we have, on account of (2·13),

$$(i\partial/\partial t_1-m)\varphi_+(1, 2)=i\delta(t)\langle 0|\boldsymbol{\alpha}_{10}(t_1)\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle$$
$$+\theta(t)\frac{g}{(2\pi)^{3/2}}\int\frac{d\boldsymbol{k}}{(2\omega)^{1/2}}\exp\left(\frac{1}{2}i\boldsymbol{k}\cdot\boldsymbol{r}\right)\langle 0|\boldsymbol{\alpha}_{10}(t_1)\boldsymbol{b}(\boldsymbol{k}, t_1)\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle.$$

In the second term, since $b(\boldsymbol{k})|\mathrm{B}\rangle=0$, it is not difficult to follow the calculation:

$$\boldsymbol{b}(\boldsymbol{k}, t_1)\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle=Z^{-1/2}\exp\{i(H-E)t_2\}\,b(\boldsymbol{k})\,e^{-i\omega t}\beta_{20}\exp(-gA_2^*)|\mathrm{B}\rangle$$
$$=-\frac{g}{(2\pi)^{3/2}}\frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r}/2)}{\sqrt{2}\,\omega^{3/2}}e^{-i\omega t}Z^{-1/2}\exp\{i(H-E)t_2\}\beta_{20}\exp(-gA_2^*)|\mathrm{B}\rangle$$
$$=-\frac{g}{(2\pi)^{3/2}}\frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r}/2)}{\sqrt{2}\,\omega^{3/2}}e^{-i\omega t}\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle. \tag{A·2}$$

Accordingly, the above equation becomes

$$(i\partial/\partial t_1-m)\varphi_+(1, 2)=i\delta(t)\langle 0|\boldsymbol{\alpha}_{10}(t_1)\boldsymbol{\alpha}_{20}(t_2)|\mathrm{B}\rangle$$
$$-\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\,e^{-i\omega t}\varphi_+(1, 2). \tag{A·3}$$

Next, operating $(i\partial/\partial t_1-m)$ on $\varphi_-(1, 2)$, we obtain in a similar way

$$(i\partial/\partial t_1-m)\varphi_-(1, 2)=i\delta(t)\langle 0|\boldsymbol{\alpha}_{20}(t_2)\boldsymbol{\alpha}_{10}(t_1)|\mathrm{B}\rangle$$
$$+\frac{g^2}{(2\pi)^3}\int\frac{d\boldsymbol{k}}{2\omega^2}\cos(\boldsymbol{k}\cdot\boldsymbol{r})\,e^{i\omega t}\varphi_-(1, 2)-g^2V(r)\varphi_-(1, 2). \tag{A·4}$$

We at once find (2·15a) by adding (A·3) and (A·4).

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A Shell Model Calculation for Scattering of Electrons by Be^9

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Elastic and inelastic scattering of 190 Mev electrons by ground and 2.43 Mev states of Be^9 is computed by using an intermediate coupling shell model. The discrepancy between the calculated and the observed inelastic cross sections reported earlier is now removed by taking the correct spin value $J=5/2^-$ for the 2.43 Mev state. It is shown that a good fit to the observed data requires that different radial extension parameters be chosen for the s- and the p-shell nucleons. The best values obtained are $a_0=1.23$ f for the s-shell and 2.0 f for the p-shell.

§ 1. Introduction

The charge distributions of nuclei in p-shell have been investigated in considerable detail by means of electron scattering experiments.[1] For Be^9, calculations for elastic and inelastic scattering cross sections have been made using a number of different nuclear models.[2] In particular, Pal and Mukherjee have applied the intermediate coupling shell model to obtain the wave functions of the various states of Be^9, and calculated from these the scattering cross sections. Elastic scattering from the ground state and inelastic scattering from 2.43 Mev and 6.8 Mev states of Be^9 has been observed for electrons of 190 Mev energy.[3] The results of the calculations of Pal and Mukherjee for this case agree well with the observed results for the elastic scattering, but there is a pronounced disagreement between the calculated and the observed results for inelastic scattering for the magnitude and also for the angular distribution. These authors have used a mixture of central, tensor and spin-orbit forces with suitable parameters to obtain the wave functions and energy levels in the intermediate coupling. No attempt has been made to obtain a precise agreement with observed energy level scheme of Be^9. Besides, it appears that they identify the 2.43 Mev level as $J=1/2^-$. It is now well known[4] that this level is $J=5/2^-$. This has prompted us to make a reevaluation of the scattering cross sections in the intermediate coupling model, and to seek a better agreement with the observed results. Our calculations show that not only an improved agreement between theory and experiment can be obtained by taking into account the correct spin value $J=5/2^-$ of the 2.43 Mev level, but a satisfactory fit with the experimental results for elastic as well as inelastic scattering can be obtained

only by taking different values of the radial parameter a_0 in the harmonic oscillator wave functions for the s-shell and the p-shell. Some information on the value of the intermediate coupling parameter ζ is also obtained.

§ 2. Calculations

French, Halbert and Pandya[5] have made an intermediate coupling calculation for the energy levels of Be^9. We choose for the wave functions of the ground and 2.43 Mev states the results given by them. For small values of the intermediate coupling parameter ζ (as defined by French, Halbert and Pandya), the components in the wave functions with symmetry other than [41] are very small, and these are neglected by Pal and Mukherjee. We have verified by explicit evaluation that the effect of these components on the scattering cross sections is indeed quite insignificant.

The scattering cross section is given by equations (1) and (9) of Tassie's paper.[2] The matrix element $\rho_{nn'}$ occurring in the expression is given by

$$\rho_{nn'}=\langle JM, TM_T|\sum_j e^{-i\boldsymbol{K}\cdot\boldsymbol{R}_j}\cdot\frac{1}{2}(1-\tau_3^{(j)})\,|J'M', T'M_T'\rangle, \tag{1}$$

where JM and $J'M'$ denote the final and the initial states of the nucleus, and the other symbols are defined in references 2). Here contributions to the scattering by the current terms (shown to be small by Pal and Mukherjee) and the magnetic moments of the nucleons are neglected. The wave functions $|JM\rangle$ are written in the intermediate coupling shell model as superposition of the suitable L-S coupling terms,

$$|JM, TM_T\rangle=\sum K'(JT, \alpha LS)\,|\alpha LS, JM, TM_T\rangle. \tag{2}$$

The coefficients $K'(JT, \alpha LS)$ are tabulated by French, Halbert and Pandya, for different values of the parameter ζ. The matrix element $\rho_{nn'}$ then contains terms of the type

$$\begin{aligned}
&\langle (l)^n \alpha LS, JM, TM_T|\sum_j e^{-i\boldsymbol{K}\cdot\boldsymbol{R}_j}\cdot\frac{1}{2}(1-\tau_3^{(j)})\,|(l)^n\alpha' L'S', J'M', T'M_T'\rangle\\
&=\sum_k(-i)^k(2k+1)\langle j_k(KR)\rangle_l\cdot\frac{1}{2}n\;\delta_{ss'}\delta_{M_TM_T'}\delta_{MM'}\\
&\times[J'LL']^{1/2}\cdot D_{ll'k}\langle J'M'k0|JM\rangle\sum\langle (l)^n\alpha LS, T|\}\,(l)^{n-1}\tilde{\alpha}\tilde{L}\tilde{S}, \tilde{T}\rangle\\
&\times\langle (l)^{n-1}\tilde{\alpha}\tilde{L}\tilde{S}, \tilde{T}|\}\,(l)^n\alpha' L'S', T'\rangle\cdot(-1)^{\tilde{L}+S-J}\cdot W(LJL'J'; Sk)\\
&\times W(lLlL; \tilde{L}k)\cdot\Big\{\delta_{TT'}\delta_{M_TM_T'}-\delta_{M_TM_T'}\cdot\sqrt{6(2T'+1)}\\
&\times(-1)^{\tilde{T}+1/2-T'}\cdot\langle T'M_T'10|TM_T\rangle\cdot W\Big(\frac{1}{2}T\frac{1}{2}T'; \tilde{T}1\Big)\Big\}
\end{aligned} \tag{3}$$

where,

$$\langle j_k(KR)\rangle_l=\int_0^\infty \psi_l^* j_k(KR)\psi_l \rho^2 d\rho,\ \rho=\frac{R}{a_0}\ ,$$

ψ_l is the radial harmonic oscillator wave function and involves the parameter a_0 as $\exp(-r^2/a_0^2)$ and $j_k(KR)$ are the spherical Bessel functions, the other notations being as follows:

$$[abc]^{1/2}=[(2a+1)(2b+1)(2c+1)]^{1/2},$$

$$D_{ll'k}=[ll'/k]^{1/2}\cdot\langle l0l'0|k0\rangle.$$

$\langle J'M'JM|k0\rangle$ are the well-known Clebsch-Gordan coefficients, $\langle\cdots\cdots|\}\cdots\cdots\rangle$ are the coefficients of fractional parentage, and are listed by Jahn and Van Wieringen,[6] $W(abcd;ef)$ are the Racah coefficients. The derivation of Eq. (3) follows standard tensor algebraic procedures developed by Racah. It is now easy to see that for scattering from the closed s-shell, the matrix element takes the simple value $2\langle j_0(KR)\rangle_{l=0}$, where

$$\langle j_0(KR)\rangle_{l=0}=\int_0^\infty \psi^*(s)\, j_0(KR)\psi(s)\rho^2 d\rho=\mathcal{A}.$$

We also write:

$$\langle j_0(KR)\rangle_{l=1}=\int_0^\infty \psi^*(p)\, j_0(KR)\psi(p)\rho^2 d\rho=\mathcal{B}\ ,$$

$$\langle j_2(KR)\rangle_{l=1}=\int_0^\infty \psi^*(p)\, j_2(KR)\psi(p)\rho^2 d\rho=\mathcal{C}.$$

The above result for s-shell scattering also follows easily from Pal and Mukherjee's equation (3a), as on carrying out the summation over $\widetilde{T}$, the second term in brackets vanishes.

§ 3. Results and Discussion

Figs. 1 and 2 show the results of the calculations as compared with the observed results for inelastic and elastic scattering for 2.43 Mev and the ground states respectively. First we discuss the inelastic scattering (Fig. 1) as this depends only on the radial integral $\mathcal{C}$ which involves the wave functions of the p-shell nucleons, and enables us to determine directly the parameter a_0 for the p-shell wave functions. It can be seen that a good agreement with the observed results is obtained for $\zeta=1.4$ and $a_0=2.0$ f. Increase in ζ leads to larger calculated cross sections, as may be seen from the curve for $\zeta=2.8$. Also shown in this figure is a curve for the energy level $J=1/2^-$, for $\zeta=1.4$ and $a_0=1.67$ f. This agrees well with the curve obtained by Pal and Mukherjee

for this state, and shows that the lack of agreement with the experimental results in their case is due to an incorrect assumption for the spin (and consequently the wave function) of the 2.43 Mev state.

In Fig. 2 is shown the calculated curve for elastic scattering using the same parameters as obtained from analysis of the inelastic scattering, i.e. $a_0=2.0$ f, $\zeta=1.4$ and the same value of a_0 for both s-shell and p-shell harmonic oscillator wave functions. The agreement with the experimental results is now seen to be rather poor in this case, and it may also be seen that variation of a_0 does not improve the matters; although with $a_0=1.7$ f, the agreement is not too bad, it is still not quite satisfactory. We have checked that for elastic scattering, variation of ζ from 1.4 to 2.8 does not alter the cross sections substantially. We therefore attempted to choose different values for the parameter a_0 in the s-shell and in the p-shell, i.e. the value of a_0 used in the radial integral $\mathcal{A}$ is different from that used in the integrals $\mathcal{B}$ and $\mathcal{C}$. It is then found that for the choice $a_0=2.0$ f. in p-shell and $a_0=1.23$ f. in s-shell, it is possible to obtain a satisfactory agreement with the observed results.

The results obtained above are similar to those of Burleson and Hofstadter,[7] and of Elton[8] for Li^6, where they also find it necessary to assume different oscillator para-

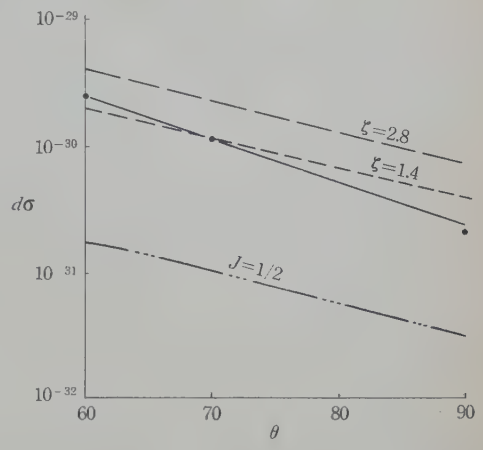


Fig. 1. Inelastic scattering of 190 Mev electrons by 2.43 Mev state ($J=5/2^-$) of Be^9. The dashed lines represent the curves for $a_0=1.67$ f., $\zeta=1.4$ and 2.8; the solid line represents the cross section for $a_0=2.0$ f, $\zeta=1.4$. Also shown in the figure is a curve for $J=1/2^-$, $a_0=1.67$ f, $\zeta=1.4$. The dots represent the experimental values.

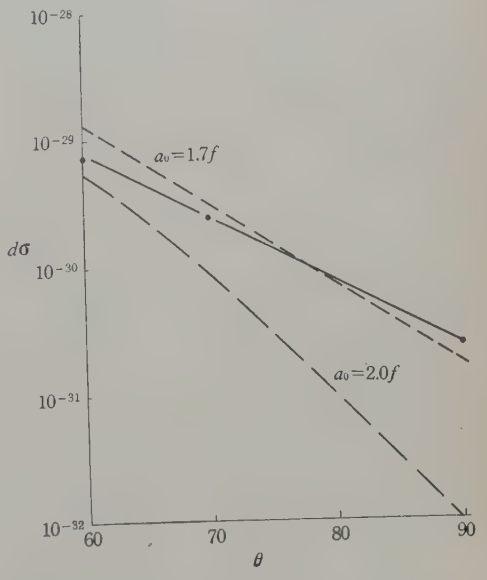


Fig. 2. Elastic scattering of 190 Mev electrons by Be^9. The dashed lines represent the curves for $a_0=1.7$ f and $a_0=2.0$ f with $\zeta=1.4$. The solid curve represents the cross section for $a_0(s)=1.23$ f, and $a_0(p)=2.0$ f with $\zeta=1.4$; the dots represent the experimental values.

meters a_0 for s-shell and p-shell nucleons. Unlike Burleson and Hofstadter, but agreeing with Elton, our results show a larger value of a_0 for p-shell than that for s-shell, which apparently signifies a more extended potential well for p-nucleons than for s-nucleons. It is also interesting to note from Figs. 1 and 2, that for an intermediate value of a_0 viz. $a_0=1.7$ f, it is possible to obtain a not too bad fit with experimental results for elastic scattering, but the discrepancy in inelastic scattering would be considerable in this case. In this connexion we also mention the recent results of Jackson[9] who has investigated the scattering of electrons from Li^6 using a modified form of the harmonic oscillator potential, but with the same parameters for s- and p-shell nucleons. One may conclude from all these results that a simple single-parameter harmonic oscillator shell model for describing the charge-distributions of p-shell nuclei seems to be inadequate, and at least a two-parameter model is needed.

The elastic scattering cross section is found to be insensitive to changes in value of ζ from 1.4 to 2.8. However, the inelastic cross section changes considerably with ζ, and gives the best fit for $\zeta=1.4$. This agrees with the earlier suggestion of French, Halbert and Pandya, but disagrees with Pinkston[10] who finds $\zeta=2.8$ to be the better value from an analysis of inelastic scattering of α-particles from Be^9.

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On the Axial Vector Current Conservation

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The conservation law of axial vector in the β-decay is investigated in the system of the pion-nucleon interacting through $ps(ps)$ and $ps(pv)$ couplings together. The system necessarily contains other types of boson beside the pion-field in order to realize the conservation law in a limit of pion mass zero. Three new bosons, scalar (iso-scalar), vector (iso-vector) and pseudo-vector (iso-vector) bosons, are at least necessary. The meaning of their coexistence is discussed from the theory of composite model.

§ 1. Introduction

The idea of the conservation of axial vector nucleon-current in the weak interaction has been pursued by many authors because of the small deviation of its constant from the one of the vector nucleon-current. As the strict conservation law would not hold in this case, its deviation should be minimized so as to relate with some constant, pion or nucleon mass. The relation proposed by Goldberger and Treiman[1]

$$\partial_\mu \boldsymbol{P}_\mu = i \cdot \text{const} \cdot \mu_\pi{}^2 \boldsymbol{\pi} \tag{1·1}$$

will be the desired one in which the deviation is related with the pion mass, where $\boldsymbol{P}_\mu$ is the axial vector current and π and μ_π are pion field and its mass.[2] One profit of this relation is that we can explain the pion lifetime without knowing its decay interaction. The theoretical justification of this relation, however, is incomplete unless we assume the special models for the strongly interacting system.

When the pion interacts with nucleon through $ps(pv)$ coupling (its constant being g_A), the relation can be automatically satisfied by

$$\boldsymbol{P}_\mu = \bar{N}\boldsymbol{\tau}\gamma_\mu\gamma_5 N - \frac{i}{g_A}\partial_\mu \boldsymbol{\pi}, \quad \partial_\mu \boldsymbol{P}_\mu = -\frac{i}{g_A}\mu_\pi{}^2 \boldsymbol{\pi}, \tag{1·2}$$

where N is the nucleon field and usual notations are used for other quantities (the gradient coupling model). While for the case of $ps(ps)$ coupling (its constant being g_P), the relation is not confined to the pion and nucleon fields. Polkinghorne[3] and then Gell-Mann-Lévy[2] stated a simple model using the scalar (iso-scalar) boson, its coupling constant being restricted by

$$f_S = g_P \tag{1·3}$$

and the proposed axial vector current is given by

$$\boldsymbol{P}_\mu = \bar{N}\boldsymbol{\tau}\gamma_\mu\gamma_5 N + \frac{2mi}{g_P}\partial_\mu\boldsymbol{\pi} + 2i(\sigma\cdot\partial_\mu\boldsymbol{\pi} - \partial_\mu\sigma\cdot\boldsymbol{\pi})\ ;\quad \partial_\mu\boldsymbol{P}_\mu = \frac{2mi}{g_P}\mu_\pi{}^2\boldsymbol{\pi}, \tag{1·4}$$

where σ and m are the scalar boson field and the nucleon mass respectively (the σ-model). The difference of mass of both bosons introduces the non-linear interactions between them.

Beside these extreme cases, we have not any theoretical background for the Goldberger-Treiman relation. The circumstance can be interpreted by two ways: The incompleteness of its deduction may be closely connected with the ignorance of the contributions from other hyperons and K-meson or it may be due to the insufficient introduction of related fields. We may answer that neither one of these viewpoints is more preferable from our present scope in field theory.

In spite of this situation, there still remains one probable analysis which is related with the idea of composite model for elementary particles. If we start with the so-called (ur-)nucleon field and its non-linear interaction, we can safely construct the desired conservation law of which deviation can be minimized by choosing the ur-nucleon mass and the type of non-linear interaction. Then when we construct the composite entities by some unknown but suitable procedure, we shall again get the real conservation law which has the similar content as (1·2) or (1·4). For this case it may be improbable that the resultant pion-nucleon interaction is reduced to one of the both extreme cases illustrated above.

Unfortunately, this procedure sensitively depends on the theory which is not yet justified. For instance, how we can remove the divergence and how we can solve the Bethe-Salpeter-like equation are still open to question.

However, we can abstract this procedure in a little phenomenological waa and introduce several boson fields without any justification from the dynamiycl procedure just as the ideal treatment in solid physics. If the original system gives the conservation law, we can also get a simlar conservation law, and inversely if we can construct the conservation law in this ideal case, we have also a suitable original system.

The restriction imposed on our consideration is that we have the pion field with the general couplings, i.e. with $ps(ps)$ and $ps(pv)$ together. And our aim is to know whether we can construct the conservation law using various kinds of the related fields which are restricted to be boson (§ 2). Since the problem stands by itself without the relation of the composite model, we then economize and make the related fields beside the pion as simple as possible (§ 3). The nature of thus introduced bosons are discussed from the

viewpoint of transformation (§ 4) and finally we come back to the theory of composite model (§ 5).

§ 2. Conserved axial vector current

The assumptions we start with are as follows:

i) Fermion in our system is restricted to be nucleon only and we neglect the effects of other hyperons. Lagrangian is

$$L_N=-\bar{N}(\gamma\partial+m)N. \tag{2·1}$$

ii) We tentatively introduce as many bosons as possible from the transformation character beside the pion. The only restriction is that their values of spin and iso-spin are not greater than 1. Possible boson fields are of the following eight kinds.

pseudoscalar (iso-vector) $L_B^{(1)}=-1/2[(\partial_\mu\boldsymbol{\pi})^2+\mu_\pi{}^2\boldsymbol{\pi}^2]$,

scalar (iso-scalar) $L_B^{(2)}=-1/2[(\partial_\mu\sigma)^2+\mu_\sigma{}^2\sigma^2]$,

pseudoscalar (iso-scalar) $L_B^{(3)}=-1/2[(\partial_\mu\rho)^2+\mu_\rho{}^2\rho^2]$, (2·2)

scalar (iso-vector) $L_B^{(4)}=-1/2[(\partial_\mu\boldsymbol{s})^2+\mu_s{}^2\boldsymbol{s}^2]$,

vector (iso-scalar) $L_B^{(5)}=-1/2[1/2(v_{\mu\nu})^2+\mu_v{}^2v_\mu{}^2]$,

$v_{\mu\nu}=\partial_\mu v_\nu-\partial_\nu v_\mu$,

pseudovector (iso-scalar) $L_B^{(6)}=-1/2[1/2(a_{\mu\nu})^2+\mu_a{}^2a_\mu{}^2]$,

$a_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$,

vector (iso-vector) $L_B^{(7)}=-1/2[1/2(\boldsymbol{u}_{\mu\nu})^2+\mu_u{}^2\boldsymbol{u}_\mu{}^2]$,

$\boldsymbol{u}_{\mu\nu}=\partial_\mu\boldsymbol{u}_\nu-\partial_\nu\boldsymbol{u}_\mu$,

pseudovector (iso-vector) $L_B^{(8)}=-1/2[1/2(\boldsymbol{w}_{\mu\nu})^2+\mu_w{}^2\boldsymbol{w}_\mu{}^2]$,

$\boldsymbol{w}_{\mu\nu}=\partial_\mu\boldsymbol{w}_\nu-\partial_\nu\boldsymbol{w}_\mu$.

The boson with half-integer iso-spin is excluded in our system.

iii) The interactions of bosons with nucleon are all of charge-independent Yukawa type, which are given by

$$\begin{aligned}
L_{BN}^{(1)}&=-ig_P\bar{N}\gamma_5(\boldsymbol{\tau}\boldsymbol{\pi})N-ig_A\bar{N}\gamma_\mu\gamma_5(\boldsymbol{\tau}\partial_\mu\boldsymbol{\pi})N,\\
L_{BN}^{(2)}&=-f_S\bar{N}\sigma N-if_V\bar{N}\gamma_\mu(\partial_\mu\sigma)N,\\
L_{BN}^{(3)}&=-if_P\bar{N}\gamma_5\rho N-if_A\bar{N}\gamma_\mu\gamma_5\partial_\mu\rho\cdot N,\\
L_{BN}^{(4)}&=-g_S\bar{N}(\boldsymbol{\tau}\boldsymbol{s})N-ig_V\bar{N}\gamma_\mu(\boldsymbol{\tau}\partial_\mu\boldsymbol{s})N,\\
L_{BN}^{(5)}&=-iF_V\bar{N}\gamma_\mu v_\mu N,\\
L_{BN}^{(6)}&=-iF_A\bar{N}\gamma_\mu\gamma_5 a_\mu N,
\end{aligned} \tag{2·3}$$

$$L_{BN}^{(7)} = -iG_V \bar{N}\gamma_\mu(\boldsymbol{\tau}\boldsymbol{u}_\mu)N,$$

$$L_{BN}^{(8)} = -iG_A \bar{N}\gamma_\mu\gamma_5(\boldsymbol{\tau}\boldsymbol{w}_\mu)N.$$

The tensor interactions are excluded in the case of spin 1 bosons.

iv) Beside these interactions, we assume that there are some unknown interactions or non-linear terms which are composed of boson fields only. They depend on boson field quantities and their first derivatives. The definite form of Lagrangian

$$L_B' \equiv L'(\boldsymbol{\pi}, \partial_\mu\boldsymbol{\pi}\,;\, \sigma, \partial_\mu\sigma\,;\, \text{etc.}) \tag{2·4}$$

will be fixed so as to realize the conservation law.

The result is easily obtained after elementary calculations: there must be the relations of the coupling constants in order to get the conserved axial vector current.

$$f_S = g_P,\quad f_P = g_S,\quad G_V = G_A \equiv G. \tag{2·5}$$

The axial vector current is obtained as

$$\begin{aligned}
\boldsymbol{P}_\mu &= \bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N + \frac{2mi}{g_P}\partial_\mu\boldsymbol{\pi} + 2i(\sigma\partial_\mu\boldsymbol{\pi} - \partial_\mu\sigma\cdot\boldsymbol{\pi}) \\
&\quad + 2i(\boldsymbol{s}\partial_\mu\rho - \partial_\mu\boldsymbol{s}\cdot\rho) + 2i\left(\frac{g_A}{G}\boldsymbol{\pi}\times\partial_\nu\boldsymbol{u}_{\nu\mu} + \frac{g_V}{G}\boldsymbol{s}\times\partial_\nu\boldsymbol{w}_{\nu\mu}\right) \\
&\quad + 2i(\boldsymbol{u}_\nu\times\boldsymbol{w}_{\mu\nu} + \boldsymbol{w}_\nu\times\boldsymbol{u}_{\mu\nu}), \qquad (2·6) \\
\partial_\mu\boldsymbol{P}_\mu &= \frac{2mi}{g_P}\mu_\pi^2\boldsymbol{\pi}.
\end{aligned}$$

The unknown non-linear terms are determined as the functional form,

$$\begin{aligned}
L_B' &= \frac{1}{2}(\mu_\sigma^2 - \mu_\pi^2)\sigma^2 + \mathcal{F}_1\left[g_P^2\left(\boldsymbol{\pi}^2 + \left(\sigma + \frac{m}{g_P}\right)^2 - \frac{m^2}{g_P^2}\right)\right] \\
&\quad + \frac{1}{2}\mu_u^2\boldsymbol{u}_\mu^2 + \frac{1}{2}\mu_w^2\boldsymbol{w}_\mu^2 + \mathcal{F}_2(G\boldsymbol{u}_\mu + g_V\partial_\mu\boldsymbol{s})^2 + (G\boldsymbol{w}_\mu + g_A\partial_\mu\boldsymbol{\pi})^2] \\
&\quad + \frac{1}{2}\mu_\rho^2\rho^2 + \frac{1}{2}\mu_s^2\boldsymbol{s}^2 + \mathcal{F}_3[g_S^2(\rho^2 + \boldsymbol{s}^2)] \\
&\quad + \frac{1}{2}\mu_v^2 v_\mu^2 + \mathcal{F}_4[(F_V v_\mu + f_V\partial_\mu\sigma)^2] \\
&\quad + \frac{1}{2}\mu_a^2 a_\mu^2 + \mathcal{F}_5[(F_A a_\mu + f_A\partial_\mu\rho)^2] \qquad (2·7)
\end{aligned}$$

(See the Appendix). The functions $\mathcal{F}_i (i=1, \cdots 5)$ are arbitrary except the general conditions

$$\mathcal{F}_i(0) = 0,\quad \mathcal{F}_1'(0) = 0. \tag{2·8}$$

However, it is interesting to see their effects.

1) The second derivative of $\mathcal{J}_1$ is determined in order to cancel out the first term,

$$\mathcal{J}_1''(0)=(\mu_\pi^2-\mu_\sigma^2)/4m^2g_P^2. \tag{2·9}$$

The function is given at least by

$$\mathcal{J}_1=\frac{g_P^2}{8m^2}(\mu_\pi^2-\mu_\sigma^2)\left[\boldsymbol{\pi}^2+\left(\sigma+\frac{m}{g_P}\right)^2-\frac{m^2}{g_P^2}\right]^2 \tag{2·10}$$

which is in agreement with the Gell-Mann-Lévy model.

2) The first derivatives of $\mathcal{J}_t$ other than $\mathcal{J}_1$ give the boson masses on one hand and their renormalization effects. Then we have

$$\mu_v^2=-2F_V^2\mathcal{J}_4'(0),\quad \mu_a^2=-2F_A^2\mathcal{J}_5'(0) \tag{2·11}$$

and pairs of $(\boldsymbol{u}_\mu, \boldsymbol{w}_\mu)$ and $(\rho, \boldsymbol{s})$ should have the same mass values

$$\begin{aligned}&\mu_u^2=\mu_w^2=-2G^2\mathcal{J}_2'(0),\\ &\mu_\rho^2=\mu_s^2=-2g_S^2\mathcal{J}_3'(0).\end{aligned} \tag{2·12}$$

The renormalization effects are

$$\begin{aligned}&Z_\pi=1-2g_A^2\mathcal{J}_2'(0),\quad Z_\rho=1-2f_A^2\mathcal{J}_5'(0),\\ &Z_\sigma=1-2f_V^2\mathcal{J}_4'(0),\quad Z_s=1-2g_V^2\mathcal{J}_2'(0),\\ &Z_a=Z_v=Z_u=Z_w=1.\end{aligned} \tag{2·13}$$

3) There appear the contact linear interactions which are

$$\begin{aligned}&2\mathcal{J}_2'(0)g_VG\boldsymbol{u}_\mu\partial_\mu\boldsymbol{s}+2\mathcal{J}_2'(0)g_AG\boldsymbol{w}_\mu\partial_\mu\boldsymbol{\pi}\\ &\quad+2\mathcal{J}_4'(0)f_VF_Vv_\mu\partial_\mu\sigma+2\mathcal{J}_5'(0)f_AF_Aa_\mu\partial_\mu\rho\\ &=-\frac{g_V}{G}\mu_u^2\boldsymbol{u}_\mu\partial_\mu\boldsymbol{s}-\frac{g_A}{G}\mu_w^2\boldsymbol{w}_\mu\partial_\mu\boldsymbol{\pi}-\frac{f_V}{F_V}\mu_v^2v_\mu\partial_\mu\sigma-\frac{f_A}{F_A}\mu_a^2a_\mu\partial_\mu\rho.\end{aligned} \tag{2·14}$$

§ 3. A model for the case of *ps* and *pv* couplings mixture

The model obtained here is too complicated to seek for its nature. We make it as simple as possible with the fixed values of g_P and g_A of pion coupling constants, taking the limiting procedure of some coupling constants and some unknown fields. Firstly we can take

$$F_V=F_A=0 \tag{3·1}$$

and simultaneously omit the vector (iso-scalar) and pseudovector (iso-scalar) fields,

$$v_\mu\to 0,\quad a_\mu\to 0.$$

This again makes the limit of

$$f_V=f_A=0. \tag{3·2}$$

Secondly, if we also take

$$g_S(=f_P)=g_V=0, \tag{3·3}$$

we can omit the scalar (iso-vector) and pseudoscalar (iso-scalar) fields,

$$\boldsymbol{s}\to 0, \quad \rho\to 0.$$

Further simplification is impossible owing to the finite values of g_P and g_A.

The resultant model is

$$\begin{aligned} L=&-\bar{N}(\gamma\partial+m)N-\frac{1}{2}[(\partial_\mu\sigma)^2+\mu_\pi{}^2\sigma^2]-\frac{1}{2}[(\partial_\mu\boldsymbol{\pi})^2+\mu_\pi{}^2\boldsymbol{\pi}^2] \\ &-\frac{1}{4}\boldsymbol{u}_{\mu\nu}{}^2-\frac{1}{4}\boldsymbol{w}_{\mu\nu}{}^2+\mathscr{J}_1\left[g_P{}^2\left(\boldsymbol{\pi}^2+\left(\sigma+\frac{m}{g_P}\right)^2-\frac{m^2}{g_P{}^2}\right)\right] \\ &+\mathscr{J}_2[G^2\boldsymbol{u}_\mu{}^2+(G\boldsymbol{w}_\mu+g_A\partial_\mu\boldsymbol{\pi})^2] \\ &-g_P N(\sigma+i(\boldsymbol{\tau}\boldsymbol{\pi})\gamma_5)N-ig_A\bar{N}\gamma_\mu\gamma_5(\boldsymbol{\tau}\partial_\mu\boldsymbol{\pi})N \\ &-iG\bar{N}\gamma_\mu((\boldsymbol{\tau}\boldsymbol{u}_\mu)+\gamma_5(\boldsymbol{\tau}\boldsymbol{w}_\mu))N \end{aligned} \tag{3·4}$$

and the axial vector current becomes

$$\begin{aligned} \boldsymbol{P}_\mu=&\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N+\frac{2mi}{g_P}\partial_\mu\boldsymbol{\pi}+2i(\sigma\partial_\mu\boldsymbol{\pi}-\partial_\mu\sigma\cdot\boldsymbol{\pi}) \\ &+2i(\boldsymbol{u}_\nu\times\boldsymbol{w}_{\mu\nu}+\boldsymbol{w}_\nu\times\boldsymbol{u}_{\mu\nu})+2i\frac{g_A}{G}\boldsymbol{\pi}\times\partial_\nu\boldsymbol{u}_{\mu\nu}. \end{aligned} \tag{3·5}$$

Then we can conclude that in order to realize the axial vector current conservation in the system of $g_P\neq 0$ and $g_A\neq 0$, the co-existence of three types of bosons, σ, $\boldsymbol{u}_\mu$ and $\boldsymbol{w}_\mu$, at least is necessary beside the pion field $\boldsymbol{\pi}$.

It is remarkable that the vector and pseudovector (iso-vector) bosons appear with the same coupling constant which magnitude is not restricted by this consideration. If we define the new fields as

$$\boldsymbol{X}_\mu=\frac{1}{\sqrt{2}}(\boldsymbol{u}_\mu+\boldsymbol{w}_\mu), \quad \boldsymbol{Y}_\mu=\frac{1}{\sqrt{2}}(\boldsymbol{u}_\mu-\boldsymbol{w}_\mu), \tag{3·6}$$

their coupling form is

$$-iG\bar{N}\gamma_\mu\frac{1+\gamma_5}{\sqrt{2}}\boldsymbol{\tau}N\cdot\boldsymbol{X}_\mu-iG\bar{N}\gamma_\mu\frac{1-\gamma_5}{\sqrt{2}}\boldsymbol{\tau}N\cdot\boldsymbol{Y}_\mu. \tag{3·7}$$

It may be a little suggestive that G could be chosen as

$$F_w=\frac{G^2}{\mu_u{}^2}, \tag{3·8}$$

where f_w is the weak coupling constant,* in order to identify $\boldsymbol{X}_\mu$ with the intermediate boson[7] but the parity non-conservation should be realized by the mysterious subsidiary condition

$$\boldsymbol{Y}_\mu \Omega_{vac}=0. \tag{3·9}$$

§ 4. Transformation characters of boson fields

The axial vector current conservation is induced by the infinitesimal transformation. This transformation makes the nucleon field change

$$N\to(1+i\gamma_5(\boldsymbol{\alpha}\boldsymbol{\tau}))N, \tag{4·1}$$

where $\boldsymbol{\alpha}$ is some constant iso-vector. We can determine the simultaneous change of boson fields with the restriction of

$$f_S=g_P,\ f_P=g_S,\ G_V=G_A\equiv G \tag{4·2}$$

as follows:

$$\begin{aligned}
&\sigma'\to\sigma'-2(\boldsymbol{\alpha}\boldsymbol{\pi}),\ \ \boldsymbol{\pi}\to\boldsymbol{\pi}+2\boldsymbol{\alpha}\sigma',\ \ \left(\sigma'=\sigma+\frac{m}{g_P}\right)\\
&\rho\to\rho+2(\boldsymbol{\alpha}\boldsymbol{s}),\ \ \boldsymbol{s}\to\boldsymbol{s}-2\boldsymbol{\alpha}\rho,\\
&v_\mu\to v_\mu-2\frac{g_V}{F_V}(\boldsymbol{\alpha}\partial_\mu\boldsymbol{\pi}),\\
&a_\mu\to a_\mu+2\frac{g_A}{F_A}(\boldsymbol{\alpha}\partial_\mu\boldsymbol{s}),\\
&\boldsymbol{u}_\mu\to\boldsymbol{u}_\mu-2\frac{g_V}{G}\boldsymbol{\alpha}\partial_\mu\rho+2\boldsymbol{\alpha}\times\left(\boldsymbol{w}_\mu+\frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right),\\
&\boldsymbol{w}_\mu\to\boldsymbol{w}_\mu+2\frac{g_A}{G}\boldsymbol{\alpha}\partial_\mu\sigma'+2\boldsymbol{\alpha}\times\left(\boldsymbol{u}_\mu+\frac{g_V}{G}\partial_\mu\boldsymbol{s}\right)
\end{aligned} \tag{4·3}$$

which make the quantities

$$\begin{aligned}
&\boldsymbol{\pi}^2+\sigma'^2,\ \ \rho^2+\boldsymbol{s}^2,\ \ \left(v_\mu+\frac{f_V}{F_V}\partial_\mu\sigma'\right)^2,\ \ \left(a_\mu+\frac{f_A}{F_A}\partial_\mu\rho\right)^2,\\
&\left(\boldsymbol{u}_\mu+\frac{g_V}{G}\partial_\mu\boldsymbol{s}\right)^2+\left(\boldsymbol{w}_\mu+\frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right)^2
\end{aligned} \tag{4·4}$$

invariant under this transformation.

We can separate the changes into five elements as

* At first glance, the identification of (3·8) makes the effect of vector boson to the system negligible. But the contribution to the axial vector current becomes inversely large from the last term of (3·5).

$$\text{I}\quad \sigma' \to \sigma' - 2(\boldsymbol{\alpha}\boldsymbol{\pi}),\quad \boldsymbol{\pi} \to \boldsymbol{\pi} + 2\boldsymbol{\alpha}\sigma'. \tag{4·5}$$

$$\text{II}\quad \rho \to \rho + 2(\boldsymbol{\alpha}\boldsymbol{s}),\quad \boldsymbol{s} \to \boldsymbol{s} - 2\boldsymbol{\alpha}\rho. \tag{4·6}$$

$$\text{III}\quad v_\mu + \frac{f_V}{F_V}\partial_\mu \sigma' \to v_\mu + \frac{f_V}{F_V}\partial_\mu \sigma'. \tag{4·7}$$

$$\text{IV}\quad a_\mu + \frac{f_A}{F_A}\partial_\mu \rho \to a_\mu + \frac{f_A}{F_A}\partial_\mu \rho. \tag{4·8}$$

$$\text{V}\quad \left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu \boldsymbol{s}\right) \to \left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu \boldsymbol{s}\right) + 2\boldsymbol{\alpha} \times \left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu \boldsymbol{\pi}\right), \tag{4·9}$$

$$\left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu \boldsymbol{\pi}\right) \to \left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu \boldsymbol{\pi}\right) + 2\boldsymbol{\alpha} \times \left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu \boldsymbol{s}\right).$$

Thus we can see that $v_\mu + f_V/F_V \cdot \partial_\mu \sigma'$ and $a_\mu + f_A/F_A \cdot \partial_\mu \rho$, $(\sigma', \boldsymbol{\pi})$ and $(\rho, \boldsymbol{s})$, and $(\boldsymbol{u}_\mu + g_V/G \cdot \partial_\mu \boldsymbol{s}, \boldsymbol{w}_\mu + g_A/G \cdot \partial_\mu \boldsymbol{\pi})$ are scalar, vector and tensor, respectively, in the four-dimensional space of this transformation. It should be noted that the scalar field σ is not covariant quantity without the cooperation of nucleon mass term m/g_P.

The conserved vector current in the general case is given by

$$\begin{aligned}\boldsymbol{V}_\mu = {} & \bar{N}\gamma_\mu \boldsymbol{\tau} N + 2i(\boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi}) + 2i(\boldsymbol{s} \times \partial_\mu \boldsymbol{s}) \\ & + 2i(\boldsymbol{u}_\nu \times \partial_\mu \boldsymbol{u}_\nu - \boldsymbol{u}_\mu \times \partial_\nu \boldsymbol{u}_\nu) + 2i(\boldsymbol{w}_\nu \times \partial_\mu \boldsymbol{w}_\nu - \boldsymbol{w}_\mu \times \partial_\nu \boldsymbol{w}_\nu) \\ & + 2i\frac{g_V}{G}\boldsymbol{s} \times \partial_\nu \boldsymbol{u}_{\nu\mu} + 2i\frac{g_A}{G}\boldsymbol{\pi} \times \partial_\nu \boldsymbol{w}_{\nu\mu},\end{aligned} \tag{4·10}$$

$$\partial_\mu \boldsymbol{V}_\mu = 0$$

and the infinitesimal transformation inducing the conservation law makes the nucleon field change

$$N \to (1 + i(\boldsymbol{\beta}\boldsymbol{\tau}))N, \tag{4·11}$$

where $\boldsymbol{\beta}$ is some constant iso-vector. The simultaneous changes of the boson fields are

$$\begin{aligned}& \sigma' \to \sigma',\quad \rho \to \rho, \\ & v_\mu \to v_\mu,\quad a_\mu \to a_\mu, \\ & \boldsymbol{\pi} \to \boldsymbol{\pi} + 2\boldsymbol{\beta} \times \boldsymbol{\pi},\quad \boldsymbol{s} \to \boldsymbol{s} + 2\boldsymbol{\beta} \times \boldsymbol{s}, \\ & \boldsymbol{u}_\mu \to \boldsymbol{u}_\mu + 2\boldsymbol{\beta} \times \boldsymbol{u}_\mu,\quad \boldsymbol{w}_\mu \to \boldsymbol{w}_\mu + 2\boldsymbol{\beta} \times \boldsymbol{w}_\mu,\end{aligned} \tag{4·12}$$

which make the quantities

$$\sigma'^2,\ \rho^2,\quad \left(v_\mu + \frac{f_V}{F_V}\partial_\mu \sigma'\right)^2,\quad \left(a_\mu + \frac{f_A}{F_A}\partial_\mu \rho\right)^2, \tag{4·13}$$

$$\boldsymbol{\pi}^2,\ \boldsymbol{s}^2,\quad \left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu \boldsymbol{s}\right)^2,\quad \left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu \boldsymbol{\pi}\right)^2$$

invariant under this transformation. The quantities σ', ρ, v_μ and a_μ are scalar and $\boldsymbol{\pi}$, $\boldsymbol{s}$, $\boldsymbol{u}_\mu$ and $\boldsymbol{w}_\mu$ vector in the three-dimensional space of this transformation.

§ 5. Interpretation of boson fields in the composite model

The boson fields including the pion field are connected with the nuleon through the equations:

$$\begin{aligned}
&\bar{N}N=\frac{1}{g_P}(\Box-\mu_\pi^2)\sigma+2g_P\left(\sigma+\frac{m}{g_P}\right)\mathcal{J}_1'\left[g_P^2\boldsymbol{\pi}^2+g_P^2\left(\sigma+\frac{m}{g_P}\right)^2-m^2\right],\\
&i\bar{N}\gamma_5\boldsymbol{\tau}N=\frac{1}{g_P}(\Box-\mu_\pi^2)\boldsymbol{\pi}+2g_P\boldsymbol{\pi}\mathcal{J}_1'\left[g_P^2\boldsymbol{\pi}^2+g_P^2\left(\sigma+\frac{m}{g_P}\right)^2-m^2\right],\\
&\bar{N}\boldsymbol{\tau}N=\frac{1}{g_S}\Box\boldsymbol{s}+2g_S\boldsymbol{s}\mathcal{J}_3'[g_S^2\boldsymbol{s}^2+g_S^2\rho^2],\\
&i\bar{N}\gamma_5N=\frac{1}{g_S}\Box\rho+2g_S\rho\mathcal{J}_3'[g_S^2\boldsymbol{s}^2+g_S^2\rho^2],\\
&i\bar{N}\gamma_\mu\boldsymbol{\tau}N=\frac{1}{G}\partial_\nu\boldsymbol{u}_{\nu\mu}+2(G\boldsymbol{u}_\mu+g_V\partial_\mu\boldsymbol{s})\mathcal{J}_2'[(G\boldsymbol{u}_\nu+g_V\partial_\nu\boldsymbol{s})^2+(G\boldsymbol{w}_\nu+g_A\partial_\nu\boldsymbol{\pi})^2],\\
&i\bar{N}\gamma_\mu\gamma_5\boldsymbol{\tau}N=\frac{1}{G}\partial_\nu\boldsymbol{w}_{\nu\mu}+2(G\boldsymbol{w}_\mu+g_A\partial_\mu\boldsymbol{\pi})\mathcal{J}_2'[(G\boldsymbol{u}_\nu+g_V\partial_\nu\boldsymbol{s})^2+(G\boldsymbol{w}_\nu+g_A\partial_\nu\boldsymbol{\pi})^2],\\
&i\bar{N}\gamma_\mu N=\frac{1}{F_V}\partial_\nu v_{\nu\mu}+2(F_V v_\mu+f_V\partial_\mu\sigma)\mathcal{J}_4'[(F_V v_\nu+f_V\partial_\nu\sigma)^2],\\
&i\bar{N}\gamma_\mu\gamma_5N=\frac{1}{F_A}\partial_\nu a_{\nu\mu}+2(F_A a_\mu+f_A\partial_\mu\rho)\mathcal{J}_5'[(F_A a_\nu+f_A\partial_\nu\rho)^2].
\end{aligned}\tag{5·1}$$

The transformation characters of the boson fields are then uniquely determined by the bilinear quantities of nucleon field.

A more direct determination of the transformation character is to interpret the bosons as the composite entities of the nucleon fields in the spirit of Sakata and Heisenberg theories. In the usual field theoretical approach, such entities would be resulted from the solutions of the so-called Bethe-Salpeter-like equation.[8)]

The quantity appearing in the equation

$$\langle 0|T(N(x)\bar{N}(y))|n\rangle \tag{5·2}$$

can generally be composed of 16×4 components which have similar characters to the boson fields. Unfortunately, since the present field theory cannot give consistent results in this approach, we shall not enter into the details in this paper but only sketch the general framework in the further progress.

Suppose that we may start with the nucleon field and its non-linear interactions. Then in order to assure the axial vector current conservation, we

must construct the whole system to be invariant under the infinitesimal transformation (4·1). The possible forms of the non-linear term in the four fermi type are

$$\begin{aligned}
\mathcal{L}_1&=\frac{1}{2}[(\bar{N}N)^2-(\bar{N}\boldsymbol{\tau}\gamma_5 N)^2],\\
\mathcal{L}_2&=\frac{1}{2}[(\bar{N}\boldsymbol{\tau}N)^2-(\bar{N}\gamma_5 N)^2],\\
\mathcal{L}_3&=\frac{1}{2}[(\bar{N}\gamma_\mu\boldsymbol{\tau}N)^2-(\bar{N}i\gamma_\mu\gamma_5\boldsymbol{\tau}N)^2],\\
\mathcal{L}_4&=\frac{1}{2}(\bar{N}\gamma_\mu N)^2,\\
\mathcal{L}_5&=\frac{1}{2}(Ni\gamma_\mu\gamma_5 N)^2,\\
\mathcal{L}_6&=\frac{1}{2}[(N\sigma_{\mu\nu}N)^2-(\bar{N}\sigma_{\mu\nu}\boldsymbol{\tau}N)^2],
\end{aligned}\tag{5·3}$$

which satisfy the relations

$$\begin{aligned}
&\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_4+\mathcal{L}_5=0,\\
&\mathcal{L}_3-\mathcal{L}_4+\mathcal{L}_5=0,\\
&\mathcal{L}_1-\mathcal{L}_2-\mathcal{L}_6=0.
\end{aligned}\tag{5·4}$$

Three terms are independent among these six quantities. On the other hand, linear terms in the system which are invariant under the transformation can be reduced to

$$\mathcal{L}_0=-\bar{N}(\gamma\partial)N \tag{5·5}$$

and then the mass term must be substituted into the non-linear terms $\mathcal{L}_i$. This circumstance is in harmony with the fact that the scalar (iso-scalar) boson σ cannot be a covariant quantity without the modification of

$$\sigma'=\sigma+\frac{m}{g_P}.$$

We may guess that the nucleon mass is related with the scalar (iso-scalar) boson-like entity which is necessitated from the existence of the pion field with $ps(ps)$ coupling. Then we may say that the occurrence of nucleon mass is closely related with the existence of pion field as pointed out by Nambu and Jona-Lasinio[9] standing on an analogy with superconductivity.

If we may take the axial-vector current conservation law and the finite nucleon mass, we can restrict the non-linear terms to be neither $\mathcal{L}_3$ only nor $\mathcal{L}_4-\mathcal{L}_5$ combination only.

Further restriction can be taken when we take our general model, because

the general model stands on the assumption of lack of the tensor interactions between the spin 1 boson and nucleon. The simple treatment of this fact is to take only the $\mathcal{L}_1+\mathcal{L}_2$ combination for $\mathcal{L}_1$ and $\mathcal{L}_2$, but it sacrifices the simpler model proposed in § 3, because of the existence of pseudoscalar (iso-scalar) and scalar (iso-vector) bosons. The general combination of non-linear terms becomes

$$\begin{aligned}&\lambda_a(\mathcal{L}_1+\mathcal{L}_2)+\lambda_b\mathcal{L}_3+\lambda_c\mathcal{L}_4+\lambda_d\mathcal{L}_5\\&\quad=(-\lambda_a+\lambda_b+\lambda_c)\mathcal{L}_4+(-\lambda_a-\lambda_b+\lambda_d)\mathcal{L}_5,\end{aligned}\tag{5·6}$$

which reduces to the combination of vector and axial-vector couplings with a restriction of

$$2\lambda_a \fallingdotseq \lambda_c+\lambda_d.\tag{5·7}$$

Finally we get the system with Lagrangian

$$\mathcal{L}=-\bar{N}(\gamma\partial)N-\frac{a}{2}(\bar{N}\gamma_\mu N)^2-\frac{b}{2}(\bar{N}i\gamma_\mu\gamma_5 N)^2,\quad a\fallingdotseq -b.\tag{5·8}$$

The approximate validity of the axial vector current conservation is reflected by the introduction of the unperturbed (bare) nucleon mass m_0

$$\delta\mathcal{L}=-m_0\bar{N}N.\tag{5·9}$$

The extension of the model to include the other hyperon, for instance, Λ-particle, is easy in Sakata's scheme. The system composed of proton N_1, neutron N_2 and Λ-particle N_3 which is consistent with the full symmetry in the approximation of equal masses[10] is obtained by

$$\mathcal{L}=-\sum_{i=1}^{3}\bar{N}_i(\gamma\partial+m_0^i)N_i-\frac{a}{2}\left(\sum_{i=1}^{3}\bar{N}_i\gamma_\mu N_i\right)^2-\frac{b}{2}\left(\sum_{i=1}^{3}\bar{N}_i i\gamma_\mu\gamma_5 N_i\right)^2.\tag{5·10}$$

There is not a definite conclusion of the ratio a/b in the present situation.

If we would assume $a/b=-1$, we could not get the $ps(ps)$ coupling of pion-nucleon interaction as the result of the theory. On the contrary, we could not have the $ps(pv)$ coupling of pion-nucleon interaction from the assumption of $a/b=+1$. When the $ps(ps)$ and $ps(pv)$ coupling constants could be derived theoretically on the one hand and the determination of coupling type could be established phenomenologically on the other hand, we might determine the value of a/b. Taketani[11] and the author offered an opinion of $b=0$ from a theoretical reason.

§ 6. Discussions

The validity of axial vector current conservation may be closely connected with the question of the existence of bosons. The existence of pion field and

its lifetime would be the most important background of this problem. While the existence of other bosons are less reliable, because of the ambiguous interpretations briefly sketched in the introduction. One may interpret them as real particles, while other may consider them to be ficticious entities. The circumstance is very similar to the existence of the intermediate vector boson in the weak interaction. The existence of neutral scalar boson is welcome theoretically[4),5)] and phenomenologically.[12)] The vector (or pseudo-vector) bosons have been considered by many authors in the strong phenomena[13)] and also in the weak phenomena.[7)] We have not a definite answer whether the vector (and pseudo-vector) boson necessitated in the conservation law is identified with the one in the strong phenomena or in the weak phenomena, because of the unknown coupling constant G.

There may also be reasons which make them the ficticious entities. In fact, the arbitrary functions appearing in the system can be interpreted as the conditions between boson fields (lagrange multiplier). They are

$$\mathcal{J}_1\left[g_P{}^2\boldsymbol{\pi}^2+g_P{}^2\left(\sigma+\frac{m}{g_P}\right)^2-m^2\right]=0,$$

$$\mathcal{J}_2[(G\boldsymbol{u}_\mu+g_V\partial_\mu\boldsymbol{s})^2+(G\boldsymbol{w}_\mu+g_A\partial_\mu\boldsymbol{\pi})^2]=0,$$

$$\mathcal{J}_3[g_S{}^2(\rho^2+\boldsymbol{s}^2)]=0,$$

$$\mathcal{J}_4[(F_V v_\mu+f_V\partial_\mu\sigma)^2]=0,$$

$$\mathcal{J}_5[(F_A a_\mu+f_A\partial_\mu\rho)^2]=0$$

and boson fields are related to each other through these functions. For instance, if we take

$$\mathcal{J}_1=\frac{g_P{}^2}{8m^2}(\mu_\pi{}^2-\mu_\sigma{}^2)\left[\boldsymbol{\pi}^2+\left(\sigma+\frac{m}{g_P}\right)^2-\frac{m^2}{g_P{}^2}\right],$$

the neutral scalar field is considered as the secondary entity

$$\sigma=\frac{m}{g_P}\left[\sqrt{1-\frac{g_P{}^2}{m^2}\boldsymbol{\pi}^2}-1\right]$$

as the non-linear model of Gell-Mann and Lévy.

From the viewpoint of composite model, for instance, of the Sakata model, the existence of bosons depends on the results of the consistent dynamical treatment for the problem whether the solutions of these entities are stable or not. If we cannot get the stable solution for some kind of boson in the lowest approximation, then such boson must be considered as a more complicated entity than other kinds of boson which result from the stable solutions. The detailed discussions of this problem will appear elsewhere.

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Appendix

Determination of non-linear term among boson fields

The equations of boson fields are given by

$$
\begin{aligned}
&(\Box-\mu_\pi^2)\boldsymbol{\pi}+\frac{\delta L_B'}{\delta\boldsymbol{\pi}}-\partial_\mu\frac{\delta L_B'}{\delta\partial_\mu\boldsymbol{\pi}}=ig_P\bar{N}\boldsymbol{\tau}\gamma_5 N-ig_A\partial_\mu(\bar{N}\boldsymbol{\tau}\gamma_\mu\gamma_5 N),\\
&(\Box-\mu_\sigma^2)\sigma+\frac{\delta L_B'}{\delta\sigma}-\partial_\mu\frac{\delta L_B'}{\delta\partial_\mu\sigma}=g_P\bar{N}N-if_V\partial_\mu(\bar{N}\gamma_\mu N),\\
&(\Box-\mu_\rho^2)\rho+\frac{\delta L_B'}{\delta\rho}-\partial_\mu\frac{\delta L_B'}{\delta\partial_\mu\rho}=ig_S\bar{N}\gamma_5 N-if_A\partial_\mu(\bar{N}\gamma_\mu\gamma_5 N),\\
&(\Box-\mu_s^2)\boldsymbol{s}+\frac{\delta L_B'}{\delta\boldsymbol{s}}-\partial_\mu\frac{\delta L_B'}{\delta\partial_\mu\boldsymbol{s}}=g_S\bar{N}\boldsymbol{\tau}N-ig_V\partial_\mu(\bar{N}\boldsymbol{\tau}\gamma_\mu N),\qquad\text{(A·1)}\\
&\partial_\nu\boldsymbol{u}_{\nu\mu}-\mu_u^2\boldsymbol{u}_\mu+\frac{\delta L_B'}{\delta\boldsymbol{u}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{u}_\mu}=iG\bar{N}\boldsymbol{\tau}\gamma_\mu N,\\
&\partial_\nu\boldsymbol{w}_{\nu\mu}-\mu_w^2\boldsymbol{w}_\mu+\frac{\delta L_B'}{\delta\boldsymbol{w}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{w}_\mu}=iGN\boldsymbol{\tau}\gamma_\mu\gamma_5 N,
\end{aligned}
$$

and the similar equations for v_μ and a_μ. The non-linear term L_B' can be determined so as to give the conservation law

$$\partial_\mu\boldsymbol{P}_\mu=\frac{2mi}{g_P}\mu_\mu^2\boldsymbol{\pi}\qquad\text{(A·2)}$$

where $\boldsymbol{P}_\mu$ is given by (2·6). L_B' must satisfy the equation

$$
\begin{aligned}
&\boldsymbol{\pi}\frac{\delta L_B'}{\delta\sigma}-\left(\sigma+\frac{m}{g_P}\right)\frac{\delta L_B'}{\delta\boldsymbol{\pi}}+\rho\frac{\delta L_B'}{\delta\boldsymbol{s}}-\boldsymbol{s}\frac{\delta L_B'}{\delta\rho}\\
&-\left(\boldsymbol{u}_\mu+\frac{g_V}{G}\partial_\mu\boldsymbol{s}\right)\times\left(\frac{\delta L_B'}{\delta\boldsymbol{w}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{w}_\mu}\right)-\left(\boldsymbol{w}_\mu+\frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right)\times\left(\frac{\delta L_B'}{\delta\boldsymbol{u}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{u}_\mu}\right)\\
&-\boldsymbol{\pi}\partial_\mu\left[\frac{\delta L_B'}{\delta\partial_\mu\sigma}-\frac{f_V}{F_V}\left(\frac{\delta L_B'}{\delta v_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu v_\mu}\right)\right]\\
&+\left(\sigma+\frac{m}{g_P}\right)\partial_\mu\left[\frac{\delta L_B'}{\delta\partial_\mu\boldsymbol{\pi}}-\frac{g_A}{G}\left(\frac{\delta L_B'}{\delta\boldsymbol{w}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{w}_\mu}\right)\right]\\
&-\rho\partial_\mu\left[\frac{\delta L_B'}{\delta\partial_\mu\boldsymbol{s}}-\frac{g_V}{G}\left(\frac{\delta L_B'}{\delta\boldsymbol{u}_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu\boldsymbol{u}_\mu}\right)\right]+\boldsymbol{s}\partial_\mu\left[\frac{\delta L_B'}{\delta\partial_\mu\rho}-\frac{f_A}{F_A}\left(\frac{\delta L_B'}{\delta a_\mu}-\partial_\nu\frac{\delta L_B'}{\delta\partial_\nu a_\mu}\right)\right]
\end{aligned}
$$

$$= (\mu_\sigma^2 - \mu_\pi^2)\sigma\boldsymbol{\pi} + (\mu_s^2 - \mu_\rho^2)\rho\boldsymbol{s}$$
$$- \mu_w^2\left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu \boldsymbol{s}\right)\times \boldsymbol{w}_\mu - \mu_u^2\left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right)\times\boldsymbol{u}_\mu$$
$$+ \frac{f_V}{F_V}\mu_v^2\boldsymbol{\pi}\partial_\mu v_\mu - \frac{g_A}{G}\mu_w^2\left(\sigma + \frac{m}{g_P}\right)\partial_\mu \boldsymbol{w}_\mu + \frac{g_V}{G}\mu_u^2\rho\partial_\mu\boldsymbol{u}_\mu - \frac{f_A}{F_A}\mu_a^2\boldsymbol{s}\partial_\mu a_\mu, \tag{A·3}$$

which can be separated into

$$L_B' = \overline{F}_1 + \overline{F}_2 + \overline{F}_3 + \overline{F}_4 + \overline{F}_5, \tag{A·4}$$

$$\boldsymbol{\pi}\frac{\delta\overline{F}_1}{\partial\sigma} - \left(\sigma + \frac{m}{g_P}\right)\frac{\delta\overline{F}_1}{\delta\boldsymbol{\pi}} = (\mu_\sigma^2 - \mu_\pi^2)\sigma\boldsymbol{\pi}, \tag{A·5}$$

$$\left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu\boldsymbol{s}\right)\times\frac{\delta\overline{F}_2}{\delta\boldsymbol{w}_\mu} + \left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right)\times\frac{\delta\overline{F}_2}{\delta\boldsymbol{u}_\mu}$$
$$= \mu_w^2\left(\boldsymbol{u}_\mu + \frac{g_V}{G}\partial_\mu\boldsymbol{s}\right)\times\boldsymbol{w}_\mu + \mu_u^2\left(\boldsymbol{w}_\mu + \frac{g_A}{G}\partial_\mu\boldsymbol{\pi}\right)\times\boldsymbol{u}_\mu,$$
$$\text{together with } \frac{\delta\overline{F}_2}{\delta\partial_\mu\boldsymbol{\pi}} - \frac{g_A}{G}\frac{\delta\overline{F}_2}{\delta\boldsymbol{w}_\mu} = -\frac{g_A}{G}\mu_w^2\boldsymbol{w}_\mu,$$
$$\frac{\delta\overline{F}_2}{\delta\partial_\mu\boldsymbol{s}} - \frac{g_V}{G}\frac{\delta\overline{F}_2}{\delta\boldsymbol{u}_\mu} = -\frac{g_V}{G}\mu_u^2\boldsymbol{u}_\mu, \tag{A·6}$$

$$\rho\frac{\delta\overline{F}_3}{\delta\boldsymbol{s}} - \boldsymbol{s}\frac{\delta\overline{F}_3}{\delta\rho} = (\mu_s^2 - \mu_\rho^2)\rho\boldsymbol{s}, \tag{A·7}$$

$$\frac{\delta\overline{F}_4}{\delta\partial_\mu\sigma} - \frac{f_V}{F_V}\frac{\delta\overline{F}_4}{\delta v_\mu} = -\frac{f_V}{F_V}\mu_v^2 v_\mu, \tag{A·8}$$

$$\frac{\delta\overline{F}_5}{\delta\partial_\mu\rho} - \frac{f_A}{F_A}\frac{\delta\overline{F}_5}{\delta a_\mu} = -\frac{f_A}{F_A}\mu_a^2 a_\mu. \tag{A·9}$$

They give the solutions

$$\overline{F}_1 = (1/2)(\mu_\sigma^2 - \mu_\pi^2)\sigma^2 + \mathscr{F}_1\left[g_P^2\boldsymbol{\pi}^2 + g_P^2\left(\sigma + \frac{m}{g_P}\right)^2 - m^2\right],$$
$$\overline{F}_2 = (1/2)\mu_u^2\boldsymbol{u}_\mu^2 + (1/2)\mu_w^2\boldsymbol{w}_\mu^2 + \mathscr{F}_2[(G\boldsymbol{u}_\mu + g_V\partial_\mu\boldsymbol{s})^2 + (G\boldsymbol{w}_\mu + g_A\partial_\mu\boldsymbol{\pi})^2],$$
$$\overline{F}_3 = (1/2)\mu_s^2\boldsymbol{s}^2 + (1/2)\mu_\rho^2\rho^2 + \mathscr{F}_3[g_S^2\boldsymbol{s}^2 + g_S^2\rho^2], \tag{A·10}$$
$$\overline{F}_4 = (1/2)\mu_v^2 v_\mu^2 + \mathscr{F}_4[(F_V v_\mu + f_V\partial_\mu\sigma)^2],$$
$$\overline{F}_5 = (1/2)\mu_a^2 a_\mu^2 + \mathscr{F}_5[(F_A a_\mu + f_A\partial_\mu\rho)^2].$$

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Letters to the Editor

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Comments on Fermi's Statistical Theory of High Energy Nuclear Events

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January 5, 1961

It is the purpose of this note to point out some minor errors that appear in Fermi's well-known paper[1] on the production of pions and nucleon-antinucleon pairs in high energy interactions and in McConnell's recent review article[2] on this subject. Correction of these errors leads to the conclusion that, in Fermi's model of extremely high energy interactions, the number of nucleons and antinucleons expected is exactly twice the number of pions.

Using standard procedures of statistical mechanics,[3] it may be shown that the ratio of the density of nucleons and antinucleons to the density of pions is

$$n_N/n_\pi = 8(1-2^{1-3})/3 = 2.$$

Fermi's equation (28) should therefore read:

$$n_\pi = 0.367\frac{(kT)^3}{\hbar^3 c^3},\quad n_N = 0.734\frac{(kT)^3}{\hbar^3 c^3}.$$

Comparing the value of n_N with Fermi's, which contained the numerical factor 0.855, it is seen that the number of nucleons and antinucleons was overestimated by about 16 percent.

With this correction, and Fermi's correction factor of 0.51 to take account of the conservation of angular momentum, the number of nucleons and antinucleons in terms of the total energy, W, of the colliding nucleons in the c.m. system becomes $1.08\,(W/Mc^2)^{1/2}$ (replacing Fermi's equation (30) in which the numerical factor was 1.3). The total number of charged particles emerging from an extremely high energy collision becomes $1.04(W'/Mc^2)^{1/4}$, where W' is the total energy in the laboratory system. This replaces Fermi's equation (30a) in which the numerical factor was 1.2.

McConnell,[2] in addition to including Fermi's error, states that $W'=2W^2/Mc^2$, whereas the correct expression is $W'=W^2/2Mc^2$. Furthermore, he appears to have quoted for the *total* number of pions and nucleons emerging from an interaction Fermi's value from the number of *charged* particles. The total number should be $1.93(W'/Mc^2)^{1/4}$.

1) E. Fermi, Prog. Theor. Phys. **5** (1950), 570.

2) J. McConnell, *Progress in Elementary Particle and Cosmic Ray Physics* (North Holland Pub. Co., 1960), Vol. V, p. 228.
3) See, for example, Landau and Lifshitz, *Statistical Physics* (Pergamon, 1958), p. 164.

The Kinetic Equation for Plasmas and Its Solution

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January 30, 1961

The transport coefficient for low density gases of short-range interaction are calculated by solving the linearized Boltzmann equation. The Boltzmann equation is, however, known to be invalid for plasmas. The purpose of this and the following note is to give the kinetic equation which is to be solved in order to calculate the conductivity of low density high temperature and high density low temperature plasmas and to indicate the method calculating the transport coefficients by its aid. In this note, the quantum statistics is not considered for the sake of brevity.

Several works[1)-3)] have recently been published which intend to derive the kinetic equation (for plasmas) by starting from the Liouville equation. Above all, Prigogine and Balescu[2)] have proposed the method in which the contributions to the distribution functions are grouped together according to their orders of magnitude. When the method is applied to the one-particle Wigner function in quantum mechanics and the contributions of 'collisions' of the orders $O((\rho\lambda)^n t)$ and $O(\lambda(\rho\lambda)^n t)$ are collected, where ρ is number density, λ order of interaction, t time $[(\rho\lambda)^n \sim 1, \lambda(\rho\lambda)^n \sim \kappa^3/\rho, \kappa^2 = 4\pi\rho e^2/kT$, for a high temperature plasma]; we get the kinetic equation of the non-Markovian type to the first order of the external field,

$$f(\boldsymbol{x}, \boldsymbol{p}, t) = f_0(\boldsymbol{p}) + f_1(\boldsymbol{x}, \boldsymbol{p}, t), \tag{1}$$

$$f_0(\boldsymbol{p}) = \rho \left(\frac{2\pi\hbar^2}{mkT} \right)^{3/2} e^{-\varepsilon_p/kT}, \tag{2}$$

$$\begin{aligned} \frac{\partial f_1(\boldsymbol{x}, \boldsymbol{p}, t)}{\partial t} &= -\boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}} f_1(\boldsymbol{x}, \boldsymbol{p}, t) \\ &+ \frac{\partial}{\partial \boldsymbol{x}} A^*(\boldsymbol{x}, t) \cdot \frac{1}{\hbar} \frac{\partial}{\partial \boldsymbol{p}} f_0(\boldsymbol{p}) \\ &+ \frac{1}{\hbar^2} \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} d\omega' e^{i\omega'(t-t')/\hbar} \\ &\times \frac{1}{\Omega} \sum_{\boldsymbol{q}} V^*(\boldsymbol{q}, \omega)^2 \\ &\times \frac{1}{\Omega} \sum_{\boldsymbol{p}'} \delta(\varepsilon_{\boldsymbol{p}'+\boldsymbol{q}} - \varepsilon_{\boldsymbol{p}'} + \omega) \\ &\times \{ f(\boldsymbol{x}, \boldsymbol{p}'+\boldsymbol{q}, t') f(\boldsymbol{x}, \boldsymbol{p}-\boldsymbol{q}, t') \\ &- f(\boldsymbol{x}, \boldsymbol{p}', t') f(\boldsymbol{x}, \boldsymbol{p}, t) \}_1, \end{aligned} \tag{3}$$

where $\omega \equiv \omega' + \varepsilon_{p-q} - \varepsilon_p$, $\varepsilon_p \equiv \hbar^2 \boldsymbol{p}^2/2m$, $v \equiv 1/\hbar \cdot \partial\varepsilon_p/\partial \boldsymbol{p}$, $1/\Omega \cdot \sum_p \equiv 1/(2\pi)^3 \int d\boldsymbol{p}$, $\{\ \}_1$ denotes the linear contribution in f_1 when f's are replaced by $f_0 + f_1$ as in (1), and

$$A^*(\boldsymbol{x}, t) = A(\boldsymbol{x}, t) + \int d\boldsymbol{x}' V(\boldsymbol{x} - \boldsymbol{x}') \times \frac{1}{\Omega} \sum_{\boldsymbol{p}'} f(\boldsymbol{x}', \boldsymbol{p}', t), \tag{4}$$

$$V^*(\boldsymbol{q}, w) = V(\boldsymbol{q}) \times \left| 1 - V(\boldsymbol{q}) \frac{1}{\Omega} \sum_{\boldsymbol{p}'} \frac{f_0(\boldsymbol{p}'+\boldsymbol{q}) - f_0(\boldsymbol{p}')}{\varepsilon_{\boldsymbol{p}'+\boldsymbol{q}} - \varepsilon_{\boldsymbol{p}'} + \omega} \right|^{-1}. \tag{5}$$

Here, $A(\boldsymbol{x}, t)$ is the potential of the external field and $V(\boldsymbol{q})$ is the Fourier tranform of the interaction potential of a pair of electrons.

The kinetic equation obtained is different from the linearized Boltzmann equation for the systems of short-range interaction, in the acceleration term and the collision term. The fact that the acceleration is performed by the internal field $A^*(\boldsymbol{x}, t)$ of (4) and not by the external field $A(\boldsymbol{x}, t)$ is convenient for the calculation of the conductivity, because the transport coefficients are defined as the ratios of the response to the internal field but not to the external field. We have only to solve (3) for prescribed $A^*(\boldsymbol{x}, t)$. Then the corresponding $A(\boldsymbol{x}, t)$ can be calculated by the aid of (5), if desired.

Now, we shall solve (3), which may be rewritten in the following form, for a prescribed $A^*(\boldsymbol{x}, t)$:

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)f_1(\boldsymbol{x}, \boldsymbol{p}, t)$$

$$+\frac{i}{2\pi}\int_{-\infty}^{t}dt'\int d\omega\, e^{i\omega(t-t')}$$

$$\times\frac{1}{\Omega}\sum_{p'}\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega)f_1(\boldsymbol{x}, \boldsymbol{p}', t')$$

$$=\frac{\partial}{\partial \boldsymbol{x}}A^*(\boldsymbol{x}, t)\cdot\frac{1}{\hbar}\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{p}). \quad (6)$$

The fact that it is non-Markovian introduces no difficulty. The solution is of the form

$$f_1(\boldsymbol{x}, \boldsymbol{p}, t)=\int d\omega\frac{1}{\Omega}$$

$$\times\sum_{q}f_1(\boldsymbol{p}; \boldsymbol{q}, \omega)\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}+i\omega t},$$

$$(\omega+\boldsymbol{q}\cdot\boldsymbol{v})f_1(\boldsymbol{p}; \boldsymbol{q}, \omega)+\frac{1}{\Omega}$$

$$\times\sum_{p'}\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega)f_1(\boldsymbol{p}'; \boldsymbol{q}, \omega)$$

$$=\boldsymbol{q}A^*(\boldsymbol{q}, \omega)\cdot\frac{1}{\hbar}\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{p}). \quad (7)$$

The last equation is formally identical with the Boltzmann equation and the same techniques will be applicable in solving it or obtaining the transport coefficients. The formal solution is

$$f_1(\boldsymbol{p}; \boldsymbol{q}, \omega)=\sum_{\mu}|\varphi_{q\omega\mu}(\boldsymbol{p})\rangle\tau_{q\omega\mu},$$

$$\tau_{q\omega\mu}$$

$$=\frac{\langle\varphi_{q\omega\mu}(\boldsymbol{p})|\boldsymbol{q}A^*(\boldsymbol{q}\cdot\omega)\cdot\frac{1}{\hbar}\frac{\partial}{\partial \boldsymbol{p}}f_0(\boldsymbol{p})\rangle}{\omega+\lambda_{q\omega\mu}},$$

where $\omega+\lambda_{q\omega\mu}$ and $\langle\varphi_{q\omega\mu}(\boldsymbol{p})|$ are the set of the eigenvalues and eigenfunctions of

$$\langle\varphi_{q\omega\mu}(\boldsymbol{p}')|(\omega+\boldsymbol{q}\cdot\boldsymbol{v}')$$

$$+\frac{1}{\Omega}\sum_{p}\langle\varphi_{q\omega\mu}(\boldsymbol{p})|\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega)$$

$$=(\omega+\lambda_{q\omega\mu})\langle\varphi_{q\omega\mu}(\boldsymbol{p}')|$$

and $|\varphi_{q\omega\mu}(\boldsymbol{p})\rangle$ is defined in such a way that

$$\langle\varphi_{q\omega\mu}(\boldsymbol{p})|\varphi_{q\omega\mu'}(\boldsymbol{p})\rangle$$

$$=\delta_{\mu,\mu'}\langle\varphi_{q\omega\mu}(\boldsymbol{p})|\varphi_{q\omega\mu}(\boldsymbol{p})\rangle.$$

$\langle f(\boldsymbol{p})|g(\boldsymbol{p})\rangle$ means $1/\Omega\cdot\sum_{p}f(\boldsymbol{p})g(\boldsymbol{p})$. Consequently, the current $\boldsymbol{j}(\boldsymbol{x}, t)=e\hbar/m\cdot\langle\boldsymbol{p}|f_1(\boldsymbol{x}, \boldsymbol{p}, t)\rangle$, $\langle\boldsymbol{p}|=\boldsymbol{p}$, is calculated by

$$\boldsymbol{j}(\boldsymbol{x}, t)=\frac{1}{\Omega}\sum_{q}\int d\omega\,\boldsymbol{j}(\boldsymbol{q}, \omega)\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}+i\omega t},$$

$$\boldsymbol{j}(\boldsymbol{q}, \omega)=\frac{e\hbar}{m}\langle\boldsymbol{p}|f_1(\boldsymbol{p}; \boldsymbol{q}, \omega)\rangle$$

$$= \frac{e\hbar}{m} \sum_{\mu} \langle \boldsymbol{p} | \varphi_{q\omega\mu}(\boldsymbol{p}) \rangle \tau_{q\omega\mu},$$

which can be evaluated if we know the values of non-vanishing $\langle \boldsymbol{p} | \varphi_{q\omega\mu}(\boldsymbol{p}) \rangle$ and the corresponding $\lambda_{q\omega\mu}$. In case that the field is uniform, the solution in the first approximation of Enskog and Chapman is obtained if we assume that the sets of $\langle \boldsymbol{p}\cdot\boldsymbol{E}| = \boldsymbol{p}\cdot\boldsymbol{E}$ and $|\boldsymbol{p}\cdot\boldsymbol{E}\rangle = f_0(\boldsymbol{p})\boldsymbol{p}\cdot\boldsymbol{E}$ belongs to sets of $\langle \varphi_{0\omega\mu}(\boldsymbol{p})|$ and $|\varphi_{0\omega\mu}(\boldsymbol{p})\rangle$; i.e.

$$\boldsymbol{j}(0, \omega) = \frac{\rho e^2}{im} \boldsymbol{E}(0, \omega) \times \left\{ \omega + \frac{\hbar^2}{\rho m k T \boldsymbol{E}^2 \Omega^2} \sum_{\boldsymbol{p},\boldsymbol{p}'} \mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega) \times (\boldsymbol{p}\cdot\boldsymbol{E})(\boldsymbol{p}'\cdot\boldsymbol{E}) f_0(\boldsymbol{p}') \right\}^{-1}$$

where $\boldsymbol{E} = \boldsymbol{E}(0, \omega)$,

$$e\boldsymbol{E}(\boldsymbol{q}, \omega) \equiv i\boldsymbol{q}A^*(\boldsymbol{q}, \omega).$$

It will be straightforward to give the higher approximations by the help of the Sonine functions.[4] The problem is equivalent to that of solving the linearized Boltzmann equation in which the collision term is somewhat complex.

In the above, we have indicated the solution of the non-Markovian kinetic equation for plasmas. It is noticed here that, if we reduced the kinetic equation to the Markovian type in the same manner as is done for the systems of short-range interaction, we had just neglected the ω-dependence of $\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega)$ and replaced $\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; \omega)$ by $\mathcal{V}(\boldsymbol{p}, \boldsymbol{p}'; 0)$ in (7). It can be shown from this fact that the non-Markovian nature is not important for the frequencies much smaller than the plasma frequency, though it is essential for the frequencies comparable with the plasma frequency.

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On the Kinetic Equation for Electrons in Metals at Low Temperatures

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The one-particle Wigner function for electron is to be calculated, to the first order of the external electric field, in order to obtain the conductivity of metals. The equation to which it obeys is obtained by the method proposed by Prigogine and Balescu.[1] The equation is found to be a linearized Boltzmann equation of the Markovian type, in which the external field is replaced by the internal field and the interactions are screened; if the frequencies we are interested in are much smaller than the plasma frequency. Then, the kinetic equation is reduced to the one usually adopted in order to

ol tain the conductivity of metals at low temperatures,[2] if we introduce the approximation that we neglect the collisions between electrons and adopt as the interactions the screened ones or neglect the screening.

Here we consider that the conduction electrons in metals are scattered by lattice vibrations (phonons), by random impurities and by each other. By considering the 'collisions' of the order $O((\rho\lambda)^n t)$, $O(\lambda(\rho\lambda)^n t)$, $O(\rho_{imp}\lambda^2_{imp}(\rho\lambda)^n t)$ and $O(\rho_{ph}\lambda_{ph}(\rho\lambda)^n t)$, where ρ, ρ_{imp} and ρ_{ph} are number densities of electrons, impurities and phonons, λ, λ_{imp} and λ_{ph} are orders of interaction of electron with each other, with impurity and with phonon, respectively, and t is time [$(\rho\lambda)^n\sim 1$, $\lambda(\rho\lambda)^n\sim\kappa^3/\rho$, $\kappa^2=4\pi\rho e^2/kT$ for high temperature case and $(\rho\lambda)^n\sim r_s$, $\lambda(\rho\lambda)^n\sim r_s^2$, $4\pi r_s^3 a_0^3/3=1/\rho$, a_0 the Bohr radius, for low temperature case]; the kinetic equation for the one-particle Wigner function of electron is, to the first order of the external field,

$$f(\boldsymbol{x},\boldsymbol{p},t)=f_0(\boldsymbol{p})+f_1(\boldsymbol{x},\boldsymbol{p},t), \tag{1}$$

$$f_0(\boldsymbol{r})=1/(1+z^{-1}e^{\varepsilon_p/kT}), \tag{2}$$

$$\begin{aligned}\frac{\partial f_1(\boldsymbol{x},\boldsymbol{p},t)}{\partial t}&=-\boldsymbol{v}\cdot\frac{\partial}{\partial\boldsymbol{x}}f_1(\boldsymbol{x},\boldsymbol{p},t)\\&+\frac{\partial}{\partial\boldsymbol{x}}A^*(\boldsymbol{x},t)\cdot\frac{1}{\hbar}\frac{\partial}{\partial\boldsymbol{p}}f_0(\boldsymbol{p})\\&+\frac{2\pi}{\hbar}\frac{1}{\Omega}\sum_{\boldsymbol{q}}V^*_{imp}(\boldsymbol{q},\omega)^2\\&\times\{f_1(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)-f_1(\boldsymbol{x},\boldsymbol{p},t)\}\\&\times\delta(\varepsilon_{p-q}-\varepsilon_p)\\&+\frac{2\pi}{\hbar}\frac{1}{\Omega}\sum_{\boldsymbol{q}}|V^*_{ph}(\boldsymbol{q},\omega)|^2\\&\times\{f(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)[1-f(\boldsymbol{x},\boldsymbol{p},t)]\\&\times[(n_{-q}+1)\delta(\omega_q-\omega)+n_q\delta(\omega_q+\omega)]\\&-f(\boldsymbol{x},\boldsymbol{p},t)[1-f(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)]\\&\times[(n_q+1)\delta(\omega_q+\omega)\\&+n_{-q}\delta(\omega_q-\omega)]\}_1\\&+\frac{2\pi}{\hbar}\frac{1}{\Omega}\sum_{\boldsymbol{q}}V^*(\boldsymbol{q},\omega)^2\frac{1}{\Omega}\\&\times\sum_{\boldsymbol{p}'}\delta(\varepsilon_{\boldsymbol{p}'+\boldsymbol{q}}-\varepsilon_{\boldsymbol{p}'}+\omega)\\&\times\{f(\boldsymbol{x},\boldsymbol{p}'+\boldsymbol{q},t)f(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)\\&\times[1-f(\boldsymbol{x},\boldsymbol{p}',t)][1-f(\boldsymbol{x},\boldsymbol{p},t)]\\&-f(\boldsymbol{x},\boldsymbol{p}',t)f(\boldsymbol{x},\boldsymbol{p},t)\\&\times[1-f(\boldsymbol{x},\boldsymbol{p}'+\boldsymbol{q},t)]\\&\times[1-f(\boldsymbol{x},\boldsymbol{p}-\boldsymbol{q},t)]\}_1,\end{aligned} \tag{3}$$

where $\omega\equiv\varepsilon_{p-q}-\varepsilon_p$; the meanings of $\boldsymbol{v}$, $1/\Omega\cdot\sum_p$ and $\{\ \}_1$ are the same as in the previous note, z is a constant to be determined so that (2) satisfies $1/\Omega\cdot\sum_p f_0(\boldsymbol{p})=\rho$, $A^*(\boldsymbol{x},t)$ is given by (4) in ref. 3), $V^*_{imp}(\boldsymbol{q},\omega)$, $V^*_{ph}(\boldsymbol{q},\omega)$ and $V^*(\boldsymbol{q},\omega)$ are equal to $V_{imp}(\boldsymbol{q})$, $V_{ph}(\boldsymbol{q})$ and $V(\boldsymbol{q})$ divided by

$$1-V(\boldsymbol{q})\frac{1}{\Omega}\sum_{\boldsymbol{p}'}\frac{f_0(\boldsymbol{p}'+\boldsymbol{q})-f_0(\boldsymbol{p}')}{\varepsilon_{\boldsymbol{p}'+\boldsymbol{q}}-\varepsilon_{\boldsymbol{p}'}+\omega}\Big|, \tag{4}$$

ε_p and ω_p are the energies of an electron and a phonon of wave vector $\boldsymbol{p}$ and $n_q=1/\{\exp(\omega_q/kT)-1\}$. Here $A(\boldsymbol{x},t)$ is the potential of the external field, $V(\boldsymbol{q})$ and $V_{imp}(\boldsymbol{q})$ are Fourier transforms of interactions of electron with each other and with an impurity and $V_{ph}(\boldsymbol{q})$ is the interaction of electron with lattice creating a phonon of wave vector $\boldsymbol{q}$.

The first term of the right-hand side of (3) is the drift term. The second term represents the acceleration by the internal field, instead of the external

field. This fact is convenient for the calculation of the conductivity.[4] The third and fourth terms represent the collisions with the phonons and impurities, respectively. The last term represents the collisions of electrons. Here, we have given the kinetic equation of the Markovian type, which is valid if the frequencies of the external field is much smaller than the plasma frequency.[4]

If we may neglect the last term which represents the collisions of electrons by means of a screened potential, the equation is reduced to the one usually adopted in order to obtain the conductivity of metals at low temperatures and the theories presented in the past[5] will be valid, if we adopt the screened potential* or we neglect the screening.

It will generally follow that the linearized Boltzmann equation of the Markovian type will be valid for low density high temperature and high density low temperature plasmas, so far as the frequency of the external field is not too large; where the interactions between electrons, with electrons and via electrons are screened. Whether the screening is effective or not is determined by whether the contributions of the interaction near $\boldsymbol{q}=0$ is appreciable or not, as the screening (4) is largest at $\boldsymbol{q}=0$; that is, the screening is more (less) effective for the interactions of which range is longer (shorter).

* Langer[6] has obtained the expression for conductivity of metals at absolute zero, where the screened potential with impurity appears.

At last, it is noticed that our results of this and the previous notes give a justification, from the diagrammatical approach, for the conjecture of Bohm and Pines[7] that the interaction between conduction electrons may be treated as the short-range one except for the part responsible for the plasma oscillation. In our cases, the latter part is included in the internal field $A^*(\boldsymbol{x}, t)$—the so-called Vlasov term—and in the screening of potentials and the former is $V^*(\boldsymbol{q}, \omega)$.[8]

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Mass of a Neutral Vector Meson and *p-p* Scattering

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High-energy *p-p* scattering strongly indicates the existence of a spin-

orbit ($\boldsymbol{L}\cdot\boldsymbol{S}$) interaction between two nucleons.[1)-3)] Since pion-theoretical calculations fail to give a sufficiently large V_{LS},[4)] several authors have proposed to account for its presence in terms of a hitherto unobserved neutral meson.[5)-8)] In particular, assuming that there exists a neutral vector meson of mass $\simeq 3m_\pi$ or $4m_\pi$ (where m_π is the pion mass), Sakurai[8)] calculated Wolfenstein's amplitude C[9)] for p-p scattering and obtained a quantitative agreement with the results of the phase-shift analysis of Stapp et al.[10)] at 310 Mev. Subsequently, however, it was pointed out by Breit[11)] that, because of his neglect of strong distorting effects caused by the vector meson and the pion interactions, Sakurai's estimates on the mass value of the vector meson are quite uncertain.

The object of the present note is to show that, aside from the uncertainty just mentioned, Sakurai's approach based on Chew's conjecture on one-pion sigularities[12)] contains a difficulty in explaining the energy dependence of C at $\theta=90°$ (θ is the p-p scattering angle) if the mass of the neutral vector meson is fixed by the angular dependence of C.

At $\theta=90°$, Sakurai's Eq. (3) for $\mathrm{Im}(C)$ reduces to a simple form:

$$\mathrm{Im}(C) \doteqdot -(f_v^2/4\pi)(3/4M) \times[1+(a^2/b)]^{-1}[1+(b/2)]^{-1/2},$$

where $(f_v^2/4\pi)$ is the coupling constant, M is the proton mass, while a and b are, respectively, the mass of the vector meson (m_v) and the kinetic energy of the incident proton in the laboratory system (T), both in units of M. In Fig. 1 are shown, for several values of $m_v=Nm_\pi$, $[\mathrm{Im}C(90°, T)/\mathrm{Im}C(90°, 90\ \mathrm{Mev})]$ based on the above formula (solid curves) and the same quantity calculated from the Gammel-Thaler (GT) potential[12)] (the broken curve). Since GT potential gives a good fit to polarization for $100\ \mathrm{Mev} \lesssim T \lesssim 300\ \mathrm{Mev}$ and the amplitude C practically determines the magnitude of polarization, the broken curve could be considered to represent the energy dependence of C reasonably well. It is then evident from Fig. 1

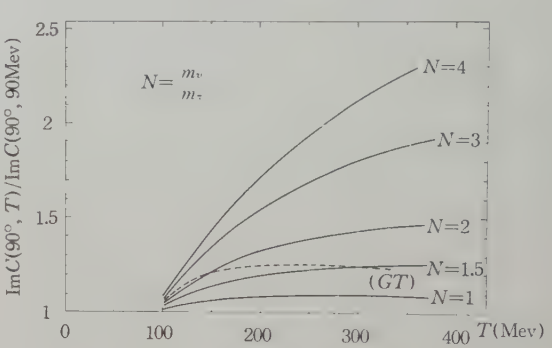


Fig. 1.

that the mass value of $3m_\pi$ or $4m_\pi$ derived by Sakurai from the angular dependence of $\mathrm{Im}(C)/\sin\theta$ at 310 Mev cannot account for the energy dependence of $\mathrm{Im}(C)$ at 90°.

Of course one might ask whether the discrepancy were due to the employment of GT potential which, strictly speaking, does not give a precise fit to the experimental data. In order to clarify this point, the results of the phase-shift analysis at 210 Mev by MacGregor and Moravcsik[13)] (set

b and set **c**) have been used to evaluate Im $C(90°, 210\ \text{Mev})$ which, together with Im $C(90°, 310\ \text{Mev})$ based on the solution 1 (or 2) of Stapp et al.,[10] would presumably give the energy dependence of C more accurately than the broken curve in Fig. 1 for 200 Mev $\lesssim T \lesssim$ 300 Mev. Theoretical values for [Im $C(90°, 310\ \text{Mev})$/Im $C(90°, 210\ \text{Mev})$] obtained from Sakurai's Eq. (3) when $m_v = m_\pi \sim 4m_\pi$ are given in Table I as well as the "experimental" value of the same quantity evaluated from the results of the phase-shift analyses at 210 and 310 Mev. Again the energy dependence

Table I.
[Im $C(90°, 310\ \text{Mev})$/Im $C(90°, 210\ \text{Mev})$]

(m_v/m_π)	1	2	3	4	"experimental"
	1.01	1.08	1.15	1.22	0.88~0.96

of Im(C) at $\theta = 90°$ favors smaller values of m_v than $3m_\pi$ or $4m_\pi$. On the other hand, the angular dependence of Im(C) at 210 Mev is consistent with the result at 310 Mev, giving $3m_\pi$ or $4m_\pi$ as the most reasonable value of m_v.

In view of this difficulty in determining the mass value, the neutral vector meson proposed by Sakurai must be considered as *a* qualitative explanation of the spin-orbit interactions between two nucleons even if the neglect of the wave function distortion pointed out by Breit turned out to be justifiable.

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Weak Interactions at High Energies

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Although the weak interaction has been almost established in decay processes, in which the momentum transfer is rather small, its behavior in high energy processes, in which the momentum transfer is very large, is quite an open question.

If the weak interaction becomes much stronger in the high energy

region, the weakly interacting particles, leptons, should be anomalously produced in high energy phenomena, say, nucleon-nucleon or nucleon-nucleus interactions in cosmic rays.

We have the experimental phenomena[1] which seem to be interpreted as multiple μ-meson production in the super high energy region above $\sim 10^{15}$ ev, though it is not so certain.

From the above point of view, Nagai and Itô[2] roughly estimated about μ-meson production caused by the weak interaction in the high energy region. As they mentioned, however, it was not clear whether their estimation had been carried out in the center of mass system or laboratory one and consequently whether these events could be found in the present cosmic-ray events or not.

Because of its experimental feasibility, we investigate in this short note the μ-meson production in high energy nucleon-nucleon collisions by using a more field-theoretical model, "the peripheral interaction model", recently developed by Chernavsky et al.[3] and F. Salzman et al.[4] independently.

Consider the following processes, corresponding to Fig. 1, (a) and (b), respectively.

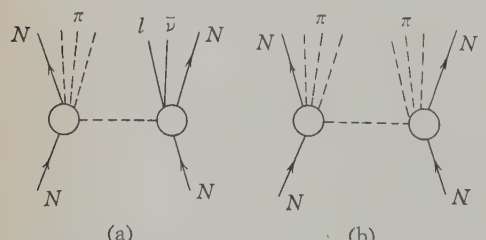


Fig. 1. Feynman diagrams pertaining to the processes (1a) and (1b).

$$N+N \to N+N+m\pi+l+\bar{\nu}, \quad \text{(1a)}$$

$$N+N \to N+N+m\pi+m'\pi. \quad \text{(1b)}$$

We can readily write down the total cross sections of the above processes as follows.

$$\sigma_{NN}(E) \cong \frac{4}{(2\pi)^3 p^2 E^2} \int_0^{(2\mu)^2} d\Delta^2 \frac{1}{(\Delta^2+1)^2} \times \int dW p_W W^2 \sigma(W, \Delta^2) \times \int dW' p_{W'}^{\prime 2} W' \sigma(W', \Delta^2), \quad (2)$$

where the notation is the same as that used in ref. 4).

In evaluating Eq. (2), it is assumed that the pion-nucleon total cross section $\sigma_{\pi N}$ is constant at high energies and that the perturbational calculation of lepton pair production cross section by pion-nucleon collisions is enough to estimate the order of the process (1a), so far as we are concerned with the order of magnitude. The latter assumption may be somewhat questionable and may result rather overestimation for the process (1a) because of the clothed structure of the nucleons, but this will not be discussed in detail here.

We estimate about the process (1b) using Eq. (2) in order to investigate the consistency of our model. Numerical estimates, shown in Table I, both for pseudoscalar pion-nucleon coupling and for pseudovector coupling, are derived with assumption $(\sigma_{\pi N})_{av} \approx 30$ mb.

According to the result shown in Table I, we may conclude that the anomalous production of leptons by

Table I. $\sigma^{l}_{NN}(ps)$ is the lepton production cross section (pion-nucleon coupling is of ps type), $\sigma^{l}_{NN}(pv)$ the lepton production cross section (pion-nucleon coupling is of pv type) and σ^{π}_{NN} the pure pion production cross section as functions of incident nucleon energy in the laboratory system.

E_{lab} (Bev)	$\sigma^{l}_{NN}(ps)$ (mb)	$\sigma^{l}_{NN}(pv)$ (mb)	σ^{π}_{NN} (mb)
10^4	10^{-14} ($g^2_{ps}/4\pi$)	10^{-7} ($g^2_{pv}/4\pi$)	1.2
10^5	10^{-13} 〃	10^{-5} 〃	1.6
10^6	10^{-12} 〃	10^{-3} 〃	1.9
10^8	10^{-10} 〃	10 〃	2.6
10^{10}	10^{-8} 〃	10^5 〃	3.2

nucleon-nucleon collisions in cosmic-ray events would be scarcely found at present, so far as our partially perturbational treatment is admitted. Although the above conclusion, based on the peripheral interaction model, seems to be intuitively reasonable, it should be noted that this conclusion might be altered if many-particle exchange contribution dominates in the high energy region.

However, it seems to be reasonable that the pure pion production cross section gives several millibarns, since we take account of only single-pion exchange contribution.

Further discussions on this model will appear in a later issue of this journal.

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Magnetoresistance in Nondegenerate Semiconductors Considering Electron-Electron Scattering

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Shockley[1] has shown that the time of relaxation under the validity of equipartition law for lattice phonons is given by

$$\tau_L = \frac{l}{x}$$

where l is a constant at a given temperature.

$$x = (m/2kT)^{1/2} v$$

v is the velocity of electron,
m is the effective mass of electron,
k is the Boltzmann constant,
T is the temperature of the crystal.

Sodha and Eastman[2] have applied the analysis of Spitzer and Harm[3] of velocity distribution of electrons in the presence of an electric field in an ionized gas to semiconductors and have shown that the time of relaxation due to both electron-ion and electron-electron scattering is given by

$$\tau' = \frac{B\phi(x)}{x},$$

where $\phi(x) = ZD(x)/A$ is a function, tabulated by Spitzer and Harm.[3] In most of semiconductors of interest the ions carry single electronic charge $Z=1$.

B is almost a constant at a given

temperature inversely proportional to impurity density. The effective time of relaxation is given by

$$\tau=\left\{\frac{1}{\tau_L}+\frac{1}{\tau'}\right\}^{-1}=\frac{l}{x}\psi(x),$$

where

$$\psi(x)=\phi(x)/\{\phi(x)+a^2\}$$

and

$$a^2=\frac{l}{B}.$$

Sodha and Varshni[2] have dealt the problem of magnetoresistance of non-degenerate semiconductors taking electron-lattice, electron-ion and electron-electron scattering by assuming that the Hall voltage is zero. However, if we use the condition that the Hall current is zero by putting as Sodha[4]

$$I_y=\frac{\partial T}{\partial x}=\frac{\partial T}{\partial y}=0,$$

one can obtain

$$I_x=\frac{nq^2}{3}\frac{E_x}{m}\left[\phi(1,3)+\omega^2\frac{\phi^2(2,3)}{\phi(1,3)}\right].$$

Expanding to powers in ω^2 and using the notation of Sodha and Varshni,[2] one gets coefficient of magnetoresistance

$$\frac{\Delta\rho}{\rho_0}=\frac{-\Delta\mu}{\mu_0}=\frac{9\pi H^2\mu_L^2}{16c^2}$$

$$\times\left[\left\{\frac{\int_0^\infty\psi^3 x e^{-x^2}dx}{\int_0^\infty\psi x^3 e^{-x^2}dx}\right\}-\left\{\frac{\int_0^\infty\psi^2 x^2 e^{-x^2}dx}{\int_0^\infty\psi x^3 e^{-x^2}dx}\right\}^2\right].$$

Table I and Fig. 1 give the variation of coefficient of magnetoresistance $(\Delta\rho/\rho_0)$ with 'a',

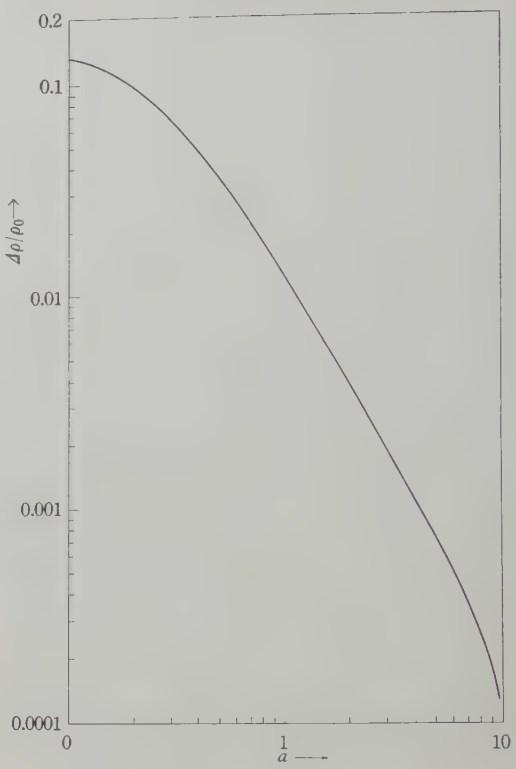


Fig. 1. Variation of $\Delta\rho/\rho_0$ with a.

Table I

a	$\Delta\rho/\rho_0$
0.1.	0.1333
0.2.	0.1034
0.4.	0.0546
0.6.	0.0282
1.0.	0.0068
2.0.	0.00208
4.0.	0.00109
6.0.	0.00531
10.0.	0.000132

Thanks are due to Dr. M. S. Sodha for his interest and Dr. B. N. Singh for his permission to publish the note.

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Interaction between Alpha Particles

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The problem on the system of two α-particles (Be^8) is the most typical one in the cluster model,[1] because the behaviors of low energy (low-lying) states of this system are mainly determined by the relative motion of α-particles due to their extreme tightness. In this note, by applying the pion-theoretical potentials recently verified in two-nucleon problems,[2] interactions between α-particles are investigated from the viewpoint of the cluster model, without taking account of the polarization effects of α-particles.

The α-α interaction has the following characteristic features:

(1) The outside attractive force ($R=2\sim5$y, where R is the distance between two α-particles and y=1 Yukawa$=10^{-13}$ cm),
(2) the inside repulsive core due to the Pauli principle ($R\lesssim 2$y), and
(3) the velocity dependence indicated by a stronger attraction for higher l, where $l=0, 2, 4, \cdots$ represent the orbital angular momentum of the relative motion.[3]

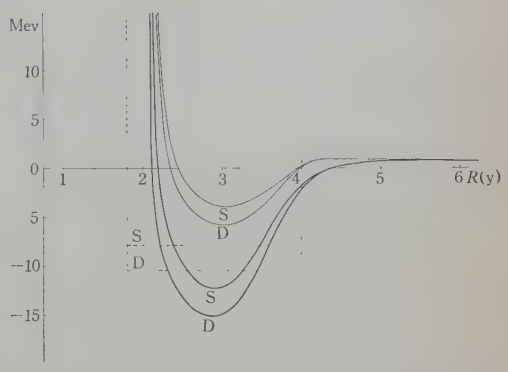


Fig. 1. The dashed lines represent the phenomenologically determined α-α interactions for S- and D-states.[3] The upper curves are obtained from the potential with an attractive TPEP only in the singlet even state and the lower curves from the potential with an additional attractive TPEP in the triplet even and triplet odd states.

Many authors discussed the origin of the outside attraction.[4] The forces arising from the polarization of α-particles (the van der Waals attractions) are attractive but too weak.[5] Therefore, only the average potential (the 1st order force) is to be responsible for the feature (1). We show the pion-theoretical potential reproduces this attraction very well. Concerning the features (2) and (3), no satisfactory explanation starting from two-nucleon forces have been made. In order to discuss these points, we need to introduce explicitly the relative wave function consistent with the

requirement of the Pauli principle. The advantage of using the cluster wave function lies in this point.

We assume the wave function in the center of mass system as follows:

$$\Psi = A\{\psi_1^{(0)}(12, 34)\psi_2^{(0)}(56, 78)\psi(\boldsymbol{R})\}, \tag{1}$$

$$\psi(\boldsymbol{R}) = \chi(R)\, Y_{lm}(\Omega_R), \tag{2}$$

$$\psi_1^{(0)} = \exp[-(a/2)\sum_{i=1}^{4}(\boldsymbol{r}_i - \boldsymbol{R}_1)^2]$$

and $$\psi_2^{(0)} = \exp[-(a/2)\sum_{j=5}^{8}(\boldsymbol{r}_j - \boldsymbol{R}_2)^2] \tag{3}$$

with $$\boldsymbol{R}_1 = (1/4)\sum_{i=1}^{4}\boldsymbol{r}_i,$$

$$\boldsymbol{R}_2 = (1/4)\sum_{j=5}^{8}\boldsymbol{r}_j$$

and $$\boldsymbol{R} = \boldsymbol{R}_1 - \boldsymbol{R}_2. \tag{4}$$

Here $\psi_i^{(0)}(i=1, 2)$ and $\psi(\boldsymbol{R})$ represent the ground state of α-particles and their relative motion, respectively. A means the operator of the antisymmetrization. $a=0.58$ f^{-2} are determined from the observed r. m. s. radius of α-particles. The Pauli principle requires for $\chi(R)$ to damp at small distance where two α-particles overlap. In fact, the harmonic oscillator shell model wave function gives $\chi(R) = R^4 \exp(-aR^3)$ independent of $l(l=0, 2, 4)$.

We start from the Schrödinger equation

$$H\Psi = E\Psi \tag{5}$$

with

$$H = -\sum_{i=1}^{8}(\hbar^2/2M)\nabla_i^2 + \sum_{1=i<j}^{8}V_{ij}. \tag{6}$$

Here, only the central parts of two-nucleon potentials V_{ij} are taken into account, because the ground state of α-particles is the spin-saturated system, and tensor and spin-orbit forces play no essential role in the α-α interaction, as far as we are concerned with low energy phenomena. For convenience' sake, radial dependence of V_{ij} are approximated by the Gaussian form.

To find the α-α interaction, we derive the Schrödinger equation for $\chi(R)$ by multiplying Ψ^* on the left side of Eq. (5) and integrating the six independent internal coordinates $\bar{\boldsymbol{r}}_i = \boldsymbol{r}_i - \boldsymbol{R}_1$ $(i=2, 3, 4)$ and $\bar{\boldsymbol{r}}_j = \boldsymbol{r}_j - \boldsymbol{R}_2$ $(j=6, 7, 8)$:

$$\{(-\hbar^2/4M)(1/R^2)\partial/\partial R(R^2\partial/\partial R) + (\hbar^2/4M)l(l+1)/R^2 + V_D(R)]\chi(R) + \int W_l(R, R')\chi(R')dR' = E'\chi(R)$$

with $E' = E - 2E_\alpha^{(0)}$, $E_\alpha^{(0)}$ being the ground state energy of α-particles.

The main results obtained are summarized as follows.

(1) Direct potential $V_D(R)$ is sensitive to the exchange character of two-nucleon forces. The one-pion-exchange potential (OPEP) does not contribute to $V_D(R)$ because it contains the factor $(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)(\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j)$. The two-pion-exchange potential (TPEP), especially the strong attractive one in the singlet even state, plays an essential role to reproduce the outside attraction. We show the results of calculation in Fig. 1.

(2) Exchange interaction term, $\int W_l(R, R')\chi(R')dR'$, is repulsive. The main contribution to it comes from the exchange terms of the two-pion-exchange potential and of the

kinetic energy. To interpret this term as a part of the effective α-α interaction, we need to replace this integral by $V_l^{exch}(R)\chi(R)$. This procedure is allowable in this case since $\chi(R)\neq 0$ for the region ($R\lesssim 5$) where the integral is appreciable. In order to find qualitative features of $V_l^{exch}(R)$, we calculate it for the assumed $\chi(R)=[1-\exp(-0.15R^2)]^3/R$ which is the approximated form to the zero energy solutions obtained from the phenomenologically determined potentials shown in Fig. 1. We obtain the reasonably strong repulsive core representing the effect of the Pauli principle.

(3) Fig. 1 shows the reasonable l-dependence of the α-α interaction. The origin of this l-dependence comes from that of the exchange integral $\int W_l(R, R')\chi(R')dR'$. The repulsion becomes weaker for larger l, reflecting the fact that the correlations in angle between the exchange and the non-exchange wave functions such as $\exp(-\text{const.}\, R\cdot R')$ are rather weak.

Thus we can reproduce all the features essential to the α-α interaction without polarizing α-clusters by use of the pion-theoretical potentials. This means a strong support to the cluster model. Full report will be published in this journal.

The authors are obliged to Professor M. Taketani for the interest he has shown in this problem and to Mr. Y. Shôno and Mr. J. Hiura of our laboratories for their helpful discussions.

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Two-Nucleon Spin-Orbit Forces and the Doublet Splitting in Low Energy n-α Scattering*

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The recent p-p scattering experiments at 310 Mev have shown us the necessity of the spin-orbit forces ($\boldsymbol{L}\cdot\boldsymbol{S}$ forces) which are strong and of short range.[1),2)] At low energies (below about 100 Mev), however, we have no clear evidence for such strong $\boldsymbol{L}\cdot\boldsymbol{S}$ forces, at present.[3)] To study their effects at *low energies*, the doublet splitting in n-α scattering is investigated.

* This work has been done as a part of the 1960-61 Annual Research Project on the Theory of Nuclear Forces organized by the Research Institute for Fundamental Physics, Kyoto University.

The pion-theory of nuclear forces predicts $\boldsymbol{L}\cdot\boldsymbol{S}$ potentials as a part of TPEP (two-pion-exchange potential). Hoshizaki and Machida[4] derived the nonstatic TPEP with full recoil, basing on the methodological standpoint proposed by Taketani and Machida.[5] The spin-orbit potential obtained by them is of the same order of magnitude with the previously derived ones, and is too small to reproduce the high energy data.

There may be other origins of $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling effects than TPEP. So it is desirable to clarify the conditions to be imposed on strong $\boldsymbol{L}\cdot\boldsymbol{S}$ effects by phenomenological analyses.

Problems left undetermined are on the necessity of strong $\boldsymbol{L}\cdot\boldsymbol{S}$ forces at low energies and on their exchange character.

We write $\boldsymbol{L}\cdot\boldsymbol{S}$ forces between i-th and j-th nucleons in the following form:

$$V_{LS}(ij)=\{(3+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)/4 +\lambda(1-\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)/4\}\,(\boldsymbol{L}_{ij}\cdot\boldsymbol{S}_{ij})\,V_{LS}(r_{ij}),$$

$$\boldsymbol{L}_{ij}\cdot\boldsymbol{S}_{ij}=(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}_j)/2 \times\{(\boldsymbol{r}_i-\boldsymbol{r}_j)\times(\boldsymbol{p}_i-\boldsymbol{p}_j)/2\}.$$

Here $V_{LS}(r_{ij})$ is $V_{LS}(T=1)$, the potential in $T=1$ state and λ is the ratio, $V_{LS}(T=0)/V_{LS}(T=1)$ specifying the T-dependence, where T is the magnitude of the total isotopic spin.

$\boldsymbol{L}\cdot\boldsymbol{S}$ forces contribute to the spin-orbit forces in nuclei (hereafter we call this the $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ forces) directly, hence strong $\boldsymbol{L}\cdot\boldsymbol{S}$ forces produce strong $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ forces. At least, a half of the $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ forces is reproduced by the strong tensor force of the one-pion-exchange potential through the effects of the Pauli principle.[6] Therefore, contributions of $\boldsymbol{L}\cdot\boldsymbol{S}$ forces are to be less than a half. This is the severe restriction imposed on $\boldsymbol{L}\cdot\boldsymbol{S}$ forces.

Many authors discussed the $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ forces due to $\boldsymbol{L}\cdot\boldsymbol{S}$ forces, mostly in the Fermi gas model or in the shell model. Unfortunately, in these treatments, it is difficult to get reliable quantitative results, since the effects are estimated by such an integral as $\int x^4 V_{LS}(x)\,dx$ which depends delicately on details of $V_{LS}(x)$, especially for short-range and singular $V_{LS}(x)$.

In n-α scattering, we can avoid this difficulty and estimate the doublet splitting due to $\boldsymbol{L}\cdot\boldsymbol{S}$ forces by their strength as a whole. The effective potential acting on the incident neutron is written in the form

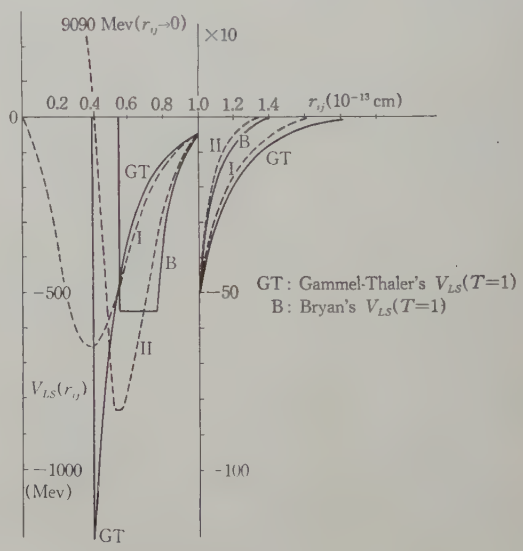


Fig. 1. $\boldsymbol{L}\cdot\boldsymbol{S}$ potentials, $V_{LS}(r_{ij})$ in $T=1$ state. GT and B represent Gammel-Thaler's and Bryan's $\boldsymbol{L}\cdot\boldsymbol{S}$ potentials, respectively.

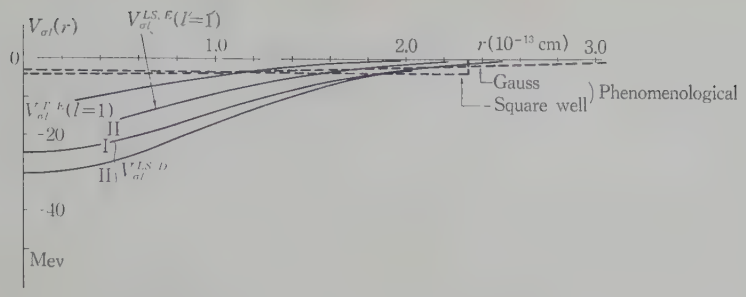


Fig. 2. $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ potentials, $V_{\sigma l}(r)$ calculated from $V_{LS}(r_{ij})$ in Fig. 1. The numbers attached correspond to those in Fig. 1.

$$V(r)=V_0(r)+\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|V_{\sigma l}(r),$$

where

$$\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|=\begin{cases}0 & (l=0)\\ l & (j=l+1/2)\\ -(l+1) & (j=l-1/2).\end{cases}$$

$V_0(r)$ is the spin-independent average potential. $V_{\sigma l}$ consists of three terms: the direct (non-exchange) contribution of $\boldsymbol{L}\cdot\boldsymbol{S}$ forces ($V_{\sigma l}^{LS,D}$), the exchange contribution of $\boldsymbol{L}\cdot\boldsymbol{S}$ forces ($V_{\sigma l}^{LS,E}$), and the exchange contribution of tensor forces ($V_{\sigma l}^{T,E}$):

$$V_{\sigma l}(r)=(\lambda+3)/4\cdot V_{\sigma l}^{LS,D}(r)+(3-\lambda)/2\times V_{\sigma l}^{LS,E}(l,r)+V_{\sigma l}^{T,E}(l,r).$$

The contributions due to $\boldsymbol{L}\cdot\boldsymbol{S}$ forces are calculated by the method used by Hochberg et al.[7] The exchange contributions are l-dependent. For the p-wave in n-α scattering, three terms contribute additively.

We adjust $V_{LS}(r_{ij}=1\times10^{-13}_{\mathrm{cm}})=-50$ Mev. The strength of this order of magnitude is indispensable to explain the p-p data at 310 Mev.[2] Then we vary range-parameters. Some of these potentials are shown in Fig. 1. For convenience' sake, we use the Gaussian form for $V_{LS}(r_{ij})$ and the spatial part of the wave function of α-particle.

The calculated $V_{\sigma l}^{LS,D}$ and $V_{\sigma l}^{LS,E}$ ($l=1$) are shown in Fig. 2, with $V_{\sigma l}$ phenomenologically determined[8] and $V_{\sigma l}^{T,E}(l=1)$.[6] Unreasonably strong $V_{\sigma l}^{LS,D}$ is obtained in any form of $V_{LS}(r_{ij})$. Furthermore, $V_{\sigma l}^{LS,E}(l=1)$ is attractive and rather large. Changing λ to $\lambda=-3$ to decrease the direct contributions, i.e. $V_{LS}(ij)=(\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j)V_{LS}(r_{ij})$, we obtain the factor 3 in the exchange contributions. Thus, the strong $\boldsymbol{L}\cdot\boldsymbol{S}$ forces necessary at high energy produce the unreasonably strong $\boldsymbol{\sigma}\cdot\boldsymbol{l}$ potentials.

We may say that the strong $\boldsymbol{L}\cdot\boldsymbol{S}$ forces playing an important role at high energies become weaker at low energies, and therefore it is not appropriate to describe such effects of two-nucleon spin-orbit coupling besides the $\boldsymbol{L}\cdot\boldsymbol{S}$ potential of TPEP by $\boldsymbol{L}\cdot\boldsymbol{S}$ *potentials*. The full report will be published in this journal in near future.

The authors wish to express their sincere thanks to Prof. M. Taketani, Dr. S. Otsuki and Dr. W. Watari for their valuable discussions, and to Prof. H. Tanaka for his continuous encouragement during this work.

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Cosmological Implication of Physical Constants

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In the fundamental equations of physics there appear physical constants, such as the gravitation constant in mechanics, the Planck constant in quantum mechanics and so forth. In the present framework of theory these constants are given *ab initio* as parameters, but the progress in physics would eventually explore their physical implication ; their numerical values would be derived from something more fundamental. As a step along this philosophy we dare to make speculation which may explain the numerical values of the gravitation constant and the Planck constant in cooperation with cosmology.

Our idea is very close to that in our previous paper.[1] In the expanding universe some amount of mass, m, is lost from a region within the cosmic radius, R. This might be thought to be violation of the energy conservation, but the energy loss, mc^2, is compensated by the change in the gravitational potential energy due to the total mass, M, inside this region. Hence the enegy conservation results in a relation

$$mc^2=GMm/R. \qquad (1)$$

Taking $M\simeq 10^{56}$g and $R\simeq 10^{28}$ cm, which are not incompatible to observational evidences, we obtain the value of the gravitation constant as

$$G\simeq 10^{-7}\ \mathrm{cm^3\,sec^{-2}\,g^{-1}}.$$

This is in essential agreement with the observed value.

The relation (1) is not new, but is essentially the same as that based on the virial theorem. However, our interpretation may serve to resolve the apparent contradiction in steady state cosmology, according to which the creation of matter might violate the conservation of energy. Again in this case the

relation (1) ensures the energy conservation: the increase in mass energy is compensated by the decrease in potential energy.

The total mass M in (1) is shared mostly with nucleons in the universe, their total number being as large as $N_n \simeq 10^{80}$. The loss of particles by expansion and (or) the creation of particles (if one prefers to rely upon steady state cosmology) may result in that the number N_n is not exaclty constant but fluctuates by an amount of about $\sqrt{N_n}$. This gives rise to the gravitational shift of energy through the fluctuations of total mass M, so that the mass value of the nucleon fluctuates by Δm_n according to

$$\Delta m_n / m_n \simeq 1/\sqrt{N_n}. \qquad (2)$$

On the other hand, the uncertainty in the mass value arises from the quantum fluctuations, because the carrier of the mass, a particle, has a finite lifetime corresponding to the cosmic age, τ. This is expressed as

$$\Delta m_n c^2 \simeq \hbar/\tau. \qquad (3)$$

From (2) and (3) we can derive the value of the Plank constant divided by 2π as

$$\hbar \simeq m_n c^2 \tau/\sqrt{N_n}. \qquad (4)$$

Putting $m_n c^2 \simeq 10^{-3}$ erg (nucleon mass), $\tau \simeq 3\times 10^{17}$ sec and $\sqrt{N_n} \simeq 10^{40}$, we have $\hbar \simeq 3\times 10^{-26}$ erg sec, which is too large, by a factor of some tens. On account of various uncertainties in the above derivation, this may be regarded as rather good agreement.

The above argument may be modified in the following way. Following the spirit of steady state cosmology, we may assume $m_i N_i = $ constant, where suffix i indicates a family of particles whose number is conserved in the sense of particle physics. Therefore, this applies either to the baryon family or to the lepton family. This again leads us to a relation similar to (2):

$$|\Delta m_i/m_i||\Delta N_i/N_i|^{-1} \lesssim a, \qquad (5)$$

where a is a numerical constant and may be of the order of unity on account of (2). The inequality in (5) is introduced, because the fluctuations should not exceed a certain limit. This relation (5) may be called the cosmological uncertainty relation, and is analogous to the statistical uncertainty relation in thermodynamics.

On account of another inequality in the quantum mechanical uncertainty relation (3), (5) results in

$$\hbar\sqrt{N_i}/m_i c^2 \tau \lesssim a. \qquad (6)$$

This relation becomes severer, as $\sqrt{N_i}/m_i$ increases. This seems to be the case for the neutrino, and the value of $\hbar$ may be determined from the equality in this case. Estimating N_i from the thermonuclear reaction rate in stars and taking $m_i c^2$ equal to the average energy of neutrinos, $N_i \simeq 10^{77}$ and $m_i c^2 \simeq 10^{-6}$ erg, we obtain

$$\hbar \simeq 10^{-27} \text{ erg sec.}$$

The agreement with the observed value is surprisingly good.

Although one might think the above coincidences as purely accidental, we feel that something essential seems to be implied in our argument. Indeed, it is not imposible to obtain the

reasonable values of some coupling constants on the basis of a similar argument. We would like to say that the interpretation of physical constants cannot be made within the present framework of physics but on the higher hierarchy, such as cosmology.

One of the present authors (S.H.) would like to express his thanks to the members of Department of Physics, Hokkaido University. Their hospitality extended to him during his stay there in last September created the atmosphere that bore the idea of the present work. The other author (H. T.) wishes to express his gratitude to Prof. S. Miyahara and Prof. T. Iwazaki for many stimulating discussions on natural philosophy.

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Invariance of Planck's Constant in Revised Relativity of J. Palacios

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A revised theory of relativity has been proposed recently by J. Palacios,[1] which avoids the clock paradox of special relativity by regarding a reference system A to be in the state of absolute rest. If the undashed coordinates refer to A, Palacios takes the special transformations to a moving system S^1 to be

$$x'=\beta^2(x-vt),\quad y'=\beta y,\quad z'=\beta z,$$
$$t'=\beta^2\left(t-\frac{vx}{c^2}\right), \tag{1}$$

where

$$\beta=\left(1-\frac{v^2}{c^2}\right)^{1/2}.$$

It is clear that Maxwell's equations in free space remain invariant under this transformation from A to S^1.[2] Moreover, the components of electric and of magnetic intensity vectors transform A to S^1 according to the relations of special relativity. If, however, their phase ϕ is invariant:

$$\phi=\omega\left\{t-\frac{1}{c}(lx+my+nz)\right\}$$
$$=\omega'\left\{t'-\frac{1}{c}(l'x'+m'y'+n'z')\right\}, \tag{2}$$

we obtain, using transformations (1), the nonrelativistic Doppler effect

$$\omega'=\omega\left(1-\frac{lv}{c}\right).$$

Apart from this, the equations of transformation of the direction cosines

$$(l,\ m,\ n)\rightarrow(l',\ m',\ n')$$

are the same as in special relativity, and, in particular,

$$l'^2+m'^2+n'^2=l^2+m^2+n^2=1$$

as required. Using these results, Palacios claims that, if his theory is to be adopted, it is necessary for the Planck constant h to transform according to the relation

$$h'=\beta h.$$

Now, Palacios secures the invariance of the velocity of light c, since it follows from (1) that

$$\beta^2(c^2t^2-x^2-y^2-z^2)$$
$$=c^2t'^2-x'^2-y'^2-z'^2,$$

both sides of this equation vanishing simultaneously. Thus, in his theory, the fundamental interval ds is not an invariant, but

$$ds'=\beta ds.$$

The determinant of the transformation (1) being β^4, the fundamental interval appears as a scalar density of weight 1/4. In the same way, we may regard the phase (2) as a scalar density of the same weight. Then

$$\phi'=\beta\phi. \qquad (3)$$

This restores the relativistic Doppler effect

$$\omega'=\beta\omega\left(1-\frac{lv}{c}\right).$$

If the transformation (3) is postulated, the noninvariance of Planck's constant is no longer required even if the transformations (1) of Palacios are correct.

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An Isobaric State K^* in K-π System and $p+\bar{p}\to K+\bar{K}+\pi+\pi$ Process

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Recent measurements on $K^-+p\to\bar{K}^0+\pi^-+p$ reaction have shown that $\bar{K}^0$ and π^- may be regarded as the decay products of an isobaric state K^* in $\bar{K}$-π system.[1] In other words this reaction mainly takes place through process $K^-+p\to K^*+p\to\bar{K}^0+\pi^-+p$. According to the experimental results $m_{K^*}\cong 884$ Mev, where m_{K^*} is the mass of K^*. And the assumption of $I_{K^*}=1/2$ (I_{K^*} indicates the isotopic spin of K^*) rather than that of $I_{K^*}=3/2$ seems to be consistent with the experimental results for the branching ratio of the decay of $K^*(K^*\to\bar{K}^0+\pi^-/K^*\to K^-+\pi^0)$. If $\bar{K}$-π interaction is invariant under the transformation of charge conjugation, there should also be an isobaric state $\bar{K}^*$ in K-π system. In order to make sure of this $\bar{K}^*$ we want to get the experimental data for energy spectrum (or momentum spectrum) of nucleon in the reaction $K+N\to K+\pi+N$ (for example, $K^++p\to K^++\pi^0+p$), although the well-known (3/2, 3/2) interaction in pion-nucleon system may have some effect on this reaction.

It is said that an experiment in a hydrogen bubble chamber will be made at CERN in which the reactions

$$p+\bar{p}\rightarrow K+\bar{K}+\pi+\pi$$

will be measured and that there should probably be about 300 such cases in the experimental data. Both by this information and by the experimental evidence of K^* we are motivated to make some predictions with respect to this reaction. In this paper we should like to study this reaction under the assumption that the reaction mainly takes place through the process

$$p+\bar{p} \rightarrow K^*+\bar{K}^*\rightarrow K+\bar{K}+\pi+\pi.$$

Now let us try to estimate the branching ratios for this reaction. Since p-$\bar{p}$ system consists of the total isotopic spin $I=1$ state and $I=0$ state, this reaction can be described in terms of the reaction amplitudes for $I=1$ and $I=0$ states. At first we assume $I_{K^*}=1/2$, then the branching ratios in the case in which only the $I=1$ state reacts strongly are the same with those in the case in which only the $I=0$ state reacts strongly, and they are given by

$p+\bar{p}\rightarrow K^++K^-+\pi^++\pi^-$	2/9
$K^++K^-+\pi^0+\pi^0$	1/18
$K^++\bar{K}^0+\pi^-+\pi^0$	2/9
$K^0+\bar{K}^0+\pi^0+\pi^0$	1/18
$K^0+\bar{K}^0+\pi^++\pi^-$	2/9
$K^0+K^-+\pi^++\pi^0$	2/9.

Next we assume $I_{K^*}=3/2$ although it may be difficult to expect this case. Then the results are shown in Table I.

In our model K^* and $\bar{K}^*$ will be emitted in the opposite direction to each other in the center of mass system. As K meson is much heavier than pion, at low energy we may expect the large probability with which K meson and $\bar{K}$ meson are emitted almost in the opposite direction.

Table I

reactions	branching ratios when only $I=1$ state reacts strongly	branching ratios when only $I=0$ state reacts strongly
$p+\bar{p}\rightarrow K^++K^-+\pi^++\pi^-$	41/90	5/18
$K^++K^-+\pi^0+\pi^0$	1/45	1/9
$K^++\bar{K}^0+\pi^-+\pi^0$	1/45	1/9
$K^0+\bar{K}^0+\pi^0+\pi^0$	1/45	1/9
$K^0+\bar{K}^0+\pi^++\pi^-$	41/90	5/18
$K^0+K^-+\pi^++\pi^0$	1/45	1/9

With respect to the cross sections for $p+\bar{p}\rightarrow K+\bar{K}+\pi+\pi$, it may also be expected that the theoretical values based on the simple statistical theory will turn out to be large when the effects of K^* are taken into account. Another note will be given on this problem.

From the recent experimental results[1), 2)] for $K^-+p\rightarrow\Lambda+\pi^++\pi^-$ at high energy it may follow that we have some doubts about the existence of an isobar π^* in pion-pion system. But, for reference, we try to estimate the branching ratios for $p+\bar{p}\rightarrow K+\bar{K}+\pi+\pi$ reaction under the assumption of strong pion-pion interaction. The results are shown in Table II, where T, T' and I stand for the isotopic spins of π-π system, K-$\bar{K}$ system and total system respectively.

On account of the effects of interference terms in addition to the existence of various states relevant to this reaction, it may be difficult with-

Table II. Branching ratios in the case in which each isotopic spin state mentioned in this table reacts strongly

final state \ isotopic spin state	branching ratios					
	$I=1$				$I=0$	
	$T=2$	$T=1$		$T=0$	$T=1$	$T=0$
	$T'=1$	$T'=1$	$T'=0$	$T'=1$	$T'=1$	$T'=0$
K^+, K^-, π^+, π^-	1/15	0	1/2	1/3	1/6	1/3
K^+, K^-, π^0, π^0	2/15	0	0	1/6	0	1/6
K^+, $\bar{K}^0$, π^-, π^0	3/10	1/2	0	0	1/3	0
K^0, $\bar{K}^0$, π^0, π^0	2/15	0	0	1/6	0	1/6
K^0, $\bar{K}^0$, π^+, π^-	1/15	0	1/2	1/3	1/6	1/3
K^0, K^-, π^+, π^0	3/10	1/2	0	0	1/3	0

out a special assumption to derive any more definite conclusion except the following results.

(i) The branching ratios in the case of $T=2$ state resonance are shown in Table II.

(ii) If the isotopic spin of π^* is equal to 1, both $p+\bar{p}\to K^++K^-+\pi^0+\pi^0$ and $p+\bar{p}\to K^0+\bar{K}^0+\pi^0+\pi^0$ are forbidden.

(iii) If the isotopic spin of π^* is equal to zero, both $p+\bar{p}\to K^++\bar{K}^0+\pi^-+\pi^0$ and $p+\bar{p}\to K^0+K^-+\pi^++\pi^0$ are forbidden and

$$\sigma(p+\bar{p}\to K^++K^-+\pi^++\pi^-)=2\sigma(p+\bar{p}\to K^++K^-+\pi^0+\pi^0),$$

$$\sigma(p+\bar{p}\to K^0+\bar{K}^0+\pi^++\pi^-)=2\sigma(p+\bar{p}\to K^0+\bar{K}^0+\pi^0+\pi^0).$$

(iv) If the event of $p+\bar{p}\to K^++K^-+\pi^++\pi^-$ is not observed in experiment in spite of the fact that this process ought to be observed easily, the result may be consistent with the largeness of the reaction amplitude for the state $T=1$, $T'=1$ and $I=1$.

The author would like to express his thanks to Dr. J. Ballam for his kind information on the experiment at CERN.

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2) 10th Rochester Conference on High Energy Physics (1960).

K Meson-Nucleon Interaction at High Energy

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March 11, 1961

The existence of new isobaric states K^* and Y^* seems to be confirmed by the recent experiment.[1),2)] As was pointed out previously[3)] there should also be an isobaric state $\bar{K}^*$ in K-π system if $\bar{K}$-π interaction is invariant under the transformation of

charge conjugation. It may be expected that K meson physics will be developed remarkably by the appearance of these new isobaric states. In this note we should like to discuss what kind of prediction in $\bar{K}$-N collision or in K-N collision at high energy may be made by taking into account the effects of K^* (or $\bar{K}^*$).*

Takeda et al.[4] have tried to explain the second, the third and the fourth resonance phenomena in pion-nucleon collision in terms of the effects both of the well-known (3/2, 3/2) isobar in pion-nucleon system and of an isobar** ρ in pion-pion system.*** If such an explanation as this is generally valid, we may develop our discussion along the same line with these authors with respect to the resonance phenomena in $\bar{K}$-N or K-N collision, although it is of course necessary to examine this assumption. $\bar{K}$-N scattering or K-N scattering can be described in terms of the scattering amplitudes T_1 and T_0 for isotopic spin states $I=1$ and $I=0$ respectively. We now assume $I_{K^*}=1/2$ (I_{K^*} indicates the isotopic spin of K^*) because the assumption of $I_{K^*}=1/2$ rather than that of $I_{K^*}=3/2$ seems to be consistent with the experimental results for the branching ratio of the decay of $K^*(K^*\to\bar{K}^0+\pi^-/K^*\to K^-+\pi^0)$.

Let us discuss the K meson-nucleon scattering processes at high energy by paying attention to the long-range $\bar{K}$-N or K-N interaction caused by two-pion exchange on which the K^* has an important effect. Among them we take into account only the process shown in Fig. 1 so far as $\bar{K}$-N or

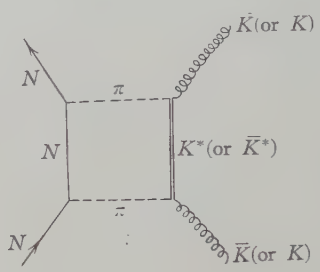


Fig. 1.

K-N scatterings at about 670 Mev (lab.) are concerned, because energies of intermediate K^*-N state (or $\bar{K}^*$-N state) can be very near or identical with the initial energy of the colliding system and, as the result, the contributions from this process may be much larger than those from other processes. Then the following result for scattering amplitudes can be derived from our perturbation calculation,

$$|T_0|^2=81|T_1|^2. \qquad (1)$$

If the model suggested by Takeda et al.* can also be applied to the case of $\bar{K}$-N or K-N interaction, the model may make it possible to predict the following property of $\bar{K}$-N or K-N interactions at high energy. In the neighborhood of the energy 675 Mev

* With regard to the effect of Y^* on $\bar{K}$-N collision, another note will be given.

** The existence of this ρ has not been confirmed by experiment.

*** Their conclusions are based on the perturbation theory. They said that the qualitative conclusions do not suffer any remarkable change even if the Tamm-Dancoff approximation is adopted.

* The result in the case $I_{K^*}=3/2$ will be mentioned below.

(lab.) which corresponds to the threshold energy of K^*(or $\bar{K}^*$) production,* we may expect the large cross sections for $\bar{K}$-N scattering (or K-N scattering) and these resonance phenomena may be due to the strong interaction in $I=0$ state (cf. Eq. (1)). Thus these resonance phenomena may not be expected in K^+-p scattering, but be expected in K^--p scattering.

Next let us consider the process

$$\bar{K}+N\rightarrow K^*+N^*\rightarrow\bar{K}+\pi+\pi+N$$

or

$$K+N\rightarrow\bar{K}^*+N^*\rightarrow K+\pi+\pi+N.$$

The $\bar{K}+N$ (or $K+N$)$\rightarrow K^*+N^*$ (or $\bar{K}^*+N^*$) transition by one-pion exchange potential is illustrated in Fig. 2, where N^* indicates the well-known

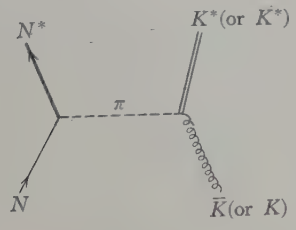


Fig. 2

(3/2, 3/2) isobar in pion-nucleon system. It may be possible to expect a maximum of the inelastic scattering cross sections** due to the final state interactions in the neighborhood of the energy 1290 Mev (lab.) which corresponds to the threshold energy of the reaction $\bar{K}+N\rightarrow K^*+N^*$ (or $K+N\rightarrow\bar{K}^*+N^*$). Since the isotopic spin of K^*-N^* system cannot have the value of $I=0$ under the assumption $I_{K^*}=1/2$, this maximum will be due to the strong interaction in $I=1$ state.

Finally we want to make some discussion about the case $I_{K^*}=3/2$, although it may be difficult to expect this case. The isotopic spin of K^*-N system (or $\bar{K}^*$-N system) in the intermediate state for $\bar{K}$-N scattering (or K-N scattering) cannot have the value of $I=0$ under the assumption $I_{K^*}=3/2$ (cf. Fig. 1). Then the resonance phenomena which may be expected in the neighborhood of 670 Mev will be attributed to the strong interaction in $I=1$ state. With regard to the transition amplitudes for the process $\bar{K}+N\rightarrow K^*+N^*$ (or $K+N\rightarrow\bar{K}^*+N^*$), we can obtain the following result by means of perturbation calculation,

$$|T_1'|^2:|T_0'|^2=5:9, \qquad (2)$$

where T_1' and T_0' are the transition amplitudes for the isotopic spin $I=1$ and $I=0$ states respectively. It may be difficult from the relation (2) to predict which state of isotopic spin plays the most important role in $\bar{K}$-N (or K-N) collision at about 1290 Mev.

* If we apply their theory to our case, the $\bar{K}$ meson-nucleon interaction may become attractive, since most intermediate K^*-N states have an energy larger than the excitation energy of K^*. In near future we will examine how the $\bar{K}$ meson-nucleon interaction depends on the spin of K^*.

** These resonance phenomena may not be so remarkable.

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2) 10th Rochester Conference on High Energy Physics (1960).
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Quantum Theory of Spin Diffusion

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It has been recognized that the notion of spin diffusion due to the exchange-type interaction between spins is very useful to describe a number of experiments on the nuclear spin relaxation in a crystal containing paramagnetic impurities,[1] and on the critical scattering of neutrons by a ferromagnet.[2] It has been assumed there that the z-component of the magnetization density, $M^0(\boldsymbol{R})$, would obey the diffusion equation

$$\partial M^0(\boldsymbol{R})/\partial t = \Lambda^0 \nabla^2 M^0(\boldsymbol{R}),$$

Λ^0 being the spin diffusion constant.

In this short note, we shall outline a general formulation of the macroscopic equations for the three components of the magnetization density $\boldsymbol{M}(\boldsymbol{R})$ with the use of a quantum theory of transport processes developed recently by one of the authors.[3]

The principal part of the Hamiltonian of the system is assumed to be the exchange-type interaction:

$$H = -\omega S^0 + N^{-1} \sum_q J(\boldsymbol{q}) \times [S^+(-\boldsymbol{q}) S^-(\boldsymbol{q}) + S^0(-\boldsymbol{q}) S^0(\boldsymbol{q})],$$

where ω is the Zeeman frequency, N the number of magnetic ions, and $J(\boldsymbol{q})$ the Fourier component of the exchange-type interaction with the wave number $\boldsymbol{q}$. The z-component of the total spin of the system, $S^0 = S^0(\boldsymbol{0})$, commutes with the Hamiltonian, and is therefore an approximate constant of motion, performing a small fluctuation due to the remaining parts of the Hamiltonian, such as the non-adiabatic part of the dipolar interaction between spins and the interaction with the neighboring systems.

According to the principle of statistical mechanics, it is convenient for the macroscopic description of such a system to introduce a thermodynamic quantity conjugate to the S^0, and to employ a grand canonical ensemble $\rho_0 = C_0 \exp\{-\beta[H - \mu^0 S^0]\}$, where actually the equilibrium value μ^0 becomes zero.

Now the system is inhomogeneous in space, and the density matrix of the system, $\rho(t)$, deviates from the ρ_0 to yield the inhomogeneous magnetization density, its Fourier components being defined by $\boldsymbol{M}(\boldsymbol{k}) \equiv \mathrm{Tr}\rho(t)\boldsymbol{S}(\boldsymbol{k})$. The equations of motion for the spin operators $\boldsymbol{S}(\boldsymbol{k})$ take the form,

$$\begin{aligned} dS^0(\boldsymbol{k})/dt &= i\boldsymbol{k}\cdot\boldsymbol{j}^0(\boldsymbol{k}), \\ dS^\pm(\boldsymbol{k})/dt &= \mp i\omega(\boldsymbol{k}) S^\pm(\boldsymbol{k}) \\ &\quad + i\boldsymbol{k}\cdot\boldsymbol{j}^\pm(\boldsymbol{k}), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \boldsymbol{j}^0(\boldsymbol{k}) &\equiv N^{-1} \sum_q \boldsymbol{u}(\boldsymbol{k}, \boldsymbol{q}) \\ &\quad \times \{S^+(\boldsymbol{q}), S^-(\boldsymbol{k}-\boldsymbol{q})\}, \\ \boldsymbol{j}^\pm(\boldsymbol{k}) &\equiv \pm 2N^{-1} \sum_{q \neq 0} \boldsymbol{u}(\boldsymbol{k}, \boldsymbol{q}) \\ &\quad \times \{S^0(\boldsymbol{q}), S^\pm(\boldsymbol{k}-\boldsymbol{q})\}, \end{aligned} \tag{2}$$

where $\lim_{k\to 0} \boldsymbol{u}(\boldsymbol{k}, \boldsymbol{q}) = \partial J(\boldsymbol{q})/\partial \boldsymbol{q}$. $\omega(\boldsymbol{k})$ is the frequency spectrum and is, in the first approximation, equal to $\omega + 2J(\boldsymbol{k}, \boldsymbol{0})(\langle S^0 \rangle_0 / N)$, where $\langle \cdots \rangle_0$

means the average with the equilibrium ensemble ρ_0. The ω-terms of Eq. (1) express the rotation of the spins with the angular frequency $\omega(\boldsymbol{k})$ about the z-axis, which has no relation to the dissipative mechanism. Therefore it is convenient to remove this part by making the transformation

$$\overline{\boldsymbol{S}}(\boldsymbol{k}) \equiv \exp[it\omega(\boldsymbol{k})S^0] \times \boldsymbol{S}(\boldsymbol{k}) \exp[-it\omega(\boldsymbol{k})S^0].$$

The local equilibrium ensemble of the system can be obtained as follows:

$$\rho_l = C \exp\{-\beta[H - \textstyle\sum_{q\neq 0} \boldsymbol{\mu}(-\boldsymbol{q}) \cdot \times \boldsymbol{S}(\boldsymbol{q})]\}.$$

The non-equilibrium magnetic chemical potential $\boldsymbol{\mu}(-\boldsymbol{q})$ conjugate to $\boldsymbol{M}(\boldsymbol{q})$ can be determined from the ρ_l with the aid of the consistency condition $\boldsymbol{M}(\boldsymbol{k}) = \mathrm{Tr}\, \rho_l \boldsymbol{S}(\boldsymbol{k})$.

Application of the statistical mechanics of irreversible processes[3] leads to the following expression for the density matrix of the system:

$$\begin{aligned}\rho(t) = \rho_l - \rho_0 \int_{\tau}^{\tau} ds \int_0^{\beta} d\lambda N^{-1} \textstyle\sum_{q\neq 0} i\boldsymbol{q} \cdot \\ \times \{\mu^0(-\boldsymbol{q}) \boldsymbol{j}^0(\boldsymbol{q}, -s - i\hbar\lambda) \\ + \frac{1}{2} \mu^-(-\boldsymbol{q}) \exp[-is\omega(\boldsymbol{k})] \\ \times \boldsymbol{j}^+(\boldsymbol{q}, -s - i\hbar\lambda) \\ + \frac{1}{2} \mu^+(-\boldsymbol{q}) \exp[is\omega(\boldsymbol{k})] \\ \times \boldsymbol{j}^-(\boldsymbol{q}, -s - i\hbar\lambda)\},\end{aligned}$$

where $\tau_c \ll \tau \ll \tau_r$, $1/|\omega(\boldsymbol{k})|$, where τ_r is the macroscopic relaxation time of the system and τ_c a microscopic correlation time characterizing the decay of the integrands of Eqs. (4). Thus, taking the average of Eq. (1) with the $\rho(t)$, we arrive at

$$\begin{aligned} dM^0(\boldsymbol{k})/dt &= -k^2 \Lambda^0 M^0(\boldsymbol{k}), \\ dM^{\pm}(\boldsymbol{k})/dt &= [\mp i\omega(\boldsymbol{k}) - k^2 \Lambda^{\pm}] M^{\pm}(\boldsymbol{k}), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Lambda^0 &\equiv \frac{1}{3\beta \langle (S^0 - \langle S^0 \rangle_0)^2 \rangle_0} \int_0^{\infty} ds \int_0^{\beta} d\lambda \\ &\quad \times \langle \boldsymbol{j}^0 \cdot \boldsymbol{j}^0(s + i\hbar\lambda) \rangle_0, \\ \Lambda^{\pm} &\equiv \frac{1}{3\beta \langle S^{\mp} S^{\pm} \rangle_0} \int_0^{\infty} ds \int_0^{\beta} d\lambda \\ &\quad \times \langle \boldsymbol{j}^{\mp} \cdot \boldsymbol{j}^{\pm}(s + i\hbar\lambda) \rangle_0 e^{\pm is\omega(k)}. \end{aligned} \tag{4}$$

It can be shown that, above the Curie temperature, the diffusion constants Λ^0 and $\Lambda^{\pm}$ become equal to each other in the absence of external magnetic field. The expression for Λ^0 turns out to be identical with the one which has been shown in a previous note[4] to yield a good agreement with the experiments on the inelastic critical scattering of neutrons by iron above T_C.

Detailed account and other applications will be published shortly.

The authors wish to express their thanks to Professor T. Matsubara for his enlightening discussions and encouragement.

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The τ^+-decay and the Boson Isobar with $I=2$ of the Sakata Model

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Some analyses of the single pion production in π-N collision[1] have been made to find the evidence of the boson isobars[2] of the Sakata model. From these analyses it has been suggested that the boson isobar with $I=2$, i.e. $B_4^1(0, 2)$ of the Sakata model, exists and has the 2π-decay mode.

The $B_4^1(0, 2)$ isobar is, then, expected to have the effect to the pionic decay of kaon. Especially the 3π decay process of kaon is very interesting since the present experimental data of pion energy spectra in τ^+-decay process[3] show a small deviation from the theoretical spectrum for spinless

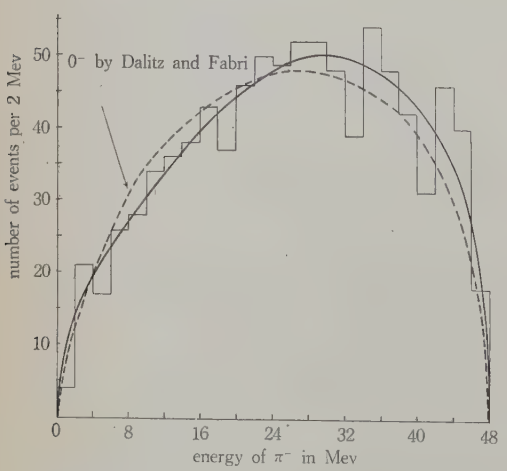


Fig. 1. π^- energy spectrum in $K^+\to\pi^+ + \pi^+ + \pi^-$.

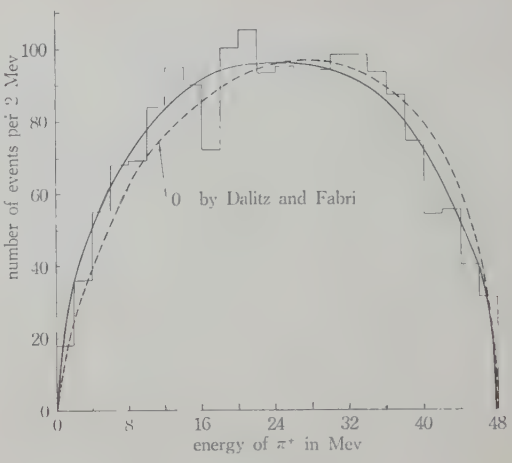


Fig. 2. π^+ energy spectrum in $K^+\to\pi^+ + \pi^+ + \pi^-$.

kaon given by Dalitz and Fabri.[4] (See Figs. 1 and 2.)
Some studies have already been made to understand the deviation as the effect due to the possible strong interaction among the final pions.[5]

In this short note we also study the problem by taking the effect of $B_4^1(0, 2)$ which can be corresponded to the strong π-π interaction, but our approach is a little different from these analyses.

Now let us assume the following for the moment.

1) The τ^+ and $\tau^{+\prime}$ decay mainly consist of the following *cascade* decay.

$$K^+ \xrightarrow[\text{slow}]{} B_4^1(0, 2) + \pi \text{ (recoil pion)}, \quad \Big| \xrightarrow[\text{rapid}]{} 2\pi \text{ (decay pions)}. \qquad (1)$$

2) $B_4^1(0, 2)$ is 0^+ and can be treated in the calculation as though it is a *metastable particle*, except that it has not a definite mass, but mass distribution $\rho(m_B)$.

3) The decay interaction for

$K^+ \to B_4^1(0, 2) + \pi$ and $K_1^0 \to 2\pi$ are effectively of the type

$$g\phi_B\phi_\pi\phi_K \quad \text{and} \quad g'\phi_\pi\phi_\pi\phi_K \qquad (2)$$

with $g \sim g'$ (Universality of weak interaction).

The theoretical energy spectra of the final pions in τ^+-decay and $\tau^{+\prime}$-decay through the decay scheme (1) will be given by the following expressions when the $\Delta I = 1/2$ rule holds for these processes :

π^--spectrum in τ^+-decay

$$I^-(E_\pi)dE_\pi \propto [1/30\ I_1(E_\pi) + 18/30\ I_2(E_\pi)]dE_\pi, \qquad (3a)$$

π^+-spectrum in τ^+-decay

$$I^+(E_\pi)dE_\pi \propto 2[18/30\ I_1(E_\pi) + 1/30\ I_2(E_\pi)]dE_\pi, \qquad (3b)$$

π^+-spectrum in $\tau^{+\prime}$-decay

$$I'(E_\pi)dE_\pi \propto [9/30\ I_1(E_\pi) + 2/30\ I_2(E_\pi)]dE_\pi. \qquad (3c)$$

$I_1(E_\pi)$ and $I_2(E_\pi)$ are the energy distribution function of the decay pions and the recoil pions respectively and expressed by

$$I_1(E_\pi)dE_\pi = N_1 \int_{m_B^{\min}}^{m_B^{\max}} dm_B \rho(m_B) m_B / p_{\pi B} dE_\pi, \qquad (4)$$

$$I_2(E_\pi)dE_\pi = N_2 \rho(m_B) m_K p_B / m_B dE_\pi,$$

where m_K and m_B are the mass of the kaon and $B_4^1(0, 2)$ respectively and $p_{\pi B} = \sqrt{(m_B/2)^2 - m_\pi^2}$ (for further notations, see reference 1)).

The dominant factor for the pion energy spectra is the mass distribution of $B_4^1(0, 2)$; $\rho(m_B)$. However, we know little about it now. Some information will be obtained from the consideration of the decay rate of kaons. From the decay interactions (2) we shall obtain

$$R = \frac{W(K^+ \to B_4^1(0, 2) + \pi)}{W(K_1^0 \to 2\pi)} = \frac{g^2 \int_{2m_\pi}^{m_K - m_\pi} dm_B \rho(m_B) p(m_B)}{g'^2 p'},$$

where $p(m_B)\,(= p_B)$ and p' is the pion momentum in $K^+ \to B_4^1(0, 2) + \pi$ and $K_1^0 \to 2\pi$ process respectively. If we take $280\ \text{Mev} < m_B^0 < 354\ \text{Mev}$ $(= m_K - m_\pi)$, where m_B^0 is the parameter characterizing the maximum of the mass distribution, and put $g \sim g'$, we shall obtain

$$R \simeq 0.1 \sim 0.6$$

which is to be compared with the experimental value $R < 10^{-3}$.* This fact indicates that the maximum of the $\rho(m_B)$ will probably be at > 354 Mev. The tendency of the $\rho(m_B)$ may manifest itself in the pion energy spectrum. Neglecting the term $I_1(E_\pi)$ in π^--spectrum, we obtain

$$\rho(m_B) \propto I_2(E_\pi) \cdot (p_B/m_B)^{-1} \simeq I^-(E_\pi)(p_B/m_B)^{-1}.$$

Then assuming that the τ^+-decay occurs entirely through (1), we shall obtain the experimental $\rho(m_B)$. It is noted that such a $\rho(m_B)$ can be safely represented by the monotonic in-

* The mass distribution $\rho(m_B)$ with the parameters $m_B^0 = 400$ Mev and $\Gamma_B = 30$ Mev which was used in reference 1) gives this ratio $\sim 10^{-3}$,

creasing curve (the solid curve in Fig. 3).

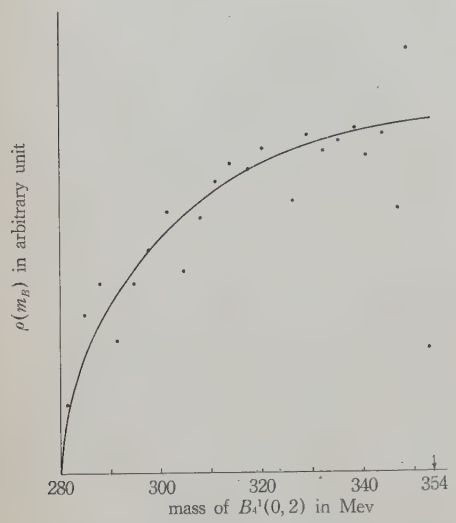


Fig. 3. The mass distribution $\rho(m_B)$ fitted to the experimental data of π^- spectrum.

Using thus obtained $\rho(m_B)$, we have calculated π^+-energy spectra which is given in Fig. 2. The agreement of the calculated spectrum with the experiment is fairly satisfactory.

As for the direct 3π decay process we shall only remark that the consistent interpretation of the π-spectrum is also possible even if this process is comparable to the $B_4^1(0, 2)$ channel, by taking the slightly changed $\rho(m_B)$. The branching ratio $\tau^+/\tau^{+\prime}$ is worthwhile to notice. With no direct process the ratio is 19/11 for $\Delta I=1/2$. This value seems to be a little smaller than the present experimental value 2.9 ± 0.5. In order to increase the branching ratio from 19/11 it is necessary to introduce either the $\Delta I=3/2$ decay amplitude or the direct 3π decay process.

The possible existence of $B_4^1(0, 2)$ may bring the effect in $K\to2\pi$, but the problem essentially needs information about the large mass part of $\rho(m_B)$ and the strength of the reaction with $\Delta I=3/2$, and will require further investigation. It is noted that the process $K\to B_4^1(0, 2)+e+\nu$ is forbidden in the Sakata model.

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A Note on Baryon-Baryon Interaction

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Okun et al.[1] have considered Ξ^- capture by a proton leading to a $(\Lambda^0\Lambda^0)$ system. In this note we study the process of a (Ξ^-n) collision giv-

ing rise to hyperon systems, the possibility of a bound state in the final system being also envisaged. The implication arising from the various parity assignments for the hyperons is discussed and the decay distribution of the possible bound system is analysed.

We consider in particular the reactions

$$\Xi^- + n \rightarrow \Xi^- + n \quad \text{Threshold} \quad (1)$$
$$\rightarrow \Sigma^- + \Lambda^0 \quad (52.5 \text{ Mev}) \quad (2)$$
$$\rightarrow \Lambda^0 + \Lambda^0 + \pi^- \quad (110.8 \text{ Mev}) \quad (3)$$
$$\rightarrow \Sigma^- + \Sigma^0 \quad (134 \text{ Mev}). \quad (4)$$

The branching ratios cannot be directly calculated but on assuming charge independence, and the Sakata model for the final state particles the ratios for the processes (2) and (3) are found to be

$$\Sigma^- + \Lambda^0 : \Lambda^0 + \Lambda^0 + \pi^- :: 1 : 1. \quad (5)$$

We assume the $(\Xi^- n)$ system is scattered from the S-state, the $(\Xi^- n)$ relative parity is even and the incident system has $j = s = 1$.

If $(\Lambda^0 \Lambda^0)$ is assumed to be in the S-state, this state is in ${}^1S_0$-one according to Pauli's exclusion principle, hence both angular momentum and parity cannot be conserved when the final state of π is in the S-state in respect of the center of mass of the $(\Lambda^0 \Lambda^0)$ system.

It is natural that the conservation law of parity and angular momentum is valid when P-wave π is emitted in respect of the center of mass of the $(\Lambda^0 \Lambda^0)$ system.

We shall now consider the possibility of Σ^- and Σ^0 in (4) forming a bound state. The conservation of I spin leads to the $I=1$ state for the $(\Sigma^- \Sigma^0)$ system which is antisymmetric. Thus the product of the space and spin wave functions should be symmetric, i.e. for $l=0$ it is in the triplet spin state. This system is analogous to a deuteron and since the Σ's form a multiplet the force between them can be due to the exchange of a single pion.

Assuming the values for the depth of the phenomenological potential used in the deuteron theory, i.e. $V_0 = 25$ Mev, we obtain the binding energy of the $(\Sigma^- \Sigma^0)$ system to be 4.66 Mev.

It is expected that the $(\Xi^0 p)$ collision leads to the bound system of $(\Sigma^+ \Sigma^0)$. Due to charge independence the $(\Sigma^+ \Sigma^0)$ and $(\Sigma^- \Sigma^0)$ systems will have the same energies and other parameters. However, the distribution of the decay pions of the two systems will be entirely different since Σ^+ has a decay mode, the $(p\pi^0)$, which exhibits maximal asymmetry while the $(n\pi^+)$ mode of Σ^+ and the $(n\pi^-)$ mode of the Σ^- have no asymmetry.

In both systems $(\Sigma^+ \Sigma^0$ and $\Sigma^- \Sigma^0)$ the Σ^0 will decay instantaneously into $\Lambda^0 + \gamma$ and the pions arising from the decay of the Λ^0 and the charged Σ will be correlated. Incidentally it would be of interest to determine the energy of the γ quanta arising from the Σ^0 decay since it would provide an estimate of the binding energy of the $\Sigma^\pm \Sigma^0$ system. In the case of the decay of the Λ^0 particle the distribution of the decay pion is given by

$$P(\sigma)\,d\Omega=(1+\alpha\boldsymbol{\sigma}\cdot\boldsymbol{p})\,d\Omega \qquad (6)$$

where $\boldsymbol{\sigma}$ is the polarization of the Λ, $\boldsymbol{p}$ the unit vector along the direction of the pion momentum in the rest system of the Λ and α the asymmetry parameter which is known to be $|\alpha|>0.7$ for both the neutral and charged modes. In the case of a $(\Sigma^-\Sigma^0)$ system the decay mode $(n\pi^-)$ exhibits no asymmetry, hence the joint distribution of the two pions which is given by

$$P(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\,d\Omega_1\,d\Omega_2=(1+\alpha_1\boldsymbol{\sigma}_1\cdot\boldsymbol{p}_1)\times(1+\alpha_2\boldsymbol{\sigma}_2\cdot\boldsymbol{p}_2)\,d\Omega_1\,d\Omega_2. \qquad (7)$$

reduces to

$$P(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\,d\Omega_1\,d\Omega_2=(1+\alpha_1\boldsymbol{\sigma}_1\boldsymbol{p}_1)\,d\Omega_1\,d\Omega_2, \qquad (8)$$

since the asymmetry parameter α_2 for the $n\pi^-$ decay mode of Σ^- is zero. It is obvious that the distribution of the pions of the $(\Sigma^+\Sigma^0)$ system for the $(n\pi^+)$ mode of Σ^+ is the same as that of the$(\Sigma^-\Sigma^0)$ system. For the $p\pi^0$ mode of the Σ^+ decay the distribution is given by (7).

The joint distribution function $\mathfrak{p}$ can be evaluted by constructing the spin space density matrix $\rho(\sigma_1\sigma_2)$ and

$$\mathfrak{p}=\frac{\mathrm{Tr}[\rho P(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)]}{\mathrm{Tr}\,\rho}. \qquad (9)$$

The polarization of Λ arising from a Σ^0 decay is given by[2]

$$\langle\sigma\rangle_\Lambda=-(\langle\sigma\rangle_\Sigma\cdot\boldsymbol{u})\boldsymbol{u} \qquad (10)$$

where $\boldsymbol{u}$ is the unit vector along the direction of Λ momentum in the Σ^0 rest system. Thus the Λ^0 and Σ^+ will have spins oriented in the opposite directions. Since the density matrix for the $^1S_0$ state is

$$\rho=1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \qquad (11)$$

the distribution function of the decay pions is

$$\mathfrak{p}=(1-\alpha_1\alpha_2\boldsymbol{p}_1\cdot\boldsymbol{p}_2). \qquad (12)$$

This also enables the determination of the relative signs of α_1 and α_2.

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2) R. Gatto, Phys. Rev. **109** (1958), 610.

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The Single Pion Production Process in $\pi^- p$ Collision at 1 Bev and the Sakata Model

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We analyse the experimental data of Derado and Schmitz on the single pion production process in π^--p collision at 1 Bev by the generalized isobar model. Our isobar model is a generalization of the Lindenbaum and Sternheimer 3-3 nucleon isobar model. But our standpoint is based on the Sakata model and all the possible fermion and boson states of the full symmetry theory are taken into account. It is shown that this generalized isobar model can well explain the experimental momentum spectra. The evidences of boson isobars of $I=2$ with mass $\sim$400 Mev and $I=1$ with mass $\sim$650 Mev are suggested from the analysis of the D-S data.

§ 1. Introduction

Along the line of the Sakata model, Ikeda, Ogawa and Ohnuki[1] have formulated the theory based on a possible symmetry among the basic particles p, n and Λ, which is described in terms of a unitary group $U(3)$ of degree 3 (this theory will be referred to as the $U(3)$ theory). They have assumed that an elementary particle (or a resonance level) corresponds to a basis vector of an irreducible representation space of $U(3)$. Comparing the $U(3)$ theory with experiment we have successfully assigned the theoretical states to the experimental levels in terms of the semi-empirical mass formula so far as the reliable experimental data are concerned.[2]

There are, however, many interesting levels which are not yet established owing to experimental difficulties. Among these levels we are particularly interrested in the unstable boson states (boson isobars) with strangeness 0, for they are closely related to the problem of the strong pion-pion interaction, which is one of the important topics of the current meson physics. For the investigation of such unstable boson isobars, we have once taken up the sub-Bev single pion production process in pion-nucleon collision. There we gave a rough sketch of the anticipation to this process from the standpoint of the Sakata model and a detailed discussion of the $\pi^+ + p \to N + \pi + \pi$ process at 500 Mev. (This will be referred to as A.[3])

In A we have assumed the following mechanism for the production

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* Nippon Gakujutsu Shinkokai Foundation Fellow.

of pion,

$$\pi+N\to B+N\to(\pi+\pi)+N, \tag{1a}$$

$$\to F+\pi\to(N+\pi)+\pi, \tag{1b}$$

where F and B are unstable fermion and boson isobar states of the $U(3)$ theory respectively. This is a generalization of the Lindenbaum and Sternheimer 3-3 nucleon isobar model[4] (referred to as the L-S model) from the standpoint of the Sakata model.

With the same aim we shall study here the single pion production process in π^--p collision at 1 Bev and analyse the momentum spectra of final particles.

A characteristic feature of the single pion production process at $\sim$1 Bev is the appearance of one or two peaks in the momentum spectra. Especially in the momentum spectra of produced pions this feature seems to be remarkable. The characteristic feature of the pion energy spectra which is hard to be explained by the statistical theory has been a powerful support for the L-S model, and in fact there has been some accumulation of the experimental data[5] which show the quantitative agreement with the L-S model.

Recently Derado and Schmitz[6] have reported that pion energy spectra at 1 Bev cannot be explained by the L-S model quantitatively, though the validity of the L-S model still partly remains. There are also some other experimental suggestions for that a mechanism different from the L-S model may be operative in the single pion production process, a mechanism such as (1a).[7] The situation seems more complicated than what the simple L-S model describes, it is rather just as anticipated from the Sakata model.

The level scheme of the $U(3)$ theory predicts that the following channels may open to the reaction in question, *if the participating isobar has the decay mode* $F\to N+\pi$ *or* $B\to 2\pi$:

$$\pi+N\to F_3^1(0,\,3/2)+\pi, \tag{I}*$$

$$F_3^3(0,\,1/2)+\pi, \tag{II}$$

$$B_4^1(0,\,2)+N, \tag{III}$$

$$B_2^1(0,\,0)+N, \tag{IV}**$$

$$B_4^{4,6}(0,\,1)+N. \tag{V}***$$

* For fermion and boson state of the $U(3)$ theory, we use $F_j^i(S,\,I)$ and $B_j^i(S,\,I)$ respectively, where the subscript j denotes the number of basic particles plus basic antiparticles of which the state is composed, and the superscript i the class to which this state belongs (each class corresponds to each irreducible constituent of $U(3)$). S and I in brackets are its strangeness and isotopic spin, respectively.

** $B_2^2(0,0)$ or $\pi^{0\prime\prime}$ which is an excited state of vacuum is not considered here since it is very questionable to take $B_2^2(0,0)$ as the same physical entity such as pion and nucleon.

*** Besides these channels the reaction $\pi+N\to B_4^1(0,1)+N$ is also possible if the mass of $B_4^1(0,1)$ is smaller than 800 Mev (the mass formula gives $\sim$850 Mev for its mass). However, we do not consider this channel in this paper, since $B_4^1(0,1)$ is expected to have a relatively large mass so that it will give very small contribution.

$F_3^1(0, 3/2)$ and $F_3^3(0, 1/2)$ are assigned to the first and second resonance levels observed in the pion-nucleon scattering and have the mass 1230 Mev and 1515 Mev respectively. The semi-empirical mass formula gives to $B_4^1(0, 2)$ (280 Mev), $B_2^1(0, 0)$ (615 Mev) and $B_4^{4,6}(0, 1)$ (750 Mev) the masses in parentheses. Here the attention must be paid as to the accuracy of the mass formula. There may be the possibility for the formula to give incorrect values but the deviation will be about 200 Mev at most. The spin and parity of $F_3^1(0, 3/2)$ and $F_3^3(0, 1/2)$ are $3/2^+$ and $3/2^-$ respectively. For boson isobars we now know little about their spin and parity, but if their 2π-decay mode is established, their possible spin and parity are considerably limited. $B_4^6(0, 1)$ is the antiparticle of $B_4^4(0, 1)$, and has the same mass, spin and parity. In our analysis it will be enough to treat them together without discrimination between both effects. As to these boson isobars of the Sakata model, there is still no convincing experiment, but some evidences are suggested. The branching ratio of $(\pi^+ + \pi^0 + p)/(\pi^+ + \pi^+ + n)$ in $\pi^+ p$ collision at ~ 500 Mev[8] can be explained consistently by the existence of $B_4^1(0, 2)$ isobar[3] and the evidence found by Derado and by Pickup et al.[7] can be interpreted as the evidence of $B_4^{4,6}(0, 1)$ isobars. There are also other theoretical and experimental suggestions from the investigation of the nucleon electromagnetic structure, etc.[9]

In the following analysis we shall study the experimental data of Derado and Schmitz. Since the phenomena in question are expected to be somewhat complicated, we shall take the following three steps in the analysis and discuss how the evidences of each isobar are recognized:

1) How does the $F_3^1(0, 3/2)$ model (the L-S model) deviate from the observed pion and nucleon momentum spectra?
2) How much is the situation 1) improved if the effect of $B_4^1(0, 2)$, $B_2^1(0, 0)$ and $B_4^{4,6}(0, 1)$ isobars are added to $F_3^1(0, 3/2)$ isobar?
3) If $F_3^3(0, 1/2)$ isobar is taken besides the isobars considered in 2), how much better are the D-S data described than in the case of 2)?

§ 2. Momentum spectra of pions and nucleons

In this section we shall make some preparation for the comparison of our model with the experimental data on the single pion production in $\pi^- p$ collision at 1 Bev, which will be given in the next section. The experiment of Derado and Schmitz has given the momentum spectra of pions and nucleons and we need the theoretical spectra of secondary particles in the channels (I)$\sim$(V).

The pion energy spectrum consists of three parts, i.e. that of decay pion from the fermion isobar F, $I_1(E_\pi)$, recoil pion of this isobar F, $I_2(E_\pi)$ and decay pions from boson isobar B, $I_3(E_\pi)$, and the nucleon energy spectrum consists of two parts, that of decay nucleon from the fermion isobar F, $J_1(E_N)$

and recoil nucleon emitted with the boson isobar B, $J_2(E_N)$ where, I_1, I_2, I_3, J_1 and J_2 are the normalized energy distribution functions for each case.

The theoretical expressions of these $I_1(E_\pi)$, $I_2(E_\pi)$, $I_3(E_\pi)$, $J_1(E_N)$ and $J_2(E_N)$ are given in reference A.

The momentum spectra calculated from these $I_1(E_\pi)$, $I_2(E_\pi)$, $I_3(E_\pi)$, $J_1(E_N)$ and $J_2(E_N)$ at 1 Bev incident pion kinetic energy are given in Figs. 1, 2 and 3.

Here we have assumed the same form for the mass spectrum of the isobars as in reference A, i.e.

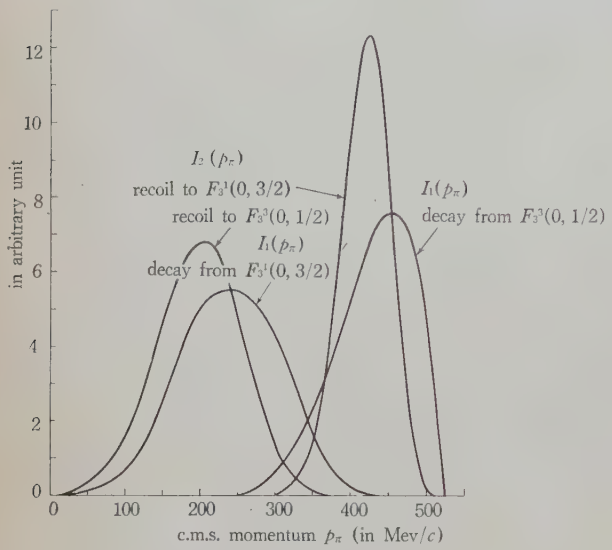


Fig. 1. Pion momentum spectra calculated from $I_1(E_\pi)$ and $I_2(E_\pi)$.

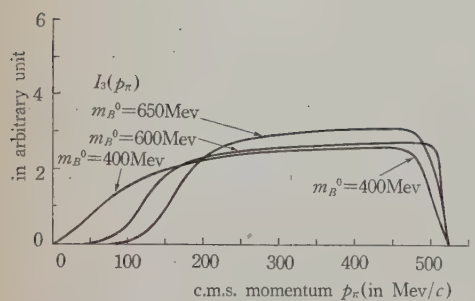


Fig. 2. Pion momentum spectra calculated from $I_3(E_\pi)$.

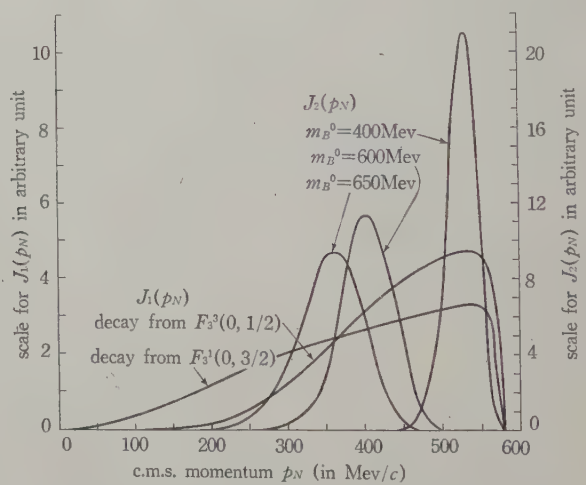


Fig. 3. Nucleon momentum spectra calculated from $J_1(E_N)$ and $J_2(E_N)$.

$\rho(m_F) \sim (m_F - m_N - m_\pi)\ \exp\left[-(m_F - m_F^0)^2/\Gamma_F^2\right]$ for fermion isobars and $\rho(m_B) \sim (m_B - 2m_\pi)\ \exp\left[-(m_B - m_B^0)^2/\Gamma_B^2\right]$ for boson isobars and the following values are chosen for m_F^0, m_B^0, Γ_F and Γ_B:

$$m_F^0 = 1230 \text{ Mev},\ \Gamma_F = 66 \text{ Mev} \quad \text{for } F_3^1(0, 3/2)$$

$$m_F^0 = 1500 \text{ Mev},\ \Gamma_F = 90 \text{ Mev} \quad \text{for } F_3^3(0, 1/2)$$

$$m_B{}^0=400\text{ Mev},\quad \Gamma_B=30\text{ Mev}\quad \text{for } B_4^1(0,2)$$
$$m_B{}^0=600\text{ Mev},\quad \Gamma_B=30\text{ Mev}\quad \text{for } B_2^1(0,0)$$
$$m_B{}^0=650\text{ Mev},\quad \Gamma_B=30\text{ Mev}\quad \text{for } B_4^{4,6}(0,1).$$

The above values for $F_3^1(0, 3/2)$ are taken so as to give approximately the total π^+-p cross section around the first resonance. For $F_3^3(0, 1/2)$ we have used the second peak at 600 Mev though it is a little ambiguous which portion of the π-N cross section is to be identified as the $F_3^3(0, 1/2)$ isobar.* We have rather poor experimental data on boson isobars. For $B_4^1(0, 2)$ the mass is expected to be larger than the value $2m_\pi$ of the mass formula. We take 400 Mev for $m_B{}^0$ of $B_4^1(0, 2)$, consistently with the observed nucleon momentum distribution. However, the change of the parameter within $\sim m_\pi/2$ around this value will not bring any essential difference into the following discussion.

Some evidences of π-π resonance of $I=1$ having the mass about $4\sim5m_\pi$ have been reported. It will be natural to associate our $B_4^{4,6}(0, 1)$ with this resonance level. In our present analysis we tentatively choose $m_B{}^0$ of $B_4^{4,6}(0, 1)$ as 650 Mev consistently with the proton momentum spectrum.

We are ignorant about the width of the mass spectrum of the boson isobars except that its magnitude would be of order 100 Mev. The above choice of relatively narrow width is taken in order to expose clearly the feature of the spectrum due to boson isobars.

Now, the reactions in question are

$$\pi^-+p\to\pi^-+\pi^0+p \qquad (p\text{-process})$$

and

$$\pi^-+p\to\pi^++\pi^-+n \qquad (n\text{-process}).$$

The energy spectra of the secondary particles in each process are given in terms of $I_1, I_2, \cdots, J_2$.

π^- spectrum in p-process:

$$\begin{aligned} I(E_\pi)dE_\pi=&[(1/135)(10\sigma_1+\sigma_3-10b)I_1^I(E_\pi)\\ &+(1/135)(10\sigma_1+16\sigma_3-40b)I_2^I(E_\pi)\\ &+(1/27)(4\sigma_1+4\sigma_3-8a)I_1^{II}(E_\pi)\\ &+(1/27)(4\sigma_1+\sigma_3+4a)I_2^{II}(E_\pi)\\ &+(9/45)\sigma_3I_3^{III}(E_\pi)\\ &+(1/9)(4\sigma_1+\sigma_3+4a)I_3^{V}(E_\pi)]dE_\pi, \end{aligned} \tag{2}$$

π^0 spectrum in p-process

* The present calculation of the pion spectrum of $F_3^3(0, 1/2)$ channel differs a little from that by Lindenbaum and Sternheimer.[10] But such a difference is not essential for the present discussion.

$$I(E_\pi)dE_\pi=[(1/135)(10\sigma_1+16\sigma_3-40b)I_1^I(E_\pi)$$
$$+(1/135)(10\sigma_1+\sigma_3-10b)I_2^I(E_\pi)$$
$$+(1/27)(4\sigma_1+\sigma_3+4a)I_1^{II}(E_\pi)$$
$$+(1/27)(4\sigma_1+4\sigma_3-8a)I_2^{II}(E_\pi)$$
$$+(9/45)\sigma_3 I_3^{III}(E_\pi)$$
$$+(1/9)(4\sigma_1+\sigma_3+4a)I_3^{V}(E_\pi)]dE_\pi, \tag{3}$$

π^+ spectrum in n-process

$$I(E_\pi)dE_\pi=[(1/135)(5\sigma_1+8\sigma_3-20b)I_1^I(E_\pi)$$
$$+(1/135)(45\sigma_1+18\sigma_3+90b)I_2^I(E_\pi)$$
$$+(1/27)(8\sigma_1+2\sigma_3+8a)I_1^{II}(E_\pi)$$
$$+(2/45)\sigma_3 I_3^{III}(E_\pi)$$
$$+(1/9)(2\sigma_1+2\sigma_3-4a)I_3^{V}(E_\pi)$$
$$+(4/9)\sigma_1 I_3^{IV}(E_\pi)]dE_\pi, \tag{4}$$

π^- spectrum in n-process

$$I(E_\pi)dE_\pi=[(1/135)(45\sigma_1+18\sigma_3+90b)I_1^I(E_\pi)$$
$$+(1/135)(5\sigma_1+8\sigma_3-20b)I_2^I(E_\pi)$$
$$+(1/27)(8\sigma_1+2\sigma_3+8a)I_2^{II}(E_\pi)$$
$$+(2/45)\sigma_3 I_3^{III}(E_\pi)$$
$$+(1/9)(2\sigma_1+2\sigma_3-4a)I_3^{V}(E_\pi)$$
$$+(4/9)\sigma_1 I_3^{IV}(E_\pi)]dE_\pi, \tag{5}$$

p spectrum in p-process

$$J(E_N)dE_N=[(1/135)(20\sigma_1+17\sigma_3-50b)J_1^I(E_N)$$
$$+(1/27)(8\sigma_1+5\sigma_3-4a)J_1^{II}(E_N)$$
$$+(9/45)\sigma_3 J_2^{III}(E_N)$$
$$+(1/9)(4\sigma_1+\sigma_3+4a)J_2^{V}(E_N)]dE_N, \tag{6}$$

n spectrum in n-process

$$J(E_N)dE_N=[(1/135)(50\sigma_1+26\sigma_3+70b)J_1^I(E_N)$$
$$+(1/27)(8\sigma_1+2\sigma_3+8a)J_1^{II}(E_N)$$
$$+(2/45)\sigma_3 J_2^{III}(E_N)$$
$$+(1/9)(2\sigma_1+2\sigma_3-4a)J_2^{V}(E_N)$$

$$+(4/9)\,\sigma_1 J_2^{IV}(E_N)]dE_N. \tag{7}$$

σ_1 and σ_3 mean the cross sections of $I=1/2$ and $I=3/2$ states and are referred to the channel which is specified by the superscript I, ···, V of the distribution functions $I_1(E_\pi)$, $I_2(E_\pi)$, $I_3(E_\pi)$, $J_1(E_N)$ and $J_2(E_N)$. $a=\sqrt{\sigma_1\sigma_3}\cos\varphi$ and $b=\sqrt{2/5}\,a$ where φ is the relative phase between $I=1/2$ and $3/2$ state.

§ 3. Comparison with experiment

In this section we shall investigate the experimental data of Derado and Schmitz. The momentum spectra of Derado and Schmitz are given in Figs. 5–10. As was noted in § 1 we shall take three steps for the analysis. Each step of the analysis will be specified as the $F_3^1(0, 3/2)$ model, $F_3^1(0, 3/2)+B$ model and $F_3^1(0, 3/2)+B+F_3^3(0, 1/2)$ model respectively.

3–1. The $F_3^1(0, 3/2)$ model (the L-S model)

Derado and Schmitz have analysed their experimental data by means of the L-S model and found that this model was incapable of explaining the pion momentum spectra quantitatively, though their spectra seem to retain the characteristic features of the L-S model.

Here we shall analyse their experimental data once more by using the $F_3^1(0, 3/2)$ model (the L-S model) to understand this situation intuitively, concerning two quantities which are convenient for our later discussions.

One is the ratio y/x, where x is the number of recoil π^- in p-process and y is the number of decay π^+ in n-process. This model gives the value 1/2 for this ratio from the charge independence. Experimentally this value is approximately given by the number of fast π^- in $\pi^-+\pi^0+p$ and slow π^+ in $\pi^++\pi^-+n$ where the discrimination between fast pions and slow pions is made at the pion momentum $\sim$350 Mev/c. The D-S experiment gives this ratio $\sim$2 and shows the contradiction with the prediction of $F_3^1(0, 3/2)$ model.

The other quantity of which we want to see the discrepancy between theory and experiment is the ratio z/x where z is the number of decay π^- in p-process. Although the requirement of charge independence does not determine this ratio uniquely, it gives a certain restrictive condition for z/x if we fix the branching ratio $R=(\pi^-+p\to\pi^-+\pi^0+p)/(\pi^-+p\to\pi^++\pi^-+n)$. z/x and R are written as

$$z/x=(10\sigma_1+\sigma_3-10b)/(10\sigma_1+16\sigma_3-40b),$$

$$R=(20\sigma_1+17\sigma_3-50b)/(50\sigma_1+26\sigma_3+70b).$$

In Fig. 4, z/x is given as a function of $\rho=\sigma_3/\sigma_1$, taking $\cos\varphi$ (solid line) or R (dashed line) as parameter.

If we take the experimental value $R=0.5$, we find the upper limit for $z/x\sim0.9$, while we have the experimental fast and slow pion ratio $\sim$3.

The calculated momentum spectra of pions corresponding to $z/x\sim0.9$,

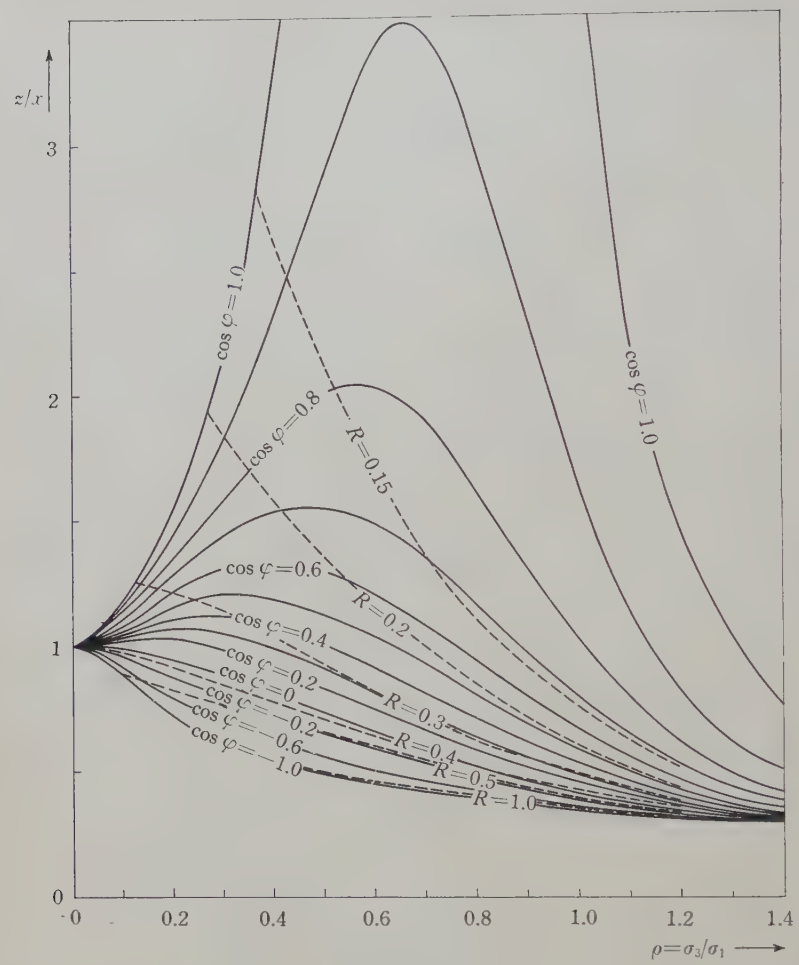


Fig. 4. z/x as a function of ρ.

$R=0.5$ are given in Figs. 5–8 (curve A).

On the contrary, in the case of nucleon spectra there will be no such contradiction between the observed spectra and the $F_3^1(0, 3/2)$ model (Figs. 9 and 10, curve A).

But if any reality exists in the observed two peaks appearing in the proton momentum distribution, it is impossible to interpret such a spectrum in terms of the $F_3^1(0, 3/2)$ model. In fact, as we shall see in the next subsection, these two peaks in the proton momentum spectrum can be reasonably identified with the evidence of boson isobars whose existence is predicted by the $U(3)$ theory.

3–2. $F_3^1(0, 3/2)+B$ model

As discussed in the previous subsection the $F_3^1(0, 3/2)$ model has failed to give a quantitative explanation of the observed pion momentum spectra.

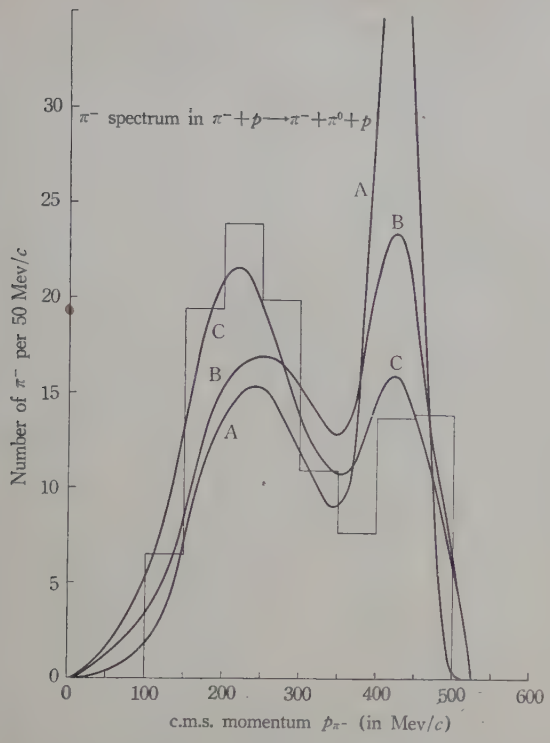


Fig. 5. Calculated and experimental momentum spectrum of π^- in $\pi^-+p\rightarrow\pi^-+\pi^0+p$

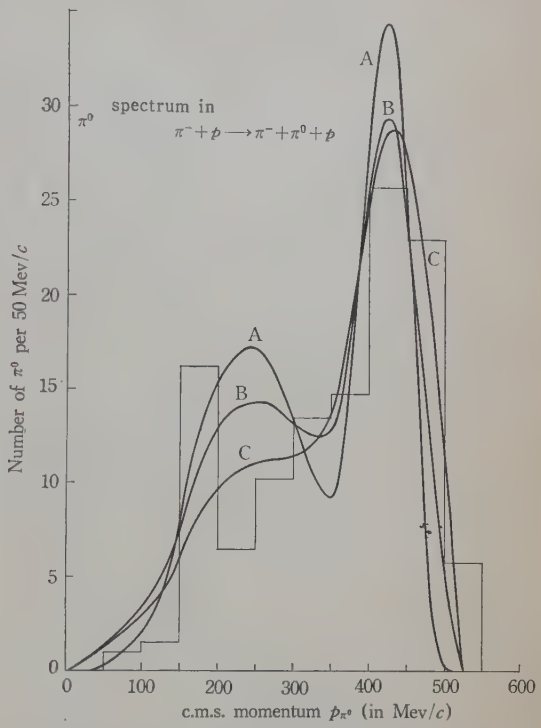


Fig. 6. Calculated and experimentalmomentum spectrum of π^0 in $\pi^-+p\rightarrow\pi^-+\pi^0+p$

However, qualitatively the dominant feature of the pion spectra seems to be described by the $F_3^1(0, 3/2)$ model. This will strongly suggest that other mechanisms will also play a role in the single pion production process in addition to the $F_3^1(0, 3/2)$ channel. Now let us examine whether it is possible to understand the discrepancy between the experimental data and the $F_3^1(0, 3/2)$ model by considering the contribution of the boson isobars of the Sakata model.

The difficulty that y/x is larger than that predicted by the theory can be easily removed if we consider the contribution of boson isobars in n-process and identify the most of the slow π^+ in $\pi^++\pi^-+n$ as the decay pions of boson isobars, since the decay pion spectra from boson isobars are such as shown in Fig. 2. The assumption of the contribution of boson isobars in n-process will necessarily require the same for p-process. If not the case, the branching ratio R (defined as to the $F_3^1(0, 3/2)$ channel) becomes larger and the upper limit of z/x (also defined as to the $F_3^1(0, 3/2)$ channel) is lowered and the deviation from the experimental spectra becomes larger.

The introduction of boson isobars gives remarkable influence on the nucleon spectrum. As mentioned in § 2, the boson isobars of the Sakata model that may play a role in the phenomenon are $B_4^1(0, 2)$, $B_2^1(0, 0)$, $B_4^4(0, 1)$ and $B_4^6(0, 1)$. The characteristic feature of the momentum spectrum given by these

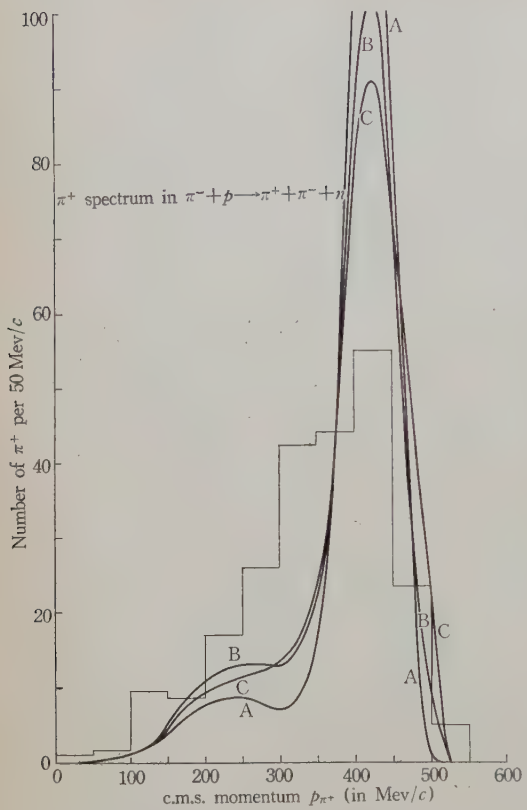


Fig. 7. Calculated and experimental momentum spectrum of π^+ in $\pi^-+p\rightarrow\pi^++\pi^-+n$

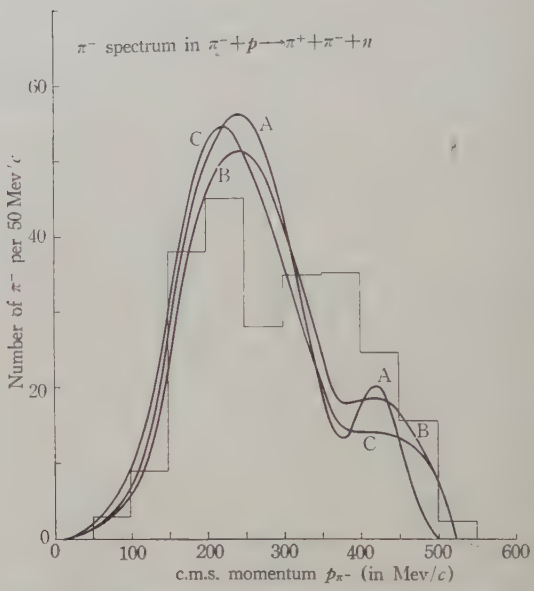


Fig. 8. Calculated and experimental momentum spectrum of π^- in $\pi^-+p\rightarrow\pi^++\pi^-+n$

boson isobars would be the appearance of the peaks in the nucleon spectrum (see Fig. 3). Taking account of their masses according to the semi-empirical mass formula,[2] the two peaks in the observed proton momentum spectrum at 350～400 Mev/c and 500～550 Mev/c can be attributed to $B_4^{4,6}(0, 1)$ and $B_4^1(0, 2)$ respectively. It is noted from the observed proton spectrum that the contributions of $B_4^{4,6}(0, 1)$ and $B_4^1(0, 2)$ will be simultaneously required with similar amount. We may observe that the neutron spectrum seems to have also two peaks, though less marked. For n-process we can expect the contribution of $B_2^1(0, 0)$ besides $B_4^{4,6}(0, 1)$ and $B_4^1(0, 2)$ if $B_2^1(0, 0)$ has 2π decay mode.

Let us discuss the situation a little quantitatively. We may have some estimation for the rough upper limit of their contribution in each process. Concerning n-process, we may estimate an upper limit of the contribution of boson isobars as 50% from the pion spectrum where the slow π^+(fast π^-) are thought to be mainly due to the decay pions of boson isobars. From the neutron momentum spectrum it is difficult to deduce an upper limit. The contribution greater than 50% seems to be improbable, since in this process the contribution

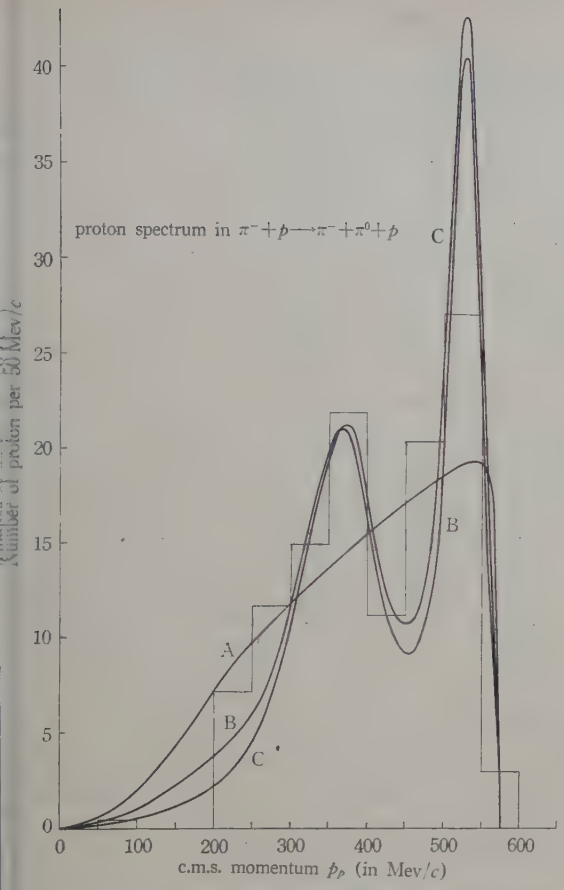


Fig. 9. Calculated and experimental momentum spectrum of p in $\pi^-+p\rightarrow\pi^-+\pi^0+p$

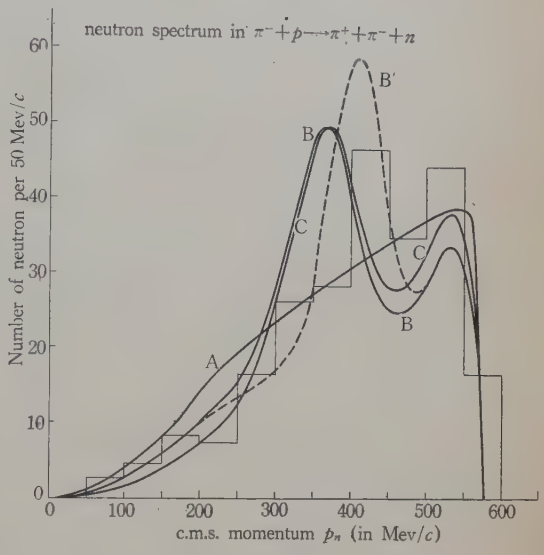


Fig. 10. Calculated and experimental momentum spectrum of n in $\pi^-+p\rightarrow\pi^++\pi^-+n$

of $B_4^1(0, 2)$ is expected to be very small (2/9 of that of p-process) so that the large contribution makes the theoretical spectrum much different from the observed one (at the high momentum region of the neutron).

As to p-process we can make a similar estimation from the pion spectra and obtain an upper limit of the contribution $\sim 70\%$. From the proton spectrum, we cannot set such an upper limit. Rather, if we recognize the two peaks as real, then an estimate for the lower limit may be possible and $\sim 20\%$.* Of course these figures are very rough and are given only as a measure of the situation.

In order to see what improvement is possible for the pion spectra within such limits we present an example of the distribution where the contribution of each isobar is taken consistently with the requirement of charge independence and the observed nucleon spectra. Theoretical curves which describe the ex-

* Derado has estimated the contribution of the strong pion-pion interaction (possibly corresponding to $B_4^{4,6}(0, 1)$ in our model) to be 50% from their experimental data of the laboratory proton momentum distribution in p-process.[7]

perimental spectra well are given with the following percentage for the contribution of each isobar. (This will not be the best one but is given only for the orientation of the later discussion.)

	p-process	n-process
$F_3^1(0, 3/2)$	50%	70%
$B_4^1(0, 2)$	25%	3%
$B_2^1(0, 0)$	—	0(27)%
$B_4^{4,6}(0, 1)$	25%	27(0)%

The momentum spectra given by these fractions are shown in Figs. 5—10 (curve B). There $\rho=\sigma_3/\sigma_1$ and $\cos\varphi$ are chosen so as to give the maximum z/x. As this example shows, the considerable improvement is achieved. But there is some discrepancy between the theory and the experimental spectra still left. The higher first peak of π^- spectrum in p-process cannot be explained by this model even if we assume the contribution of boson isobar to be larger than 50%. If the discrepancy is significant, it can be interpreted as the evidence of $F_3^3(0, 1/2)$ in this process as will be seen in the next subsection. Someone may feel unsatisfactory to the agreement of the above example with the experimental π^+ spectrum in n-process. For this we should like to note that a better agreement will be obtained, so far as the pion spectra are concerned, by increasing the contribution of boson isobars in n-process from 30% to 40%, without much influencing over the pion spectra in p-process, but resulting a little discrepancy in the neutron spectrum. However, it will be beyond the purpose of the present paper to discuss such details.

From the discussion so far made the possible contribution and the role of $B_4^1(0, 2)$ and $B_4^{4,6}(0, 1)$ have been clarified. As to $B_2^1(0, 0)$, the evidence is not clear in the present experimental data. It might have 3π decay mode and contribute to the double pion production process, etc.[11] If the location of the first peak in the neutron spectrum, which seems to be higher by about 50 Mev than the proton spectrum in the present experiment, is confirmed, it will give important information on this problem (cf. curve B' which corresponds to the case of figures in brackets).

3-3. $F_3^1(0, 3/2)+B+F_3^3(0, 1/2)$ model

As discussed in the the previous subsection the $F_3^1(0, 3/2)+B$ model can give a much better explanation of the pion and nucleon spectrum than the $F_3^1(0, 3/2)$ model. On the other hand the higher first peak of π^- spectrum in p-process has remained unexplained.

In this subsection we shall discuss how the introduction of $F_3^3(0, 1/2)$ in addition to $F_3^1(0, 3/2)$ and boson isobars will save this difficulty. $F_3^3(0, 1/2)$ appears in the part of decay $\pi^-(X)$ and recoil $\pi^-(Y)$ in p-process and in the

part of decay π^+ in n-process ($2Y$). The first peak of π^- spectrum in p-process can be made higher if we assume there is the considerable amount of Y, taking $X \sim 0$.

It may be supposed that the peak in the high momentum region of π^+ spectrum in n-process will also be made higher and it seems that the agreement might become worse. However, this is not necessarily the case. For the fixed contribution of boson isobar the replacement of some part of $F_3^1(0, 3/2)$ portion by $F_3^3(0, 1/2)$ produces little difference in the peak of π^+ spectrum in n-process. The introduction of $F_3^3(0, 1/2)$ comparable in magnitude to $F_3^1(0, 3/2)$ of p-process results in that the π^- spectrum in p-process is better fitted without much altering the π^+ spectrum in n-process.

As for the nucleon spectra the features of the distribution of $F_3^1(0, 3/2)$ channel and $F_3^3(0, 1/2)$ channel are not so much different from each other as seen in Fig. 3 except that the spectrum from $F_3^3(0, 1/2)$ favours faster nucleons than $F_3^1(0, 3/2)$. There does not appear a remarkable change by bringing the effect of $F_3^3(0, 1/2)$ isobar into the process if we are not to discuss the detailed feature of the distribution. An example of the $F_3^1(0, 3/2)+B+F_3^3(0, 1/2)$ model is also given in Figs. 5–10 (curve C). There is taken the fraction of each isobar as follows:

	p-process	n-process
$F_3^1(0, 3/2)$	25%	45%
$F_3^3(0, 1/2)$	25%	25%
$B_4^1(0, 2)$	25%	3%
$B_4^{4,6}(0, 1)$	25%	27%

In concluding this section let us discuss the cases of $F_3^1(0, 3/2)+F_3^3(0, 1/2)$ model and $F_3^3(0, 1/2)+B$ model. The former case, $F_3^1(0, 3/2)+F_3^3(0, 1/2)$ model, has recently been proposed by Sternheimer and Lindenbaum as an extension of their 3-3 isobar model (the $F_3^1(0, 3/2)$ model).[10] The model, though very reasonable and natural in their standpoint, is insufficient if the view is taken on the basis of the Sakata model, since we have boson isobars which are expected to give larger contribution than $F_3^3(0, 1/2)$ isobar. As for the D-S data, this model cannot explain the ratio of slow π^+ in n-process to fast π^- in p-process. Finally it is noted that the other combination $F_3^3(0, 1/2)+B$, though the occurrence of such a situation seems to be rather improbable, can better describe the spectra of Derado and Schmitz than the $F_3^1(0, 3/2)+B$ model.

§ 4. Summary

In the preceding sections we have analysed the experimental data of Derado and Schmitz by the isobar model which is the generalization of the L-S model

involving the possible isobar states of the $U(3)$ theory. There we have found a consistent interpretation assuming that $B_4^1(0, 2)$, $B_4^{4,6}(0, 1)$, $F_3^1(0, 3/2)$ and $F_3^3(0, 1/2)$ take part in the single pion production process in π^--p collision at 1 Bev. The characteristics of the D-S experimental results which lead to the evidence of $B_4^1(0, 2)$, $B_4^{4,6}(0, 1)$, $F_3^1(0, 3/2)$ and $F_3^3(0, 1/2)$ are

1) $R \simeq 0.5$.
2) The relatively large abundance of slow π^+ in n-process compared with fast π^- in p-process ($y/x \sim 2$).
3) The first peak is higher than the second peak in the π^- spectrum in p-process ($z/x > 0.9$).
4) The appearance of two peaks in the nucleon momentum spectra.

Among the isobars discussed so far, $F_3^1(0, 3/2)$ and $F_3^3(0, 1/2)$ are very familiar ones which are the first and the second resonance in π-N scattering. For the boson isobars, however, no clear evidence has been obtained so far, though many authors have debated this problem. Here we have pointed out from the D-S data that $B_4^1(0, 2)$ and $B_4^{4,6}(0, 1)$ should take the significant part in the single pion production process. Taking the masses 400 Mev and 650 Mev for $B_4^1(0, 2)$ and $B_4^{4,6}(0, 1)$ respectively, we can reproduce the feature of nucleon spectrum finely.

As for $B_2^1(0, 0)$ it is suggested that the relative location between the first peak in the proton momentum distribution and the possible first peak in neutron momentum distribution will give an important key. If the future experiment confirms these features 1)–4) of the Derado-Schmitz data, it indicates that all possible states of the $U(3)$ theory must be taken into consideration as to the single pion production process.

For the determination of the isotopic spin of the isobars the $\pi^+ + p$ experiment may be helpful, since in this reaction only the state with total isotopic spin $I=3/2$ contributes and the ambiguity of the interference between $I=1/2$ and $3/2$ is avoided. In order to know the spin and parity of the isobars Adair's analysis[12] will be useful.

Finally it should be noted that the data of Alles-Borelli et al. at 960 Mev[5] have shown quite different features about the pion spectra in p-process and they can be explained by the L-S model. It will require further experimental investigation to confirm whether such an abrupt change really occurs or not. But the fact that the data of Alles-Borelli et al. can be explained by the L-S model does not exclude the possibility that the boson isobars contribute to this process as well as $F_3^1(0, 3/2)$ isobar with the similar amount to that taken to explain the D-S data at 1 Bev. In this case the analysis of the nucleon momentum spectra is required for the decisive argument of the contribution of the boson isobars.

We are very grateful to Prof. K. Sakuma, Dr. S. Ogawa, Dr. K. Senba and the colleagues of our laboratory for their many helpful discussions and continual encouragement.

Note added in proof : After the completion of the present analysis, J. G. Rushbrooke and D. Radojičić have published the experimental analysis of $\pi^-+p\to\pi^-+\pi^0+p$ at 0.96 Bev (Phys. Rev. Letters **5** (1960), 567). Their data seem to serve as another support for our present conclusion, since the experimental distribution of the total energy of π^--π^0 system in its barycentric system shows the peaking of the events corresponding to the boson isobars with masses 400 Mev and 650 Mev as taken in this paper.

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The Effect of Exchange and Correlation on Binding Energy of Compressed Atoms

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In the binding energy formula the effect of exchange has been taken by making use of the T. F. D. function. Correlation has been incorporated as a perturbation and the effect of finite boundary has been taken as given by Scott. The agreement between the calculated values and the experimentally observed ones is much better than provided by Scott.

Introduction

The Thomas[1]-Fermi[2] statistical model of an atom has been used by various investigators in their pursuit to study the properties of solids and of atoms which however should be reasonably independent of the electron shell structure. The model has been used for approximate calculations of potential fields and the charge densities inside an atom, thereby leading to the calculation of equation of state, binding energy, etc., of various atoms. Milne[3] (1927) suggested a formula $20.92\,Z^{7/3}$ev representing the variation of binding energy with respect to Z. This formula, however, gives values of binding energy of atoms which are some 20 to 30 percent higher than the experimentally observed ones, derived from spectroscopic analysis. The binding energy was found to be smoothly varying as $Z^{12/5}$ for high values of Z and as $Z^{7/3}$ for low Z with different coefficients.

Scott[4] (1952) suggested that the effect of finite boundary and that of exchange have to be incorporated and he gave that the corrections are $-13.6\,Z^2$ and $6.01\,Z^{5/3}$ ev respectively. However, it appears that though the agreement between the theoretical values and those observed experimentally is better, some discrepancy has yet to be accounted for.

The purpose of the present paper is to make an attempt in the direction of accounting for the discrepancy still left. In the above derivations the correlation energy of electrons has not been taken into account. We shall incorporate the correlation energy in the analysis and calculate its contribution to the binding energy of an atom. We make use of the potential distribution inside an atom as given by Feynman, Metropolis and Teller[5] (1949) and regard the correlation energy as perturbation to calculate its effect on the binding energy of an atom.

I

Binding energy of an atom can be related with the potential distribution inside the atom by making use of Feynman's theorem:

"The partial derivative of an energy eigenvalue of a system with respect to a parameter occurring in the Hamiltonian is given by the expectation value of the partial derivative of Hamiltonian operator with respect to the parameter."

$$\left\langle \frac{\partial W}{\partial Z} \right\rangle = -V, \tag{1}$$

where V is the potential at the nucleus due to the charge distribution of the electrons,

$$-V = \operatorname*{Lim}_{r\to 0} \frac{Ze}{r}(1-\chi) \tag{2}$$

$$= -\frac{Ze}{\mu}\chi'(0),$$

where $\chi(r)$ represents the electrostatic potential at a distance r from the nucleus as a fraction of the potential due to the nucleus itself and μ is a scale factor:

$$r=\mu x \quad \mu=\left(\frac{9\pi^2}{128Z}\right)^{1/3} a_0 = 0.88534\, a_0 Z^{-1/3}. \tag{3}$$

Milne,[3] while calculating the binding energy of an atom, made use of Baker's[6] constant $\chi'(0)=-1.5881$ based on the Thomas-Fermi model of an atom. This leads to the well-known formula,

$$V=-48.80\, Z^{4/3} \tag{4}$$

which gives for the binding energy W the expression

$$W=20.92\, Z^{7/3} \text{ ev.} \tag{5}$$

In the foregoing the atom is considered to be free with its boundary extending to infinity, and the binding energy formula has to be corrected in the light of a finite boundary of the atom. Scott[4] (1952) has effectively estimated the correction to be applied to the T.F. formula by comparing the results of T.F. approximation with exact wave mechanical treatment for the case of particles moving in a Coulomb field. He gave the correction as $-13.6\, Z^2$ ev.

He also modified Milne's formula by taking the effect of exchange energy of electrons by regarding exchange simply as perturbation and taking T.F. density distribution. According to him the contribution of exchange to the binding energy is

$$\frac{4\mu Z^2}{\pi^2}\int_0^\infty \chi^2 dx, \quad \int_0^\infty \chi^2 dx = 0.616. \tag{6}$$

Thus

$$W = 20.92\, Z^{7/3} - 13.6\, Z^2 + 6.01\, Z^{5/3} \text{ ev.} \tag{7}$$

II

Dirac[7] (1930) introduced exchange into the Thomas-Fermi theory and arrived at the differential equation for potential distribution

$$\frac{d^2\psi}{dx^2} = x\left(\epsilon + \frac{\psi^{1/2}}{x^{1/2}}\right)^3. \tag{8}$$

x has the same meaning as in I and

$$\epsilon = (3/32\,\pi^2)^{1/3} \cdot Z^{-2/3} = 0.21178\, Z^{-2/3}. \tag{9}$$

The potential without exchange effect, denoted by $-V_0$, is connected with ψ by the relation

$$-eV_0 = E_0 + \frac{2me^4}{h^2} - \frac{Ze^2\psi}{\mu x}. \tag{10}$$

The foregoing equations have been obtained by including exchange energy term in the expression for the energy of an electron. The electron density is now connected to the electro-static potential energy by the relation

$$n = \frac{8\pi}{3h^3}\left[\frac{2me^2}{h} + \left\{\frac{4m^2e^4}{h^2} + 2m(E + eV_0)\right\}^{1/2}\right]^3. \tag{11}$$

Feynman, Metropolis and Teller[5] (1949) have solved the above equations numerically for the potential distribution inside an atom for carbon and uranium at absolute zero temperature. They have arrived at the result for carbon at normal density,

$$\psi'(0) = -1.686$$

and

$$x_0 = 6.28.$$

According to Feynman's theorem the T.F.D. equation leads to

$$-V = -\psi_z'(0)\,\frac{Ze}{\mu}.$$

But the exact definition of $\psi(x)$ gives

$$-V = 3.4 - \frac{Ze}{\mu}\psi_z'(0) \tag{12}$$

instead of the above equation, where 3.4 is nothing but $1/16\bar{\tau}_0^2$ in Gombás[8] notation. Thus

$$W=-13.17\,\psi_Z'(0)Z^{7/3}-13.6\,Z^2+3.4\,Z, \qquad (13)$$

where the boundary effect has also been included. For carbon one arrives at $W=993.3$ ev, and for other atoms one has simply to put the corresponding value of $\psi_Z'(0)$. Slater and Krutter[9] have solved the T.F.D. equation for $Z=3$, 11 and 29 and Jensen[10] for $Z=18$, 36 and 54.

III

We shall now calculate the contribution of correlations to the binding energy of an atom.

Wigner[11] has calculated the correlation per electron as $-0.89\,e^2/r_s$ which is equal to the difference between the exact solution of Hartree Fock's equations and the exact solution of the problem; r_s is given by $n^{-1}=4\pi/3\cdot r_s^3$. Wigner's result, however, is correct only in the low density limit ($r_s\gtrsim 20$). Lately Gell-Mann and Brueckner[12] (1957) have derived an expression for the average correlation energy per electron by writing down the important part of the perturbation expansion for the energy when the electrons are in Coulomb interaction with each other and adding both over all the electrons and perturbation series. They have arrived at the expression true for high density limit ($r_s\lesssim 1$)

$$\varepsilon_{corr}=-\left(\frac{me^4}{\pi^2\hbar^2}\right)(1-\ln 2)\ln P+\text{constant}, \qquad (14)$$

where P is the momentum at the top of Fermi distribution of the degenerate electron gas. Lewis[13] (1958) has taken account of both the above results in an interpolation formula

$$\varepsilon_{corr}=-\left(\frac{me^4}{\pi^2\hbar^2}\right)(1-\ln 2)\ln\left[1+\frac{(0.89\,\alpha\pi-1)}{(1-\ln 2)\,\hbar}\pi P a_0\right], \qquad (15)$$

where $\alpha=(4/9\pi)^{1/3}$ and the above can be put in terms of r_s as

$$\varepsilon_{corr}=-0.8464\ln\left[1+\frac{0.897}{r_s}\right].$$

When the correlation energy is incorporated into the T.F.D. equation, the equation becomes quite cumbersome and a solution can be obtained with a great pain but without much benefit for our purpose. Lewis[13] has tried to take account of correlation energy and has given a differential equation but no numerical results have so far been reported. We may, however, use the density distribution given by the unmodified T.F.D. equation and regard correlation merely as a perturbation, and calculate the contribution of correlation to the binding energy of an atom.

In the following we shall make use of a more suitable formula given by Pines[14] (1955) for the region of actual metallic densities ($r_s=1.8$ to 5.6). He has given the result as

$$\varepsilon_{corr} = -1.564 + 0.426 \ln r_s \text{ ev}, \tag{16}$$

using the plasma-oscillation picture based on the collective description of electron interactions due to Bohm and Pines[14] (1952). The contribution of correlations to the binding energy can be written in the form

$$W_C = -\int n\varepsilon_{corr} \cdot dv$$

$$= 4\pi \int nf(n) r^2 \cdot dr, \tag{17}$$

where

$$f(n) = 1.768 + 0.142 \ln n \text{ ev}. \tag{18}$$

T.F.D. analysis gives the electron density as expressed in Eq. (11). On combining the two equations (10) and (11) we get

$$n(x) = \frac{8\pi}{3h^3}\left[\frac{2me^2}{h} + \left(\frac{2mZe^2\psi}{\mu x}\right)^{1/2}\right]^3 = a[1 + bZ^{2/3}(\psi/x)^{1/2}]^3, \tag{19}$$

where $$a = \frac{1}{3\pi^5} a_0^3$$

and $$b = 4.722.$$

Substituting the above expression of n into the expression (17),

$$W_C = 4\pi\mu^3 a \int_0^{x_0} [F(x)]^3 [1.768 + 0.142 \ln (aF^3)] x^2 dx, \tag{20}$$

$$F(x) = 1 + bZ^{2/3}(\psi/x)^{1/2}. \tag{21}$$

The above integral has to be evaluated numerically and for this purpose Chebyshev integration formula is very useful. According to Chebyshev[16]

$$\int_{y_1}^{y_2} g(y)dy = \frac{y_2 - y_1}{N} \sum_{r=1}^{N} g(y_r), \tag{22}$$

where

$$y_r = \frac{y_2 + y_1}{2} + \frac{y_2 - y_1}{2} \alpha_r. \tag{23}$$

α_r is independent of $g(y)$ and depends solely on N. In the present calculations N has been chosen equal to 6 and for this N, α_r's are $\pm 0.26665, \pm 0.42252$ and ± 0.86625.

The numerical results obtained by method are shown in Table I. The Table I tells us that the agreement between the binding energy calculated by the author and that experimentally observed is better than that provided by Scott. Mention may also be made of Tomishima's result of correlation

Table I.

	Z	$-\phi_Z'(0)$	W_e	W Correlation included	W_{exp}	W Scott formula
carbon	6	1.686	2.0	37.8	37.8	36.7
argon	18	1.635	9.5	522.5	528	518
krypton	36	1.6175	11.5 (2.77*)	2718.8		

In the Table atomic units are used.
* This result is given by Tomishima.[17]

energy of krypton which seems to be less plausible than the one calculated by the method used in this paper.

It is however to be noted that the above analysis suggests that the binding energy is a function of density (and that of temperature) since the value of $\phi_Z'(0)$ depends upon density (and on temperature) under consideration, as is shown in Table II.

Table II. Variation of binding energy with density at zero degree absolute.
carbon $Z=6$

$-\psi'(0)$	x_0	ρ gm/cc	Binding energy
1.6740	3.262	8.35	37.39
1.6800	3.715	5.65	37.58
1.6840	4.378	3.45	37.71
1.6858	5.168	2.10	37.76
1.6863	5.729	1.54	37.78
1.6867	7.008	0.84	37.80

Energy is given in atomic units.

It is seen from the above Table that as the density decreases the binding energy increases and appears to tend to a particular limit.

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The Composite Model of Mesons as a Limit of Meson Theory

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In meson theory with local interactions, the probability amplitudes for bare mesons vanish in the limit of infinite bare coupling constant $g\to\infty$ and infinite bare meson mass $\mu\to\infty$, where g/μ remains finite. Then the probability amplitude for no meson state satisfies the same eigenvalue equation as the composite model with local Fermi interactions, whose coupling constant is $G=\frac{1}{2}\bar{g}^2$.

The physical meaning of such a procedure and the results thus obtained are discussed.

§ 1. Introduction

Meson theory and the composite model of baryons and mesons seem to be quite different approaches to meson physics. However, it has been shown that several composite models and field theory with Fermi interactions are derived from meson theory by the appropriate limiting processes. In meson theory, the limiting process

$$\left.\begin{aligned} &g\to\infty,\\ &\mu\to\infty,\\ &\frac{g}{\mu}\to\bar{g}, \end{aligned}\right\} \qquad (1\cdot1)$$

where g is the unrenormalized coupling constant, μ is the bare meson mass and $\bar{g}$ is the finite constant, has been able to derive the corresponding field theory with Fermi interaction, whose coupling constant is given by $G=\bar{g}^2/2$. This procedure has been introduced by Jouvet.[1] Similar discussions about the Lee model[2] and the Machida model[3] have been carried out by Houard and Jouvet[4] and the author,[5] respectively.*

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* The limiting procedures $g\to\infty$ and $\mu\to\infty$ in (1·1) are not independent. For example, we consider the Machida model, where the relation between the bare and physical mass of a meson is given by

$$(\mu-\bar{\mu})\mu=\frac{g^2}{2}\sum_q \Omega^{-1}|v(q,0)|^2[2E(q)-\bar{\mu}]^{-1}$$

from (3·7) in ref. 5). $\bar{\mu}$, the physical mass of a meson, is an observed and finite quantity. Because of $2M\equiv 2E(0)>\bar{\mu}$, it is clear that $\mu>\bar{\mu}$. Then, we get

$$\mu=\frac{1}{2}\bar{\mu}+\left[\left(\frac{1}{2}\bar{\mu}\right)^2+\frac{g^2}{2}\sum_q \Omega^{-1}|v(q,0)|^2(2E(q)-\bar{\mu})^{-1}\right]^{1/2}.$$

Thus it is easily shown that in the limit $\mu\to\infty$ the bare meson mass μ becomes infinite, and the limiting value of g/μ is finite.

It is one of the purposes of this work to obtain the corresponding Fermi interactions in meson theory. The results thus obtained are important for the practical study for the composite model.

The composite model thus derived plays the role analogous to geometrical optics derived from wave optics in the zero wave length of light. The validity of geometrical optics depends upon not only the wave length, but also the states of the physical system under consideration. Geometrical optics and wave optics have their own domains for applicability.

In the frame of the present field theory, the composite model may be considered as a special approximation theory to meson theory. The validity of the composite model depends upon not only the values of g and μ, but also the states of the physical system under consideration. Such a composite model may be effective in several phenomena—perhaps the " core " phenomena in baryons and mesons.[6] In the future theory, the composite model may be expected to include other physical contents than meson theory does.

It is the other purpose of this work to give the foundation for the approximation theory thus derived.

In § 2, it is shown that the probability amplitudes for bare mesons vanish in the limit (1·1), under the appropriate assumption for any physical-particle or bound state.

In § 3, the results in the limit (1·1) for pseudoscalar meson theory are discussed.

§ 2. Physical-particle and bound states

Any stationary (physical-particle or bound) state except scattering state in meson theory satisfies the energy eigenvalue equation

$$H\Psi=E\Psi, \tag{2·1}$$

where the Hamiltonian H is given by

$$H=H_0+H_1+H_2, \tag{2·2}$$

$$H_0=H_N+H_{meson}. \tag{2·3}$$

H_N and H_{meson} are the free Hamiltonian operators for nucleon and meson fields, respectively. H_1 is the meson-nucleon interaction Hamiltonian, and H_2 is the direct (Fermi) interaction Hamiltonian for nucleon field. As is well known, H_2 has the ambiguity. We adopt H_2 so as to satisfy the integrability condition for the interaction Hamiltonian density.

In pseudoscalar meson theory, H_{meson}, H_1 and H_2 are given as follows:

$$H_{meson}=\sum_k \omega_k c_k^\dagger c_k, \tag{2·4}$$

$$H_1=\sum_k \int \{\mathfrak{M}_k(x)c_k e^{ik\cdot x}+\mathfrak{M}_k^*(x)c_k^\dagger e^{-ik\cdot x}\}\,dv, \tag{2·5}$$

$$H_2=\frac{g_2^2}{2\mu^2}\sum_i\int\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)dv, \tag{2·6}$$

$$\mathfrak{M}_k(x)=(2\omega_k\Omega)^{-1/2}\left\{ig_1\bar{\psi}(x)\gamma_5\tau_i\psi(x)-\frac{g_2}{\mu}\bar{\psi}(x)\gamma_5\gamma\cdot k\psi(x)\right\}, \tag{2·7}$$

$$\gamma\cdot k\equiv\boldsymbol{\gamma}\cdot\boldsymbol{k}+i\gamma_4\omega_k,$$

where k stands for momentum $\boldsymbol{k}$ and isospin i, Ω is the normalization volume, $\omega_k\equiv\omega(k)=(\boldsymbol{k}^2+\mu^2)^{1/2}$ is the energy of the bare meson with the momentum $\boldsymbol{k}$. g_1 and g_2 are the unrenormalized coupling constants for the pseudoscalar and pseudovector couplings, respectively.

E in (2·1) is, for example, the physical mass of a pion or a nucleon, etc., which is considered as finite during our calculations below. (This is equivalent to the mass renormalization procedure.)

The state vector Ψ is expanded as

$$\Psi=\sum_{m=0}^{\infty}\sum_{k_1\cdots k_m}\Psi(k_1,\cdots k_m)|k_1,\cdots k_m\rangle, \tag{2·8}$$

where $|k_1,\cdots k_m\rangle$ $(m=1,\ 2,\ \cdots)$ are symmetrized eigenvectors for H_{meson} and satisfy

$$H_{meson}|k_1,\cdots k_m\rangle=\sum_{r=1}^{m}\omega(k_r)|k_1,\cdots k_m\rangle. \tag{2·9}$$

Thus $\Psi(k_1,\cdots k_m)$ $(m=0,\ 1,\ 2,\ \cdots)$ are considered as probability amplitudes for meson field operators, but state vectors for nucleon field operators. $\Psi(k_1,\cdots k_m)$ are symmetrized with respect to every permutation of $k_1,\cdots k_m$.

Using (2·4) and (2·5), we obtain the coupled integral equations for $\Psi(k_1,\cdots k_m)$ from (2·1) and (2·8) as follows:

$$\begin{aligned}&\Big(\sum_{s=1}^{m}\omega(k_s)+H_N+H_2-E\Big)\Psi(k_1,\cdots k_m)\\&\quad+\frac{1}{\sqrt{m}}\sum_{r=1}^{m}\int\mathfrak{M}_{k_r}{}^*(x)\,e^{-ik_r\cdot x}dv\,\Psi(k_1,\cdots k_{r-1},\,k_{r+1},\cdots k_m)\\&\quad+\sqrt{m+1}\sum_k\int\mathfrak{M}_k(x)\,e^{ik\cdot x}dv\,\Psi(k,\,k_1,\cdots k_m)=0,\\&\qquad(m=0,\ 1,\ 2,\cdots).\end{aligned} \tag{2·10}$$

Now, writing $\Psi(k_1,\cdots k_m)$ for $m=0$ as Ψ_N, (2·10) for $m=0$ is

$$(H_N+H_2-E)\Psi_N=-\sum_k\int\mathfrak{M}_k(x)\,e^{ik\cdot x}dv\,\Psi(k). \tag{2·11}$$

From (2·14) for $m=1$, we get

$$\Psi(k)=-(\omega_k+H_N+H_2-E)^{-1}\int \mathfrak{M}_k^*(x)\,e^{-ik\cdot x}\,dv\,\Psi_N$$
$$-\sqrt{2}\,(\omega_k+H_N+H_2-E)^{-1}\sum_{k'}\int \mathfrak{M}_{k'}(x)\,e^{ik'\cdot x}\,dv\,\Psi(k,k'). \qquad (2\cdot12)^*$$

Substituting (2·12) in (2·11), we obtain the exact relation between Ψ_N and $\Psi(k, k')$. Then performing the limiting process (1·1), i.e. in our case

$$\left.\begin{aligned} &g_1\to\infty,\\ &g_2\to\infty,\\ &\mu\to\infty,\\ &g_1/\mu\to\bar{g}_1,\\ &g_2/\mu\to\bar{g}_2, \end{aligned}\right\} \qquad (2\cdot13)$$

we get

$$(H_N+H'+H''-E)\Psi_N$$
$$=\sqrt{2}\lim\sum_k\sum_{k'}\iint \mathfrak{M}_k(x)\,e^{ik\cdot x}\,dv(\omega_k+H_N+H_2-E)^{-1}$$
$$\times\mathfrak{M}_{k'}(x')\,e^{ik'\cdot x'}\,dv'\,\Psi(k,k'), \qquad (2\cdot14)$$

where

$$H'=-\lim\sum_k\iint \mathfrak{M}_k(x)\,e^{ik\cdot x}\,dv(\omega_k+H_N+H_2-E)^{-1}$$
$$\times\mathfrak{M}_{k'}^*(x')\,e^{-ik\cdot x'}\,dv' \qquad (2\cdot15)$$

is easily calculated as

$$H'=\frac{1}{2}\bar{g}_1^{\,2}\sum_i\int\bar{\psi}(x)\gamma_5\tau_i\psi(x)\bar{\psi}(x)\gamma_5\tau_i\psi(x)\,dv$$
$$-\frac{1}{2}\bar{g}_2^{\,2}\sum_i\int\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)\,dv. \qquad (2\cdot16)$$

$H''=\lim H_2$ in (2·14) is easily written down as

$$H''=\frac{1}{2}\bar{g}_2^{\,2}\sum_i\int\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)\bar{\psi}(x)\gamma_5\gamma_4\tau_i\psi(x)\,dv. \qquad (2\cdot17)$$

Assuming that $\Psi(k_1,\cdots k_m)$ $(m=0,\ 1,\ 2,\cdots)$ *remain finite in the limit* (2·13), *it is shown from* (2·10) *that all* $\Psi(k_1,\cdots k_m)$ *except* $m=0$ *vanish in this limit.* Then the right-hand side of (2·14) vanishes and we get

$$(H_N+H'+H'')\Psi_N=E\Psi_N. \qquad (2\cdot18)$$

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* This equation is not applicable for scattering state.

In pseudoscalar meson theory, we obtain

$$H'+H''=\frac{1}{2}\bar{g}_1^2\sum_i\int\bar{\psi}(x)\gamma_5\tau_i\psi(x)\bar{\psi}(x)\gamma_5\tau_i\psi(x)dv, \qquad (2\cdot19)$$

where the pseudovector coupling does not contribute in the limit.

Similarly in scalar, vector and pseudovector meson theories, we get

$$H'+H''=-\frac{1}{2}\bar{g}_1^2\sum_i\int\bar{\psi}(x)\tau_i\psi(x)\bar{\psi}(x)\tau_i\psi(x)dv, \qquad (2\cdot20\text{s})$$

$$H'+H''=\frac{1}{2}\bar{g}_1^2\sum_\mu\sum_i\int\bar{\psi}(x)\gamma_\mu\tau_i\psi(x)\bar{\psi}(x)\gamma_\mu\tau_i\psi(x)dv$$
$$+\frac{1}{2}\bar{g}_2^2\sum_{\mu\nu}\sum_i\int\bar{\psi}(x)\sigma_{\mu\nu}\tau_i\psi(x)\bar{\psi}(x)\sigma_{\mu\nu}\tau_i\psi(x)dv, \qquad (2\cdot20\text{v})$$

$$H'+H''=\frac{1}{2}\bar{g}_1^2\sum_\mu\sum_i\int\bar{\psi}(x)\gamma_5\gamma_\mu\tau_i\psi(x)\bar{\psi}(x)\gamma_5\gamma_\mu\tau_i\psi(x)dv$$
$$-\frac{1}{2}\bar{g}_2^2\sum_{\mu\nu}\sum_i\int\bar{\psi}(x)\sigma_{\mu\nu}\tau_i\psi(x)\bar{\psi}(x)\sigma_{\mu\nu}\tau_i\psi(x)dv, \qquad (2\cdot20\text{pv})$$

respectively.

Therefore we conclude that *the energy eigenvalue equation for any stationary physical-particle or bound state in meson theory*, (2·1), *is reduced to that in field theory with Fermi interactions* $H'+H''$ *in the limit* (1·1) *or* (2·13).

§ 3. Discussions

Let us call the effects described by $\Psi(k_1,\cdots k_m)$ $(m=1, 2, \cdots)$ the "meson clouds" of the system, then *the limiting process* (1·1) *eliminates meson cloud effects for the system containing baryons and mesons.* Then the composite model derived in § 2 describes the "cores" of baryons and mesons defined by Ψ_N.

For pseudoscalar coupling in pseudoscalar meson theory, we obtain the corresponding composite model with pseudoscalar Fermi interaction, which has already been discussed by Maki.[7]

For pseudovector coupling in pseudoscalar meson theory, $H'+H''$ vanishes in the limit (1·1). In this case, the cores of baryons are reduced to the point sources. Therefore the pseudovector coupling describes only the meson clouds. However, we can adopt H_2 other than (2·6). For example, when H_2 is given by

$$H_2=-\frac{g_2^2}{2\mu^2}\sum_i\int\bar{\psi}(x)\gamma_5\boldsymbol{\gamma}\tau_i\psi(x)\cdot\bar{\psi}(x)\gamma_5\boldsymbol{\gamma}\tau_i\psi(x)dv, \qquad (3\cdot1)$$

we obtain

$$H'+H''=-\frac{1}{2}\bar{g}_2^2\sum_{\mu}\sum_{i}\int\bar{\psi}(x)\gamma_5\gamma_\mu\tau_i\psi(x)\bar{\psi}(x)\gamma_5\gamma_\mu\tau_i\psi(x)\,dv. \qquad (3\cdot2)$$

It is not clear that the composite model with the interaction (3·2) is possible or not. In general, we can add the appropriate Fermi interactions to (2·6). Then the added Fermi interactions describe the cores of elementary particles. Such a model is just that proposed in ref. 6).

The stationary treatment as in the above is not applicable for scattering state, except several simple models. It may be, however, expected that, for scattering state, the same results as in this note will be obtained.

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Internal Degrees of Freedom and Elementary Particles. I

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Quantum theory of point-like systems is established by extending the concept of relativistic particle in some respects: A point-like system means a one-parameter series of events $x_\mu(\tau)$ with substantial internal degrees of freedom concentrated upon x_μ, and indefinite metric in Hilbert space is generally taken as to the internal degrees. The theory corresponds to an extention of the usual local field equations, suitable to obtaining a unified theory of elementary particles. The rest-mass, $m^2=-p_\mu{}^2$ (with p_μ as momentum-energy vector) becomes a dynamical quantity of the system with its possible eigenspectrum, leading to uncertainty relations between rest mass value and space-time localization. The internal angular momentum tensor $S_{\mu\nu}$ is another basic dynamical quantity of the system and is responsible for spin and Zitterbewegung. Also defined is the instantaneous velocity operator v_μ, which is not generally colinear with p_μ and must be restricted by certain kinematical conditions. Three different criteria about these conditions on v_μ make point-like systems classified into various types. For "normal class" of systems, $\rho\equiv-v_\mu{}^2$ is an absolute invariant with eigenvalue 1 or 0 and is regarded to represent baryon number. Especially important are point-like systems of the first kind, i.e. the ones in which v_μ commute with the position x_μ and thus mean internal variables. Such a system generally has, besides rest mass, spin and ρ, three self-adjoint commuting invariant quantities formed out of p_μ, v_μ and $S_{\mu\nu}$ only, which are to be identified eventually with the intrinsic properties of elementary particles (isospin, hypercharge, etc.). Systems are further divided into "classical models", where velocity components are commutable (an example being relativistic rotator), and "non-classical models" where they are not ($[v_\mu, v_\nu]\neq 0$), to derive general characteristics for each of them. Dirac and Kemmer particles are special simple examples of the latter, where system has no substantial internal degrees of freedom apart from v_μ.

> It thus becomes a matter of great importance to set up new dynamical systems and see if they will better describe the atomic world.
>
> P.A.M. Dirac

§ 1. Introduction

The usual theory of elementary particles is based upon field equations corresponding to originally structureless point particles with arbitrary rest mass values with the hypothesis that an elementary particle cannot have a *substantial* internal structure apart from that due to interactions in so far as it must be a geometrical point. On such hypothesis possible relativistic wave equations are enumerated according to irreducible representations of the Lorentz group, e. g. by the method of Dirac, Fierz and Pauli or that of Wigner. Each of such wave equations is suitable to representing one kind of elementary particles with

definite spin and rest mass values. This kind of method, however, affords only a partial explanation for the existence of various elementary particles associated with that variety of apparently irreducible intrinsic properties.

On the other hand there are several lines to attempt at the unification of elementary particles. One of the lines is to find out the "unified equation" which describes not the elementary particles of a certain kind but all possible elementary particles. This will mean to renounce the above-mentioned hypothesis of structureless point particle in the ordinary theory and to assume a certain unique dynamical system with some sort of internal degrees of freedom in such a way that various elementary particles be reproduced as various possible forms of motion of that system which are mutually distinct with respect to its internal states. Introducing internal degrees of freedom within the framework of relativistic causality and quantum principles would raise a delicate problem but is by no means forbidden. Instead it will open ample possibilities beyond the conventional wave equations.

In this standpoint the wave function is a reducible representation as far as the usual Lorentz group is concerned, but each elementary-particle state corresponds to an irreducible representation with respect to the totality of the Lorentz group and the various transformation groups relevant to internal motions. This means on the one hand that spin and rest mass are not the parameters fixed for each system as in the usual theory but represent themselves dynamical quantities of a single basic system, taking their eigenvalue spectra. It means on the other hand that internal states of the system are not specified completely by rest mass and spin alone but we need some additional quantum numbers. Evidently, both points are necessary to have a unified theory along the present line; in particular the latter point is required to account for the intrinsic properties of elementary particles other than spin and rest mass (isospin, hypercharge, etc.).

Indeed a typical example of this kind of approach was given by Yukawa[1] in his non-local theory. However, in order to proceed by a less radical extension of the conventional theory we will take the hypothesis that particle is essentially local but nevertheless maintains certain degrees of freedom of internal motion. Also we start with one-particle theory following the case of Dirac equation, which was historically found out as a relativistic Schrödinger equation for the state amplitude $\psi(x)$ for a single particle. From the original Dirac's standpoint the x_k in $\psi(x_\mu)$ corresponded to position* of a particle. We extend such notion of particle in the following basic respects, and call it "*point-like system*", or "*corpuscule*".

(i) By point-like system we mean first of all a one-parameter series of events x_μ in the quantum-mechanical sense. An event is an elementary relativistic concept and is to be regarded as a set of observables x_μ's quantum-

* This is different from the position operator in the sence of Newton-Wigner.[2]

mechanically, where $t \equiv x_0$ is also an observable on the same footing as x_k. They are observed by scale and clock, and assuming that these observations do not mutually interfere, we put

$$[x_\mu, x_\nu] = 0. \tag{1}$$

(ii) A point-like system is, however, not merely events x_μ but means a certain localizable dynamical system which otherwise has certain degrees of freedom* ξ_α describing certain internal motion concentrated upon the x_μ.

First we remark that since t is itself an observable in our theory we need a time-ordering parameter τ, which is essentially a c-number, in order to describe the time evolution of the system. We shall call it "intrinsic time" or "instant parameter" to distinguish it from the observable time t. Both times must be correlated in a certain manner, but the correlation itself may differ according to the characteristics of each point-like system.

The internal motion must be concentrated upon x_μ since we assume that the real spatial extension of the system is reduced, indeed, to a single point x_μ. This assumption is required in order to be consistent with (1). Thus x_μ cannot be supposed, for instance, to mean center of mass coordinates X_μ of a certain extended continuous matter, since the latter components do not mutually commute:

$$[X_\mu, X_\nu] \neq 0,$$

as long as the center of mass is defined to be independent of reference frames.[3]** The problem of how it is physically possible and consistent for a system to be point-like but nevertheless to maintain the degrees of freedom of internal motion (rotation, etc.) is investigated in detail in Part III, where we shall consider in particular the "*condition of reality*" and the "*condition of original rest-mass null*". In this paper, however, being content with the fact that the idealization of point-like system is consistent at least on formal grounds, we shall refer only casually to those conditions.

(iii) The third important point is that we generally admit an indefinite metric in Hilbert space, because once we introduce internal degrees of freedom we may allow negative probabilities for internal configurations which are not directly observable as such.

Now the first general property of our point-like systems is that the momentum-energy 4-vector p_μ of the system become canonical conjugates of the position coordinates x_μ:

$$[x_\mu, p_\nu] = i\hbar\,\delta_{\mu\nu}. \tag{2}***$$

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* The suffix α in ξ_α does not mean a vector suffix but simply runs over all internal degrees of freedom.

** Compare also with Eq. (74).

*** In this paper Greek suffixes range from 1 to 4, and imaginary time coordinate $x_4 = ix_0 = it$ is used, together with the unit system where $c=1$. Latin suffixes run over 1 to 3. Summation convention is applied to repeated indices unless otherwise stated.

This covariant commutation relations contain, besides the usual commutation rule between coordinate and momentum, the relation

$$[x_4, p_4] = -[x_0, p_0] = -[t, E] = i\hbar . \tag{3}$$

Thus time $t = x_0$ and energy $E = p_0 = p_4/i$ are regarded as canonical conjugates like x_k and p_k (apart from negative sign).

That this is possible is related to the previously mentioned fact that in our theory the rest mass m (or more exactly the squared rest mass P) is not a fixed c-number parameter but is rather *a dynamical quantity that is defined by the very relation*

$$-p_\mu{}^2 = P = m^2, \tag{4}$$

(which is of course a constant of motion for a closed system), and can take any possible value (i. e. the eigenvalues) as the result of the "unified" wave equation, and (4) itself does not mean any condition. This is a situation different from the usual theory, where the rest mass m_0 appearing in the wave equation is a fixed parameter restricting the momentum and energy of the particle by

$$-p_\mu{}^2 = E^2 - p_k{}^2 = m_0{}^2. \tag{5}*$$

In our theory, on the other hand, all p_μ's represent independent quantities so that (2) does not involve any contradiction only if we regard t as dynamical variable on the same footing as x_k. The commutation relation (3) accounts for the uncertainty relation between E and t: $\Delta E \, \Delta t \gtrsim h$, on the same basis as for $\Delta p \, \Delta x \gtrsim h$.

Our equation (4) simply expresses the fact that the rest mass is usually measured through the equation $m^2 = -p_\mu{}^2 = E^2 - p_k{}^2$ in such state as is approximately a simultaneous eigenstate of (p_k, E). On the other hand if particle is sharply localized the rest mass is generally subject to certain indeterminacy due to the relations (2) and (4) (cf. Eq. (60)). This is a new effect but seems not to conflict with observational evidences.

The next important point for any poin-tlike system is the existence of the internal angular momentum tensor $S_{\mu\nu} = -S_{\nu\mu}$, as a direct consequence of (1) and (2), whose components are functions of internal variables of the system ξ_α alone and work as generators of homogeneous Lorentz transformations for any internal variable, with definite algebraic properties. This $S_{\mu\nu}$ produces the spin of the system as well as Zitterbewegung in the orbital motion x_μ (§ 4). We thus have the set of basic quantities

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* This is the relation to result from the wave equation and is to be regarded as subsidiary condition to be fulfilled for a permissible state. So it is better to be written as $-p_\mu{}^2 \approx m_0{}^2$, meaning the Klein-Gordon equation, $(p_\mu{}^2 + m_0{}^2)\psi = 0$. If one regards (5) as a relationship between dynamical quantities, energy E becomes a definite function of momentum p_k and it cannot satisfy $[E, x_k] = 0$ contained in (2), nor (3).

$$(x_\mu, p_\mu, S_{\mu\nu}) \qquad (6)$$

for any point-like system.

The time-evolution of the system is described, as stated already, by means of its intrinsic time τ. *New independent quantities to be supplied by the intrinsic-time derivation of the dynamical variables* (6) *are "4-velocity" components:*

$$v_\mu = \frac{dx_\mu}{d\tau} \equiv \dot{x}_\mu$$

only, on account of the basic conservation laws (cf. § 5). In (relativistic) Newtonian mechanics one tacitly imposes a stringent condition that v_μ be colinear with p_μ, but for our point-like system we generally have essential disparity between p_μ and v_μ in accordance with the existence of the internal degrees. Indeed v_μ means the instantaneous velocity of the orbital motion which contains Zitterbewegung and is intimately connected with the internal motion of the system. Especially the (negative) relativistic magnitude of the v_μ-vector,

$$\rho \equiv -v_\mu^2, \quad \text{(i.e. } dx_\mu^2 = -\rho d\tau^2\text{)}$$

represents a new scalar dimensionless quantity which is generally independent of the rest mass, $P = -p_\mu^2$.

Thus we always have the set of basic variables

$$(x_\mu, p_\mu, S_{\mu\nu}, v_\mu), \qquad (7)$$

irrespective of the properties of the pure internal variables ξ_α, although $S_{\mu\nu}$'s are certain functions of ξ_α(and v_μ), and *the interrelations and behaviours of the variables* (7) *determine already the gross characteristics of point-like systems* (§§ 4 and 5). Naturally, in order to specify the system completely it is generally required to treat all internal degrees of freedom ξ_α explicitly,* whose kinematical and dynamical properties with their physical significances are essential in defining the system. Simply speaking, they are characterized by the structure of the self-adjoint invariant hamiltonian H of the system, which is different from $p_0 = E$ and means the intrinsic-time displacement operator determining the equation of motion for any dynamical quantity F in the "Heisenberg picture" by

$$i\hbar \frac{dF}{d\tau} = [F, H] + i\hbar \frac{\partial F}{\partial \tau}.$$

However, before entering into the precise treatment of the whole internal properties by the explicit use of ξ_α, it is necessary to define the kinematical properties of v_μ.

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* In some simple cases we need not treat ξ_α explicitly. This may happen when the independent internal degrees of freedom implied by ξ_α is less than six and can completely be represented by $S_{\mu\nu}$'s.

The commutation relations within the set of the quantities (6) are universally given for point-like systems in general. On the contrary those concerning v_μ are not given in advance. Thus to have a definite theory we must always postulate certain *inhomogeneous* kinematical conditions on v_μ which determine the commutation rules concerning v_μ and fix the definition of τ at the same time. The condition must be such that the inequality

$$\langle v_0 \rangle = \frac{d\langle t \rangle}{d\tau} > 0, \tag{8}$$

is ensured consequently, where $\langle\ \rangle$ designates the expectation value. (8) means that τ goes on monotonously with the expectation value of t even though the reading of clock has a quantum-mechanical dispersion in general.

The possible form of the above envisaged condition for v_μ allows us just to classify point-like systems into various types. The following three criteria are important for this purpose.

(A) *Whether or not* $\rho = -v_\mu^2$ *is an absolute invariant* (i. e. commutable with all other quantities). We shall call the former case "the normal class".

(B) *Whether or not the commutators* $[v_\mu, v_\nu]$ *vanish identically.*

(C) *Whether or not the commutators* $[v_\mu, x_\nu]$ *vanish identically.* We shall call the former case the "system of the first kind" and the latter "the second kind".

Now, for normal class, since ρ is practically a c-number, we can always normalize it such that $\rho = 1$ through a suitable redefinition of the parameter τ, only if $\rho \neq 0$. Therefore ρ is essentially a dichotomic variable which takes only two possible values, 1 or 0. Evidently, states belonging to $\rho = 1$ and $\rho = 0$ have inherently different internal properties, and for the latter some of the internal degrees of freedom necessarily degenerate. Between both states no transition is allowed. We may consider $\rho = 1$ and $\rho = 0$ as corresponding to baryon and lepton states, respectively (§ 5).

Next let us consider the criterion (B). For the former case:

$$[v_\mu, v_\nu] = 0,$$

v_μ can be regarded as the usual 4-velocity and then τ becomes a *proper time* in the usual sense. This case corresponds to corpuscules with realistic internal structure in general, having its direct classical analogue. We therefore call them "classical models", although the treatment is always made quantum-mechanically. On the other hand we call a system for which $[v_\mu, v_\nu] \neq 0$ "non-classical model".

Finally we consider the criterion (C). For example, a Newtonian particle belongs to the second kind because $[v_\mu, x_\nu]$ does not vanish on account of the direct connection between v_μ and p_μ together with the basic commutation relation (2).

Our main interest, however, is placed on *systems of the first kind.* Here velocity components v_μ become parts of the internal variables of the system. Also, the third scalar quantity to be formed of p_μ and v_μ:

$$\varepsilon = -v_\mu p_\mu ,$$

(the first being $P=-p_\mu{}^2$ and the second $\rho=-v_\mu{}^2$), plays an essential role for the motion. This is because the invariant hamiltonian for such system has the general form

$$H=-\varepsilon+\mathcal{H}=v_\mu p_\mu+\mathcal{H}. \tag{9}$$

Here $\mathcal{H}$ is a scalar function of the internal variables $(S_{\mu\nu}, v_\mu, \xi_\alpha)$ alone, while the term $-\varepsilon$ means the existence of the coupling between internal and external motions. The ε and $\mathcal{H}$ are not separately conserved in general.

In this case the Schrödinger equation,

$$i\hbar\frac{d\psi}{d\tau}=H\psi=(v_\mu p_\mu+\mathcal{H})\psi, \tag{10}$$

becomes a linear differential equation with respect to x_μ, since $p_\mu\to\hbar/i\cdot\partial/\partial x_\mu$ due to (2). The properties of (10) depends on the kinematical characters of v_μ already mentioned and the structure of $\mathcal{H}$, but with a suitable prescription for them Eq. (10) has appropriate properties to comprise different elementary-particle states by its eigen-solutions. Then Eq. (10) means our "unified equation". It refers to one-particle states, but it provides the starting point for proceeding to the treatment of interactions by means of the subsequent second-quantization or other procedures. Such approach has an evident advantage that the theory maintains particle character clearly.

In § 6 we deduce general properties of the first kind systems, and especially we conclude the existence of six mutually commuting conserved invariant quantities which are formed of the variables $(p_\mu, S_{\mu\nu}, v_\mu)$ alone. They include, besides rest mass and the magnitude of spin, four quantities that may eventually be identified with baryon number, magnitude of isospin, and hypercharge.

Evidently, systems of the first kind are divided, according to the criterion (B), into two categories, classical and non-classical models. A typical example* belonging to the former is the relativistic rotator given in a previous paper.[4] On the other hand an important type of non-classical models is defined by the condition

$$[v_\mu, v_\nu]=\frac{i}{\kappa}S_{\mu\nu}.$$

It includes as special examples various familiar systems such as Bhabha particles.[5]

The precise treatment of both of the above types will be given in Part II and III. The main purpose of this paper (Part I) is to set up the notion

* Various examples are investigated in T. Takabayasi, Soryusiron-Kenkyu (Mimeographed circular in Japanese) **21** (1960), 723.

of point-like system and establish the general foundation of theory in order to clarify *what will generally emerge, once we introduce substantial internal degrees of freedom into relativistic quantum mechanics* in the level of one-particle theory.

§ 2. Theoretical framework

First in this section we outline the general foundation of the theory of point-like systems. As stated in the Introduction, it means a one-parameter series of events $x_\mu(\tau)$ satisfying $[x_\mu(\tau), x_\nu(\tau)]=0$, associated with possible internal degrees of freedom ξ_α.

2a. Let us consider in the "Schrödinger picture", where x_μ and other dynamical quantities do not depend on τ while the state vector ψ changes with τ. If we take the representation in which x_μ are diagonal, ψ is represented as a function

$$\psi(x_\mu, \theta_a, \tau), \tag{11}$$

where θ_a signifies the set of independent parameters* corresponding to the variables ξ_α of internal motion (rotation, etc.) of the corpuscule. We are dealing not only with the external motion (represented by x_μ) but also with the internal motion at the same time, but for the latter the internal configuration θ_a cannot be measured as such and hence we may admit negative probabilities for them. Thus, introducing an indefinite metric[6] in general, we assume that the normalization is given by

$$\psi^* \eta \psi, \tag{12}$$

where η is a τ-independent hermitian operator:

$$\eta=\eta^*, \quad \dot{\eta}=0,$$

and is related to internal variables alone. In the representation of (11),

$$\psi^*(x_\mu, \theta_a, \tau)\eta\psi(x_\mu, \theta_a, \tau)D$$

means the (hypothetical) relative probability density that the corpuscule appears as an event at $(x_k, x_0=t)$ with the internal configuration around θ_a at an intrinsic time τ. The factor D means the weight function for the parameters θ_a. Accordingly, the expectation value of any observable F is given by

$$\langle F\rangle=\psi^*\eta F\psi=\int \psi^*(x_\mu, \theta_a, \tau)\eta F\psi(x_\mu, \theta_a, \tau)\, d^3x\, dt\, D d\theta_a. \tag{13}$$

The adjoint of an operator F is defined by

* The ξ_α themselves may be subject to some constraints in general.

$$F^\dagger=\eta^{-1}F^*\eta.$$

Clearly,

$$(F^\dagger)^\dagger=F,\quad (FG)^\dagger=G^\dagger F^\dagger.$$

A physical variable must correspond to self-adjoint operator, since its expectation values must be real. Since η is independent of x_μ, we have

$$[x_\mu,\ \eta]=0,$$

so that for x_μ the hermitian conjugate and the adjoint are identical: $x_\mu{}^\dagger=x_\mu{}^*$. On the other hand, $x_k{}^*=x_k$, $x_4{}^*=-x_4$. We shall call a q-number vector "self-adjoint vector" if its space components are self-adjoint while its time component is anti-self-adjoint. The expectation value of such a vector is *real vector* (namely a vector with real space components and pure-imaginary time component). Evidently, x_μ is a self-adjoint vector.

Now a state vector ψ develops with τ by the Schrödinger equation:

$$i\hbar\frac{d\psi^{(S)}}{d\tau}=H^{(S)}\psi^{(S)}. \tag{14}*$$

Since τ is a real scalar parameter, the invariant hamiltonian $H^{(S)}$ must be a scalar, and self-adjoint:

$$\eta H=H^*\eta. \tag{15}$$

The latter fact is necessary in order that the norm (12) be conserved with τ, as is verified by employing (14) and its adjoint:

$$-i\hbar\frac{d\psi^{(S)\dagger}}{d\tau}=\psi^{(S)\dagger}H^{(S)},\qquad (\psi^\dagger=\psi^*\eta). \tag{16}$$

The invariant hamiltonian represents the structure of the point-like system and we assume that it does not involve τ explicitly for closed system. Under this assumption there still remains certain arbitrariness in the definition of τ which will be discussed later. The evolution of the expectation values of a not explicitly τ-dependent quantity F is given by

$$\frac{d}{d\tau}\langle F\rangle=\frac{d}{d\tau}(\psi^{(S)\dagger}F^{(S)}\psi^{(S)})=\frac{i}{\hbar}\psi^{(S)\dagger}(H^{(S)}F^{(S)}-F^{(S)}H^{(S)})\psi^{(S)}$$

$$=\frac{i}{\hbar}\langle HF-FH\rangle. \tag{17}$$

A conserved quantity (or constant of motion) in our theory is a quantity whose expectation value for any state does not change with τ; i. e.

* In this section we associate the index S or H to distinguish between the quantities in Schrödinger picture and those in Heisenberg picture wherever confusion may arise.

$$\frac{d}{d\tau}\langle F\rangle=0, \quad \text{so} \quad \frac{d\langle f(F)\rangle}{d\tau}=0.$$

Eq. (17) means that if F commutes with H and is not explicitly τ-dependent it is a conserved quantity. Clearly, H is a conserved quantity for closed system.

2b. Next we consider a linear transformation in the Hilbert space which leaves the norm (12) as well as the hermiticity of η invariant:

$$\left.\begin{aligned}&\psi'=\Lambda\psi,\\&\eta'=\Lambda^{-1*}\eta\Lambda^{-1}, \quad \text{so} \quad \psi^{\dagger\prime}=\psi^{\dagger}\Lambda^{-1}.\end{aligned}\right\} \tag{18}$$

At the same time an observable transforms by

$$F'=\Lambda F\Lambda^{-1} \tag{19}$$

to leave expectation values invariant. We shall call such transformation "canonical" if Λ is restricted by the condition

$$\Lambda^{\dagger}\Lambda=\Lambda\Lambda^{\dagger}=1, \tag{20}$$

which means

$$\Lambda^{*}\eta\Lambda=\eta, \text{ i.e. } \eta'=\eta=\text{invariant}. \tag{20'}$$

In this case

$$(F')^{\dagger}=\Lambda F^{\dagger}\Lambda^{-1}=(F^{\dagger})',$$

so that the self-adjoint (or anti-self adjoint) property of any observable is maintained by a canonical transformation. If A is self-adjoint $\Lambda=e^{iA}$ means a canonical transformation.

The theory of *2a* can now be transferred to the "Heisenberg picture" by a canonical transformation with

$$\Lambda=e^{(i/\hbar)H\tau} \tag{21}$$

namely,

$$\left.\begin{aligned}&\psi^{(\mathrm{H})}=e^{(i/\hbar)H\tau}\psi^{(\mathrm{S})},\\&\eta^{(\mathrm{H})}=\eta^{(\mathrm{S})},\\&F^{(\mathrm{H})}=e^{(i/\hbar)H\tau}F^{(\mathrm{S})}e^{(-i/\hbar)H\tau}, \quad (H^{(\mathrm{H})}=H^{(\mathrm{S})}).\end{aligned}\right\} \tag{22}$$

We then have

$$\dot{\psi}^{(\mathrm{H})}=0, \quad \dot{\eta}^{(\mathrm{H})}=0, \tag{23}$$

and

$$\dot{F}^{(\mathrm{H})}=\frac{i}{\hbar}[H, F^{(\mathrm{H})}], \tag{24}$$

for a not explicitly τ-dependent quantity F. Eqs. (23) and (24) reproduce (17):

$$\frac{d}{d\tau}\langle F\rangle=\psi^{(\mathrm{H})*}\eta^{(\mathrm{H})}\frac{i}{\hbar}[H, F^{(\mathrm{H})}]\psi^{(\mathrm{H})}=\frac{i}{\hbar}\langle[H, F]\rangle. \tag{17'}$$

From x_μ we can always define the 4-velocity v_μ by (24):

$$v_\mu=\frac{dx_\mu}{d\tau}=\frac{i}{\hbar}[H, x_\mu]. \tag{25}$$

Since x_μ is a self-adjoint vector, so is v_μ. However,

$$[v_\mu, \eta]\neq 0$$

in general, hence v_k and $v_0\equiv v_4/i$ are not necessarily hermitian.

2c. Next we consider inhomogeneous Lorentz transformations. If we take the "Schrödinger standpoint", ψ changes under such transformation by a canonical transformation:

$$\left.\begin{aligned}&\psi'=\Lambda\psi,\\&\Lambda^\dagger\Lambda=\Lambda\Lambda^\dagger=1,\quad\text{so}\quad\eta'=\eta=\Lambda^*\eta\Lambda,\end{aligned}\right\} \tag{26a}$$

while physical quantities remain unchanged:

$$F=\text{invariant}. \tag{26b}$$

Evidently the Lorentz invariance of the norm (12) and of the self-adjoint characters of physical quantities is ensured by (26). On the other hand, from the "Heisenberg standpoint" the same situation is represented by

$$\left.\begin{aligned}&\psi=\text{invariant},\quad\eta=\text{invariant},\\&F'=\Lambda F\Lambda^{-1},\end{aligned}\right\} \tag{27}$$

with the same Λ. The transformations of expectation values are identical in both cases. In any case Λ forms a representation (satisfying the canonical condition (20)) of the inhomogeneous Lorentz group, and the set of infinitesimal operators corresponding to Λ defines the components of momentum-energy vector, p_μ, and of the total angular momentum skew-tensor around the origin, $J_{\mu\nu}$, of the system. They satisfy the well-known commutation relations for the generators of the inhomogeneous Lorentz group:

$$\begin{cases}[p_\mu, p_\nu]=0, & (28)\\ [J_{\mu\nu}, p_\rho]=i\hbar\,\delta_{\rho[\mu}p_{\nu]}, & (29)\\ [J_{\mu\nu}, J_{\rho\sigma}]=i\hbar(\delta_{\mu[\rho}J_{\nu\sigma]}+\delta_{\nu[\sigma}J_{\mu\rho]}). & (30)\end{cases}$$

First we consider about Lorentz transformations: $x_\mu'=a_{\mu\nu}x_\nu$. The expectation values of any scalar quantity S and of any vector quantity V_μ must, respectively, transform such that

$$\begin{aligned}\langle S\rangle'&=\psi'^*\eta S\psi'=\psi^*\eta\Lambda S\Lambda^{-1}\psi\\&=\langle S\rangle=\psi^*\eta S\psi,\end{aligned}$$

$$\langle V_\mu\rangle' = \psi'^* \eta V_\mu \psi' = \psi^* \eta \Lambda V_\mu \Lambda^{-1} \psi$$
$$= a_{\mu\nu}\langle V_\nu\rangle = a_{\mu\nu}\psi^* \eta V_\nu \psi,$$

(where we are considering from the Schrödinger standpoint), so that Λ satisfies

$$\begin{cases} \Lambda S \Lambda^{-1} = S, & \text{(31a)} \\ \Lambda V_\mu \Lambda^{-1} = a_{\mu\nu} V_\nu, & \text{(31b)} \end{cases}$$

and analogous relations for tensor quantities. If we consider an infinitesimal rotation in the $(\rho\sigma)$-plane:

$$a_{\mu\nu}^{(\rho\sigma)} = \delta_{\mu\nu} + \varepsilon(\delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu}),$$

the corresponding operator is written as

$$\Lambda^{(\rho\sigma)} = 1 + \frac{i}{\hbar}\varepsilon J_{\rho\sigma}. \qquad (32)$$

Taking into account that ε is real if $(\rho\sigma) = (ij)$ but is pure imaginary if $(\rho\sigma) = (k4)$, the canonical condition (20) leads to

$$J_{ij}^* \eta = \eta J_{ij}, \quad J_{k4}^* \eta = -\eta J_{k4}, \qquad (33)$$

namely $J_{\mu\nu}$ has self-adjoint space components and anti-self-adjoint space-time components. We call such tensor the " self-adjoint tensor ". On the other hand the relations (31) yield the commutation rules:

$$\begin{cases} [J_{\mu\nu}, S] = 0, & \text{(34a)} \\ [J_{\mu\nu}, V_\rho] = i\hbar\,\delta_{\rho[\mu} V_{\nu]}. & \text{(34b)} \end{cases}$$

The latter is the generalization of (29).

Next we consider space-time displacement by an amount a_μ. From the fact that x_μ are observables representing the space-time location of the particle, the corresponding transformation matrix in the Hilbert space, Λ, must satisfy

$$\Lambda x_\mu \Lambda^{-1} = x_\mu - a_\mu. \qquad (35)$$

On the other hand Λ is written

$$\Lambda = 1 + \frac{i}{\hbar} a_\mu p_\mu, \qquad (36)$$

if we assume the displacement a_μ to be infinitesimal. Eq. (35) together with (36) gives the canonical commutation relation formerly mentioned:

$$[x_\mu, p_\nu] = i\hbar\,\delta_{\mu\nu},$$

which is characteristic for point-like systems in general. Like x_μ, the p_μ mean external variables, hence

$$\begin{cases} [p_\mu, \eta] = 0, \\ p_\mu^\dagger = p_\mu^* = g(\mu) p_\mu, \end{cases} \qquad (37)$$

with

$$g(\mu)=\begin{cases} 1 & \text{for} \quad \mu=1, 2, 3, \\ -1 & \text{for} \quad \mu=4. \end{cases}$$

Finally, an intrinsic-time displacement is independent of translations and Lorentz transformations for a closed system and this means

$$\begin{cases} \dot{p}_\mu=0, & \text{(38a)} \\ \dot{J}_{\mu\nu}=0. & \text{(38b)} \end{cases}$$

These represent in our theory the covariant conservation laws of the total momentum-energy and the total angular momentum.

2d. Particularly simple are "stationary states", which are the eigenstates of H given by

$$H\psi=\lambda\psi, \quad (\psi^\dagger H=\lambda^*\psi^\dagger). \tag{39}$$

Since H is self-adjoint, λ is real provided that the state is not "isotropic" (i. e. $\psi^*\eta\psi\neq 0$), and then the state vector has the simple τ-dependence:

$$\psi^{(S)}(\tau)=e^{-(i/\hbar)\lambda\tau}\psi^{(S)}(0).$$

Accordingly, the expectation value of any not explicitly τ-dependent quantity F, $\langle F\rangle_\lambda$, remains unchanged with τ. Thus

$$\langle\dot{F}\rangle_\lambda=0. \tag{40}$$

This theorem, however, does not apply to an unbounded operator such as x_μ, so that $\langle\dot{x}_\mu\rangle_\lambda=\langle v_\mu\rangle_\lambda$ does not necessarily vanish.* This fact is important especially for $\langle dt/d\tau\rangle_\lambda=\langle v_0\rangle_\lambda$. Indeed, $\langle v_0\rangle_\lambda$ must be positive definite in order that τ really means a time-ordering parameter (cf. (8)). We have the general relation (cf. § 5):

$$\langle v_\mu\rangle_\lambda=\left\langle\frac{\varepsilon}{m^2}p_\mu\right\rangle_\lambda.$$

2e. The physical definition of τ, however, is fixed sufficiently only when we supplement the basic equations (14) or (24) with certain inhomogeneous kinematical conditions on v_μ. To make it explicit we here mention two simple possibilities.

(i) As mentioned in § **1**, a simple possibility is

$$\left.\begin{aligned} &[v_\mu, v_\nu]=0, \\ &\rho\equiv -v_\mu{}^2=1, \end{aligned}\right\} \tag{41}$$

* The situation is analogous to the case of the elementary non-relativistic theory of a free particle, where "stationary state" (eigenstate of the energy) includes, for instance, a plane wave $e^{i(k_l x_l-\omega t)}$ with $\omega=\hbar k^2/2m$. For this state the expectation value of the velocity $dx_i/dt=p_i/m$ does not vanish but rather equals $\hbar k_i/m$.

which define systems with classical analogue. In this case τ becomes proper time in the usual sense and v_μ becomes unitary velocity, its components taking continuous spectrum between ± 1. Consistent with (41) we may put the condition

$$\langle v_0 \rangle \geqq 1,$$

in accordance with the condition (8). The ordinary 3-dimensional velocity components V_k can be defined by

$$V_k = v_k / v_0, \quad v_0 = (1 + v_k^2)^{1/2}.$$

(ii) Another simple possibility for the kinematical condition on v_μ is given by

$$\{v_\mu, v_\nu\} = -2\delta_{\mu\nu}. \tag{42}*$$

For this case we must take

$$\eta = v_0 \equiv v_4 / i \tag{43}$$

as the indefinite metric operator, so that the v_μ under (42) forms a self-adjoint vector, and also fulfills (8):

$$\langle v_0 \rangle = \psi^* \eta v_0 \psi = \psi^* \psi \geqq 0.$$

The solution of (42) is expressed as

$$v_\mu = i\gamma_\mu, \tag{44}$$

with the use of Dirac matrices γ_μ's which can all be taken hermitian simultaneously, and then

$$\eta = \gamma_4^{(S)}. \tag{43'}$$

(As a dynamical quantity γ_4 is constant in the Schrödinger picture which we denote $\gamma_4^{(S)}$, and η must be taken to equal this constant $\gamma_4^{(S)}$ even in the Heisenberg picture.) The velocity is quantized in this case and $\rho = 4$.

If we take $\mathcal{H} = 0$ in Eq. (10) for simplicity, the self-adjoint invariant hamiltonian becomes

$$H = -\mathcal{E} = v_\mu p_\mu = i\gamma_\mu p_\mu, \tag{45}$$

with the Schrödinger equation

$$i\hbar \frac{d\psi}{d\tau} = i\gamma_\mu p_\mu \psi.$$

A corresponding stationary state to be given by

$$H\psi = i\gamma_\mu p_\mu \psi = \lambda\psi \tag{46}$$

represents a Dirac particle, since (46) is exactly the Dirac equation with λ for the rest-mass constant. In fact for the present case an eigenstate of H is always

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* { } means the anticommutator.

a simultaneous eigenstate of rest mass since the squared rest mass operator P satisfies $P=H^2$ identically.

Various other possibilities for the condition on v_μ will be systematically considered in later sections.

§ 3. Preliminary considerations about conserved quantities

On the basis of the theoretical foundation given in § **2** we shall in this section summarize some preliminary considerations about conserved quantities of relativistic dynamical systems.

The existence of ten basic dynamical quantities $(p_\mu, J_{\mu\nu})$ satisfying the commutation rules (28)–(30) is merely the general consequence of special relativity, i. e. the invariance under inhomogeneous Lorentz transformations: We shall first give the consequences following therefrom. The particular situation for a point-like system is the fact that p_μ and $J_{\mu\nu}$ (and other possible dynamical quantities F) are quantities assigned to one-particle state by the nature of the theory and that their conservation means $dF/d\tau=0$ (cf. (38)) instead of $dF/dt=0$. Those points, however, do not essentially influence the general arguments to follow.

3a. First we write

$$P\equiv -p_\mu{}^2=m^2=\text{const.}, \tag{4'}$$

then m represents the rest mass of the system, since it signifies the total energy in the reference frame in which $p_k=0$. Such state and frame exist, unless $m=0$, owing to (28), and means an inertial frame because of (38a). We call this frame "mean rest frame" and denote it as Π.

Next we introduce the pseudovector

$$\begin{aligned} &W_\mu=\tilde{J}_{\mu\nu}p_\nu=p_\nu\tilde{J}_{\mu\nu}=\text{const.},\\ &(W_\mu p_\mu=p_\mu W_\mu=0) \end{aligned} \tag{47}$$

or

$$w_\mu=\frac{1}{m}\tilde{J}_{\mu\nu}p_\nu=\text{const.}, \tag{47'}$$

provided that $m\neq 0$. Here the tilde signifies the dual for a skew-tensor, such that

$$\tilde{J}_{\mu\nu}=\frac{1}{2i}\varepsilon_{\mu\nu\kappa\lambda}J_{\kappa\lambda},$$

with the alternating symbol $\varepsilon_{\mu\nu\kappa\lambda}$. The commutation rules for W_μ, which are derived from (28)–(30), read

$$\begin{cases} [W_\mu, p_\nu]=0, & (48) \\ [J_{\mu\nu}, W_\rho]=i\hbar\delta_{\rho[\mu}W_{\nu]}, & (49) \\ [W_\mu, W_\nu]=\hbar\varepsilon_{\mu\nu\kappa\lambda}W_\kappa p_\lambda. & (50) \end{cases}$$

As is known, w_μ gives the covariant definition[6] of the spin of the total system for case* $m\neq 0$, since it is independent of the origin (cf. Eq. (48)), and in the *II*-frame its space components coincide with the angular momentum components while its time component vanishes:

$$w_\mu^{(\pi)}=(J_{23}^{(\pi)}, J_{31}^{(\pi)}, J_{12}^{(\pi)}, 0).$$

Indeed it is well-known that (4′) and

$$\Sigma\equiv w_\mu{}^2=\text{const.}, \qquad (51)$$

represents the two group-invariants of the inhomogeneous Lorentz group generated by the basic quantities p_μ and $J_{\mu\nu}$, and also that Σ takes integer and half-integer eigenvalues:

$$\hbar^2\sigma(\sigma+1), \qquad \sigma=0, \frac{1}{2}, 1, \cdots.$$

Generally, for the product of non-commuting quantities we need to take a symmetrized one to have the result self-adjoint. The W_μ of (47) is a self-adjoint vector since $[\tilde{J}_{\mu\nu}, p_\nu]=0$ by (29).

3b. We have seen the existence of rest mass and spin as dynamical quantities, which are particle quantities in our theory. However, in order to attain a unified theory of elementary particles, the problem is immediately raised whether we can really find out a point-like system which has, besides rest mass and spin, just a set of additional invariant conserved quantities $T^{(1)}$, $\cdots$, $T^{(\nu)}$ that are identifiable with various known intrinsic properties of elementary particles (isospin, strangeness, etc.), so that the quantum states of the system may just be classified in terms of a set of quantum numbers taken by those quantities.

Now in order that such identification be possible for certain dynamical quantities, they must satisfy the following general requirements:

(i) They must be self-adjoint.

(ii) They must be *invariant quantities* (under Lorentz transformations and translations), being commutable with all p_μ and $J_{\mu\nu}$ components. Thus, for instance, $J_{\mu\nu}^2$ and $J_{\mu\nu}\tilde{J}_{\mu\nu}$ are not such quantities since they depend on the origin, being non-invariant under displacement. Since we know that we can form only two such invariants (4′) and (51) as far as we use the basic dynamical quantities p_μ and $J_{\mu\nu}$ alone, the other invariant quantities, if any, must be constructed

* The case with $m=0$ needs special consideration.

by using the kinematical variables of the system such as v_μ and ξ_α as well as p_μ and $J_{\mu\nu}$.

(iii) They must be conserved at least for closed system. We are calling the quantities satisfying the conditions (ii) and (iii) *invariant conserved quantities*. The existence of such quantities, besides rest mass and spin, is made possible if the configuration and dynamics of the system involve any "internal" symmetry property independent of the inhomogeneous Lorentz transformations.

(iv) They must satisfy due commutation rules. In particular the quantities $T^{(r)}$, $(r=1, 2, 3)$, to be identified with isospin components must satisfy $[T^{(r)}, T^{(s)}]=i\varepsilon_{rst}T^{(t)}$, and those corresponding to hypercharge and baryon number must commute with each other and with $T^{(r)}$. (They, of course, commute with p_μ and w_μ by the requirement of (ii)).

(v) By the above commutation rules the isospin certainly takes integer and half-integer eigenvalues, but it is further required that the other quantities also have discrete eigenvalues. In particular the quantities corresponding to hypercharge and baryon number need to take eigenvalues of preferably limited integer values, and the eigenvalues of the rest mass m need to agree qualitatively with the empirical mass spectrum of elementary particles.

§ 4. Dynamical quantities of point-like systems

In this section we shall further analyse the general properties of dynamical quantities of point-like systems. The analysis in the velocity operator will be given separately in the next section.

The defining property of point-like system was the existence of the position coordinate x_μ as kinematical variables satisfying

$$[x_\mu, x_\nu]=0, \tag{1}$$

and this led to the identity of the momentum-energy vector of the system with the canonical momentum conjugate to x_μ:

$$[x_\mu, p_\nu]=i\hbar\delta_{\mu\nu}, \tag{2}$$

as far as the system is closed from external interactions. By point-like system we are considering a system of which spatial extension is really reduced to a single point x_μ. However, such coordinates x_μ under (1) exist, e. g. for a bi-local system,[1] and so the arguments in this section may formally be applicable to such systems also,* in so far as interactions are disregarded.

4a. The immediate consequence of the above basic relations, (1) and (2), is the existence of the internal angular momentum skew-tensor $S_{\mu\nu}$, as is defined by

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* Compare, e.g. with D. R. Yennie, Phys. Rev. **85** (1952), 877.

$$J_{\mu\nu}=S_{\mu\nu}+x_{[\mu}p_{\nu]}=S_{\mu\nu}-p_{[\mu}x_{\nu]}=\text{const.}, \tag{52}$$

which satisfies the commutation rules:

$$[S_{\mu\nu}, p_\rho]=0, \quad [S_{\mu\nu}, x_\rho]=0, \tag{53}$$

$$[S_{\mu\nu}, S_{\rho\sigma}]=i\hbar(\delta_{\mu[\rho}S_{\nu\sigma]}+\delta_{\nu[\sigma}S_{\mu\rho]}), \tag{54}$$

as the result of (28)–(30), (1) and (2). Eqs. (53) indicate that $S_{\mu\nu}$'s are independent of x_μ and p_μ and are therefore functions of internal degrees of freedom ξ_α only. We shall call such quantities *proper internal quantities.**

Furthermore, (53) and (54) indicate that $S_{\mu\nu}$ represent the generators of homogeneous Lorentz transformations for any proper internal tensor variable. Thus, noting (34), for a proper internal scalar quantity s we have

$$[J_{\mu\nu}, s]=[S_{\mu\nu}, s]=0, \tag{55}$$

and for any possible proper internal vector quantity ξ_μ,

$$[J_{\mu\nu}, \xi_\rho]=[S_{\mu\nu}, \xi_\rho]=i\hbar\,\delta_{\rho[\mu}\xi_{\nu]}. \tag{56}$$

Eq. (52) gives the covariant separation of $J_{\mu\nu}$ into mutually commutable parts, the orbital part $x_{[\mu}p_{\nu]}$ and the internal part $S_{\mu\nu}$. Each part, however, is not separately conserved in general.** Also $S_{\mu\nu}$ must be a self-adjoint tensor.

We observed that the fundamental commutation relations for a point-like system split into the two independent sets, the canonical commutation rules (28), (1), and (2) for external quantities (x_μ, p_μ), and the commutation rules (54) and (56) for internal quantities $(S_{\mu\nu}, \xi_\mu)$. Thus the problem of setting up a new point-like system may be regarded as finding a new solution of (54) and (56) by constructing $S_{\mu\nu}$ as suitable functions of internal variables ξ_α.

We here write some relations following from (54) and (56) for future references:

$$\left.\begin{aligned}
&[S_{\mu\nu}, \widetilde{S}_{\rho\sigma}]=\hbar\,\varepsilon_{\mu\nu\kappa\lambda}(\delta_{\kappa\rho}S_{\lambda\sigma}+\delta_{\lambda\sigma}S_{\kappa\rho}),\\
&[\widetilde{S}_{\mu\nu}, \widetilde{S}_{\rho\sigma}]=-i\hbar(\delta_{\mu[\rho}S_{\nu\sigma]}+\delta_{\nu[\sigma}S_{\mu\rho]}),\\
&[S_{\mu\nu}, S_{\mu\lambda}]=2i\hbar S_{\nu\lambda}, \qquad [S_{\mu\nu}, \widetilde{S}_{\mu\lambda}]=2i\hbar\widetilde{S}_{\nu\lambda},\\
&[S_{\mu\nu}, \widetilde{S}_{\mu\nu}]=0 \text{ (N. S.***)}\ ; \ [S_{ij}, S_{kl}]=0. \qquad (i\neq k,\ j\neq k)
\end{aligned}\right\} \tag{57}$$

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* A *proper* internal quantity commutes with both p_μ and x_μ but can be a tensor or spinor quantity or an even more complicated parameter (such as an Eulerian angle). It must not be confused with the notion of the *invariant quantity* previously mentioned in (ii) in *3b*. The latter is externally invariant, i.e. independent of translations and Lorentz transformations and may therefore be considered as *internal* quantity of the system, but it need not commute with x_μ.

** In fact it is always possible to split the tensor $J_{\mu\nu}$ into the orbital part and spin part for any relativistic dynamical system. The main feature for the case of point-like systems is rather the fact that they are particle quantities.

*** N. S. indicates that summation convention is not applied to repeated indices.

$$\left.\begin{aligned}&[\widetilde{S}_{\mu\nu}, \xi_\rho]=\hbar\varepsilon_{\mu\nu\rho\sigma}\xi_\sigma, \quad [\widetilde{S}_{\mu\nu}, \xi_\nu]=0, \ (\text{N. S.})\\&[S_{\mu\nu}, \xi_\nu]=-3i\hbar\xi_\mu.\end{aligned}\right\} \tag{58}$$

4b. Although we eventually adopt a system which has *originally* null rest mass, it takes in ordinary states non-zero rest mass in virtue of its internal motion. For simplicity* we shall therefore consider states with $m\neq 0$. We can then employ the spin pseudovector w_μ, which is now written

$$w_\mu=\frac{1}{m}\widetilde{S}_{\mu\nu}p_\nu=\text{const.} \tag{47''}$$

This contains two mutually commuting quantities. One is its magnitude (51), which is written

$$\Sigma=w_\mu{}^2=\frac{1}{m^2}(\widetilde{S}_{\mu\nu}p_\nu)^2, \tag{51'}$$

while the other may be taken to be either the fourth component:

$$w_0=w_4/i=\sum_{\text{cyc}} S_{ij}p_k/m$$

representing the helicity, or else the third component in the Π-frame:

$$w_3^{(\Pi)}=S_{12}^{(\Pi)}.$$

In any case the latter depends on the reference frame employed** and does not correspond to an intrinsic property of elementary particles. It means a property to distinguish between different states of the same elementary particle, which just works to specify the space-quantization of the internal motion.

As stated before the rest mass and the magnitude of spin, (4′) and (51′), are two invariant conserved quantities commuting with all p_μ and $J_{\mu\nu}$. The rest mass commutes furthermore with $S_{\mu\nu}$. However, the fact that m is a dynamical quantity and not a mere c-number appears in the fact that it does not commute with x_μ. We obtain

$$\left.\begin{aligned}&[P, x_\mu]=2i\hbar p_\mu, \quad [m, x_\mu]=\frac{i\hbar}{m}p_\mu,\\&[m, x_\mu]^2=\hbar^2,\end{aligned}\right\} \tag{59}$$

which imply uncertainty relations concerning the rest mass:

$$\Delta m\,\Delta x_k\gtrsim\frac{h|\bar{p}_k|}{\bar{m}c^2}, \quad \Delta m\,\Delta t\gtrsim\frac{h\bar{E}}{\bar{m}c^4}, \tag{60}$$

(where we have restored c.) These relations indicate that a sharp localization

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* Most of the relations to be derived below can be easily rewritten in the forms valid for the case $m=0$ included, if we employ W_μ instead of w_μ, and R_μ instead of r_μ (cf. *4d.*).

** As is known, helicity becomes independent of the frame for the special case of neutrino.

of the particle in space or in time not only accompanies the well-known indeterminacy of its momentum or energy but also induces superposition of states of different rest masses. Especially the former relation of (60) means that the rest mass of the particle, when running nearly with the light velocity, becomes indeterminate to an extent comparable with its value ($\Delta m \sim \bar{m}$) if it is localized to within a region of its compton wavelength. (The other unusual consequence is that under external interaction the rest mass is no longer conserved in general as we shall discuss later, but this effect will not necessarily conflict with experimental evidences either).

On the other hand Σ commutes neither with x_μ nor with $S_{\mu\nu}$ in general as we shall see presently.

4c. Next we consider the scalar and pseudoscalar

$$\left.\begin{aligned} \phi &= \frac{1}{4} S_{\mu\nu}^2 = -\frac{1}{4} \tilde{S}_{\nu\mu}^2, \\ \chi &= \frac{1}{4} S_{\mu\nu} \tilde{S}_{\mu\nu} = S_{23} S_{10} + S_{31} S_{20} + S_{12} S_{30}. \end{aligned}\right\} \tag{61}$$

(Note that χ is self-adjoint owing to the last equation of (57).) They are the group-invariants of the *homogeneous* Lorentz transformations for proper internal variables, hence

$$[S_{\mu\nu}, \phi] = 0, \quad [S_{\mu\nu}, \chi] = 0. \tag{62}$$

Since they are proper internal variables themselves, they commute with all of $(x_\mu, p_\mu, J_{\mu\nu}, S_{\mu\nu})$.

They do not, however, commute with any proper internal vector quantity ξ_μ, since (56) yields

$$[\phi, \xi_\mu] = \frac{i\hbar}{2} \{S_{\mu\nu}, \xi_\nu\}, \tag{63}$$

$$[\chi, \xi_\mu] = i\hbar \tilde{S}_{\mu\nu} \xi_\nu = i\hbar \xi_\nu \tilde{S}_{\mu\nu}. \tag{64}$$

As we shall see this fact is related to the fact that ϕ and χ are not constants of motion in general.

For the $S_{\mu\nu}$ obeying (54) we have further two important formulas*:

$$S_{\mu\rho} S_{\nu\rho} - \tilde{S}_{\mu\rho} \tilde{S}_{\nu\rho} = 2(\delta_{\mu\nu} \phi + i\hbar S_{\mu\nu}), \tag{65}$$

$$S_{\mu\rho} \tilde{S}_{\nu\rho} = \tilde{S}_{\mu\rho} S_{\nu\rho} = \delta_{\mu\nu} \chi + i\hbar \tilde{S}_{\mu\nu}. \tag{66}$$

Using the former, Σ is expressed as

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* For instance, to derive (65) we use the general formula

$$A_{\mu\rho} A_{\nu\rho} - \tilde{A}_{\nu\rho} \tilde{A}_{\mu\rho} = \frac{1}{2} \delta_{\mu\nu} A_{\rho\sigma}^2,$$

for an arbitrary skew-tensor $A_{\mu\nu}$, and the third equation of (57).

$$\Sigma=\frac{p_\rho p_\sigma}{2m^2}\{\widetilde{S}_{\mu\rho},\ \widetilde{S}_{\mu\sigma}\}=2\phi+\frac{p_\rho p_\sigma}{2m^2}\{S_{\mu\rho},\ S_{\mu\sigma}\}. \tag{67}$$

Also, from $S_{\mu\nu}$ we may construct further scalars:

$$\begin{cases}\phi^{(3)}=\dfrac{1}{4}S_{\mu\nu}S_{\nu\lambda}S_{\lambda\mu},\\ \phi^{(4)}=\dfrac{1}{4}S_{\mu\nu}S_{\nu\kappa}S_{\kappa\lambda}S_{\lambda\mu},\quad \text{etc.}\end{cases}$$

By this notation, $\phi=-\phi^{(2)}$. All those $\phi^{(n)}$ are proper internal scalar quantities, hence $[\phi^{(n)},\ S_{\mu\nu}]=0$.

Since, however, we know that there exist only two group-invariants, ϕ and χ, corresponding to the generators $S_{\mu\nu}$, all $\phi^{(n)}$ must be reduced to ϕ and χ. In fact, using (65) and (66), we get the identities:

$$\begin{cases}\phi^{(3)}=i\hbar\phi,\\ \phi^{(4)}=\chi^2+2\phi^2+\hbar^2\phi,\quad \text{etc.}\end{cases} \tag{68}$$

4d. We now introduce the vector

$$R_\mu\equiv S_{\mu\nu}p_\nu,$$

or, for case $m\neq 0$,

$$r_\mu=\frac{1}{m^2}S_{\mu\nu}p_\nu,\quad (r_\mu p_\mu=0)$$

which represents the radius vector of the Zitterbewegung as we shall see in *5b*. The commutation rules are derived to be

$$\begin{cases}[r_\mu,\ r_\nu]=-\dfrac{\hbar}{m^3}\varepsilon_{\mu\nu\kappa\lambda}w_\kappa p_\lambda, & (69)\\ [w_\mu,\ r_\nu]=\dfrac{\hbar}{m}\varepsilon_{\mu\nu\kappa\lambda}r_\kappa p_\lambda, & (70)\end{cases}$$

which reduce in the *II*-frame to the well-known commutation rules for S_{ij} and S_{k0}.

In terms of w_μ and r_μ the skew-tensor $S_{\mu\nu}$ is decomposed in the following form:[4]

$$S_{\mu\nu}=-\frac{i}{m}\varepsilon_{\mu\nu\kappa\lambda}w_\kappa p_\lambda-r_{[\mu}p_{\nu]}. \tag{71}$$

Thus the set of quantities $(p_\mu,\ S_{\mu\nu})$ is equivalent to the set of vectors:

$$(p_\mu,\ w_\mu,\ r_\mu), \tag{72}$$

where both w_μ and r_μ are "space-like" vectors $(w_\mu p_\mu=r_\mu p_\mu=0)$.

In (71) the first term is the constant part due to the spin w_μ, and the

second term is the fluctuating part which has the form of an orbital angular momentum tensor with the radius vector r_μ.

Using (71) the conservation law for $J_{\mu\nu}$ is expressed in the form

$$J_{\mu\nu}=-\frac{i}{m}\varepsilon_{\mu\nu\kappa\lambda}w_\kappa p_\lambda+y_{[\mu}p_{\nu]}=\text{const.}, \tag{73}$$

with

$$y_\mu=x_\mu-r_\mu,\ ([y_\mu, p_\nu]=i\hbar\delta_{\mu\nu})$$

where both terms in the left side of (73) are separately conserved. The conserving "orbital" part $y_{[\mu}p_{\nu]}$ is defined not with the particle position x_μ but with the proper center of mass* y_μ of the system, which is the position that excludes from x_μ the Zitterbewegung r_μ. Taken in the *II*-frame, the space components of (73) are nothing but $w_k^{(II)}=\text{const.}$ while the time components are

$$y_k^{(II)}=x_k^{(II)}-r_k^{(II)}=\text{const.}$$

This corresponds to the relation usually known as the conservation of center of mass (cf. Eq. (95)). It must be noted that since we are considering in the *II*-frame where p_k's have sharp values (i. e. zero), $y_k^{(II)}$'s are completely indeterminate. More important is the fact that *the components of* r_μ (*and so of* y_μ) *are not mutually commuting in contrast with those of* x_μ. We have the simple relation:

$$[y_\mu, y_\nu]=-[r_\mu, r_\nu]=\frac{1}{m^2}[w_\mu, w_\nu]. \tag{74}$$

Corresponding to the decomposition (71) the scalars ϕ and χ are expressed as

$$\phi=\frac{1}{2}(\Sigma-m^2A), \tag{75}$$

$$\chi=-mw_\mu r_\mu=-mr_\mu w_\mu, \tag{76}$$

where A denotes the magnitude of Zitterbewegung:

$$A\equiv r_\mu^2. \tag{77}$$

It is worthwhile to note that the minus sign appearing in (75) and (76) is due to the Lorentz metric. The pseudoscalar χ, (76), represents the non-orthogonality between spin and Zitterbewegung and we called it internal chirality[4]. The squared radius of the Zitterbewegung, (77), will also play an important role, and obeys

* The proper center of mass is such point in reference to which the total angular momentum tensor $J_{\mu\nu}'$ satisfies $J_{\mu\nu}'\,p_\nu=0$. The above y_μ really satisfies this condition since

$$J_{\mu\nu}'=J_{\mu\nu}-y_{[\mu}p_{\nu]}=-\frac{i}{m}\varepsilon_{\mu\nu\kappa\lambda}w_\kappa p_\lambda.$$

$$[A, r_\mu]=-\frac{2\hbar}{m^3}\varepsilon_{\mu\nu\kappa\lambda}p_\kappa w_\lambda r_\nu+\frac{2\hbar^2}{m^2}r_\mu, \qquad [A, w_\mu]=0. \tag{78}$$

4e. Starting from the variables $(S_{\mu\nu}, p_\mu)$, i.e. (w_μ, r_μ, p_μ), we thus have the commuting set of four independent invariant quantities:

$$(P, \Sigma, \phi, \chi), \tag{79}$$

taking account of the identical relations $p_\mu w_\mu=0$, $p_\mu r_\mu=0$, (75) and (76). In (79) ϕ may be replaced by A. Of those four invariant quantities, P and Σ are conserved, but ϕ (or A) and χ in general are not so.

Finally we consider the commutation relation between spin and position coordinate. First we get

$$[w_\mu, x_\nu]=-\frac{i\hbar}{m}\widetilde{S}_{\mu\rho}\left(\delta_{\rho\nu}+\frac{1}{m^2}p_\rho p_\nu\right)=\frac{1}{m}p_\mu w_\nu+i\varepsilon_{\mu\nu\kappa\lambda}r_\kappa p_\lambda, \tag{80}$$

taking account of (59). This leads to

$$[\Sigma, x_\mu]=-\frac{i\hbar}{m^2}\{\widetilde{S}_{\nu\lambda}, \widetilde{S}_{\nu\rho}\}p_\lambda\left(\delta_{\rho\mu}+\frac{1}{m^2}p_\rho p_\mu\right), \tag{81}$$

which contains

$$[\Sigma, x_\rho p_\rho]=0.$$

If in particular the condition

$$\{S_{\mu\rho}, S_{\nu\rho}\}=\delta_{\mu\nu}\cdot(\text{scalar quantity}) \tag{82}$$

be satisfied, it must be

$$\{S_{\mu\rho}, S_{\nu\rho}\}=-\{\widetilde{S}_{\mu\rho}, \widetilde{S}_{\nu\rho}\}=2\delta_{\mu\nu}\phi, \tag{82'}*$$

by noting the formula (65). In this case, (81) reduces to

$$[\Sigma, x_\mu]=0. \tag{83}$$

This fact is evident because under the condition (82), Eq. (67) reduces to

$$\Sigma=\phi, \tag{84}$$

i.e. Σ becomes a proper internal quantity ϕ, independent of p_μ. This means, conversely speaking, that under this special condition ϕ coincides with the magnitude of spin and hence is conserved. At the same time we have the relation

$$m^2A=-\Sigma=-\phi. \tag{85}$$

The condition (82) represents a quite particular one, but we shall see that it is satisfied in the case of Dirac particle as it should (cf. Part II).

Eqs. (81) and (83) show that a sharp localization of the particle generally accompanies an indeterminacy of the magnitude of spin, Σ, but for a special

* This relation is equivalent to $\chi S_{\mu\nu}=\phi\widetilde{S}_{\mu\nu}$.

system satisfying (82) Σ remains to be a definite value under a sharp localization.

§ 5. Velocity operator

Now we enter into the analysis of the properties directly connected with the time evolution of the system. We consider in the "Heisenberg picture", where every dynamical quantity develops with τ.

5a. Taking the derivative of any quantity F with respect to τ we get $\dot{F}$, but essentially new quantity thus obtained is the instantaneous 4-velocity corresponding to x_μ:

$$\frac{dx_\mu}{d\tau}=\dot{x}_\mu=v_\mu \tag{25'}$$

only, because the derivation of p_μ gives

$$\dot{p}_\mu=0, \tag{38a}$$

while the derivative of $S_{\mu\nu}$ is related to the other quantities by

$$\dot{S}_{\mu\nu}=-v_{[\mu}p_{\nu]}=p_{[\mu}v_{\nu]}, \tag{86}$$

which is the re-expression of the basic conservation law (38b), taking account of (25') and (38a). On the other hand, the derivation of (2) gives

$$[v_\mu, p_\nu]=0, \quad [v_\mu, P]=0, \tag{87}$$

and

$$[\dot{v}_\mu, p_\nu]=0, \tag{88}$$

noting (38a), while the derivation of (1) gives

$$[v_\mu, x_\nu]=[v_\nu, x_\mu]\equiv h_{\mu\nu}, \tag{89}$$

and

$$-[\dot{v}_\mu, x_\nu]=[v_\mu, v_\nu]\equiv ik_{\mu\nu}. \tag{90}$$

The relation (89) means that $h_{\mu\nu}$ is a symmetric tensor, while $k_{\mu\nu}$ in (90) is a skew-symmetric tensor. As stated in the Introduction, whether $h_{\mu\nu}$ vanishes or not is the criterion (C), and whether $k_{\mu\nu}$ vanishes or not is the criterion (B), for classifying point-like systems.

Adding the new variables v_μ, we thus have the set of basic variables (7) for any point-like system. Among them the set of quantities (6) has universal commutation rules among themselves as we have seen in § 4, while those relating to v_μ are not definite in advance apart from (87). Also, there is arbitrariness in the definition of τ, since any monotonously increasing function of τ, $\tau'=f(\tau)$, may equally be employed as the ordering parameter. As stated before we must postulate certain inhomogeneous conditions on v_μ in order to define the kinematical properties of v_μ sufficiently as well as to suppress the

arbitrariness of τ, and those conditions work to characterize point-like systems into various types. Here we consider the properties of v_μ, taking the criterion (A) together with (B) into consideration, (while the criterion (C) is considered in *5c* and thereafter).

(i) First we assume $\rho \equiv -v_\mu^2$ is an absolute invariant, i.e. it commutes with v_μ and all other quantities. We have called such a system "of the normal class", which includes many familiar systems.* Then we can practically suppress the arbitrariness in the definition of τ by requiring that

$$\dot{\rho}=0, \quad \text{i.e.} \quad \rho=\text{const.} \tag{91}$$

Since ρ is dimensionless it thus takes numerical constant eigenvalues, but their absolute values have no physical significance and the only meaningful distinction is that between the cases $\rho \neq 0$ and $\rho=0$. For the former case we may always normalize v_μ (i.e. τ) such that

$$v_\mu^2=-\rho=-1. \tag{92}^{**}$$

In other words, for normal class ρ is regarded to obey the idempotent condition

$$\rho^2=\rho, \qquad (\rho^\dagger=\rho) \tag{93}$$

taking eigenvalues 1 and 0. We get from (91)

$$\{v_\mu, \dot{v}_\mu\}=0. \tag{94}$$

The case $\rho=0$ is quite different from the case $\rho=1$. It is to be noted that, since in our theory v_μ and p_μ are essentially different, the condition of zero rest mass, $p_\mu^2=0$, and the present condition, $v_\mu^2=0$, are also different in general. Thus we may have states in which $\rho=0$ but $m \neq 0$. If we consider about the case of "classical model" ($[v_\mu, v_\nu]=0$), $\rho=1$ means a state where particle has the unitary velocity (cf. *2e*), while $\rho=0$ means a state where particle runs always with the light velocity.*** (This refers to the instantaneous velocity including the Zitterbewegung; the velocity of the mean motion is still less than c in general.) We anticipate that two distinct families of states, $\rho=1$ and $\rho=0$, for a system of normal class correspond to baryons and leptons, respectively. Then, speaking for the case of classical model, the conservations of baryon and lepton numbers correspond just to the impossibility of transitions between a state where the instantaneous velocity is less than c and a state where it is c.

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* Relativistic rotator[4] as well as the Dirac particle are examples of normal class, but not the Kemmer particle (cf. Part II).

** For the case of "non-classical model" the normalization (92) is not necessarily the most suitable one, and, for example, for Dirac particle we take $\rho=4$ (cf. *2e*).

*** In pure classical theory such kind of particle was considered by Weyssenhoff.[8]

(ii) Next, we consider systems for which ρ is not an absolute invariant. Then we cannot impose the condition (91) in advance. However, we shall see that there generally exists an absolute invariant ρ', which is intimately related to $\rho=-v_\mu^2$. This ρ' may then be regarded as the operator corresponding to baryon number (cf. *6f.* (ii)).

5b. We are now in a position to show that the general aspect of the motion of any point-like system is governed by the conservation equations (38a) and (86). Taking account of (47″) and (71), Eq. (86) is equivalent to $w_\mu=$const. (Eq. (47″)) together with

$$\dot{y}_\mu=\frac{\varepsilon}{m^2}p_\mu, \quad \text{i.e.} \quad \dot{r}_\mu=v_\mu-\frac{\varepsilon}{m^2}p_\mu, \tag{95}$$

where

$$\varepsilon\equiv -v_\mu p_\mu \tag{96}$$

is connected with $\dot{r}_\mu$ through the relation

$$\dot{r}_\mu^{\,2}=\frac{\varepsilon^2}{m^2}-\rho, \tag{97}$$

which is a consequence of (95) itself. The ε means the rest mass times the magnitude of the velocity of Zitterbewegung subsisting in the *II*-frame:

$$\varepsilon=mv_0^{(II)},$$

and plays an important physical role as will be seen later. Like Σ, it is an invariant quantity (so $[\varepsilon, p_\mu]=0$, $[\varepsilon, J_{\mu\nu}]=0$) but does not commute with x_μ nor with $S_{\mu\nu}$.

Eq. (95) is expressed in the integrated form:

$$\left.\begin{aligned} &x_\mu=y_\mu+r_\mu, \quad y_\mu=\left(\int^\tau \varepsilon\, d\tau\right)\frac{p_\mu}{m^2}-r_\mu^{\,0}, \\ &\left(r_\mu^{\,0}=\frac{1}{m^2}J_{\mu\nu}p_\nu=\text{const.}\right) \end{aligned}\right\} \tag{98}$$

which directly shows that particle makes Zitterbewegung r_μ around the mean rectilinear motion of the direction of p_μ. The motion of any point-like system thus consists of three parts with mutual correlations: The first is the mean rectilinear translation represented by p_μ, the second is a certain orbital Zitterbewegung represented by r_μ or v_μ, and the third is pure internal motion (rotation, etc.) to be described by the other internal kinematical variables ξ_α (which have not yet appeared explicitly.)

It is important to note that r_μ always exists as dynamical variables except the trivial case of structureless Newtonian particle, since if all r_μ's identically vanish, all w_μ's must also vanish identically because of the commutation rule (69), and we are left with $S_{\mu\nu}=0$. The appearance of Zitterbewegung is thus

a general and necessary consequence due to the coupling between relativity and quantum mechanics,*† and it will play an important role in our theory.

To see the motion more clearly we consider about the expectation values. First, for a "stationary state" satisfying (39), we had $\langle \dot{F} \rangle_\lambda = 0$, (Eq. (40)), where F is any bounded operator that is not explicitly τ-dependent. Applying this to Eq. (95), we get

$$\langle v_\mu \rangle_\lambda = \left\langle \frac{\varepsilon}{m^2} p_\mu \right\rangle_\lambda . \tag{99}$$

This is rewritten, for an eigenstate of p_μ, as

$$\langle v_\mu \rangle_\lambda = \frac{\langle \varepsilon \rangle_\lambda}{m^2} p_\mu , \tag{99'}$$

and in particular in the Π-frame

$$\langle v_k^{(\Pi)} \rangle_\lambda = 0. \tag{99''}$$

To consider for a general state, we construct an intrinsic-time average over an interval T of the expectation value of $\dot{F}$:

$$\frac{1}{T} \int_0^T \psi^* \dot{F} \psi \, d\tau = \frac{1}{T} [\psi^* (F(\tau) - F(0)) \psi].$$

Then for any bounded operator F we have

$$\langle\langle \dot{F} \rangle\rangle_\tau = 0, \tag{100}$$

where $\langle\ \rangle_\tau$ denotes the intrinsic-time average over a sufficiently long interval. Thus for a general state, (99') and (99'') are replaced by

$$\langle\langle v_\mu \rangle\rangle_\tau = \frac{\langle\langle \dot{\varepsilon} \rangle\rangle_\tau}{m^2} p_\mu ,$$

$$\langle\langle v_k^{(\eta)} \rangle\rangle_\tau = 0,$$

respectively. This is a kind of virial theorem indicating that the velocity subsisting in the Π-frame is essentially of quasi-periodical character.

5c. Now we take up the criterion (C) which concerns the commutator $h_{\mu\nu}$ defined in (89). The one is systems of the first kind for which

$$h_{\mu\nu} \equiv [v_\mu, x_\nu] = 0, \tag{101}$$

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* The Zitterbewegung for Dirac particle is merely a special example of this general situation.

† In pure classical theory w_μ need not vanish even if $r_\mu = 0$. This is the case which we called "classical case" previously[4] and is realized by a relativistic rotator as a constant rectilinear translation associated with pure spatial internal rotation in the rest frame of the translational motion but without any Zitterbewegung. Such classical motion does not have good quantum-mechanical correspondence.

while the other is systems of the second kind for which the commutator does not vanish identically: $h_{\mu\nu}\neq 0$. In the latter case, if it belongs to normal class, $h_{\mu\nu}$ needs to satisfy

$$\{h_{\mu\nu}, v_\nu\}=0.$$

A simple example belonging to the second kind (and to the normal class) is given by relativistic Newtonian particle which is defined by imposing a strict connection between p_μ and v_μ, i.e. the colinearity:

$$v_\mu=\nu p_\mu.$$

This corresponds to an invariant hamiltonian *quadratic* in p_μ:

$$H=\frac{\nu}{2}p_\mu{}^2+\mathcal{H},$$

and the corresponding stationary state equation is of Klein-Gordon type.

In the following section we accord our attention to systems of the first kind. This corresponds to invariant hamiltonian *linear* in p_μ and will represent more important systems with internal degrees of freedom.

§ 6. Systems with velocity as internal variables

Hereafter we consider systems of the first kind exclusively, so that the *velocity components commute with position coordinates*:

$$[v_\mu, x_\nu]=0. \tag{101}$$

Then this condition leads to various general and remarkable consequences as we shall now see.

6a. The assumption (101) means that *v_μ's are proper internal variables* (although they represent instantaneous orbital velocity) commuting with all external variables (note (87)).

Thus, according to (56), we have the new basic commutation rule:

$$[S_{\mu\nu}, v_\rho]=i\hbar\delta_{\rho[\mu}v_{\nu]}. \tag{102}*$$

We also note the following relations derivable from (102):

$$\left.\begin{aligned}&[\tilde{S}_{\mu\nu}, v_\rho]=\hbar\varepsilon_{\mu\nu\rho\sigma}v_\sigma,\quad [\tilde{S}_{\mu\kappa}, v_\nu]=0,\ \text{(N. S.)}\\&[S_{\mu\nu}, v_\nu]=-3i\hbar v_\mu.\end{aligned}\right\} \tag{103}$$

Since we now have the set of proper internal variables:

$$(S_{\mu\nu}, v_\mu), \tag{104}$$

we can construct, by their combination, two proper internal vectors: the pseudovector

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* This can be verified directly by taking derivative of (53) and using (86), (2), and (101).

$$H_\mu \equiv \tilde{S}_{\mu\nu} v_\nu = v_\nu \tilde{S}_{\mu\nu}, \tag{105}$$

and the vector

$$K_\mu \equiv \frac{1}{2}\{S_{\mu\nu}, v_\nu\} = S_{\mu\nu} v_\nu + \frac{3}{2} i\hbar v_\mu = v_\nu S_{\mu\nu} - \frac{3}{2} i\hbar v_\mu. \tag{106}$$

Then, like v_μ itself, each of them must obey, according to (56), the same type of commutation rules as (102) :

$$\left.\begin{aligned} [S_{\mu\nu}, H_\rho] &= i\hbar \delta_{\rho[\mu} H_{\nu]}, \\ [S_{\mu\nu}, K_\rho] &= i\hbar \delta_{\rho[\mu} K_{\nu]}. \end{aligned}\right\} \tag{107}$$

These H_μ and K_μ are self-adjoint vectors and satisfy

$$\{H_\mu, v_\mu\} = 0, \quad \{K_\mu, v_\mu\} = 0, \tag{108}$$

meaning that they are space-like in the classical limit.

Using the above relations we can verify that $S_{\mu\nu}$ is decomposed in the form

$$\rho S_{\mu\nu} = \frac{1}{2i} \varepsilon_{\mu\nu\kappa\lambda} \{H_\kappa, v_\lambda\} - \frac{1}{2}\{K_{[\mu}, v_{\nu]}\} - \frac{i\hbar}{2}[v_\mu, v_\nu]. \tag{109}$$

Thus, unless $\rho = 0$, the set (104) is equivalent to the set of three proper internal vectors :

$$(v_\mu, H_\mu, K_\mu), \tag{110}$$

obeying the relations (108). This is the second triple of vectors which is important for any system under (101), as well as the first triple (72), and brings to light a new aspect different from that already brought by means of the first triple in § *4*. The physical meaning of H_μ and K_μ will become clearer later.

The commutation rules between one of the vectors (110) and ϕ or χ are given by the general formula (63) and (64). First we obtain

$$[\phi, v_\mu] = i\hbar K_\mu, \tag{111}$$

$$[\chi, v_\mu] = i\hbar H_\mu, \tag{112}$$

indicating that ϕ is the generator of the transformation turning v_μ to the direction K_μ "normal" to v_μ, keeping $(S_{\mu\nu}, p_\mu)$ invariant, and the analogous meaning for χ.

Here we insert the formulas

$$\left.\begin{aligned} -\frac{1}{2}\{S_{\mu\nu}, H_\nu\} &= \tilde{S}_{\mu\nu} K_\nu = K_\nu \tilde{S}_{\mu\nu} \\ &= -\frac{1}{2}\{\chi, v_\mu\} = -\chi v_\mu + \frac{i\hbar}{2} H_\mu, \\ S_{\mu\nu} H_\nu &= -(\chi v_\mu + i\hbar H_\mu), \end{aligned}\right\} \tag{113}$$

(which are obtained by using (66), (107), and (112)). Utilizing (113), Eqs. (63) and (64) give

$$[\phi, H_\mu]=[\chi, K_\mu]=-\frac{i\hbar}{2}\{\chi, v_\mu\}. \tag{114}$$

Further we get the relation

$$[H_\mu, \chi]+i\hbar\{\phi, v_\mu\}=[K_\mu, \phi]-\frac{3}{4}i\hbar^3 v_\mu=\frac{i\hbar}{4}\{\{S_{\mu\rho}S_{\lambda\rho}\}v_\lambda\}-\frac{3}{2}i\hbar^3 v_\mu. \tag{115}$$

The commutation rule between w_μ(or r_μ) and any proper internal vector ξ_μ easily follow from (56). Thus we get

$$\begin{cases} [w_\mu, v_\nu]=-\dfrac{\hbar}{m}\varepsilon_{\mu\nu\rho\sigma}p_\rho v_\sigma=-\dfrac{i\hbar}{m}(\tilde{S}_{\mu\nu})^{\cdot}, & (116)\\ [w_\mu, H_\nu]=-\dfrac{\hbar}{m}\varepsilon_{\mu\nu\rho\sigma}p_\rho H_\sigma, & (117)\\ [r_\mu, v_\nu]=-\dfrac{i\hbar}{m^2}(\delta_{\mu\nu}\varepsilon+v_\mu p_\nu), & (118)\\ [r_\mu, H_\nu]=\dfrac{i\hbar}{m^2}(\delta_{\mu\nu}p_\rho H_\rho-H_\mu p_\nu), & (119) \end{cases}$$

and analogous relations for $[w_\mu, K_\nu]$ and $[r_\mu, K_\nu]$. Especially, (116) and (118) supply

$$\begin{cases} [w_\mu, \varepsilon]=0, & (120)\\ [r_\mu, \varepsilon]=-i\hbar\left(v_\mu-\dfrac{\varepsilon}{m^2}p_\mu\right)=-i\hbar\dot{r}_\mu. & (121) \end{cases}$$

6b. The next important fact for systems of the first kind is that $-\varepsilon$ works as the generator of the intrinsic-time displacement for any quantity *that is function of* x_μ, p_μ *and* $S_{\mu\nu}$ *only*. This is because, as was stated in § *1*, the invariant hamiltonian of the system is linear in p_μ, taking the general form

$$H=-\varepsilon+\mathcal{H}, \tag{9}$$

where $\mathcal{H}$ is a scalar function of proper internal variables alone and hence it commutes with x_μ, p_μ, and $S_{\mu\nu}$, due to (55). The preceding equations (120) and (121) are simply examples of this general theorem (compare with (47″) and (95)). Also we really reproduce, e.g.

$$\frac{i}{\hbar}[S_{\mu\nu}, \varepsilon]=p_{[\mu}v_{\nu]}=\dot{S}_{\mu\nu}, \tag{122}$$

due to (102) and (86), and

$$\frac{i}{\hbar}[x_\mu, \varepsilon]=v_\mu=\dot{x}_\mu. \tag{123}$$

Noting (111) and (112), we also get

$$\dot{\phi}=-\frac{m^2}{2}\dot{A}=\frac{i}{\hbar}[\phi,\varepsilon]=p_\mu K_\mu=-\frac{m^2}{2}\{v_\mu, r_\mu\}, \tag{124}$$

$$\dot{\chi}=\frac{i}{\hbar}[\chi,\varepsilon]=p_\mu H_\mu=-mv_\mu w_\mu=-mw_\mu v_\mu. \tag{125}$$

It is to be noted that the equation of motion for any function of the variables x_μ, p_μ and $S_{\mu\nu}$ alone gives a relation universal for the first kind systems, such as (120) – (125), because of its commutability with $\mathcal{H}$, and of the general relations (101) and (102). On the contrary, only v_μ among the variables (7) does not necessarily commute with $\mathcal{H}$, and the actual form of its equation of motion:

$$\dot{v}_\mu=\frac{i}{\hbar}[v_\mu, v_\nu]p_\nu+\frac{i}{\hbar}[\mathcal{H}, v_\mu] \tag{126}$$

depends on further characteristics of the system (see Part II).

Finally we write the following commutation rules for future references:

$$[r_\mu, \dot{r}_\nu]=-i\hbar\frac{\varepsilon}{m^2}\left(\delta_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}\right), \tag{127}$$

so that

$$[r_i^{(n)}, \dot{r}_j^{(n)}]=-i\hbar\frac{v_0^{(n)}}{m}\delta_{ij}, \tag{127'}$$

and

$$[A, \dot{r}_\mu]=-\frac{i\hbar}{m^2}\{\varepsilon, r_\mu\}. \tag{128}$$

We also have

$$[\Sigma, v_\mu]=i\hbar\left(-\{\varepsilon, r_\mu\}+\frac{2}{m^2}\dot{\phi}p_\mu+2K_\mu\right), \tag{129}$$

$$[A, v_\mu]=-\frac{i\hbar}{m^2}(\{\varepsilon, r_\mu\}+\dot{A}p_\mu), \tag{130}$$

obtained from (116) and (118). The former indicates that Σ does not generally commute with v_μ either.

6c. Now the set of proper internal variables (104) or (110) allows us to construct, besides ϕ and χ, the following six self-adjoint proper internal scalars:

$$\left.\begin{aligned} &\rho=-v_\mu^2, && H_\mu^2, && K_\mu^2, \\ &H_\mu K_\mu=K_\mu H_\mu, && iH_\mu v_\mu=-iv_\mu H_\mu, && iK_\mu v_\mu=-iv_\mu K_\mu. \end{aligned}\right\} \tag{131}$$

Due to (55) each of (131) must commute with $S_{\mu\nu}$:

$$\left.\begin{aligned}&[S_{\mu\nu}, \rho]=0, \quad [S_{\mu\nu}, \theta]=0, \cdots,\\ &[S_{\mu\nu}, H_\rho K_\rho]=0, \quad [S_{\mu\nu}, H_\rho v_\rho]=0, \cdots,\end{aligned}\right\} \tag{132}$$

where we have introduced the notation

$$\theta \equiv H_\mu^2. \tag{133}$$

Thus each of (131) must also commute with w_μ and r_μ:

$$\left.\begin{aligned}&[w_\mu, \rho]=0, \quad [w_\mu, \theta]=0, \cdots,\\ &[r_\mu, \rho]=0, \quad [r_\mu, \theta]=0, \cdots.\end{aligned}\right\} \tag{134}$$

Of the quantities (131), however, only four are independent, e.g.

$$(\rho, \theta, iH_\mu v_\mu, iK_\mu v_\mu), \tag{135}$$

because we can prove that there hold the two important identical relations:

$$H_\mu K_\mu = K_\mu H_\mu = -\left(\rho\chi + \frac{i\hbar}{2} H_\mu v_\mu\right), \tag{136}$$

$$H_\mu^2 - K_\mu^2 = 2\rho\phi + i\hbar K_\mu v_\mu + \frac{3}{4}\hbar^2\rho. \tag{137}$$

The four quantities of (135) do not necessarily commute with each other. However, if the commutators $[v_\mu, v_\nu]$ are specified, we can generally form of (135) two independent quantities mutually commuting, and the other two of (135) are represented in terms of the former two (together with ϕ and χ) or simply reduce to c-numbers (cf. Part II). We regard those two independent quantities to be ρ and θ. (For normal class ρ always commutes with others.) Thus, starting from the quantities $(S_{\mu\nu}, v_\mu)$ we have the set of four independent invariant quantities which commute with one another and with p_μ and $S_{\mu\nu}$:

$$(\phi, \chi, \rho, \theta). \tag{138}$$

On the other hand, starting from $(S_{\mu\nu}, p_\mu)$, we had the set of four independent invariant quantities (79). Then, combining (79) and (138), we get the *set of six independent invariant quantities commutable with one another*:

$$(P, \Sigma, \rho, \theta, \phi, \chi), \tag{139}$$

which represents the set of invariant quantities to be formed of all basic variables:*

$$(p_\mu, S_{\mu\nu}, v_\mu), \tag{140}$$

without using internal variables ξ_α explicitly.

It remains to consider about the invariant quantities which mix the

* Besides (140) we had the variable x_μ, but this does not contribute in the construction of the invariant quantities because of the requirement of the translational invariance (cf. (ii) of §3*b*).

(p_μ, w_μ, r_μ) system and the (v_μ, H_μ, K_μ) system. The examples of such "mixed" scalar quantities have already occurred. They are ε, $\dot{\phi}$, and $\dot{\chi}$. (see Eqs. (124) and (125)). The remaining mixed scalars are the following three:

$$\left.\begin{aligned} &\Omega = H_\mu w_\mu = w_\mu H_\mu, \\ &\Omega' = \frac{m}{2}\{r_\mu, K_\mu\}, \\ &\Omega'' = K_\mu w_\mu = w_\mu K_\mu = \frac{m}{2}\{r_\mu, H_\mu\}, \end{aligned}\right\} \tag{141}$$

which are all self-adjoint, and especially Ω will play an important physical role.

Those quantities, however, can be represented in terms of

$$(\phi, \chi, \varepsilon), \tag{142}$$

or

$$(\phi, \chi, \dot{\phi}, \dot{\chi}), \tag{142'}$$

in the following form:

$$\Omega = -\frac{1}{m\hbar^2}[\chi[\chi\varepsilon]] = \frac{i}{m\hbar}[\chi, \dot{\chi}], \tag{143a}$$

$$\Omega' = -\frac{1}{m\hbar^2}[\phi[\phi\varepsilon]] = \frac{i}{m\hbar}[\phi, \dot{\phi}], \tag{143b}$$

$$\Omega'' = -\frac{1}{2m}\{\chi, \varepsilon\} = -\frac{1}{m\hbar^2}[\chi[\phi\varepsilon]] = -\frac{1}{m\hbar^2}[\phi[\chi\varepsilon]]. \tag{144}$$

Thus Ω's are commutable with p_μ and w_μ, since all of the quantities (142) are so.

We can furthermore derive the following relation:

$$m(\Omega - \Omega') = \frac{1}{\hbar^2}([\phi[\phi, \varepsilon]] - [\chi[\chi, \varepsilon]]) = \{\phi, \varepsilon\} + \frac{3}{4}\hbar^2\varepsilon, \tag{145}$$

by the aid of the formula (65). Eqs. (144) and (145) mean two non-linear relations between the three quantities of (142). Again, Ω has the following remarkable property:

$$[\Omega, \dot{\phi}] = -\frac{1}{2m}[\varepsilon, \chi^2] = -\frac{i\hbar}{2m}(\chi^2)^{\cdot} = [\Omega'', \chi], \tag{146}$$

which is obtained by (114) and (125).

From the foregoing discussions we see that all mixed invariant quantities can be generated out of the basic quantities (142), and therefore that, if we add

$$(\varepsilon, iH_\mu v_\mu, iK_\mu v_\mu) \tag{147}$$

to the set (139), all invariant quantities constructed of our variables (140) are represented by them.

We can adopt the set of commutable quantities, (139), as part of the quantities to specify eigenstates of the system, but this is not adequate since ϕ and χ are generally not conserved. On the contrary θ is a constant of motion under a certain general assumption (cf. *6f*—(i)). Thus the important problem is to replace ϕ and χ by two independent conserved quantities. To be general we have been developing the theory without entering into the more precise characteristics of the system (i.e. the commutator $[v_\mu, v_\nu]$ and the form of $\mathcal{H}$, or the equation of motion for v_μ), but if they are given we should be able to construct two independent quantities starting from the set (142), in such a way that they commute with θ and with each other and are constants of motion. Evidently, one of them is nothing but the invariant hamiltonian H, while we write the other as Z. Thus we finally get the set of six conserved invariant quantities mutually commuting:

$$(P, \Sigma, \rho, \theta, H, Z). \tag{148}$$

The physical significance of θ, H, and Z will be seen later. It is to be remarked that, speaking most generally, ρ and θ also may not be conserved, and in such a case we may construct four independent conserved invariant quantities, ρ', θ', H and Z, replacing $(\rho, \theta, \phi, \chi)$, so that we have

$$(P, \Sigma, \rho', \theta', H, Z) \tag{148'}$$

in place of (148). Here ρ' and θ' are intimately related to ρ and θ, respectively (cf. Eq. (160)).

In any case the set of those six quantities exhausts possible independent conserved invariant quantities to be obtained by using our basic variables (140) alone. It is possible that some of those conserved quantities degenerate according to the special characteristics of the system. On the other hand, the pure internal variables $\tilde{\xi}_\alpha$ of the system are not necessarily implied by the variables $(S_{\mu\nu}, v_\mu)$ completely, and accordingly it is possible that the system has independent invariant conserved quantities over and above (148') which explicitly depend on $\tilde{\xi}_\alpha$.

6d. To see the relation between H_μ and w_μ, we insert the dual of (71) into (105) and take (125) into account. We get

$$H_\mu = \frac{\varepsilon}{m} w_\mu - \frac{\dot{\chi}}{m^2} p_\mu + \Lambda_\mu, \tag{149}$$

where

$$\Lambda_\mu = -i\varepsilon_{\mu\nu\kappa\lambda} r_\kappa v_\lambda p_\nu = i\varepsilon_{\mu\nu\kappa\lambda} p_\kappa v_\lambda r_\nu = -i\varepsilon_{\mu\nu\kappa\lambda} r_\kappa \dot{r}_\lambda p_\nu, \tag{150}$$

which satisfies

$$\Lambda_\mu p_\mu = 0, \quad \{\Lambda_\mu, v_\mu\} = 0, \quad \{\Lambda_\mu, r_\mu\} = 0. \tag{151}$$

The Λ_μ represents the *kinematical angular momentum* of the Zitterbewegung, since in the *II*-frame

$$\Lambda_i^{(II)}=m\varepsilon_{ijk}r_j^{(II)}v_k^{(II)}, \quad (\Lambda_4^{(II)}=0) \tag{150'}$$

and (149) becomes

$$v_0^{(II)}w_k^{(II)}=H_k^{(II)}-\Lambda_k^{(II)}, \tag{149'}$$

indicating that the spin $w_k^{(II)}$ is the difference of the angular momentum related to pure internal motion, $H_k^{(II)}$, and the kinematical angular momentum of the orbital precession, $\Lambda_k^{(II)}$ (apart from the factor $v_0^{(II)}$). The long-time average of (149) yields

$$\langle H_\mu\rangle_\tau=\frac{\langle\mathcal{E}\rangle_\tau}{m}w_\mu+\langle\Lambda_\mu\rangle_\tau.$$

We have the commutation rule:

$$[\Lambda_\mu, w_\nu]=-m\dot{r}_{[\mu}r_{\nu]}. \tag{152}$$

The Λ_μ does not provide new independent invariant quantities: First the projection of Λ_μ on w_μ, defined by

$$\Lambda_w\equiv\Lambda_\mu w_\mu=w_\mu\Lambda_\mu=-i\varepsilon_{\mu\nu\kappa\lambda}p_\kappa v_\lambda r_\mu w_\nu=-i\varepsilon_{\mu\nu\kappa\lambda}w_\mu r_\kappa\dot{r}_\lambda p_\nu, \tag{153}$$

stands in an intimate connection with Ω defined in (141), since we get from (149):

$$\Omega=\frac{\mathcal{E}}{m}\Sigma+\Lambda_w. \tag{154}$$

Similarly we get

$$\Lambda_\mu H_\mu=\Theta-\frac{\mathcal{E}}{m}\Omega+\frac{1}{m^2}\dot{\chi}^2, \tag{155}$$

$$\Lambda_\mu^2=\Theta+\frac{\mathcal{E}^2}{m^2}\Sigma+\frac{1}{m^2}\dot{\chi}^2-\frac{1}{m}\{\mathcal{E},\Omega\}. \tag{156}$$

These relations indicate that the invariant quantities involving Λ_μ are always reduced to the mixed invariant quantities, Ω and $\dot{\chi}$, already considered. The last relation (156) is especially important as we shall see in Part II.

6e. Finally we briefly mention the Schrödinger equation. It has the general form

$$i\hbar\frac{d\psi}{d\tau}=(v_\mu p_\mu+\mathcal{H})\psi, \tag{10}$$

and in particular the stationary state equation becomes

$$(v_\mu p_\mu+\mathcal{H})\psi=\lambda\psi, \tag{157}$$

or in short

$$\mathcal{E}+\lambda \approx \mathcal{H}. \qquad (157')$$

This equation has similarity with the usual Dirac equation but is more general. A usual wave equation always satisfies, when iterated, the Klein-Gordon equation, $p_\mu{}^2 \approx -m_0{}^2$, meaning that it corresponds to an irreducible representation of the inhomogeneous Lorentz transformations, to describe particles of a unique rest mass m_0. On the other hand our wave equation (157) need not lead to the Klein-Gordon equation by iteration but it is compatible with the latter since H and P commute. A general solution of (10) is a superposition of waves corresponding to different rest mass states, and those possible rest mass values are to be determined by (157). If we classify the solutions of (157) by taking simultaneous eigenstates of all the quantities of (148), they are expected just to represent various possible elementary-particle states.

6f. In this section we have analysed general properties of point-like systems of the first kind. However, (101) does not yet determine the kinematical properties of v_μ completely, and includes different types of systems, to be characterized according to the criteria (A) and (B) and possibly still to other ones (see below). As already mentioned it is most convenient to apply the criterion (B) first so as to divide systems of the first kind into two categories, namely into "classical models":

$$[v_\mu, v_\nu] \equiv ik_{\mu\nu} = 0, \qquad (158)$$

and "non-classical models": $k_{\mu\nu} \neq 0$.

(i) Now for classical models we can consider simultaneous eigenstates of velocity components and so v_μ's are not only internal variables but also the *kinematical variables* describing the internal configuration of the system. This is just part of the general conditions required for a system to be of *realistic* internal structure. Since in this case $[\rho, v_\mu]=0$, it is appropriate to regard ρ as an absolute invariant restricted by $\rho^2=\rho$, namely to restrict the system to normal class (cf. *5a*).

The remarkable fact for the present case is that the quantities $(S_{\mu\nu}, v_\mu)$ exactly satisfy among themselves the commutation rules for the generators of an *inhomogeneous Lorentz group*, taking account of (54), (102), and (158). Thus H_μ plays for the set of quantites $(S_{\mu\nu}, v_\mu)$ the same role as W_μ does for $(J_{\mu\nu}, p_\mu)$, hence takes integer and half-integer eigenvalues again; ρ and Θ represent the two group-invariants of this inhomogeneous Lorentz group generated by $(S_{\mu\nu}, v_\mu)$. However, we then have the general restriction that Σ and Θ take either both integer or both half-integer eigenvalues at the same time.

This kind of systems will be treated more closely in Part II and Part III. (Simple examples are the Hönl-Papapetrou particle and relativistic rotators.)

(ii) On the other hand non-classical models are also important, of which the simplest type is given by assuming Bhabha's relation:

$$[v_\mu, v_\nu] \equiv ik_{\mu\nu} = \frac{i}{\kappa} S_{\mu\nu}, \tag{159}$$

where κ is an absolute constant of the dimension of action. Eq. (159) means the assumption that the internal angular momentum tensor $S_{\mu\nu}$ is produced out of v_μ only and has no contribution from the other internal degrees of freedoms of the system even if it has any.

In this case the totality of $S_{\mu\nu}$ and

$$S_{\mu 5} = -S_{5\mu} \equiv (\hbar\kappa)^{1/2} v_\mu$$

constitutes the generators of a *homogeneous 5-dimensional Lorentz group*, and

$$\begin{cases} \rho' = \rho - \dfrac{2}{\hbar\kappa}\phi, & \text{(160a)} \\ \theta' = \theta + \dfrac{1}{\hbar\kappa}\chi^2 & \text{(160b)} \end{cases}$$

represent the two group-invariants of that group, giving general conserved invariant quantities.

If we take the limit $\kappa \to \infty$, a system of this type goes over to the case of classical model, but then $S_{\mu\nu}$ becomes decoupled from v_μ and this fact allows us to have broad varieties of possible "classical models".

Systems obeying (159) will also be analysed in Part II. They contain as its special examples the Dirac and Kemmer particles, and particle analogue corresponding to Heisenberg's non-linear field,[9] etc. Indeed we shall see that if we further specialize the system by adding the simple relation

$$\{\chi, v_\mu\} = 0, \tag{161}$$

we get the system of Dirac type, while by assuming another relation

$$\{\rho, v_\mu\} = -5\frac{\hbar}{\kappa} v_\mu \tag{162}$$

we get the system of Kemmer type.

Finally we mention two other criteria which are useful for characterizing systems of the first kind and are employed later. The previous equation (126) indicates that $\dot{v}_\mu$ depends not only on $k_{\mu\nu}$ but also on $[\mathcal{H}, v_\mu]$. Since $\mathcal{H}$ is in general a function of the proper internal variables $(v_\mu, S_{\mu\nu}, \xi_\alpha)$ alone, if $[v_\mu, v_\nu]$ is specified, the form of $[\mathcal{H}, v_\mu]$ is restricted to a certain extent, but not entirely, and this commutator gives our fourth criterion: (D) Whether or not $[\mathcal{H}, v_\mu]$ vanishes identically. If it vanishes, ε (and so $\mathcal{H}$) becomes a constant of motion. Our final criterion is given by the formerly mentioned condition (cf. Eqs. (81) and (83)): (E) Whether or not $[\Sigma, x_\mu] = 0$.

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General Theory of Superexchange Interaction

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Mechanisms of superexchange interaction in a magnetic compound are investigated by using the second quantization method which was originally developed by Bogoljubov in the polarized-ion model for metals. We expand the electron-field operator by a set of orthogonalized atomic orbitals which are constructed from the original non-orthogonal ones. By taking into account the configuration interaction we evaluate the energy of the crystal up to the fourth order of the overlap integral between the nearest neighbour orbitals. The coupling between spin pairs, which is obtained in the order of ε^4, is shown to be describable by the familiar spin-operator formalism. The effective interaction constant $\boldsymbol{J}$ has really a very complicated structure owing to non-orthogonality of the original orbitals. We notice that $\boldsymbol{J}$ contains the interaction in $\pi/2$-direction besides the usual interaction in π-direction.

§ 1. Introduction

As is well known, the Heitler-London approximation which is based on solutions of Hartree-Fock's equation for the free ions is fairly good to describe the electronic structure of ionic crystals. In this approximation, the cohesive energy is a quantity of the order of ε^2, where ε is an overlap integral between neighbouring free ions'AO. In crystals like rock salt both positive and negative ions have a closed-shell structure in the ground state, so there remains no freedom about electronic spins. In magnetic compounds like MnO, however, there exists freedom of electronic spin in positive ions, which causes various magnetic properties of the system through mutual interaction. As for the spin-dependent energy of magnetic compounds many works have recently been published.[1)]

In this paper a method of computing the spin-dependent energy is developed from a more general standpoint. We start from the Hamiltonian of many-electron system and compute the energy up to the order of ε^4 by taking into account the configuration interaction.

In order to treat the interacting many-electron system, we have a method of configuration interaction through many Slater determinants. In this method, however, difficulty increases rapidly with increasing number of electrons because of non-orthogonality of one-electron wave functions. Thus, this approach has not yet succeeded to describe the interaction energy in a familiar "spin operator" form, although it has done for the 4-electron problem.[1)]

There is another method of approach to this problem, that is, the method of second quantization. The second quantization approach, however, needs the expansion of electronic field by a complete orthonormal set of functions. It has the disadvantage, as criticized by Slater,[2] that the wave function of the zeroth approximation contains too much ionic parts and they work to raise the energy when the ionic distance is large. In order to avoid this defect we must consider all possible interactions between configurations. Since the second method is more general and consistent than the first, we shall follow it in this paper.

Recently, Bogoljubov has developed a new approach to this problem[3] on the basis of the second-quantization formalism. The so-called polarized-ion model of metals seems to be very convenient to treat the many-electron problem when interionic distance is sufficiently large. Therefore, we extend the method so as to be applicable to magnetic compounds. We expand the total energy by an ascending power of a parameter, which corresponds to the overlap integral between neighbouring orbitals. The spin-dependent energy, which is the energy of the so-called superexchange interaction, appears first at the fourth order terms, and we can describe the interaction energy in a familiar form

$$-\frac{1}{2}\boldsymbol{J}_{ij}(1+\sigma_i\cdot\sigma_j)$$

where $\boldsymbol{J}_{ij}$ is a very complicated function of integrals between atomic orbitals. Thus, we find that, even when the basic atomic orbitals are non-orthogonal, the mutual interaction energy is still describable by the spin-operator formalism. The difficult question that has been raised by Slater[2] has found a solution here.

§ 2. Outline of the method

In order to formulate the problem in a concrete way, we shall consider a simple model, that is, the magnetic compound of AB type with the simple cubic lattice like MnO, in which an A-ion has one magnetic electron, and a B-ion has two paired-electrons per each orbital in the normal state. The Hamiltonian in the form of second quantization to this system is expressed as follows:

$$\boldsymbol{H}=\boldsymbol{U}_0+\int\varphi^\dagger(\boldsymbol{x}_1)\,\mathrm{H}_1(\boldsymbol{x}_1)\varphi(\boldsymbol{x}_1)\,d\boldsymbol{x}_1+(1/2)\int\varphi^\dagger(\boldsymbol{x}_1)\varphi^\dagger(\boldsymbol{x}_2)\Phi(1,2)\varphi(\boldsymbol{x}_2)\varphi(\boldsymbol{x}_1)\,d\boldsymbol{x}_1 d\boldsymbol{x}_2. \tag{1}$$

$\boldsymbol{U}_0$ is the interaction potential between ion cores, $\mathrm{H}_1(\boldsymbol{x}_1)$, sum of the kinetic energy and the potential energies due to the ion cores periodically arrayed in lattice, and $\Phi(1,2)$ is the Coulomb repulsion between electrons 1 and 2.

Let us consider the AO's φ_f for the isolated ions. Here f is the lattice vector, and when f corresponds to the A-lattice, we take an A-type AO at the

point, and when f corresponds to the B-lattice we take a B-type AO there, respectively. Localized AO's φ_f of different f's, however, are not orthogonal to each other, so it is not convenient to expand the quantized electronic field $\varphi(\boldsymbol{x})$ on the basis of φ_f's. Therefore, we shall use as the basic functions the OAO θ_f which we construct by the orthogonalization procedure due to Löwdin,

$$\varphi(\boldsymbol{x})=\sum_{f,\sigma} a_{f\sigma}\theta_f(\boldsymbol{r})\delta(s-\sigma), \quad \boldsymbol{x}=(\boldsymbol{r} \text{ and } s), \tag{2}$$

where θ_f is the OAO constructed from the *orbital part* of φ_f, and $a_{f\sigma}$'s are the well-known Fermion operators.

Through expansion (2) our Hamilitonian is expressed as

$$\boldsymbol{H}=\boldsymbol{U}_0+\sum_{(f,f',\sigma)}\mathrm{L}(f;f')a^{\dagger}_{f\sigma}a_{f'\sigma}+(1/2)\sum_{(f_1,f_2,f_1',f_2',\sigma_1,\sigma_2)}\mathrm{F}(f_1f_2;f_1'f_2')a^{\dagger}_{f_1\sigma_1}a^{\dagger}_{f_2\sigma_2}a_{f_2'\sigma_2}a_{f_1'\sigma_1},$$

where

$$\mathrm{L}(f,f')=\int\theta_f(\boldsymbol{r})\mathrm{H}_1\theta_{f'}(\boldsymbol{r})\,d\boldsymbol{r}$$

and

$$\mathrm{F}(f_1f_2;f_1'f_2')=\int\theta_{f_1}(\boldsymbol{r}_1)\theta_{f_2}(\boldsymbol{r}_2)\Phi(1,2)\theta_{f_1'}(\boldsymbol{r}_1)\theta_{f_2'}(\boldsymbol{r}_2)\,d\boldsymbol{r}_1\,d\boldsymbol{r}_2. \tag{3}$$

Then, let us classify the matrices L and F according to the order of the magnitude. As θ_f's and φ_f's are connected by Löwdin's procedure, we can appraise the order with the following expressions.

$$\begin{aligned}\mathrm{L}(f,f)=&(f|\mathrm{H}_1|f)+(1/4)\{3(f|\mathrm{H}_1|f)S_{f,f}^2+\sum_g S_{f,g}(g|\mathrm{H}_1|g)S_{g,f}\}\\&-(1/2)\{\sum_g S_{f,g}(g|\mathrm{H}_1|f)+\sum_g(f|\mathrm{H}_1|g)S_{g,f}\}+\cdots\end{aligned}$$

and

$$\begin{aligned}\mathrm{L}(f,f')=&(f|\mathrm{H}_1|f')-(1/2)\{S_{f,f'}(f'|\mathrm{H}_1|f')+(f|\mathrm{H}_1|f)S_{f,f'}\}\\&+(1/8)\{3S_{f,f'}^2(f'|\mathrm{H}_1|f')+2\sum_g S_{f,g}(g|\mathrm{H}_1|g)S_{g,f'}+3(f|\mathrm{H}_1|f)S_{f,f'}^2\}\\&-(1/2)\{\sum_g{}' S_{f,g}(g|\mathrm{H}_1|f')+\sum_g{}'(f|\mathrm{H}_1|g)S_{g,f'}\}+\cdots \quad \text{for } (f\neq f'),\end{aligned} \tag{4}$$

where $(f|\cdots|f')$ means the matrix elements as to the AO φ_f and $\varphi_{f'}$. L is a quantity of the order of ε^0 in case of $f=f'$, and of ε^1 in $f\neq f'$. Further, φ_f being the AO of free ions, $(f|\mathrm{H}_1|f')$ can be expressed in terms of orbital energies and Madelung energies as follows:

$$(f|\mathrm{H}_1|f')=E_{f'}(\delta_{f,f'}+S_{f,f'})+(f|V-U_{f'}|f')-(\mathrm{N}_{f'}-1)(ff'|f'f'). \tag{5}$$

Here, $V=\sum_f U_f$, where U_f is the potential due to the ion-core located at f. N_f is equal to one when f corresponds to an A-ion, and it is equal to two when f corresponds to a B-ion, that is, it gives the number of electrons belonging to each ion in its normal state. We can proceed in the same way for the

F-matrix and the results are: F(fg; fg)'s are O(1), F(fg; $f'g$)'s or F(gf; gf')'s are O(ε) and F(f_1f_2; $f_1'f_2'$)'s are O(ε^2).

$$\begin{aligned}
\mathrm{F}(f_1f_2;f_1f_2)&=(f_1f_2|f_1f_2)+(3/4)(S_{f_1,f_1}^2+S_{f_2,f_2}^2)(f_1f_2|f_1f_2)\\
&+(1/4)\{\sum S_{f_1,g}(gf_2|gf_2)S_{g,f_1}+\sum S_{f_2,g}(f_1g|f_1g)S_{g,f_2}\}\\
&-(1/2)\{\sum S_{f_1,g}(gf_2|f_1f_2)+\sum(f_1f_2|gf_2)S_{g,f_1}+\sum S_{f_2,g}(f_1g|f_1f_2)\\
&+\sum(f_1f_2|f_1g)S_{g,f_2}\}+\cdots\\
\mathrm{F}(f_1f_2;f_1'f_2)&=\mathrm{F}(f_2f_1;f_2f_1')\\
&=(f_1f_2|f_1'f_2)-(1/2)S_{f_1,f_1'}(f_1'f_2|f_1'f_2)-(1/2)(f_1f_2|f_1f_2)S_{f_1,f_1'}\\
&+(1/8)\{3S_{f_1,f_1'}^2(f_1'f_2|f_1'f_2)+2\sum S_{f_1,g}(gf_2|gf_2)S_{g,f_1'}+3(f_1f_2|f_1f_2)S_{f_1,f_1'}^2\}\\
&-(1/2)\{\sum{}'S_{f_1,g}(gf_2|f_1'f_2)+\sum{}'(f_1f_2|gf_2)S_{g,f_1'}\}+\cdots\quad\text{for } f_1\neq f_1'
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{F}(f_1f_2;f_1'f_2')&=(f_1f_2|f_1'f_2')\\
&+(1/4)\{S_{f_1,f_1'}S_{f_2,f_2'}(f_1'f_2'|f_1'f_2')+S_{f_1,f_1'}(f_1'f_2|f_1'f_2)S_{f_2,f_2'}\\
&\quad+S_{f_2,f_2'}(f_1f_2'|f_1f_2')S_{f_1,f_1'}+(f_1f_2|f_1f_2)S_{f_1,f_1'}S_{f_2,f_2'}\}\\
&-(1/2)\{S_{f_2,f_2'}(f_1f_2'|f_1'f_2')+(f_1f_2|f_1'f_2)S_{f_2,f_2'}+S_{f_1,f_1'}(f_1'f_2|f_1'f_2')\\
&\quad+(f_1f_2|f_1f_2')S_{f_1,f_1'}\}+\cdots\quad\text{for } f_1\neq f_1' \text{ and } f_2\neq f_2'. \qquad (6)
\end{aligned}$$

We mention here the order of magnitude of nondiagonal L's and F's. When f and f' are both A-ions and they are connected by the intervenient B-ion, L is a quantity of the order of ε^2 in considering the overlap between nearest neighbouring ions only.

Now our task is to arrange the spin-dependent energies in order of ε. For this purpose we shall extend the method of Vonsovskij and Bogoljubov[3] which was originally intended to apply to metals.* Our Hamiltonian is separated in the form of expansion by overlap integral.

$$\begin{aligned}
&\boldsymbol{H}=\boldsymbol{H}_0+\varepsilon\boldsymbol{H}_1+\varepsilon^2\boldsymbol{H}_2,\\
&\boldsymbol{H}_0=\sum\mathrm{L}(f;f)a_{f\sigma}^{\dagger}a_{f\sigma}+(1/2)\sum\mathrm{F}(f_1f_2;f_1f_2)a_{f_1\sigma_1}^{\dagger}a_{f_2\sigma_2}^{\dagger}a_{f_2\sigma_2}a_{f_1\sigma_1}+\boldsymbol{U}_0,\\
&\varepsilon\boldsymbol{H}_1=\sum_{f\neq f'}\mathrm{L}(f;f')a_{f\sigma}^{\dagger}a_{f'\sigma}+(1/2)\sum_{f_1\neq f_1'}\mathrm{F}(f_1f_2;f_1'f_2)a_{f_1\sigma_1}^{\dagger}a_{f_2\sigma_2}^{\dagger}a_{f_2\sigma_2}a_{f_1'\sigma_1}\\
&\quad+(1/2)\sum_{f_2\neq f_2'}\mathrm{F}(f_1f_2;f_1f_2')a_{f_1\sigma_1}^{\dagger}a_{f_2\sigma_2}^{\dagger}a_{f_2'\sigma_2}a_{f_1\sigma_1}
\end{aligned}$$

and

$$\varepsilon^2\boldsymbol{H}_2=(1/2)\sum_{f_1\neq f_1';\ f_2\neq f_2'}\mathrm{F}(f_1f_2;f_1'f_2')a_{f_1\sigma_1}^{\dagger}a_{f_2\sigma_2}^{\dagger}a_{f_2'\sigma_2}a_{f_1'\sigma_1}. \qquad (7)$$

Here, it must be pointed out that the separation by ε in (7) does not cor-

* The details of the treatment for metals will be found in the literature by Bogoljubov. See reference 3).

respond to the very expansion by overlap integral. For example, $\boldsymbol{H}_0$ contains the quantity higher than ε and $\varepsilon\boldsymbol{H}_1$ contains also terms higher than ε^2. Therefore, the expression (7) means what degree is the lowest order of each term. After introduction of the number operator[4] of electrons to each ion f, $\mathrm{N}_f=\Sigma_\sigma a_{f\sigma}^\dagger a_{f\sigma}$, we obtain the simple form of (7)

$$\boldsymbol{H}_0=\sum \mathrm{L}(f;f)\mathrm{N}_f+(1/2)\sum \mathrm{F}(f_1f_2;f_1f_2)\{\mathrm{N}_{f_1}\mathrm{N}_{f_2}-\delta(f_1-f_2)\mathrm{N}_{f_1}\}+\boldsymbol{U}_0,$$

$$\varepsilon\boldsymbol{H}_1=\sum_{f\neq f'}a_{f\sigma}^\dagger\{\mathrm{L}(f;f')+\sum_{f''}\mathrm{F}(ff'';f'f'')\mathrm{N}_{f''}\}a_{f'\sigma}. \tag{7'}$$

The normal state of $\boldsymbol{H}_0$, where we start from, is described by the value of N_f only. The spin freedom of magnetic ions remains free at all. The perturbation to this, i.e. $\varepsilon\boldsymbol{H}_1$, and $\varepsilon^2\boldsymbol{H}_2$ will induce the transitions to excited states from the normal state and thus the spin-degeneracy will be removed partly. The increment of unperturbed energy, $\Delta\langle\boldsymbol{H}_0\rangle$ in the transition $\mathrm{N}_f\longrightarrow\mathrm{N}_f+\Delta\mathrm{N}_f$ is obtained as follows:

$$\Delta\langle\boldsymbol{H}_0\rangle\cong\sum\{E_f+(f|\widetilde{V}-\widetilde{U}_f|f)+(1/2)(ff|ff)\}\Delta\mathrm{N}_f$$
$$+(1/2)\sum(f_1f_2|f_1f_2)\Delta\mathrm{N}_{f_1}\Delta\mathrm{N}_{f_2}$$

where

$$\widetilde{U}_f(\boldsymbol{r})=U_f(\boldsymbol{r})+\mathrm{N}_f\int\Phi(\boldsymbol{r},\boldsymbol{r}')\varphi_f^2(\boldsymbol{r}')d\boldsymbol{r}'$$

and

$$\widetilde{V}(\boldsymbol{r})=\sum\widetilde{U}_f(\boldsymbol{r}). \tag{8}$$

The change of energy for each case is shown in Table I. In order to investigate the effect of perturbations $\varepsilon\boldsymbol{H}_1$ and $\varepsilon^2\boldsymbol{H}_2$, we follow the perturbation treatment for degenerate states developed by Bogoljubov.[3]

The equation to be solved,

$$(\boldsymbol{H}_0+\varepsilon\boldsymbol{H}_1+\varepsilon^2\boldsymbol{H}_2-E)\mathrm{C}=0, \tag{9}$$

is projected into the space of state C_0 having the lowest energy E_0, and the equivalent Schrödinger equation in C_0-space is obtained:

$$(E-E_0)\mathrm{C}_0=\widetilde{\boldsymbol{H}}\mathrm{C}_0, \tag{10}$$

where $\widetilde{\boldsymbol{H}}$ operates on the state in C_0 and trasnforms it in the space of C_0. Then, we have

$$\widetilde{\boldsymbol{H}}=\varepsilon\widetilde{\boldsymbol{H}}_1+\varepsilon^2\widetilde{\boldsymbol{H}}_2+\varepsilon^3\widetilde{\boldsymbol{H}}_3+\cdots,$$

where

$$\varepsilon\widetilde{\boldsymbol{H}}_1=\varepsilon\mathrm{P}\boldsymbol{H}_1\mathrm{P},$$

$$\varepsilon^2\widetilde{\boldsymbol{H}}_2=\varepsilon^2\{\mathrm{P}\boldsymbol{H}_2\mathrm{P}-\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}\},$$

$$\varepsilon^3\widetilde{\boldsymbol{H}}_3=\varepsilon^3\{\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}$$
$$-\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P})\mathrm{P}-\mathrm{P}\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}\}$$

and

$$\begin{aligned}\varepsilon^4\widetilde{\boldsymbol{H}}_4=\varepsilon^4\{&\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}\\&+\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P})\mathrm{P}\\&-\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}\\&+\mathrm{P}\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1\mathrm{P})\mathrm{P}\\&-\mathrm{P}\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P})\mathrm{P}\}.\end{aligned}\tag{11}$$

The application of the above general formula to our case requires to treat the products of the Fermion operators a, $a^\dagger$ and the projection operator P for the space C_0. The consideration of the properties of the space C_0 projected by P will make the physical meaning of the following expressions clear,

$$\mathrm{P}a^\dagger_{f_1\sigma_1}a^\dagger_{f_2\sigma_2}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\mathrm{P}=0 \qquad \text{for } (f_1\cdots f_s)\neq(f_1'\cdots f_s')$$

and

$$\begin{aligned}\mathrm{P}a^\dagger_{f_1\sigma_1}a^\dagger_{f_2\sigma_2}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\mathrm{P}&=a^\dagger_{f_1\sigma_1}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\mathrm{P}\\&\text{for } (f_1\cdots f_s)=(f_1'\cdots f_s')\end{aligned}\tag{12}$$

where $(f_1\cdots f_s)=(f_1'\cdots f_s')$ means that the set of points $f_1\cdots f_s$ is equal to the set of points $f_1'\cdots f_s'$ disregarding their order. The left side of (12) is the operator which creates holes at points $f_1'\cdots f_s'$ and electrons at points $f_1\cdots f_s$. $(\boldsymbol{H}_0-E_0)^{-1}$ can be handled by the same way.

$$\begin{aligned}&(\boldsymbol{H}_0-E_0)^{-1}a^\dagger_{f_1\sigma_1}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\mathrm{P}\\&\quad=(1/\Delta(f_1\cdots f_s\,;f_1'\cdots f_s'))a^\dagger_{f_1\sigma_1}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\mathrm{P}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{P}a^\dagger_{f_1\sigma_1}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}(\boldsymbol{H}_0-E_0)^{-1}\\&\quad=(1/\Delta(f_1'\cdots f_s'\,;f_1\cdots f_s))\mathrm{P}a^\dagger_{f_1\sigma_1}\cdots a^\dagger_{f_s\sigma_s}a_{f_{s'}\sigma_{s'}}\cdots a_{f_{1'}\sigma_{1'}}\\&\qquad\text{for } (f_1\cdots f_s)\neq(f_1'\cdots f_s').\end{aligned}\tag{13}$$

Now let us obtain the effective Hamiltonian in each order. In the first approximation, we obtain

$$\varepsilon\widetilde{\boldsymbol{H}}_1=0.\tag{14}$$

The spin degeneracy remains unremoved in this approximation, as Bogoljubov showed for metals. The second approximation consists in two terms. One is expressed as

$$\varepsilon^2\mathrm{P}\boldsymbol{H}_2\mathrm{P}=(1/2)\sum_{f_1\neq f_2}\mathrm{F}(f_1f_2\,;f_2f_1)a^\dagger_{f_1\sigma_1}a^\dagger_{f_2\sigma_2}a_{f_1\sigma_2}a_{f_2\sigma_1}\mathrm{P}.\tag{15}$$

For the other term, the following is obtained:

$$-\varepsilon^2\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1\mathrm{P}=-\sum_{f_1\neq f_2}\frac{\vec{\Delta}(f_1\,;f_2)\overleftarrow{\Delta}(f_2\,;f_1)}{\Delta(f_2\,;f_1)}a^\dagger_{f_1\sigma_1}a_{f_2\sigma_1}a^\dagger_{f_2\sigma_2}a_{f_1\sigma_2}\mathrm{P}.\tag{16}$$

The quantity Λ introduced here is the so-called Hartree-Hamiltonian for a single electron, whose definitions are as follows,

$$\overleftarrow{\Lambda}(f;f')=\mathrm{L}(f;f')+\sum_{f''}\mathrm{F}(ff'';f'f'')\{\mathrm{N}_{f''}-\delta(f''-f')\}$$

and

$$\overrightarrow{\Lambda}(f;f')=\mathrm{L}(f;f')+\sum_{f''}\mathrm{F}(ff'';f'f'')\{\mathrm{N}_{f''}-\delta(f''-f)\}. \qquad (17)$$

From Eqs. (15) and (16), $\varepsilon^2\boldsymbol{H}_2$ required is expressed as follows.

$$\varepsilon^2\widetilde{\boldsymbol{H}}_2=-\sum_{f_1\neq f_2}\left\{\frac{1}{2}\mathrm{F}(f_1f_2;f_2f_1)-\frac{\overrightarrow{\Lambda}(f_1;f_2)\overleftarrow{\Lambda}(f_2;f_1)}{\Delta(f_2;f_1)}\right\}a^\dagger_{f_1\sigma_1}a_{f_1\sigma_2}a^\dagger_{f_2\sigma_2}a_{f_2\sigma_1}\mathrm{P}$$
$$-\sum_{f_1\neq f_2}\frac{\overrightarrow{\Lambda}(f_1;f_2)\overleftarrow{\Lambda}(f_2;f_1)}{\Delta(f_2;f_1)}\mathrm{N}_{f_1}\mathrm{P}. \qquad (18)$$

Table I. Change of unperturbed energies by the transition.

Excited state		Change of energy
hole	electron	
b	a	$\Delta(a;b)\cong E_a-E_b+(a\|\widetilde{V}-\widetilde{U}_a\|a)-(b\|\widetilde{V}-\widetilde{U}_b\|b)+(aa\|aa)-(ab\|ab)$
a_1	a_2	$\Delta(a_2;a_1)\cong(aa\|aa)-(a_1a_2\|a_1a_2)$
$b,\ b$	$a_1,\ a_2$	$\Delta(a_1a_2;bb)=2[E_a-E_b+(a\|\widetilde{V}-\widetilde{U}_a\|a)-(b\|\widetilde{V}-\widetilde{U}_b\|b)]+2(aa\|aa)+(a_1a_2\|a_1a_2)+(bb\|bb)-2(a_1b\|a_1b)-2(a_2b\|a_2b)$
$b_1,\ b_2$	$a_1,\ a_2$	$\Delta(a_1a_2;b_1b_2)=2[E_a+(a\|\widetilde{V}-\widetilde{U}_a\|a)-E_b-(b\|\widetilde{V}-\widetilde{U}_b\|b)]+2(aa\|aa)+(a_1a_2\|a_1a_2)+(b_1b_2\|b_1b_2)-\sum_{\substack{i=1,2\\j=1,2}}(a_ib_j\|a_ib_j)$

Before transforming the above results into the form of spin Hamiltonian to the magnetic compound, let us consider the cases of the metal and the hydrogen molecule as simple example. Here, the space of states, C_0 is defined as $\mathrm{N}_f=a_f^\dagger(+)a_f(+)+a_f^\dagger(-)a_f(-)\Rightarrow 1$, thus the Fermion operators are connected with Pauli's spin operators by the relation[5]

$$a_f^\dagger(+)a_f(+)=(1/2)(1+\sigma_{f,z})\ ;\ a_f^\dagger(+)a_f(-)=(1/2)(\sigma_{f,x}+i\sigma_{f,y}),$$
$$a_f^\dagger(-)a_f(-)=(1/2)(1-\sigma_{f,z})\ ;\ a_f^\dagger(-)a_f(+)=(1/2)(\sigma_{f,x}-i\sigma_{f,y}).$$

The operator part of $\varepsilon^2\widetilde{\boldsymbol{H}}_2$ may be replaced by the spin Hamiltonian as follows,

$$\sum_{\sigma_1,\sigma_2}a^\dagger_{f_1\sigma_1}a_{f_1\sigma_2}a^\dagger_{f_2\sigma_2}a_{f_2\sigma_1}\mathrm{P}\Rightarrow(1/2)(1+\sigma_{f_1}\cdot\sigma_{f_2}). \qquad (19)$$

Thus, an application of the expression (18) in case of metals gives the so-called exchange coupling, and as an expression of its strength we obtain the following formula,

$$\boldsymbol{J}(f_1;f_2)=\mathrm{F}(f_1f_2;f_2f_1)-\frac{1}{\Delta(f_2;f_1)}\left\{\mathrm{L}(f_1;f_2)+\sum_{f''\neq f_1}\mathrm{F}(f_1f'';f_2f'')\right\}$$
$$\times\left\{\mathrm{L}(f_2;f_1)+\sum_{f''\neq f_1}\mathrm{F}(f_2f'';f_1f'')\right\}-\frac{1}{\Delta(f_1;f_2)}\left\{\mathrm{L}(f_2;f_1)+\sum_{f''\neq f_2}\mathrm{F}(f_2f'';f_1f'')\right\}$$
$$\times\left\{\mathrm{L}(f_1;f_2)+\sum_{f''\neq f_2}\mathrm{F}(f_1f'';f_2f'')\right\}. \tag{20}$$

Let us further transform the above formula into the expression with molecular integrals between two atomic orbitals a and b in case of hydrogen molecule. Defining the necessary integrals as follows,

$$K=(a|U_B|a),\quad K'=(ab|ab),\quad L=(aa|aa),\quad M=(b|U_A|a),$$
$$M'=(ba|aa),\quad S=(b|a),\qquad J=(ab|ba), \tag{21}$$

we obtain the following results:

$$\mathrm{F}(ab;ba)=J+(1/2)S^2(K'+L)-2SM',$$
$$\Delta(a;b)=L-K',\quad \mathrm{L}(a;b)=-SK+M,$$
$$\mathrm{F}(ab;bb)=M'-(1/2)S(L+K'),$$
$$\mathrm{L}(a;b)+\mathrm{F}(ab;bb)=M+M'-S(K+K')-(1/2)S(L-K'),$$

and

$$\boldsymbol{J}(a;b)=J+2SM-2S^2K-S^2K'-\{2/(L-K')\}(M+M'-SK-SK')^2. \tag{22}$$

Comparing the above with the result obtained by the usual Heitler-London treatment, we see that the first four terms in (22) is just half of the difference of energy between singlet and triplet states when we considered the homopolar binding only, and the last term corresponds just to the effect of the ionic binding. In other words our approximation is equivalent to the Heitler-London treatment including further the effect of configuration interaction for the hydrogen molecule.

In our ionic crystal, in contrast with the metals and hydrogen molecule, there appear paired orbitals when f corresponds to a B-ion. Taking into account the fact that it vanishes unless $\sigma_1=\sigma_2$ when f_1 or f_2 corresponds to B-ion in the summation, we have

$$\varepsilon^2\widetilde{\boldsymbol{H}}_2=-\sum_{a_1\neq a_2}\frac{\vec{\Lambda}(a_1;a_2)\overleftarrow{\Lambda}(a_2;a_1)}{\Delta(a_2;a_1)}-\sum\frac{\vec{\Lambda}(b;a)\overleftarrow{\Lambda}(a;b)}{\Delta(a;b)}$$
$$-\sum\frac{1}{2}\{\mathrm{F}(ab;ba)+\mathrm{F}(ba;ab)\}+\sum_{b_1\neq b_2}\mathrm{F}(b_1b_2;b_2b_1)$$
$$-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{1}{2}\mathrm{F}(a_1a_2;a_2a_1)-\frac{\vec{\Lambda}(a_1;a_2)\overleftarrow{\Lambda}(a_2;a_1)}{\Delta(a_2;a_1)}\right\}. \tag{23}$$

The terms except for the last in (23) do not depend upon the spins of magnetic ions. The physical meaning of each term in the last bracket in (23) is easily seen. The first term correspond to the direct ferromagnetic exchange coupling between OAO θ_{a_1} and θ_{a_2}. The second term comes from the configuration like $[Mn^{+}-O^{--}-Mn^{+3}]$. This type of interaction was first pointed out by Yamashita and Kondo[1] and was investigated also by Anderson[1),6)] as the antiferromagnetic delocalization effect.

§ 3. Spin dependent energies

Following Eq. (11), we divide $\varepsilon^3\tilde{\boldsymbol{H}}_3$ into three parts

$$\varepsilon^3\tilde{\boldsymbol{H}}_3=A_1+A_1^{\dagger}+A_2,$$

where

$$A_1=-\varepsilon^3 P\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1 P, \text{ and } A_2=\varepsilon^3 P\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1 P. \quad (24)$$

From the above we immediately obtain

$$A_1=-\frac{1}{2}\sum\frac{F(f_1f_2;f_1'f_2')\tilde{A}(f;f')}{\Delta(f;f')}P\,a^{\dagger}_{f_1\sigma_1}a^{\dagger}_{f_2\sigma_2}a_{f_2'\sigma_2}a_{f_1'\sigma_1}a^{\dagger}_{f\sigma}a_{f'\sigma}P.$$

In the above summation we put conditions $f_1\neq f_1'$, $f_2\neq f_2'$ $f\neq f'$ and $(f_1,f_2,f)=(f_1',f_2',f')$. As a result it contributes only in cases (1) $f_1'=f_2$, $f_2'=f$, $f'=f_1$, and $f_1\neq f_2\neq f\neq f_1$ or (2) $f_1'=f$, $f_2'=f_1$, $f'=f_2$, and $f_1\neq f_2\neq f\neq f_1$. After rearrangement of the indices of summation we gather these up,

$$A_1=\sum_{f_1\neq f_2\neq f\neq f_1}\frac{F(f_1f_2;f_2f)\tilde{A}(f;f_1)}{\Delta(f;f_1)}P\,a^{\dagger}_{f_1\sigma_1}a_{f,\sigma}a^{\dagger}_{f_2\sigma_2}a_{f_2\sigma_1}a_{f\sigma_2}a^{\dagger}_{f\sigma}P. \quad (25)$$

Thus, it is found that the terms in which f corresponds to a B-ion give a vanishing result. Although it may give three-body-force between spins when f_1 and f_2 as well as f correspond to A-ions, the coefficients of them are $O(\varepsilon^6)$ and we do not consider these cases here. After all, spin dependent forces are obtainable when one of f_1 and f_2 corresponds to an A-ion and the other to a B-ion, respectively. We obtain the coefficient of $O(\varepsilon^4)$ when two A-ions have a B-ion as a common nearest neighbour. In the configuration a_1——f——a_2* the spin-dependent force in A_1 is

$$A_1=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{F(fa_1;a_1a_2)\tilde{A}(a_2;f)}{\Delta(a_2;f)}+\frac{F(a_1f;fa_2)\tilde{A}(a_2;a_1)}{\Delta(a_2;a_1)}\right\}. \quad (26)$$

The transition process which corresponds to each term in (26) will be easily traced. The reverse processes appear in $A_1^{\dagger}$.

* Of course we include the case in which the angle $\angle a_1fa_2$ is equal to a right angle.

$$A_1^{\dagger}=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\Lambda}(f\,;\,a_2)\mathrm{F}(a_2a_1\,;\,a_1f)}{\Delta(a_2\,;\,f)}+\frac{\vec{\Lambda}(a_1\,;\,a_2)\mathrm{F}(a_2f\,;\,fa_1)}{\Delta(a_2\,;\,a_1)}\right.. \tag{27}$$

Since we meet the configurations corresponding to the Kramers-Anderson model, i.e. $[\mathrm{Mn}^{+}-\mathrm{O}^{-}]$, and to the delocalization effect in the band approximation, we shall obtain the antiferromagnetic coupling in (26) and (27).

Next, the explicit expression of A_2 is

$$A_2=\sum\frac{\vec{\mathrm{R}}(f_1f_2\,;\,f_2f)\overleftarrow{\Lambda}(f\,;\,f_1)}{\Delta(f\,;\,f_1)}\mathrm{P}\,a^{\dagger}_{f_1\sigma_1}a_{f_2\sigma_1}a^{\dagger}_{f_2\sigma_2}a_{f\sigma_2}a^{\dagger}_{f\sigma}a_{f_1\sigma}\mathrm{P}$$
$$+\sum\frac{\vec{\mathrm{R}}(f_1f_2\,;\,ff_1)\overleftarrow{\Lambda}(f\,;\,f_2)}{\Delta(f\,;\,f_2)}\mathrm{P}\,a^{\dagger}_{f_1\sigma_1}a_{f\sigma_1}a^{\dagger}_{f_2\sigma_2}a_{f_1\sigma_2}a^{\dagger}_{f\sigma}a_{f_2\sigma}\mathrm{P},$$

where

$$\vec{\mathrm{R}}(f_1f_2\,;\,f_1'f_2')=\frac{\vec{\Lambda}(f_1\,;\,f_1')}{\Delta(f_1'\,;\,f_1)}\{\mathrm{L}(f_2\,;\,f_2')+\sum_{f''}\mathrm{F}(f_2f''\,;\,f_2'f'')$$
$$\times[\mathrm{N}_{f''}-\delta(f_1-f'')+\delta(f_1'-f'')-\delta(f_2-f'')]\},$$

and

$$\overleftarrow{\mathrm{R}}(f_1f_2\,;\,f_1'f_2')=\Big[\mathrm{L}(f_1\,;\,f_1')+\sum_{f''}\mathrm{F}(f_1f''\,;\,f_1'f'')\{\mathrm{N}_{f''}-\delta(f_1'-f'')$$
$$+\delta(f_2-f'')-\delta(f_2'-f'')\}\Big]\frac{\overleftarrow{\Lambda}(f_2\,;\,f_2')}{\Delta(f_2\,;\,f_2')}. \tag{28}$$

After similar treatments as in A_1, we obtain the expression of A_2:

$$A_2=\sum_{\sigma_1\neq\sigma_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\mathrm{R}}(fa_2\,;\,a_2a_1)\overleftarrow{\Lambda}(a_1\,;\,f)}{\Delta(a_1\,;\,f)}+\frac{\vec{\mathrm{R}}(a_2f\,;\,a_1a_2)\overleftarrow{\Lambda}(a_1\,;\,f)}{\Delta(a_1\,;\,f)}\right.$$
$$\left.+\frac{\vec{\mathrm{R}}(fa_2\,;\,a_1f)\overleftarrow{\Lambda}(a_1\,;\,a_2)}{\Delta(a_1\,;\,a_2)}\right\}. \tag{29}$$

Here the first term does not contain the coupling in the $\pi/2$-direction, but other terms do. The Kramers-Anderson model configurations interact mutually in the first term and furthermore they interact with the band-approximation configurations in the other terms by the effect of delocalization.

We shall divide $\varepsilon^4\widetilde{\boldsymbol{H}}_4$ as in the case of $\varepsilon^3\widetilde{\boldsymbol{H}}_3$,

$$\varepsilon^4\widetilde{\boldsymbol{H}}_4=B_1+B_2+B_3+B_4+B_4^{\dagger},$$

where

$$B_1=-\varepsilon^4\mathrm{P}\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P})\mathrm{P},$$
$$B_2=\varepsilon^4\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_2(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1\mathrm{P},$$
$$B_3=-\varepsilon^4\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}\boldsymbol{H}_1\mathrm{P},$$

and

$$B_4=\varepsilon^4 \mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_1-\mathrm{P}\boldsymbol{H}_1)(\boldsymbol{H}_0-E_0)^{-1}(\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P})\mathrm{P}. \tag{30}$$

As an example, we shall first investigate B_2. We express $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ in (30) with Fermion operators, and we have

$$B_2=\frac{1}{2}\sum\frac{\Delta(f_1;f_1')\mathrm{F}(f_2f_3;f_2'f_3')\vec{\Delta}(f;f')}{\Delta(f_1';f_1)\Delta(f;f')}\mathrm{P}a^\dagger_{f_1\sigma_1}a_{f_1'\sigma_1}a^\dagger_{f_2\sigma_2}a^\dagger_{f_3\sigma_3}a_{f_1'\sigma_3}a_{f_2'\sigma_2}a^\dagger_{f\sigma}a_{f'\sigma}\mathrm{P}. \tag{31}$$

From the requirement $(f_1,f_2,f_3,f)=(f_1',f_2',f_3',f')$ and, $f_1'\neq f_1'\cdots$, we obtain the sets listed in Table II for the permitted sets of combinations in the

Table II. The permitted set of combinations in the summation (31).

case	f_1	f_2	f_3	f
(1)	f_2'	f_1'	f'	f_3'
(2)	f_2'	f'	f_1'	f_3'
(3)	f_2'	f_3'	f'	f_1'
(4)	f_3'	f_1'	f'	f_2'
(5)	f_3'	f'	f_1'	f_2'
(6)	f_3'	f'	f_2'	f_1'
(7)	f'	f_1'	f_2'	f_3'
(8)	f'	f_3'	f_1'	f_2'
(9)	f'	f_3'	f_2'	f_1'

summation (31). We have here nine cases in contrast with two for A's. We have to investigate them one by one, separately. For the case (1) we have

$$\frac{1}{2}\sum\frac{\Delta(f_1;f_2)\mathrm{F}(f_2f_3;f_1f)\vec{\Delta}(f;f_3)}{\Delta(f_2;f_1)\Delta(f;f_3)}\mathrm{P}a^\dagger_{f_1\sigma_1}a_{f_3\sigma_1}a^\dagger_{f_2\sigma_2}a_{f_3\sigma_3}a_{f\sigma_3}a_{f_1\sigma_2}a^\dagger_{f\sigma}a_{f_3\sigma}\mathrm{P}.$$

By considering the order of Fermion operators, it is found that f and f_2 must correspond to A-ions and thus, f_3 and f_1 correspond to the nearest neighbour B-ion, respectively. If $f=f_2$, we obtain only the spin-independent force, as is easily seen. We obtain the spin-independent operators when $f_3\neq f_1$, i.e. the case of two different B-ions by the following relation.

$$\sum_{spin}\mathrm{P}a^\dagger_{f_1\sigma_1}a_{f_2\sigma_1}a^\dagger_{f_2\sigma_2}a^\dagger_{f_3\sigma_3}a_{f\sigma_3}a_{f_1\sigma_2}a^\dagger_{f\sigma}a_{f_3\sigma}\mathrm{P}=\sum_{\sigma,\sigma_1}a_{f\sigma}a^\dagger_{f\sigma}a_{f_2\sigma_1}a^\dagger_{f_2\sigma_1}\mathrm{P}.$$

As a result, we obtain the coupling between $f(\Rightarrow a_1)$ and $f_2(\Rightarrow a_2)$ through an intervenient B-ion in the following form.

$$\text{Case (1)}: \quad -\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{2}\frac{\vec{\Delta}(f;a_2)\mathrm{F}(a_2f;fa_1)\vec{\Delta}(a_1;f)}{\Delta(a_2;f)\Delta(a_1;f)} \tag{32}$$

which contains the exchange interaction between two Kramers-Anderson configurations and therefore vanishes in the $\pi/2$-interaction.

Case (2): After similar considerations, especially for the order of Fermion operators and the position of f-indices in F, we see that it gives only the spin-

independent energies. Since other cases are treated by similar ways, we shall write down only the results.

$$\text{Case (3)}: \quad -\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{2}\frac{\vec{\Lambda}(f;a_2)\mathrm{F}(a_1g;fa_1)\overleftarrow{\Lambda}(a_2;g)}{\Delta(a_2;f)\Delta(a_2;g)}, \tag{33}$$

where we have the spin-dependent force even if $g\neq f$ in which we obtain only right-angled coupling. If $g=f$ we obtain a weak right-angled coupling in addition to linear coupling, which corresponds to the naive Kramers-Anderson model and is expected to be ferromagnetic because the orbitals located at a_1 and f, i.e. θ_{a_1} and θ_f, are orthogonal.

Case (4) contributes nothing.

Cases (5) and (6): We obtain the same coupling as (1) and (3), respectively. Thus, the factor 1/2 in (32) and (33) is dropped.

Cases (7) and (8): Here we obtain the terms that we have already considered in (1) and (5), respectively. We must take care not to count the same terms twice.

Case (9): We obtain the expression

$$\sum\frac{1}{2}\frac{\vec{\Lambda}(f_1;f)\mathrm{F}(f_2f_3;f_3f_2)\overleftarrow{\Lambda}(f;f_1)}{\Delta(f;f_1)\Delta(f;f_1)}\mathrm{P}\,a^{\dagger}_{f_1\sigma_1}a_{f\sigma_1}a^{\dagger}_{f_2\sigma_2}a^{\dagger}_{f_3\sigma_3}a_{f_2\sigma_3}a_{f_3\sigma_2}a^{\dagger}_{f\sigma}a_{f_1\sigma}\mathrm{P}\,.$$

This expression vanishes unless f corresponds to an A-ion and f_1 to a B-ion as its neighbour. Even though we make f_2, and f_3 correspond to A, B ions suitably, we obtain only the terms already considered in (3) or (6).

After summing up the terms from (1) through (9) we obtain the final expression for B_2:

$$B_2=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\Lambda}(f;a_2)\mathrm{F}(a_2f;fa_1)\overleftarrow{\Lambda}(a_1;f)}{\Delta(a_2;f)\Delta(a_1;f)}\right.$$
$$\left.+\frac{\Lambda(f;a_2)\mathrm{F}(a_1g;fa_1)\overleftarrow{\Lambda}(a_2;g)}{\Delta(a_2f)\Delta(a_2;g)}\right\}. \tag{34}$$

Next, let us investigate B_1. We easily see that the excited states induced by perturbation $\boldsymbol{H}_2-\mathrm{P}\boldsymbol{H}_2\mathrm{P}$ consist of those states with one pair of electron and hole, and with two such pairs. Here we have also nine sets of combinations listed up in Table II. We shall investigate them in order.

Case (1): In this case the intermediate state has no pairs; it is the state induced by $\mathrm{P}\boldsymbol{H}_2\mathrm{P}$ in $\boldsymbol{H}_2$. This must be omitted.

Cases (2), (3), (4) and (7): We obtain for case (2)

$$\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{4}\frac{\mathrm{F}(fa_2;a_1f)\mathrm{F}(a_1g;ga_2)}{\Delta(a_1;a_2)}, \tag{35}$$

where the configuration appearing in the band approximation interacts through the exchange coupling, and it is expected to be a kind of delocalization effects

acting as antiferromagnetic interaction. Further, we obtain the same expressions in cases (3), (4) and (7) and the factor 1/4 above is dropped.

Case (5): Here we meet with the two pair excitations with a result

$$\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{4}\frac{\mathrm{F}(ff;a_1a_2)\mathrm{F}(a_1a_2;ff)}{\Delta(a_1a_2;ff)}. \tag{36}$$

Here appears the configuration due to Goodenough, i.e. $[\mathrm{Mn}^{+}-\mathrm{O}-\mathrm{Mn}^{+}]$, which acts as antiferromagnetic interaction.[6]

Case (6) gives

$$\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{4}\frac{\mathrm{F}(gf;a_1a_2)\mathrm{F}(a_2a_1;gf)}{\Delta(a_1a_2;gf)}, \tag{37}$$

where the expression becomes identical with the above when $f=g$. The configuration $[\mathrm{Mn}^{+}-\mathrm{O}-\mathrm{Mn}^{+}]$ plays a role in the expression (37), which gives an antiferromagnetic interaction in both cases $f=g$ and $f\neq g$.

Cases (8) and (9); We have just the same terms as those appearing in (6) and (5), respectively. After summing up these we have an expression of B_1:

$$B_1=\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\sum_{f,g}\frac{\mathrm{F}(fa_2;a_1f)\mathrm{F}(a_1g;ga_2)}{\Delta(a_1;a_2)}\right.$$
$$\left.+\sum_{f\neq g}\frac{1}{4}\frac{\mathrm{F}(gf;a_1a_2)\mathrm{F}(a_1a_2;fg)}{\Delta(a_1a_2;fg)}+\frac{1}{2}\sum_{f}\frac{\mathrm{F}(ff;a_1a_2)\mathrm{F}(a_1a_2;ff)}{\Delta(a_1a_2;ff)}\right\}. \tag{38}$$

By using the R matrix defined by (28) we have an expression for B_3.

$$B_3=-\sum\frac{\vec{\mathrm{R}}(f_1f_2;f_1'f_2')\overleftarrow{\mathrm{R}}(f_3f;f_3'f')}{\Delta(f_3f;f_3'f')}\mathrm{P}a^{\dagger}_{f_1\sigma_1}a_{f_1'\sigma_1}a^{\dagger}_{f_2\sigma_2}a_{f_2'\sigma_2}a^{\dagger}_{f_3\sigma_3}a_{f_3'\sigma_3}a^{\dagger}_{f\sigma}a_{f'\sigma}\mathrm{P}. \tag{39}$$

Again we shall consider the nine cases in order.

Case (1) contributes nothing.

Cases (2), (4) and (7) do not give coupling of $\mathrm{O}(\varepsilon^4)$, although terms of $\mathrm{O}(\varepsilon^6)$ may appear.

$$\text{Case (3):}\quad \sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{\vec{\mathrm{R}}(fa_2;a_1f)\overleftarrow{\mathrm{R}}(ga_1;a_2g)}{\Delta(a_1;a_2)}. \tag{40}$$

This seems to give weak $\pi/2$-directed couplings in both cases of $g=f$ and $g\neq f$ form the symmetry of R.

$$\text{Case (8):}\quad \sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{\vec{\mathrm{R}}(fg;a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1;gf)}{\Delta(a_1a_2;gf)} \tag{41}$$

This gives no $\pi/2$-directed couplings.

$$\text{Case (6):}\quad \sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{\vec{\mathrm{R}}(fg;a_1a_2)\overleftarrow{\mathrm{R}}(a_2a_1;fg)}{\Delta(a_1a_2;fg)} \tag{42}$$

Here we obtain also a $\pi/2$-directed coupling besides a π-directed coupling.

Cases (5) and (9) give only repeat of (8) and (6), respectively.

Thus we obtain the expression for B_3:

$$B_3=\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\mathrm{R}}(fa_2\,;\,a_1f)\overleftarrow{\mathrm{R}}(ga_1:a_2g)}{\Delta(a_1\,;\,a_2)}\right.$$
$$\left.+\frac{\vec{\mathrm{R}}(fg\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,gf)}{\Delta(a_1a_2\,;\,fg)}+\frac{\vec{\mathrm{R}}(fg\,;\,a_1a_2)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,fg)}{\Delta(a_1a_2\,;\,fg)}\right\}. \qquad (43)$$

In these fourth order processes the two Kramers-Anderson configurations interact through the band approximation- and Goodenough-configurations, respectively. They seem to act antiferromagnetically.

Finally we shall treat B_4 and $B_4^\dagger$. We consider $B_4^\dagger$, which has a form $\mathrm{F}\cdot\overleftarrow{\mathrm{R}}$, and is written as

$$B_4^\dagger=\frac{1}{2}\sum\frac{\mathrm{F}(f_1f_2\,;\,f_1'f_2')\overleftarrow{\mathrm{R}}(f_3f\,;\,f_3'f')}{\Delta(f_3f\,;\,f_3'f')}\mathrm{P}a^\dagger_{f_1\sigma_1}a^\dagger_{f_2\sigma_2}a_{f_2'\sigma_2}a_{f_1'\sigma_1}a^\dagger_{f_3\sigma_3}a_{f_3'\sigma_3}a^\dagger_{f\sigma}a_{f'\sigma}\mathrm{P}.$$

Cases (1), (2) and (7) contribute nothing to the spin Hamiltonian.

Cases (3) and (4) give the same terms after rearrangement of f-indices,

$$-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{\mathrm{F}(fa_2\,;\,a_1f)\overleftarrow{\mathrm{R}}(ga_1\,;\,a_2g)}{\Delta(a_1\,;\,a_2)}, \qquad (44)$$

where the $\pi/2$-directed coupling remains both in $f\neq g$ and $f=g$.

Case (5) and (6): We obtain similar terms.

$$-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{\mathrm{F}(ff\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,ff)}{\Delta(a_2a_1\,;\,ff)}$$
$$-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\frac{1}{2}\frac{\mathrm{F}(gf\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,fg)}{\Delta(a_1a_2\,;\,fg)} \qquad (45)$$

Cases (8) and (9): We obtain the terms that we have already considered. After all the expression of $B_4^\dagger$ is

$$B_4^\dagger=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\mathrm{F}(fa_2\,;\,a_1f)\overleftarrow{\mathrm{R}}(ga_1\,;\,a_2g)}{\Delta(a_1\,;\,a_2)}\right.$$
$$\left.+\frac{\mathrm{F}(ff\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,ff)}{\Delta(a_1a_2\,;\,ff)}+\sum_{f\neq g}\frac{1}{2}\frac{\mathrm{F}(gf\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,fg)}{\Delta(a_1a_2\,;\,fg)}\right\}. \qquad (46)$$

The term B_4 is easily determined, because it is regarded as the reverse process of $B_4^\dagger$.

$$B_4=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\mathrm{R}}(ga_2\,;\,a_1g)\mathrm{F}(fa_1\,;\,a_2f)}{\Delta(a_1\,;\,a_2)}\right.$$
$$\left.+\frac{\vec{\mathrm{R}}(ff\,;\,a_1a_2)\mathrm{F}(a_1a_2\,;\,ff)}{\Delta(a_1a_2\,;\,ff)}+\sum_{f\neq g}\frac{1}{2}\frac{\vec{\mathrm{R}}(fg\,;\,a_1a_2)\mathrm{F}(a_1a_2\,;\,gf)}{\Delta(a_1a_2\,;\,gf)}\right\}. \qquad (47)$$

Thus, the configuration due to the Kramers-Anderson model interacts with the one in the band approximation or the Goodenough model in the expression of B_4 and $B_4^{\dagger}$.

After all we obtain the final equation up to the order of ε^4

$$(E-E_0)C_0=\{\Delta E-\sum_{a_1\neq a_2}\boldsymbol{J}(a_1\,;\,a_2)\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\}C_0, \tag{48}$$

where ΔE is a constant energy independent of spins, and $\boldsymbol{J}(a_1\,;\,a_2)$ is the effective exchange coupling.

$$\begin{aligned}
\boldsymbol{J}(a_1\,;\,a_2)=&\frac{1}{2}\mathrm{F}(a_1a_2\,;\,a_2a_1)-\frac{\vec{\Lambda}(a_1\,;\,a_2)\overleftarrow{\Lambda}(a_2\,;\,a_1)}{\Delta(a_2\,;\,a_1)}\\
&+2\frac{\mathrm{F}(fa_1\,;\,a_1a_2)\overleftarrow{\Lambda}(a_2\,;f)}{\Delta(a_2\,;f)}+2\frac{\mathrm{F}(a_1f\,;fa_2)\overleftarrow{\Lambda}(a_2\,;\,a_1)}{\Delta(a_2\,;\,a_1)}\\
&-\frac{\vec{\mathrm{R}}(fa_2\,;\,a_2a_1)\overleftarrow{\Lambda}(a_1\,;f)}{\Delta(a_1\,;f)}-2\frac{\vec{\mathrm{R}}(a_2f\,;\,a_1a_2)\overleftarrow{\Lambda}(a_1\,;f)}{\Delta(a_1\,;f)}\\
&+\frac{\vec{\Lambda}(f\,;\,a_2)\mathrm{F}(a_2f\,;fa_1)\overleftarrow{\Lambda}(a_1\,;f)}{\Delta(a_2\,;f)\Delta(a_1\,;f)}+\sum_g\frac{\vec{\Lambda}(f\,;\,a_2)\mathrm{F}(a_1g\,;fa_1)\overleftarrow{\Lambda}(a_2\,;\,g)}{\Delta(a_2\,;f)\Delta(a_2\,;\,g)}\\
&-\sum_g\frac{\mathrm{F}(fa_2\,;\,a_1f)\mathrm{F}(a_1g\,;ga_2)}{\Delta(a_1\,;\,a_2)}-\frac{1}{2}\frac{\mathrm{F}(ff\,;\,a_1a_2)\mathrm{F}(a_1a_2\,;ff)}{\Delta(a_1a_2\,;ff)}\\
&-\frac{1}{4}\sum_{g\neq f}\frac{\mathrm{F}(gf\,;\,a_1a_2)\mathrm{F}(a_1a_2\,;fg)}{\Delta(a_1a_2\,;fg)}-\sum_g\Big\{\frac{\vec{\mathrm{R}}(fa_2\,;\,a_1f)\overleftarrow{\mathrm{R}}(ga_1\,;\,a_2g)}{\Delta(a_1\,;\,a_2)}\\
&+\frac{\vec{\mathrm{R}}(fg\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;\,gf)}{\Delta(a_1a_2\,;fg)}+\frac{\vec{\mathrm{R}}(fg\,;\,a_1a_2)\overleftarrow{\mathrm{R}}(a_2a_1\,;fg)}{\Delta(a_1a_2\,;fg)}\Big\}\\
&+\sum_g 2\frac{\mathrm{F}(fa_2\,;\,a_1f)\overleftarrow{\mathrm{R}}(ga_1\,;\,a_2g)}{\Delta(a_1\,;\,a_2)}+2\frac{\mathrm{F}(ff\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;ff)}{\Delta(a_1a_2\,;ff)}\\
&+\sum_{g\neq f}\frac{\mathrm{F}(gf\,;\,a_2a_1)\overleftarrow{\mathrm{R}}(a_2a_1\,;fg)}{\Delta(a_1a_2\,;fg)}.
\end{aligned}\tag{49}$$

Of course we have to take the value $\boldsymbol{J}(a_1\,;\,a_2)+\boldsymbol{J}(a_2\,;\,a_1)$, instead of $\boldsymbol{J}$ in the above, as the usual exchange coupling constant. The orbital located at f or g must belong to an intervenient B-ion connecting the orbitals at a_1 and a_2. The directional property comes from the nature of the f-orbital. We note that in the terms of $g\neq f$, only the $\pi/2$-directed interactions are contained. Although the $\pi/2$-directed interaction may disappear or become weak sometimes by symmetrical reason, both types of couplings may exist in general. We may expect rather strong $\pi/2$-directed couplings, because we have many terms which cause it $(g\neq f)$ in the summation.

§ 4. Discussions

In the last section we showed that the spin dependent energy of an N-body system can be expressed by a familiar spin operator formalism. Our method is sufficiently accurate, because we consider all the configurations and configuration interactions, as long as we limit the orbitals to the ground state orbitals of the atoms. The effective exchange integral $\boldsymbol{J}$, however, is very complicated. As mentioned in the last section we can roughly assign the physical meaning of each term in (49). It contains all terms which correspond to the terms in Yamashita-Kondo's paper. However, the expressions in this paper are more complicated than those in Yamashita and Kondo's work. In fact, a variational method was adopted in Yamashita and Kondo's paper and the trial orbitals are chosen from intuitive considerations. Therefore, they are rather simple, but not quite general. In this paper we adopted a more general and accurate approach; consequently the result is inevitably too complicated to be interpreted by simple models. Thus, we conclude that the intuitive models of superexchange interaction are very useful for getting a physical picture of the interaction, but they are rather idealized concepts and the real situation of the interaction seems to be much more complicated.

Among the many terms in (49) some terms may be more important and some less important. But we did not attempt to estimate the order of magnitude of these terms, because we do not find any importance in making a quantitative estimate in the present simplified model. Although we content ourselves here with developing a formal theory by a simplified model, we did not introduce impracticable ideas, for example, to get the Hartree-Fock wave functions of the crystal, into the theory.[5),6)] We believe, however, that we can apply the present formalism to simple real crystals, for example, MnO or MnF_2, in order to make a quantitative estimate of superexchange interaction in these crystals. This problem is not difficult, if we are allowed to adopt the Hartree-Fock wave functions of the free ions as the basic orbitals. Such a computation will be carried out in near future.

Finally we shall mention a little about the $\pi/2$-directed interaction. In the π-directed interaction the orbital φ_b of the interveinient negative ion plays an essential role except for the direct interaction between magnetic ions in $\mathcal{E}^2\widetilde{\boldsymbol{H}}_2$. On the other hand, it might be thought that we have no or weak couplings in the $\pi/2$-direction because of the directional properties of φ_b. But there may be couplings through φ_{b_e}* and further we have more terms in the summation which give the $\pi/2$-directed interaction. Thus we may expect a rather strong coupling in spite of the directional properties of p-orbitals.[7)]

The author wishes to express his sincere thanks to Prof. J. Yamashita for

* We shall treat in the next paper the role of φ_{be} which is the excited state orbital belonging to a B-ion.

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General Theory of Superexchange Interaction. II

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The method developed in the previous paper to treat superexchange interaction in a magnetic compound is extended to include the excited-state orbitals, and new contributions to the spin-dependent energy is obtained.

§ 1. Introduction

In the previous paper, which will be referred to as I,[1] we have developed a method to treat superexchange interaction. The method, however, is still restricted in a sense that the basic orbitals are only ground-state orbitals. In the present paper, we shall show that the previous method can be extended to include the excited-state orbitals. As a simple illustration we shall discuss superexchange interaction through the excited state of the negative ion.

§ 2. Treatment of the excited-state orbitals

Let φ_{fe} be an excited-state orbital of a negative ion located at the lattice point f. For simplicity, we assume it to be an s-orbital. Then, let us construct the basic function θ_{fe} corresponding to φ_{fe}, for which we shall require that θ_{fe} is orthogonal to all $\theta_{f'}$ and $\theta_{f'e}$ orbitals where θ_f is an orthogonalized ground-state orbital. Hereafter we shall denote the overlap integral between φ_{fe} and $\theta_{f'}$ as $T_{f',fe}$, which vanishes when f and f' coincide, because they have different symmetries. The order of magnitude of the other elements of T-matrix is easily estimated. For the nearest neighbour f' and f, we obtain

$$T_{f',fe}=(\theta_{f'},\varphi_{fe})=(\varphi_{f'},\varphi_{fe})-(1/2)\sum S_{f',g}(\varphi_g,\varphi_{fe})+0(\varepsilon^3), \qquad (1)$$

and for the second neighbour, we get the quantity of the order of ε^2, though it may happen to vanish by symmetry.

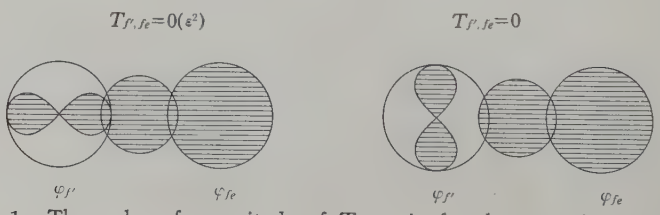


Fig. 1. The order of magnitude of T-matrix for the second neighbour.

By using T-matrix we get a function u_f, which is orthogonal to all $\theta_{f'}$ ($\theta_{f'}$ is the basis orbitals for the ground state):

$$u_f = M_f(\varphi_{fe} - \sum T_{f'',fe}\theta_{f''}), \tag{2}$$

where M_f is a factor for normalization. The desired basic function θ_{fe} is thus obtained after the process of symmetric orthogonalization to the set of the functions u_f.

Now our expansion formula to the electron-field operator is extended to

$$\varphi(\boldsymbol{x}) = \sum a_{\alpha\sigma}\theta_\alpha(\boldsymbol{r})\delta(s-\sigma),$$

where α is used to designate f and f_e together.

In Eq. (7) of I we separated the Hamiltonian by the degree of difference of suffices f and f' in matrices L and F. Here we may treat α instead of f and obtain the expressions which correspond to Eqs. (7) and (7′) in I as follows:

$$\begin{aligned}\boldsymbol{H}_0 = &\sum \mathrm{L}(f;f)\mathrm{N}_f + (1/2)\sum \mathrm{F}(f_1f_2;f_1f_2)\{\mathrm{N}_{f_1}\mathrm{N}_{f_2} - \delta(f_1-f_2)\mathrm{N}_{f_1}\} \\ &+ \sum \mathrm{L}(f_e;f_e)\mathrm{N}_{fe} + (1/2)\sum \mathrm{F}(f_{1e},f_{2e};f_{1e}f_{2e})\{\mathrm{N}_{f_{1e}}\mathrm{N}_{f_{2e}} - \delta(f_1-f_2)\mathrm{N}_{f_{1e}}\} \\ &+ \sum \mathrm{F}(ff_e';ff_e')\mathrm{N}_f\mathrm{N}_{f'e},\end{aligned} \tag{3}$$

and

$$\begin{aligned}\varepsilon\boldsymbol{H}_1 = &\sum_{f\neq f'} a_{f\sigma}^\dagger\{\mathrm{L}(f;f') + \sum_{f''}\mathrm{F}(ff'';f'f'')\mathrm{N}_{f''} + \sum_{f''}\mathrm{F}(ff_e'';f'f_e'')\mathrm{N}_{f''e}\}a_{f'\sigma} \\ &+ \sum a_{fe\sigma}^\dagger\{\mathrm{L}(f_e;f') + \sum_{f''}\mathrm{F}(f_ef'';f'f'')\mathrm{N}_{f''} + \sum_{f''}\mathrm{F}(f_ef_e'';f'f_e'')\mathrm{N}_{f''e}\}a_{f'\sigma} \\ &+ \sum a_{f\sigma}^\dagger\{\mathrm{L}(f;f_e') + \sum_{f''}\mathrm{F}(ff'';f_e'f'')\mathrm{N}_{f''} + \sum_{f''}\mathrm{F}(ff_e'';f_e'f_e'')\mathrm{N}_{f''e}\}a_{f'e\sigma} \\ &+ \sum_{f\neq f'} a_{fe\sigma}^\dagger\{\mathrm{L}(f_e;f_e') + \sum_{f''}\mathrm{F}(f_ef'';f_e'f'')\mathrm{N}_{f''} + \sum_{f''}\mathrm{F}(f_ef_e'';f_e'f_e'')\mathrm{N}_{f''e}\}a_{f'e\sigma},\end{aligned} \tag{4}$$

where we shall designate each term as $\varepsilon\boldsymbol{H}_1(f,f')$, $\varepsilon\boldsymbol{H}_1(f_e,f')$, $\varepsilon\boldsymbol{H}_1(f,f_e')$ and $\varepsilon\boldsymbol{H}_1(f_e,f_e')$, respectively. The terms corresponding to $f=f'$, do not appear in $\varepsilon\boldsymbol{H}_1(f,f')$ and $\varepsilon\boldsymbol{H}_1(f_e,f_e')$ in general. Moreover, if the distribution of $\mathrm{N}_{f''}$ and $\mathrm{N}_{fe''}$ has cubic symmetry, $\varepsilon\boldsymbol{H}_1(f_e,f')$ and $\varepsilon\boldsymbol{H}_1(f,f_e')$ do not contain the terms of $f=f'$ also.* Next, $\varepsilon^2\boldsymbol{H}_2$ is given by:

$$\begin{aligned}\varepsilon^2\boldsymbol{H}_2 = &(1/2)\sum_{f_1\neq f_1';\,f_2\neq f_2'}\mathrm{F}(f_1f_2;f_1'f_2')a_{f_1\sigma_1}^\dagger a_{f_2\sigma_2}^\dagger a_{f_2'\sigma_2}a_{f_1'\sigma_1} \\ &+ \sum_{f_2\neq f_2'}\mathrm{F}(f_{1e}f_2;f_1'f_2')a_{f_1e\sigma_1}^\dagger a_{f_2\sigma_2}^\dagger a_{f_2'\sigma_2}a_{f_1'\sigma_1} + \text{c.c.} \\ &+ (1/2)\sum\mathrm{F}(f_{1e}f_{2e};f_1'f_2')a_{f_1e\sigma_1}^\dagger a_{f_2e\sigma_2}^\dagger a_{f_2'\sigma_2}a_{f_1'\sigma_1} + \text{c.c.} \\ &+ \sum_{f_1\neq f_1';\,f_2\neq f_2'}\mathrm{F}(f_{1e}f_2;f_{1e}'f_2')a_{f_1e\sigma_1}^\dagger a_{f_2\sigma_2}^\dagger a_{f_2'\sigma_2}a_{f_1'\sigma_1} +\end{aligned}$$

* Thus $\varepsilon\boldsymbol{H}_1$ contain the terms of $f'=f$ in view of the above situations. However, we shall find this evaluation of order of magnitude does not lead in error in the following calculations.

$$+\sum \mathrm{F}(f_{1e}f_2\,;f_1'f_{2e}')\,a^{\dagger}_{f_{1e}\sigma_1}a^{\dagger}_{f_{2}\sigma_2}a_{f'_{2}e\sigma_2}a_{f_1'\sigma_1}$$

$$+\sum_{f_1\neq f_1'}\mathrm{F}(f_{1e}f_2\,;f_{1e}'f_{2e}')\,a^{\dagger}_{f_{1e}\sigma_1}a^{\dagger}_{f_{2}\sigma_2}a_{f_2'e\sigma_2}a_{f'_1e\sigma_1}+\mathrm{c.c.}$$

$$+(1/2)\sum_{1\neq f_1';\,f_2\neq f_2'}\mathrm{F}(f_{1e}f_{2e}\,;f_{1e}'f_{2e}')\,a^{\dagger}_{f_{1e}\sigma_1}a^{\dagger}_{f_{2e}\sigma_2}a_{f_2'e\sigma_2}a_{f_1'e\sigma_1}, \tag{5}$$

where each term is designated as $\varepsilon^2\boldsymbol{H}_2(f_1f_2\,;f_1'f_2')$, $\varepsilon^2\boldsymbol{H}_2(f_{1e}f_2\,;f_1'f_2')$, and so on. The space of the states C_0 has 0-eigenvalue for N_{fe}, which results into $a_{fe\sigma}a_{f_1\sigma_1}\cdots a_{f_n\sigma_n}\mathrm{P}=0$. Only the combinations of

$$a_{f_1e\sigma_1}\cdots a_{f_se\sigma s}\,a^{\dagger}_{f'_se\sigma_s'}\cdots a^{\dagger}_{f'_1e\sigma_1'}\cdots\mathrm{P}\quad\text{for}(f_1\cdots f_s)=(f_1'\cdots f_s')$$

give nonvanishing contributions.

The structure of unperturbed energy levels is easily analysed by the formula

$$\Delta\langle\boldsymbol{H}_0\rangle\cong\sum_f[\mathrm{E}_f+(f|\widetilde{V}-\widetilde{U}_f|f)+\frac{1}{2}\,(ff|ff)]\,\Delta\mathrm{N}_f+\frac{1}{2}\sum(f_1f_2|f_1f_2)\,\Delta\mathrm{N}_{f_1}\Delta\mathrm{N}_{f_2}$$

$$+\sum_f[\mathrm{E}_{fe}+(f_e|\widetilde{V}-\widetilde{U}_f|f_e)+(f_ef|ff_e)-\frac{1}{2}(f_ef_e|f_ef_e)]\Delta\mathrm{N}_{fe}$$

$$+\frac{1}{2}\sum(f_{1e}f_{2e}|f_{1e}f_{2e})\,\Delta\mathrm{N}_{f_{1e}}\Delta\mathrm{N}_{f_{2e}}+\sum(f_1f_{2e}|f_1f_{2e})\,\Delta\mathrm{N}_{f_1}\Delta\mathrm{N}_{f_{2e}}, \tag{6}$$

from which Table I is composed for the particular transitions. Thus, we have finished the necessary preparation to investigate the contribution from excited orbitals.

§ 3. Spin dependent energies to be added

We can easily handle the operators for excited-state orbitals in considering their relation to the projection operator P. For example, the number of terms to connect P is much reduced in $\boldsymbol{H}_1$, and $\boldsymbol{H}_2$,

$$\varepsilon\boldsymbol{H}_1\mathrm{P}=\sum_{f\neq f'}\overleftarrow{\Lambda}(f\,;f')\,a^{\dagger}_{f\sigma}a_{f'\sigma}\mathrm{P}+\sum_{f\neq f'}\overleftarrow{\Lambda}(f_e\,;f')\,a^{\dagger}_{fe\sigma}a_{f'\sigma}\mathrm{P},$$

$$\varepsilon^2\boldsymbol{H}_2\mathrm{P}=\varepsilon^2\{\sum_{f_1'\neq f_1;\,f_2''\neq f_2}\boldsymbol{H}_2(f_1f_2\,;f_1'f_2'')+\sum_{f_2'\neq f_2}\boldsymbol{H}_2(f_{1e}f_2\,;f_1'f_2')+\sum\boldsymbol{H}_2(f_{1e}f_{2e}\,;f_1'f_2')\}\mathrm{P}$$

$$\varepsilon\mathrm{P}\boldsymbol{H}_1=\mathrm{P}\{\sum_{f\neq f'}\overrightarrow{\Lambda}(f\,;f')\,a^{\dagger}_{f\sigma}a_{f'\sigma}+\sum_{f\neq f'}\Lambda(f\,;f_e')\,a^{\dagger}_{f\sigma}a_{fe'\sigma}\},$$

and

$$\varepsilon^2\mathrm{P}\boldsymbol{H}_2=\varepsilon^2\mathrm{P}\{\sum_{f_1'\neq f_1;\,f_2\neq f_2''}\boldsymbol{H}_2(f_1f_2\,;f_1'f_2'')+\sum_{f_2\neq f_2'}\boldsymbol{H}_2(f_1f_2\,;f_{1e}'f_2')+\sum\boldsymbol{H}_2(f_1f_2\,;f_{1e}'f_{2e}')\}, \tag{7}$$

where Λ is the matrix of order of ε and expressed as

$$\overleftarrow{\Lambda}(\alpha\,;f')=\mathrm{L}(\alpha\,;f')+\sum_{f''}\mathrm{F}(\alpha f''\,;f'f'')\mathrm{N}_{f''}-\mathrm{F}(\alpha f'\,;f'f')$$

and

$$\Lambda(f\,;\alpha')=\mathrm{L}(f\,;\alpha')+\sum_{f''}\mathrm{F}(ff''\,;\alpha'f'')\mathrm{N}_{f''}-\mathrm{F}(ff\,;\alpha'f). \tag{8}$$

Table I. The increment of unperturbed energies for particular transitions including the excited-state orbital of negative ions.

Position of hole	Position of electron	Increment of energy
b	b_e	$\Delta(b_e;\, b)\cong \mathrm{E}_{b_e}+(b_e\|\tilde{V}-\tilde{U}_b\|b_e)-\mathrm{E}_b-(b\|\tilde{V}-\tilde{U}_b\|b)+(b_eb\|bb_e)-(bb_e\|bb_e)$
b_1	b_{2e}	$\Delta(b_{2e};\, b_1)\cong \mathrm{E}_{b_e}+(b_e\|\tilde{V}-\tilde{U}_b\|b_e)-\mathrm{E}_b-(b\|\tilde{V}-\tilde{U}_b\|b)+(b_eb\|bb_e)-(b_1b_{2e}\|b_1b_{2e})$
a	b_e	$\Delta(b_e;\, a)\cong \mathrm{E}_{b_e}+(b_e\|\tilde{V}-\tilde{U}_b\|b_e)-\mathrm{E}_a-(a\|\tilde{V}-\tilde{U}_a\|a)+(b_eb\|bb_e)-(ab_e\|ab_e)$
b, b	b_e, b_e	$\Delta(b_eb_e;\, bb)\cong 2[\mathrm{E}_{b_e}+(b_e\|\tilde{V}-\tilde{U}_b\|b_e)-\mathrm{E}_b-(b\|\tilde{V}-\tilde{U}_b\|b)]+(bb\|bb)+(b_eb_e\|b_eb_e)+2(b_eb\|bb_e)-4(bb_e\|bb_e)$
a_1, a_2	b_e, b_e	$\Delta(b_eb_e;\, a_1a_2)\cong 2[\mathrm{E}_{b_e}+(b_e\|\tilde{V}-\tilde{U}_b\|b_e)-\mathrm{E}_a-(a\|\tilde{V}-\tilde{U}_a\|a)]+(b_eb_e\|b_eb_e)+(a_1a_2\|a_1a_2)+2(b_eb\|bb_e)-2(a_1b_e\|a_1b_e)-2(a_2b_e\|a_2b_e)$

In the expression

$$\varepsilon\tilde{\boldsymbol{H}}_1=\varepsilon\mathrm{P}\boldsymbol{H}_1\mathrm{P}$$

and

$$\varepsilon^2\tilde{\boldsymbol{H}}_2=\varepsilon^2\mathrm{P}\boldsymbol{H}_2\mathrm{P}-\varepsilon^2\mathrm{P}\boldsymbol{H}_1(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1\mathrm{P},\ \text{(Eq. (11) in I)}$$

we see that no spin-dependent force is obtained. We shall designate the term which appears in A_1 afresh as ΔA_1, whose spin-dependent part is given by

$$\Delta A_1=\sum_{a_1\neq a_2}\frac{\mathrm{F}(a_1a_2;\, f_ea_1)\overleftarrow{\Delta}(f_e;\, a_2)}{\Delta(f_e;\, a_2)}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2}). \tag{9}$$

We obtain similar terms for $A_1^\dagger$ which correspond to the reverse process.

$$\Delta A_1^\dagger=\sum_{a_1\neq a_2}\frac{\Delta(a_2;\, f_e)\mathrm{F}(a_1f_e;\, a_2a_1)}{\Delta(f_e;\, a_2)}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2}) \tag{10}$$

In the expressions (9) and (10) there appear the triply ionized negative and positive ions, i.e. the configuration $[\mathrm{O}^{-3*}-\mathrm{Mn}^{+3}]$. We have four new contributions in A_2 (Eq. (24) in I),

$$\begin{aligned}\Delta A_2=\varepsilon^3\mathrm{P}\{&\sum\boldsymbol{H}_1(f_1;\, f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e};\, f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f;\, f')+\text{c.c.}\\&+\sum\boldsymbol{H}_1(f_1;\, f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2;\, f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_e;\, f')\\&+\sum\boldsymbol{H}_1(f_1;\, f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e};\, f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_e;\, f')\}\mathrm{P},\end{aligned} \tag{11}$$

which are treated by using the extended R-matrix with suffices α

$$\begin{aligned}\bar{\mathrm{R}}(\alpha_1\alpha_2;\, \alpha_1'f_2')=[&\mathrm{L}(\alpha_1;\, \alpha_1')+\sum_{f''}\mathrm{F}(\alpha_1f'';\, \alpha_1'f'')\ \{\mathrm{N}_{f''}+\delta(\alpha_2-f'')\\&-\delta(\alpha_1'-f'')-\delta(f_2'-f'')\}\end{aligned}$$

$$+\sum_{f''}F(\alpha_1 f_e'';\alpha_1' f_e''\{\delta(\alpha_2-f_e'')-\delta(\alpha_1'-f_e'')\}]\frac{\vec{\Lambda}(\alpha_2;f_2')}{\Delta(\alpha_2;f_2')}$$

and

$$\vec{\mathrm{R}}(f_1\alpha_2;\alpha_1'\alpha_2')=\frac{\vec{\Lambda}(f_1;\alpha_1')}{\Delta(\alpha_1';f_1)}[\mathrm{L}(\alpha_2;\alpha_2')+\sum_{f''}\mathrm{F}(\alpha_2 f'';\alpha_2' f'')$$
$$\times\{\mathrm{N}_{f''}+\delta(\alpha_1'-f'')-\delta(\alpha_2-f'')-\delta(f_1-f'')\}$$
$$+\sum_{f''}\mathrm{F}(\alpha_2 f_e'';\alpha_2' f_e'')\{\delta(\alpha_1'-f_e'')-\delta(\alpha_2-f_e'')\}]. \tag{12}$$

The four terms in (11) give the following spin-dependent force,

$$\Delta A_2=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\vec{\mathrm{R}}(a_1a_2;f_e a_1)\vec{\Lambda}(f_e;a_2)}{\Delta(f_e;a_2)}+2\frac{\vec{\mathrm{R}}(a_1f_e;f_e a_2)\vec{\Lambda}(a_2;a_1)}{\Delta(a_2;a_1)}\right\}. \tag{13}$$

In this expression (13) the configuration $[Mn^{+3}-O^{-3*}]$ interacts mutually and with the configuration $[Mn^{+3}-O^{--}-Mn^{+}]$, respectively.

Next let us proceed to $\varepsilon^4\tilde{\boldsymbol{H}}_4$ which consists of five B's. B_1 has a new contribution which is expressed as follows,

$$\Delta B_1=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\mathrm{F}(a_1g;f_e a_1)\mathrm{F}(f_e a_2;a_2 g)}{\Delta(f_e;g)}\right.$$
$$+2\frac{\mathrm{F}(a_1a_2;f_e a_1)\,\mathrm{F}(f_e f;fa_2)}{\Delta(f_e;a_2)}-2\frac{\mathrm{F}(fa_2;f_e a_1)\,\mathrm{F}(f_e a_1;fa_2)}{\Delta(f_e a_1;fa_2)}$$
$$\left.+2\frac{\mathrm{F}(a_1f;f_e a_2)\,\mathrm{F}(f_e a_2;fa_1)}{\Delta(f_e a_2;fa_1)}-\frac{1}{2}\frac{\mathrm{F}(a_1a_2;f_e g_e)\,\mathrm{F}(g_e f_e;a_1a_2)}{\Delta(g_e f_e;a_1a_2)}\right\}, \tag{14}$$

where the first term acts ferromagnetically through the configuration in the Slater-model, i.e. $[Mn^{+2}-O^{-2*}-Mn^{+2}]$, the second term through the configuration $[O^{-3*}-Mn^{+3}]$ and in the other terms the configurations $[Mn^{+}-O^{-2*}-Mn^{+3}]$, $[Mn^{+3}-O^{-4*}-Mn^{+3}]$ or $2[O^{-3*}-Mn^{+3}]$ interact through a kind of antiferromagnetic delocalization effects, respectively.

We obtain the following new contribution for B_2,

$$\Delta B_2=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\left\{\frac{\Lambda(a_1;f_e)\mathrm{F}(f_e a_2;a_2g_e)\vec{\Lambda}(g_e;a_1)}{\Delta(f_e;a_1)\Delta(g_e;a_1)}\right.$$
$$-\frac{\Lambda(a_1;f_e)\mathrm{F}(a_2g;ga_1)\vec{\Lambda}(f_e;a_2)}{\Delta(f_e;a_1)\Delta(f_e;a_2)}-2\frac{\vec{\Lambda}(a_1;f_e)\,\mathrm{F}(f_e f;fa_2)\vec{\Lambda}(a_2;a_1)}{\Delta(f_e;a_1)\Delta(a_2;a_1)}$$
$$\left.-2\frac{\vec{\Lambda}(a_1;f_e)\mathrm{F}(f_e g;a_2a_1)\vec{\Lambda}(a_2;g)}{\Delta(f_e;a_1)\Delta(a_2;g)}\right\}. \tag{15}$$

In the above expression, the configuration $[Mn^{+3}-O^{-3*}]$ interacts mutually, and with the band approximation one, i.e. $[Mn^{+3}-O^{--}-Mn^{+}]$, and the Kramers-Anderson's one, i.e. $[O^{-}-Mn^{+}]$, respectively.

We shall proceed to treat B_3 in which many contributions appear (Eq. (30) in I).

$$\begin{aligned}
\Delta B_3 = -\varepsilon^3 \mathrm{P}\sum{}'\{ & \boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e}; f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{3e}; f_{3e}') \\
& (\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_e; f') \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e}; f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_3; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f_e; f')+\text{c.c.} \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e}; f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{3e}; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f; f')+\text{c.c.} \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2; f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{3e}; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f_e; f') \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e}; f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_3; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f; f')+\text{c.c.} \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2; f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{3e}; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f; f')+\text{c.c.} \\
& +\boldsymbol{H}_1(f_1; f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2; f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_3; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f_e; f') \\
& +\boldsymbol{H}_1(f_1; f_1')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2; f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{3e}; f_3')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1} \\
& \boldsymbol{H}_1(f; f')\}\mathrm{P} \qquad (16)
\end{aligned}$$

The first term is certainly spin-independent. The four terms from the second to the 5th contain only higher order terms than $0(\varepsilon^6)$. The 6th term gives the coupling in which the right-angled case seems to give stronger coupling than in the linear case, and the configuration $[\mathrm{Mn}^{+3}-\mathrm{O}^{-3*}]$ interacts with $[\mathrm{Mn}^{+3}-\mathrm{O}^{-4*}-\mathrm{Mn}^{+3}]$ or $2[\mathrm{Mn}^{+3}-\mathrm{O}^{-3*}]$. From the 7th and 8th terms we obtain the configuration $[\mathrm{Mn}^{+3}-\mathrm{O}^{-3*}]$ instead of the Kramers-Anderson configuration (Eq. (43) in I). In the 9th and 10th, the Slater-model configuration, i.e. $[\mathrm{Mn}^{+2}-\mathrm{O}^{-2*}]$, plays an important role. In the 11th term, the two configurations $[\mathrm{Mn}^{+3}-\mathrm{O}^{-3*}]$ interact through the Slater-model configuration. In the last term the Slater-model configuration interacts with the band approximation configurations and the Kramers-Anderson's, respectively.

Thus on summing up we obtain for ΔB_3,

$$\begin{aligned}
\Delta B_3 = -\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\Big\{ & -\frac{\vec{\mathrm{R}}(a_1a_2; f_eg_e)\vec{\mathrm{R}}(f_eg_e; a_2a_1)}{\Delta(f_eg_e; a_1a_2)} \\
& -\frac{\vec{\mathrm{R}}(a_1a_2; f_eg_e)\vec{\mathrm{R}}(g_ef_e; a_1a_2)}{\Delta(g_ef_e; a_1a_2)}+2\frac{\vec{\mathrm{R}}(a_1f_e; f_ea_2)\vec{\mathrm{R}}(ga_2; a_1g)}{\Delta(a_2; a_1)}
\end{aligned}$$

$$+2\frac{\vec{R}(a_1f;f_ea_2)\overleftarrow{R}(f_ea_2;fa_1)}{\Delta(f_ea_2;fa_1)}+2\frac{\vec{R}(a_1g;f_ea_1)\overleftarrow{R}(f_ea_2;a_2g)}{\Delta(f_e;g)}$$
$$+\frac{\vec{R}(a_1g;f_ea_1)\overleftarrow{R}(a_2f_e;ga_2)}{\Delta(f_e;g)}-2\frac{\vec{R}(a_1f;a_2f_e)\overleftarrow{R}(f_ea_2;fa_1)}{\Delta(f_ea_2;fa_1)}$$
$$+\frac{\vec{R}(fa_2;a_2f_e)\overleftarrow{R}(f_ea_1;a_1f)}{\Delta(f_e;f)}+2\frac{\vec{R}(fa_2;a_1f_e)\overleftarrow{R}(f_ea_1;fa_2)}{\Delta(f_ea_1;fa_2)}\Big\}. \quad (17)$$

Finally we shall treat ΔB_4 and $\Delta B_4^{\dagger}$.

$$\Delta B_4=\varepsilon^4\mathrm{P}\sum\{\boldsymbol{H}_1(f_1;f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e};f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_2(f_{3e}f;f_3'f')$$
$$+\boldsymbol{H}_1(f_1;f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_{2e};f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}(1-\mathrm{P})\boldsymbol{H}_2(f_3f;f_3'f')$$
$$+\boldsymbol{H}_1(f_1;f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2;f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_2(f_{3e}f_e;f_3'f')$$
$$+\boldsymbol{H}_1(f_1;f_{1e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_1(f_2;f_2')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_2(f_{3e}f;f_3'f')$$
$$+\boldsymbol{H}_1(f_1;f_1')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}(1-\mathrm{P})\boldsymbol{H}_1(f_2;f_{2e}')(\boldsymbol{H}_0-\mathrm{E}_0)^{-1}\boldsymbol{H}_2(f_{3e}f;f_3'f')\}\mathrm{P} \quad (18)$$

The first gives the coupling between the spins of $0(\varepsilon^6)$. The second term gives the coupling in which the right-angled case seems weak by directional property of g-orbital in F. In the third term we may expect stronger coupling in the right-angled case, and the excited configuration $[Mn^{+3}-O^{-3*}]$ interacts with the configuration $[Mn^{+3}-O^{-2}-Mn^{+}]$ and $[Mn^{+3}-O^{-4*}-Mn^{+3}]$, respectively. For the 4th term we obtain the interaction in which the Slater-model configuration interacts with $[Mn^{+3}-O^{-2*}-Mn^{+}]$ and $[Mn^{+3}-O^{-3*}]$, and the configurations $[O^{-3*}-Mn^{+3}]$ interact mutally. In the last term for ΔB_4, we see that the configurations $[Mn^{+3}-O^{-2}-Mn^{+}]$ and $[Mn^{+3}-O^{-2*}-Mn^{+}]$ interact with the configuration $[Mn^{+3}-O^{-3*}]$ or $[Mn^{+}-O^{-}]$, respectively.

We shall obtain the similar coefficients for $\Delta B_4^{\dagger}$ and gather them up to obtain

$$\Delta B_4+\Delta B_4^{\dagger}=-\sum_{a_1\neq a_2}\frac{1}{2}(1+\sigma_{a_1}\cdot\sigma_{a_2})\,2\Big\{-\frac{\vec{R}(a_1f_e;f_ea_2)\,\mathrm{F}(a_2g,ga_1)}{\Delta(a_2;a_1)}$$
$$+\frac{\vec{R}(a_1a_2;f_eg_e)\,\mathrm{F}(g_ef_e;a_1a_2)}{\Delta(g_ef_e;a_1a_2)}-\frac{\vec{R}(a_1g;g_ea_2)\mathrm{F}(g_ea_2;ga_1)}{\Delta(g_ea_2;ga_1)}$$
$$-\frac{\vec{R}(a_1g;g_ea_1)\mathrm{F}(g_ea_2;a_2g)}{\Delta(g_e;g)}-\frac{\vec{R}(a_1a_2;g_ea_1)\mathrm{F}(g_eg;ga_2)}{\Delta(g_e;a_2)}$$
$$-2\frac{\vec{R}(a_1g;a_2g_e)\mathrm{F}(g_ea_2;ga_1)}{\Delta(g_ea_2;ga_1)}-\frac{\vec{R}(ga_2;a_2f_e)\mathrm{F}(f_ea_1;a_1g)}{\Delta(f_e;g)}$$
$$-\frac{\vec{R}(a_1a_2;a_2g_e)\mathrm{F}(g_eg;ga_1)}{\Delta(g_e;a_1)}-\frac{\vec{R}(a_1g;a_2g_e)\,\mathrm{F}(g_ea_2;a_1g)}{\Delta(g_ea_2;a_1g)}$$
$$-\frac{\vec{R}(ga_2;a_1g_e)\,\mathrm{F}(g_ea_1;ga_2)}{\Delta(g_ea_1;ga_2)}\Big\}. \quad (19)$$

§ 4. Discussions

Thus, we have obtained all the new contributions from the excited state orbitals for superexchange interactions: Eqs. (9), (10), (13), (14), (15), (17) and (19). It is shown that the spin-dependent energy in a magnetic compound can still be expressed in terms of a familiar spin operator formalism, even if we consider all the configurations and configuration-interactions derived from the excited state orbitals for negative ions. We note that in this approximation the configurations which have higher negative ions, i.e. $[O^{-3*}]$ or $[O^{-4*}]$ appear in addition to the Slater-model configuration $[Mn^{+2}-O^{-2*}-Mn^{+2}]$.

As we have mentioned in the previous paper, the effective exchange integral has a very complicated expression, because we consider the problem not by simple intuitive models, but from a quite general standpoint, and moreover we take up all the terms that appear in the order of ε^4. In the present treatment of the excited state orbitals as well as in the previous paper for the ground state orbitals, however, we have formulated the theory in terms of practicable quantities. Therefore, we may expect that the present formalism is still applicable to simple real crystals.

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An Expansion Theorem for the Electric Conductivity of Metals. I

——*Electric Conductivity for Longitudinal Electric Field*——

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A systematic diagram representation in a composite 4-dimensional space is developed for Kubo's response function which describes the electric response currents of metals for longitudinal electric fields. Proper diagrams are defined as the Feynman type linked diagrams which cannot be decomposed into simpler diagrams connected only by one Coulomb line. The greatest care is exercised with reference to the fact that Kubo's formula for the conduction phenomena gives the transport coefficient $\chi(\boldsymbol{q}, \omega)$ defined as the ratio of the electric current vector to the electric displacement vector $\boldsymbol{D}(\boldsymbol{q}, \omega)$, while the electric conductivity $\sigma(\boldsymbol{q}, \omega)$ of a metal is defined as the electric current vector divided by the electric field vector $\boldsymbol{E}(\boldsymbol{q}, \omega)$ in the metal. Thus $\sigma(\boldsymbol{q}, \omega)$ is written as the product of $\chi(\boldsymbol{q}, \omega)$ and the dielectric constant of the metal. It is shown that, the product is reduced to a simple form. In the reduced form, $\sigma(\boldsymbol{q}, \omega)$ is expressed as the sum of the proper diagrams. In this expression the lowest order term in respect to the Coulomb interaction includes the usual sum on ring diagrams and, moreover, constitutes a much better approximation than the ring approximation.

§ 1. Introduction

An exact formal expression for electric conductivity in, say, a metal has been given by Kubo and Nakano.[1] In their theory the electric current is interpreted as the response current for an external force which is adiabatically applied to the system and in the Taylor expansion of the response current in terms of the external force, the coefficient of the linear term which has been interpreted as the electric conductivity can be written down in an exact formal expression. The observed conductivity, however, is the quotient of the response current divided by the macroscopic electric field in the system.

This fact has been overlooked in actual calculations hitherto made theoretically on the basis of Kubo's formalism; Kubo's formula which was intended to give the coefficient of the external field has been regarded as a formula for the coefficient of the electric field. Nevertheless reasonable results have been derived through such calculations. It should be noted here that in these calculations one has completely neglected the Coulomb interaction between electrons as well as one has replaced the external field by the electric field.

The electric field in a metal is much different from the external field. A finite external field induces a large polarization of the electron cloud in a metal

and is screened almost completely by the polarization charge. Thus, the strength of the electric field in the metal is almost zero in spite of the presence of the finite external field.†

Several works[2] have recently appeared which are intended to develop a general scheme to calculate exactly the electric conductivity of a many-electron system on the basis of Kubo's formalism. In such treatments, however, one must be very careful about the difference between the two fields: One should not regard the transport coefficient given by Kubo's formula as a real conductivity of the many-electron system. Otherwise, one would obtain an absurd result, because the Coulomb interaction between electrons together with the interaction between the electron system and the external field leads to a strong screening for the field and, consequently, one would obtain a vanishing conductivity especially in the limit of infinitely long wave length of the external field.

In this paper we will develop a general scheme to calculate directly the electric conductivity of a metal without going into a calculation of the coefficient of the external field. The conventional treatment in which the external field is replaced by the electric field and the Coulomb interaction is neglected is shown to be a fairly good approximation.

In this paper the external field is taken to be a longitudinal field which may be oscillating. The electro-magnetic responses for a general electro-magnetic field will be considered in a forthcoming paper. A system composed of an electron gas and a phonon assembly is investigated here. A system which contains impurities will be treated elsewhere.

§ 2. Kubo's formalism

The electron-phonon system, whose electric properties will be investigated, is taken to be in thermal equilibrium before the application of the external field. In the equilibrium state, the system is described by the grand canonical ensemble with the following Hamiltonian:

$$H=H_0+H', \tag{2·1}$$

$$H_0=\sum_{k}\varepsilon_k a_k^* a_k+\sum_{k}\omega_k b_k^* b_k, \tag{2·2}$$

$$H'=\sum_{k\neq 0}\frac{4\pi e^2}{k^2}\sum_{l,m} a_{l+k}^* a_l a_{m-k}^* a_m + \sum_{k,l}\{v(\boldsymbol{k}:\boldsymbol{l})a_l^* a_{l+k} b_k^*+\text{C.C.}\}, \tag{2·3}$$

$(\hbar=1)$

† This fact was pointed out by Prof. Y. Toyozawa (private communications). A careful discussion on this problem is found in the note of S. Nakajima: Busseiron Kenkyu II 8 (1960), 340 (a mimeographed circular in Japanese).

where a_k is the destruction operator for an electron of momentum $\boldsymbol{k}$, ε_k denotes the energy of this electron, b_k is the destruction operator for a phonon of momentum $\boldsymbol{k}$, ω_k denotes the frequency of the normal mode described by this phonon, and $v(\boldsymbol{k}:\boldsymbol{l})$ represents the strength of the electron-phonon interaction. In the above, we have used the assumption that the negative charge of our electron system is cancelled completely by a uniform positive charge which is not a constant but is equal to the total number of electrons multiplied by e.

The external field which is applied to the system is produced by some external charges. These external charges are placed on the two plates of a condenser between which our system is held, or they are produced in the batteries. These charges are called "true charges". The electric field produced only by the true charges is called the electric displacement $\mathbf{D}(\boldsymbol{x}, t)$, which is nothing but the external field. Thus, the system in the presence of the external field is described by the following Hamiltonian:

$$\mathcal{H}=H+\int\phi(\boldsymbol{x}, t)\,\mathfrak{N}(\boldsymbol{x})\,d\boldsymbol{x}, \tag{2·4}$$

where

$$\mathfrak{N}(\boldsymbol{x})=-e(\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})-N), \tag{2·5}$$

and

$$\mathbf{D}(\boldsymbol{x}, t)=-\nabla\phi(\boldsymbol{x}, t). \tag{2·6}$$

In the above, $\psi(\boldsymbol{x})$ is the quantized wave function for the electron cloud, N is the total number of electrons, and then eN is the density of the uniform positive charge; the volume of the system is taken to be unity.

The density matrix $\rho(t)$ describing our system obeys the following equation of motion,

$$i\dot{\rho}(t)=[\mathcal{H}, \rho(t)].$$

Further, writing $\rho(t)$ as

$$\rho(t)=\rho+\Delta\rho(t),$$

where ρ represents the equilibrium density matrix for the grand canonical ensemble, we must impose on $\Delta\rho(t)$ the following boundary condition

$$\Delta\rho(t)\xrightarrow{t\to-\infty}0$$

$$\left(\phi(\boldsymbol{x}, t)\xrightarrow{t\to-\infty}0\right),$$

due to the assumption mentioned previously. Then we get in the linear approximation,

$$\Delta\rho(t)=\frac{1}{i}\int_{-\infty}^{t}dt'\int d\boldsymbol{x}'\,\phi(\boldsymbol{x}', t')\,e^{iH(t'-t)}[\mathfrak{N}(\boldsymbol{x}'), \rho]e^{-iH(t'-t)}. \tag{2·7}$$

Now, the current operator is denoted by

$$\mathfrak{J}(\boldsymbol{x})=\frac{e}{2i\mathfrak{m}}\left\{\phi^*(\boldsymbol{x})(\nabla\phi(\boldsymbol{x}))-(\nabla\phi^*(\boldsymbol{x}))\phi(\boldsymbol{x})\right\}. \tag{2·8}$$

Then the observed current is given by

$$\mathbf{J}(\boldsymbol{x}, t)=\mathrm{Tr}\{\rho(t)\mathfrak{J}(\boldsymbol{x})\}=\mathrm{Tr}\{\Delta\rho(t)\,\mathfrak{J}(\boldsymbol{x})\}.$$

Inserting Eq. (2·7) into the above expression, we get

$$\mathbf{J}(\boldsymbol{x}, t)=\frac{1}{i}\int_{-\infty}^{t}dt'\int d\boldsymbol{x}'\phi(\boldsymbol{x}', t')\langle[\mathfrak{J}(\boldsymbol{x}, t-t'),\ \mathfrak{N}(\boldsymbol{x}')]\rangle, \tag{2·9}$$

where

$$\langle\mathfrak{A}\rangle\equiv\mathrm{Tr}\{\rho\mathfrak{A}\},$$

and

$$\mathfrak{J}(\boldsymbol{x}, t)\equiv e^{iHt}\mathfrak{J}(\boldsymbol{x})\,e^{-iHt}.$$

If we define

$$\mathfrak{N}(\boldsymbol{x}, t)\equiv e^{iHt}\mathfrak{N}(\boldsymbol{x})\,e^{-iHt}, \tag{2·10}$$

it can be easily seen that the equation of continuity is satisfied by $\mathfrak{N}(\boldsymbol{x}, t)$ and $\mathfrak{J}(\boldsymbol{x}, t)$ given in Eqs. (2·10) and (2·8) respectively. The Fourier transform of the spatial component of this continuity equation is written as

$$\dot{\mathfrak{N}}(\boldsymbol{q}, t)=-i\boldsymbol{q}\cdot\mathfrak{J}(\boldsymbol{q}, t). \tag{2·11}$$

If the applied field is periodic, i.e.,

$$\phi(\boldsymbol{x}, t)=e^{i\boldsymbol{q}\cdot\boldsymbol{x}+i\omega t}\phi(\boldsymbol{q}, \omega),^{\dagger} \tag{2·12}$$

then

$$\mathbf{J}(\boldsymbol{x}, t)=e^{i\boldsymbol{q}\cdot\boldsymbol{x}+i\omega t}\mathbf{J}(\boldsymbol{q}, \omega),$$

and Eq. (2·9) is expressed as

$$\mathbf{J}(\boldsymbol{q}, \omega)=\frac{1}{i}\int_{0}^{\infty}dt\,e^{-i\omega t}\langle[\mathfrak{J}(\boldsymbol{q}, t),\ \mathfrak{N}(-\boldsymbol{q})]\rangle\phi(\boldsymbol{q}, \omega). \tag{2·13}$$

Noting that $\mathbf{J}(\boldsymbol{q}, \omega)$ is parallel to $\boldsymbol{q}$ and

$$\mathbf{D}(\boldsymbol{q}, \omega)=-i\boldsymbol{q}\phi(\boldsymbol{q}, \omega)$$

and inserting Eq. (2·11) into Eq. (2·13), we obtain

$$\mathbf{J}(\boldsymbol{q}, \omega)=\chi(\boldsymbol{q}, \omega)\,\mathbf{D}(\boldsymbol{q}, \omega), \tag{2·14}$$

where[3)]

† The convergence factor is omitted here. It will be omitted sometimes in order to save notations.

$$\chi(\boldsymbol{q}, \omega) = \lim_{\delta \to +0} \frac{i}{q^2} \int_0^\infty dt\, e^{-i\omega t - \delta t} \langle [\dot{\mathfrak{N}}(\boldsymbol{q}, t),\ \mathfrak{N}(-\boldsymbol{q})] \rangle$$

$$= \lim_{\delta \to +0} \frac{\omega}{q^2} \int_0^\infty dt\, e^{-i\omega t - \delta t} \langle [\mathfrak{N}(-\boldsymbol{q}),\ \mathfrak{N}(\boldsymbol{q}, t)] \rangle. \qquad (2\cdot15)$$

In the above we have used the following fact,

$$[\mathfrak{N}(\boldsymbol{q}),\ \mathfrak{N}(-\boldsymbol{q})] = 0.$$

It should be noted that the usual expression (2·15) is a formula for the transport coefficient χ defined by Eq. (2·14) and does not give the conductivity in itself.

Now, the dielectric constant is defined by

$$\mathbf{D}(\boldsymbol{q}, \omega) = \epsilon(\boldsymbol{q}, \omega)\, \mathbf{E}(\boldsymbol{q}, \omega), \qquad (2\cdot16)$$

where $\mathbf{E}$ is the macroscopic electric field in the metal. Then, the conductivity σ, which is defined by

$$\mathbf{J}(\boldsymbol{q}, \omega) = \sigma(\boldsymbol{q}, \omega)\, \mathbf{E}(\boldsymbol{q}, \omega), \qquad (2\cdot17)$$

is expressed as

$$\sigma(\boldsymbol{q}, \omega) = \epsilon(\boldsymbol{q}, \omega)\, \chi(\boldsymbol{q}, \omega). \qquad (2\cdot18)$$

As we shall see in the next section,

$$\epsilon(\boldsymbol{q}, \omega) = \left[1 - \frac{4\pi}{i\omega} \chi(\boldsymbol{q}, \omega)\right]^{-1}, \qquad (2\cdot19)$$

and therefore we get the relation

$$\sigma(\boldsymbol{q}, \omega) = \frac{\chi(\boldsymbol{q}, \omega)}{1 - (4\pi/i\omega)\chi(\boldsymbol{q}, \omega)}. \qquad (2\cdot20)$$

§ 3. Dielectric constant

Eq. (2·19) is easily verified, if we are allowed to use the phenomenological relation

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{J} = \dot{\mathbf{P}}.$$

However, it may be inadequate to make use of the phenomenological equation without any proof. Then we shall prove Eq. (2·19) on the basis of the fundamental equations by means of Kubo's linear approximation.

The electric field $\hat{\mathbf{E}}(\boldsymbol{x}, t)$ produced by the matter field as well as the external field is expressed as an operator, and the macroscopic electric field $\mathbf{E}(\boldsymbol{x}, t)$ is given by

$$\mathbf{E}(\boldsymbol{x}, t) = \mathrm{Tr}\{\rho(t)\, \hat{\mathbf{E}}(\boldsymbol{x}, t)\}. \qquad (3\cdot1)$$

This may be written as

$$\varphi(\boldsymbol{x}, t)=\mathrm{Tr}\{\rho(t)\hat{\varphi}(\boldsymbol{x}, t)\}, \tag{3·2}$$

where

$$\hat{\mathbf{E}}(\boldsymbol{x}, t)=-\nabla\hat{\varphi}(\boldsymbol{x}, t).$$

and

$$\mathbf{E}(\boldsymbol{x}, t)=-\nabla\varphi(\boldsymbol{x}, t).$$

Let us denote the free charge, the true charge and the polarization charge by $\mathfrak{N}_f$, $\mathfrak{N}_r$ and $\mathfrak{N}$, respectively. Then

$$\mathrm{div}\,\hat{\mathbf{E}}(\boldsymbol{x}, t)=4\pi\mathfrak{N}_f(\boldsymbol{x}, t)=4\pi\mathfrak{N}_r(\boldsymbol{x}, t)+4\pi\mathfrak{N}(\boldsymbol{x})=\mathrm{div}\,\mathbf{D}(\boldsymbol{x}, t)+4\pi\mathfrak{N}(\boldsymbol{x}).$$

Therefore,

$$\hat{\varphi}(\boldsymbol{x}, t)=\phi(\boldsymbol{x}, t)+\int\frac{d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathfrak{N}(\boldsymbol{x}'). \tag{3·3}$$

Substituting Eq. (3·3) in Eq. (3·2), we get

$$\varphi(\boldsymbol{x}, t)=\phi(\boldsymbol{x}, t)+\int\frac{d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{Tr}\{\rho\mathfrak{N}(\boldsymbol{x}')\}+\int\frac{d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{Tr}\{\Delta\rho(t)\mathfrak{N}(\boldsymbol{x}')\}. \tag{3·4}$$

It is easily shown that Fourier coefficients of the second term in the right-hand side of Eq. (3·4) vanish, so far as macroscopic phenomena are concerned.† Therefore, inserting Eq. (2·7) into Eq. (3·4), we obtain

$$\varphi(\boldsymbol{x}, t)=\phi(\boldsymbol{x}, t)+\frac{1}{i}\int_{-\infty}^{t}dt''\int d\boldsymbol{x}'\int d\boldsymbol{x}''\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}$$
$$\times\langle[\mathfrak{N}(\boldsymbol{x}', t-t''),\ \mathfrak{N}(\boldsymbol{x}'')]\rangle\phi(\boldsymbol{x}'', t''). \tag{3·5}$$

The Fourier transform of this is

$$\varphi(\boldsymbol{q}, \omega)=\phi(\boldsymbol{q}, \omega)+\frac{4\pi}{iq^2}\int_0^\infty d\tau e^{-i\omega\tau}\langle[\mathfrak{N}(\boldsymbol{q}, \tau),\ \mathfrak{N}(-\boldsymbol{q})]\rangle\phi(\boldsymbol{q}, \omega). \tag{3·6}$$

Now, the dielectric constant has been defined by Eq. (2·16) or

$$\epsilon(\boldsymbol{q}, \omega)\varphi(\boldsymbol{q}, \omega)=\phi(\boldsymbol{q}, \omega).$$

Therefore, we obtain from Eq. (3·6)

$$\epsilon(\boldsymbol{q}, \omega)=\left[1+\frac{4\pi}{iq^2}\int_0^\infty dt\, e^{-i\omega t}\langle[\mathfrak{N}(\boldsymbol{q}, t),\ \mathfrak{N}(-\boldsymbol{q})]\rangle\right]^{-1}.$$

Thus, Eq. (2·19) has been verified.

† If microscopic phenomena are concerned, on the other hand, it is necessary to consider Fourier components whose wave vectors are comparable to reciprocal lattice vectors in magnitude. The Fourier component of the second term whose wave vector is equal to a reciprocal lattice vector does not vanish.

§ 4. Diagram representation

Here we shall develop a diagram representation of the expression for χ given by Eq. (2·15), by making use of the technique developed by C. Bloch et C. de Dominicis[4] (hereafter referred to as BDD) in the statistical mechanics of equilibrium states.

The following functions are introduced at first:

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, it) \equiv \langle \alpha_{\boldsymbol{l}}^{-\boldsymbol{q}} e^{iHt} \alpha_{\boldsymbol{k}}^{\boldsymbol{q}} e^{-iHt} \rangle = \frac{1}{\mathrm{Z}} \mathrm{Tr}\{e^{-\beta(H-\mu N)} \alpha_{\boldsymbol{l}}^{-\boldsymbol{q}} e^{iHt} \alpha_{\boldsymbol{k}}^{\boldsymbol{q}} e^{-iHt}\}, \quad (4\cdot1)$$

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, it) \equiv \langle e^{iHt} \alpha_{\boldsymbol{k}}^{\boldsymbol{q}} e^{-iHt} \alpha_{\boldsymbol{l}}^{-\boldsymbol{q}} \rangle = \frac{1}{\mathrm{Z}} \mathrm{Tr}\{e^{-\beta(H-\mu N)} e^{iHt} \alpha_{\boldsymbol{k}}^{\boldsymbol{q}} e^{-iHt} \alpha_{\boldsymbol{l}}^{-\boldsymbol{q}}\}, \quad (4\cdot2)$$

where Z is the grand partition function and

$$\alpha_{\boldsymbol{k}}^{\boldsymbol{q}} \equiv a_{\boldsymbol{k}+\boldsymbol{q}}^* a_{\boldsymbol{k}}. \quad (4\cdot3)$$

Thus,

$$\langle [\mathfrak{N}(-\boldsymbol{q}), \mathfrak{N}(\boldsymbol{q}, t)] \rangle = e^2 \sum_{\boldsymbol{k},\boldsymbol{l}} \{\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, it) - \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}. \quad (4\cdot4)$$

Next the following notations are introduced:

$$\left.\begin{aligned} &\alpha_{\boldsymbol{k}}^{\boldsymbol{q}}(u) \equiv \alpha_{\boldsymbol{k}}^{\boldsymbol{q}} e^{u(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon_{\boldsymbol{k}})} = a_{\boldsymbol{k}+\boldsymbol{q}}^* a_{\boldsymbol{k}} e^{u(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon_{\boldsymbol{k}})}, \\ &b_{\boldsymbol{k}}^*(u) \equiv b_{\boldsymbol{k}}^* e^{u\omega_{\boldsymbol{k}}}, \\ &b_{\boldsymbol{k}}(u) \equiv b_{\boldsymbol{k}} e^{-u\omega_{\boldsymbol{k}}}, \end{aligned}\right\} \quad (4\cdot5)$$

and

$$H'(u) \equiv e^{uH_0} H' e^{-uH_0} = \sum_{\boldsymbol{k}\neq0} \frac{4\pi e^2}{k^2} \sum_{\boldsymbol{l},\boldsymbol{m}} \alpha_{\boldsymbol{l}}^{\boldsymbol{k}}(u) \alpha_{\boldsymbol{m}}^{-\boldsymbol{k}}(u)$$
$$+ \sum_{\boldsymbol{k}\neq0} \sum_{\boldsymbol{l}} \{v(\boldsymbol{k}:\boldsymbol{l}) \alpha_{\boldsymbol{l}+\boldsymbol{k}}^{-\boldsymbol{k}}(u) b_{\boldsymbol{k}}^*(u) + v^*(\boldsymbol{k}:\boldsymbol{l}) \alpha_{\boldsymbol{l}}^{\boldsymbol{k}}(u) b_{\boldsymbol{k}}(u)\}. \quad (4\cdot6)$$

Then

$$\left.\begin{aligned} &e^{-\beta H} = e^{-\beta H_0} \cdot \underset{(+)}{e^{-\int_0^\beta du H'(u)}}, \\ &e^{iHt} = \underset{(-)}{e^{-i\int_t^0 du H'(iu)}} \cdot e^{iH_0 t}, \\ &e^{-iHt} = e^{-iH_0 t} \cdot \underset{(+)}{e^{-i\int_0^t du H'(iu)}}, \end{aligned}\right\} \quad (4\cdot7)$$

where we denote by $e_{(+)}$ and $e_{(-)}$ the ordered exponentials

$$\underset{(+)}{e^{\int_a^b du \mathfrak{A}(u)}} \equiv 1 + \sum_{n=1}^{\infty} \int_a^b du_1 \int_a^{u_1} du_2 \cdots \int_a^{u_{n-1}} du_n \mathfrak{A}(u_1) \mathfrak{A}(u_2) \cdots \mathfrak{A}(u_n)$$

and

$$e^{\int_b^a du\,\mathfrak{A}(u)}_{(-)} \equiv 1+\sum_{n=1}^{\infty}\int_b^a du_1\int_b^{u_1} du_2\cdots\int_b^{u_{n-1}} du_n\cdot\mathfrak{A}(u_1)\,\mathfrak{A}(u_2)\cdots\mathfrak{A}(u_n)$$

$(a<b)$,

for any operator $\mathfrak{A}(u)$. Substituting Eq. (4·7) into Eqs. (4·1) and (4·2), we obtain, respectively,

$$\mathrm{F}(\boldsymbol{l},\boldsymbol{k}|0^+,it)=\frac{\mathrm{Z}_0}{\mathrm{Z}}\langle e^{-\int_0^\beta duH'(u)}_{(+)}\cdot\alpha_{\boldsymbol{l}}^{-q}\cdot e^{-i\int_t^0 duH'(iu)}_{(-)}\cdot\alpha_{\boldsymbol{k}}^{q}(it)\cdot e^{-i\int_0^t duH'(iu)}_{(+)}\rangle^0, \qquad (4\cdot1')$$

and

$$\mathrm{F}(\boldsymbol{l},\boldsymbol{k}|0^-,it)=\frac{\mathrm{Z}_0}{\mathrm{Z}}\langle e^{-\int_0^\beta duH'(u)}_{(+)}\cdot e^{-i\int_t^0 duH'(iu)}_{(-)}\cdot\alpha_{\boldsymbol{k}}^{q}(it)\cdot e^{-i\int_0^t duH'(iu)}_{(+)}\cdot\alpha_{\boldsymbol{l}}^{-q}\rangle^0. \qquad (4\cdot2')$$

In the above,

$$\langle\mathfrak{A}\rangle^0\equiv\mathrm{Tr}\,\rho_0\mathfrak{A},$$

where

$$\rho_0\equiv\mathrm{Z}_0^{-1}\,e^{-\beta(H_0-\mu N)}$$

is the grand canonical density matrix for the unperturbed system, whose grand partition function is denoted by Z_0.

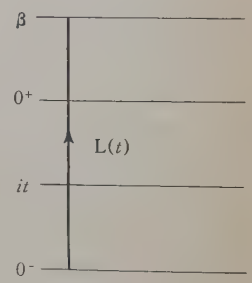


Fig. 1.

In order to describe the "time-ordered" expansion in Eqs. (4·1′) and (4·2′), we here introduce a composite 4-dimensional space as shown in Fig. 1. The abscissa of this figure represents the 3-dimensional configuration space. A path $\mathrm{L}(t)$ is taken along the vertical axis. This path is composed of the following three parts. The first part which is denoted by $[[0^-, it]]$ is composed of the imaginary numbers ranging from zero to it. An imaginary number $i\tau$ $(0\leqq\tau<t)$ is represented by $i\tau^-$ on this part. The second part which is denoted by $[[it, 0^+]]$ is the reflection of $[[0^-, it]]$ with respect to the point it. An imaginary number $i\tau$ $(0\leqq\tau<t)$ is represented by $i\tau^+$ on $[[it, 0^+]]$. The last part which is denoted by $[[0^+, \beta]]$ corresponds to an interval $[0, \beta]$ composed of real numbers.

Referring to the path $\mathrm{L}(t)$, we may write Eqs. (4·1′) and (4·2′) as

$$\mathrm{F}(\boldsymbol{l},\boldsymbol{k}|0^\pm,it)=\frac{\mathrm{Z}_0}{\mathrm{Z}}\langle\mathrm{P}[e^{-\int_{\mathrm{L}(t)}duH'(u)}\cdot\alpha_{\boldsymbol{l}}^{-q}(0^\pm)\cdot\alpha_{\boldsymbol{k}}^{q}(it)]\rangle^0, \qquad (4\cdot8)$$

where P represents Dyson's chronological ordering on the path $\mathrm{L}(t)$, and $\int_{\mathrm{L}(t)}du$ is the path integral along this path.

According to BDD, the contraction between two operators a and a' is defined as

$$a\cdot a'=\langle aa'\rangle^0.$$

In the evaluation of Eq. (4·8), we are met by the statistical average of a product

of some creation and annihilation operators. The average is taken in the unperturbed system, where the density matrix can be written as

$$\rho_0 = \mathrm{II}\left(\frac{e^{\beta(\mu-\varepsilon_k)a_k^* a_k}}{1+e^{\beta(\mu-\varepsilon_k)}}\right)\mathrm{II}\left((1-e^{-\beta\omega_k})\,e^{-\beta\omega_k b_k^* b_k}\right).$$

Therefore, according to BDD, the statistical average of the product of the operators under consideration can be decomposed into a unique sum of all possible sets of the complete contractions one can indicate.

The contractions which do not vanish are classified into the following four types :

$$a_k \cdot a_k^* \cdot = f_k^{(+)} \equiv \frac{1}{1+e^{\beta(\mu-\varepsilon_k)}},$$

$$a_k^* \cdot a_k \cdot = f_k^{(-)} \equiv \frac{1}{1+e^{\beta(\varepsilon_k-\mu)}},$$

$$b_k \cdot b_k^* \cdot = g_k^{(+)} \equiv \frac{1}{1-e^{-\beta\omega_k}},$$

$$b_k^* \cdot b_k \cdot = g_k^{(-)} \equiv \frac{1}{e^{\beta\omega_k}-1}.$$

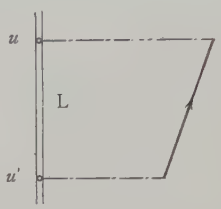


Fig. 2·1.

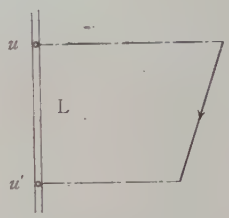


Fig. 2·2.

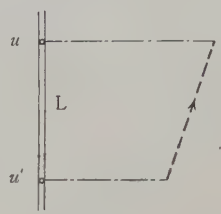


Fig. 3·1.

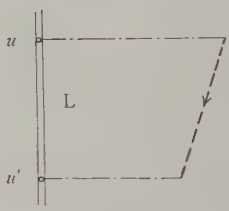


Fig. 3·2.

With use of the notations of BDD, we represent $a_k(u)\cdot a_k^*(u')\cdot$ and $a_k^*(u)\cdot a_k(u')\cdot$ by the respective arrows shown in Fig. 2·1 and Fig. 2·2, and $b_k(u)\cdot b_k^*(u')\cdot$ and $b_k^*(u)\cdot b_k(u')\cdot$ by the "dotted" arrows shown in Fig. 3·1 and Fig. 3·2, respectively.

The Coulomb interaction

$$\frac{4\pi e^2}{q^2}\, a_{l+k}^* a_l\, a_{m-k}^* a_m$$

is written as in Fig. 4. The electron-phonon interactions

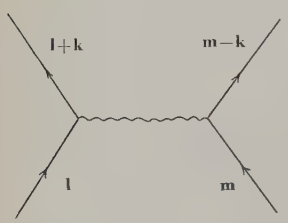


Fig. 4.

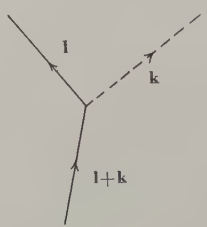


Fig. 5.

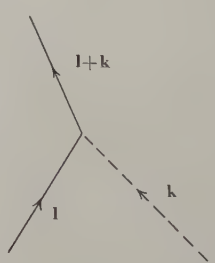


Fig. 6.

$$v(\boldsymbol{k}:\boldsymbol{l})a_{\boldsymbol{l}}^{*}a_{\boldsymbol{l}+\boldsymbol{k}}b_{\boldsymbol{k}}^{*}$$

and

$$v^{*}(\boldsymbol{k}:\boldsymbol{l})a_{\boldsymbol{l}+\boldsymbol{k}}^{*}a_{\boldsymbol{l}}b_{\boldsymbol{k}}$$

are expressed as in Fig. 5 and Fig. 6, respectively.

Thus, $F(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it)$ given by Eq. (4·8) can be expressed as a unique sum of Feynman diagrams in the composite 4-dimensional space. The rule for calculating the contribution of each diagram to the F-function is similar to the rule given by BDD, except that the path of the u-integration is now $L(t)$ instead of the interval $[0, \beta]$.

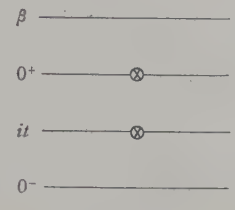


Fig. 7.

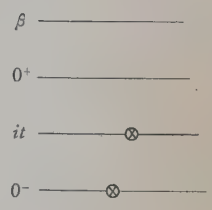


Fig. 8.

§ 5. Elimination of disconnected diagrams

A diagram which falls into two or more unconnected parts is said to be a disconnected diagram, and otherwise it is called connected. A disconnected diagram has such a form as shown in Fig. 9. There are no diagrams like that in Fig. 10.

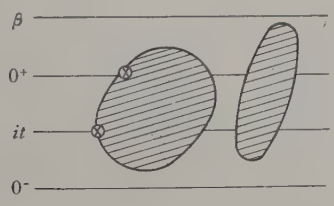


Fig. 9.

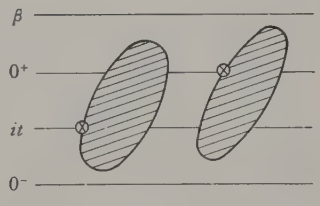
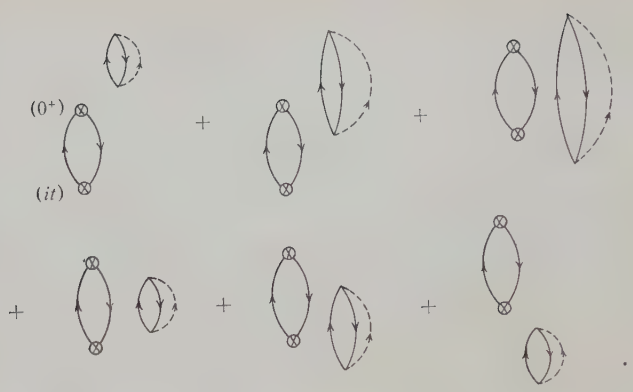


Fig. 10.

Let us consider a disconnected diagram. Its connected part, i.e., the part connected to the ⊗ vertices, is indicated as Γ_1, and its disconnected part composed of one or more unconnected parts which do not contain the ⊗ vertex is denoted by Γ_2. Retaining the order† of vertices of Γ_1 and that of Γ_2, respectively, we shall change the order between the vertices of Γ_1 and those of Γ_2. Then we obtain a family of diagrams. We shall consider the contribution of all the diagrams belonging to this family. For example,

† The order of vertices on the path L is simply called the order from now on.



The contribution of the above sum is equal to the product of "the contribution of  (Γ_1) to the F-function" and "the contribution of  (Γ_2) to $\langle \mathrm{P}[e^{-\int_{\mathrm{L}} du H'(u)}] \rangle^0$". Accordingly, we get

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) = \frac{\mathrm{Z}_0}{\mathrm{Z}} \langle \mathrm{P}[e^{-\int_{\mathrm{L}} du H'(u)} \cdot \alpha_k^q(it) \alpha_l^{-q}(0^{\pm})] \rangle_{\mathrm{C}}^0 \langle \mathrm{P}[e^{-\int_{\mathrm{L}} du H'(u)}] \rangle^0, \quad (5\cdot1)$$

where the suffix C indicates the sum of all connected diagrams. Further,

$$\frac{\mathrm{Z}_0}{\mathrm{Z}} \langle \mathrm{P}[e^{-\int_{\mathrm{L}} du H'(u)}] \rangle = \frac{\mathrm{Z}_0}{\mathrm{Z}} \langle e_{(+)}^{-\int_0^{\beta} du H'(u)} \rangle^0 = \frac{1}{\mathrm{Z}} \mathrm{Tr}\, e^{-\beta(H-\mu N)} = 1.$$

Inserting this into Eq. (5·1), we obtain

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) = \langle \mathrm{P}[e^{-\int_{\mathrm{L}} du H'(u)} \cdot \alpha_k^q(it) \cdot \alpha_l^{-q}(0^{\pm})] \rangle_{\mathrm{C}}^0. \quad (5\cdot2)$$

§ 6. Definitions and lemmas

(Definition)

For two arbitrary points ξ and η on $\mathrm{L}(t)$, we define

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|\xi, \eta) \equiv \frac{\mathrm{Z}_0}{\mathrm{Z}} \langle \mathrm{P}[e^{-\int_{\mathrm{L}(t)} du H'(u)} \cdot \alpha_l^{-q}(\xi) \alpha_k^q(\eta)] \rangle^0$$

$$= \langle \mathrm{P}[e^{-\int_{\mathrm{L}(t)} du H'(u)} \alpha_l^{-q}(\xi) \alpha_k^q(\eta)] \rangle_{\mathrm{C}}^0. \quad (6\cdot1)$$

(Corollary)

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) = \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it).$$

(Lemma 1)

i) $\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau^{+}) = \mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau^{-}) = \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau)$.

ii) $\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau^{+}) = \mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau^{-}) = \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau)$.

$(0 < \tau < t)$

[Proof]

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^+, i\tau^+)=\frac{\mathrm{Z}_0}{\mathrm{Z}}\langle e_{(+)}^{-\int_0^\beta duH'(u)}\alpha_l^{-q}\, e_{(-)}^{-i\int_\tau^0 duH'(iu)}\alpha_k^{q}(i\tau)$$

$$\times e_{(-)}^{-i\int_t^\tau duH'(iu)}\cdot e_{(+)}^{-i\int_0^t duH'(iu)}\rangle^0$$

$$=\frac{\mathrm{Z}_0}{\mathrm{Z}}\langle e_{(+)}^{-\int_0^\beta duH'(u)}\cdot\alpha_l^{-q}\cdot e_{(-)}^{-i\int_\tau^0 duH'(iu)}\alpha_k^{q}(i\tau)\, e_{(+)}^{-i\int_0^\tau duH'(iu)}\rangle^0$$

$$=\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, i\tau).$$

The remaining relations can be proved quite in the same way.

(Lemma 2)

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^-, \xi)=\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|0^+, \xi)$$

for $\xi\in[[0^+, \beta]]$.

[Proof]

This is a straightforward consequence of the identity

$$e_{(-)}^{-i\int_t^0 duH'(iu)}\, e_{(+)}^{-i\int_0^t duH'(iu)}=1.$$

(Lemma 3)

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|i\tau^-, it)=\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, i(t-\tau)),$$

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|i\tau^+, it)=\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, i(t-\tau)).$$

[Proof]

$$\mathrm{F}_t(\boldsymbol{l}, \boldsymbol{k}|i\tau^-, it)=\frac{1}{\mathrm{Z}}\,\mathrm{Tr}\{e^{-\beta(H-\mu N)}\cdot e^{iHt}\cdot\alpha_k^{q}\cdot e^{-iH(t-\tau)}\cdot\alpha_l^{-q}\cdot e^{-iH\tau}\}$$

$$=\frac{1}{\mathrm{Z}}\,\mathrm{Tr}\{e^{-\beta(H-\mu N)}\cdot e^{iH(t-\tau)}\cdot\alpha_k^{q}\cdot e^{-iH(t-\tau)}\cdot\alpha_l^{-q}\}=\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, i(t-\tau)).$$

The other part of this lemma can also be proved in the same way.

(Definition)

A connected diagram is called improper whenever it can be made to fall into two parts by eliminating a single Coulomb line. A connected diagram which is not improper is called proper: A proper diagram can never be transformed into a disconnected diagram by cutting a single Coulomb line.

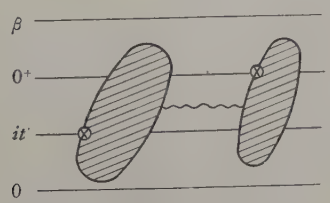


Fig. 11.

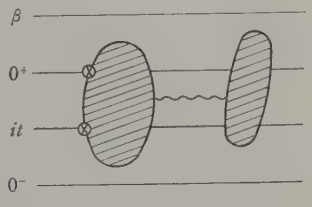


Fig. 12.

An improper diagram has such a form as shown in Fig. 11. There are no diagrams like that in Fig. 12. Examples of proper and improper diagrams are given in Fig. 13.

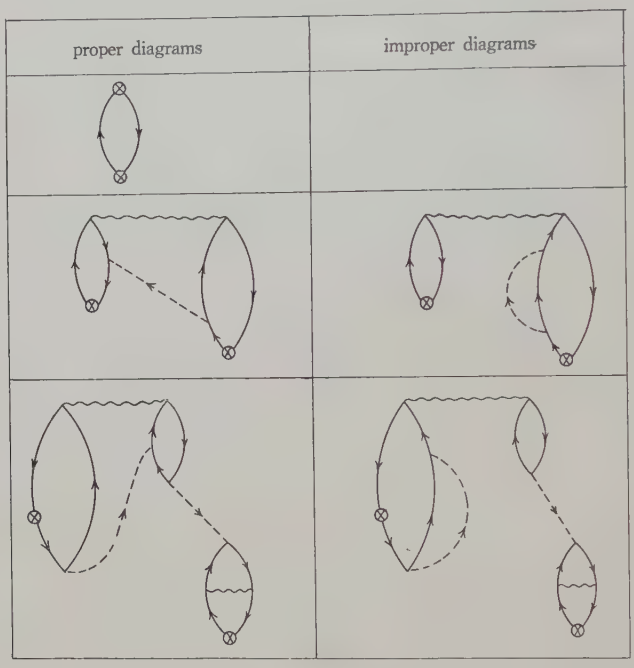


Fig. 13.

(Definition)
We shall define the following function of ξ and η on $\mathrm{L}(t)$

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|\xi, \eta) \equiv \langle \mathrm{P}[e^{-\int_{\mathrm{L}(t)} duH'(u)} \cdot \alpha_l^{-q}(\xi) \cdot \alpha_k^{q}(\eta)] \rangle^0_{proper}, \tag{6·2}$$

where the suffix "proper" indicates that the expression is a sum of proper diagrams. In the following the suffix "t" of Δ_t shall be omitted in the case

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it). \tag{6·3}$$

(Lemma 1′)

i) $\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau^{+}) = \Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau^{-}) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau).$

ii) $\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau^{+}) = \Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau^{-}) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{-}, i\tau).$

[Proof]

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^{+}, i\tau^{+}) = \langle e^{-\int_0^{\beta} duH'(u)}_{(+)} \alpha_l^{-q}\, e^{-i\int_{\tau}^{0} duH'(iu)}_{(-)} \alpha_k^{q}(i\tau)\, \mathrm{P}[e^{-\int_{[[0^-, i\tau^+]]} duH'(u)}] \rangle^0_{proper}.$$

In the above expression,

$$\mathrm{P}[e^{-\int_{[[0^-,i\tau^+]]}duH'(u)}]\sim e^{-i\int_{(-)}^{\tau}duH'(iu)}\cdot e^{-i\int_{(+)}^{t}duH'(iu)}$$

is shown to be reduced to

$$e^{-i\int_{(+)}^{\tau}duH'(iu)}.$$

This is a consequence of the following fact: The contribution of each diagram which has one or more vertices (except $\otimes$) lying in the interval $[[i\tau^-, i\tau^+]]$ vanishes after the integration over u. Thus, it is proved that

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^+, i\tau^+)=\Delta(\boldsymbol{l}, \boldsymbol{k}|0^+, i\tau).$$

The remaining relations of this lemma are proved quite similarly.
(Lemma 2′)

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^+, \tilde{\xi})=\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^-, \tilde{\xi}),$$

for $\tilde{\xi}\in[[0^+, \beta]]$.
[Proof]

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^+, \tilde{\xi})=\langle e^{-\int_{(+)}^{\beta}duH'(u)}\cdot\alpha_k{}^q(\tilde{\xi})\cdot e^{-\int_{(+)}^{\xi}duH'(u)}\cdot\alpha_l^{-q}\mathrm{P}[e^{-\int_{[[0^-,0^+]]}duH'(u)}]\rangle^0_{proper}.$$

In the above expression,

$$\mathrm{P}[e^{-\int_{[[0^-,0^+]]}duH'(u)}]\sim e^{-i\int_{(-)}^{0}duH'(iu)}\cdot e^{-i\int_{(+)}^{t}duH'(iu)}$$

is reduced to 1. This is a special case of the fact mentioned in the proof of (Lemma 1′). Thus, we obtain

$$\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^+, \tilde{\xi})=\langle e^{-\int_{(+)}^{\beta}duH'(u)}\cdot\alpha_k{}^q(\tilde{\xi})\cdot e^{-\int_{(+)}^{\xi}duH'(u)}\cdot\alpha_l^{-q}\rangle^0_{proper}.$$

It can be shown in the same manner that $\Delta_t(\boldsymbol{l}, \boldsymbol{k}|0^-, \tilde{\xi})$ is reduced to the right-hand side of the above equation.

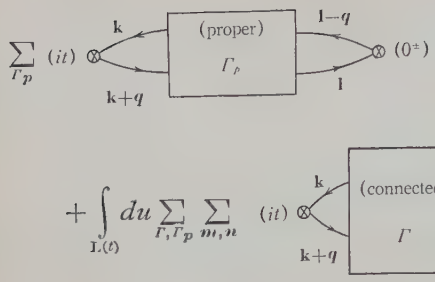
§ 7. Elimination of improper diagrams

Inserting Eq. (2·15) into Eq. (2·20) and using Eq. (4·4), we obtain

$$\sigma(\boldsymbol{q}, \omega)=\frac{\left(\frac{e^2\omega}{q^2}\right)\sum_{k,l}\int_0^\infty dt\cdot e^{-i\omega t}\{\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, it)-\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}}{1+i\left(\frac{4\pi e^2}{q^2}\right)\sum_{k,l}\int_0^\infty dt\cdot e^{-i\omega t}\{\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, it)-\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}}. \quad (7\cdot1)$$

It will be shown in the forthcoming discussion that this expression for σ can be reduced to a simple form.

According to Eq. (5·2), $\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^\pm, it)$ can be expressed as follows:



This expression indicates that the F-function satisfies the following integral equation:

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{\pm}, it) - \left(\frac{4\pi e^2}{q^2}\right) \int_{\mathrm{L}(t)} du \sum_{m,n} \Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{\pm}, u)\mathrm{F}_t(\boldsymbol{m}, \boldsymbol{k}|u, it). \tag{7·2}$$

The minus sign in front of the last term has its origin in the minus sign found in the exponential function on the right-hand side of Eq. (5·2).

From the integral equation Eq. (7·2) we get

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{+}, it) - \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{-}, it) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{+}, it) - \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{-}, it)$$
$$- \left(\frac{4\pi e^2}{q^2}\right) \sum_{m,n} \int_{\mathrm{L}(t)} du \{\Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{+}, u) - \Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{-}, u)\} \mathrm{F}_t(\boldsymbol{m}, \boldsymbol{k}|u, it).$$

Based upon Lemma 2′ the above path integral can be rewritten as

$$\int_{\mathrm{L}(t)} du \{\Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{+}, u) - \Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{-}, u)\} \cdot \mathrm{F}_t(\boldsymbol{m}, \boldsymbol{k}|u, it)$$
$$= i \int_0^t d\tau \{\Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{+}, i\tau^{-}) - \Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{-}, i\tau^{-})\} \cdot \mathrm{F}_t(\boldsymbol{m}, \boldsymbol{k}|i\tau^{-}, it)$$
$$- i \int_0^t d\tau \{\Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{+}, i\tau^{+}) - \Delta_t(\boldsymbol{l}, \boldsymbol{n}|0^{-}, i\tau^{+})\} \cdot \mathrm{F}_t(\boldsymbol{m}, \boldsymbol{k}|i\tau^{+}, it).$$

This is further reduced to

$$i \int_0^t d\tau \{\Delta(\boldsymbol{l}, \boldsymbol{n}|0^{+}, i\tau) - \Delta(\boldsymbol{l}, \boldsymbol{n}|0^{-}, i\tau)\}$$
$$\times \{\mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^{-}, i(t-\tau)) - \mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^{+}, i(t-\tau))\},$$

by means of Lemma 1′ and Lemma 3. Consequently, we obtain

$$\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{+}, it) - \mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^{-}, it) = \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{+}, it) - \Delta(\boldsymbol{l}, \boldsymbol{k}|0^{-}, it)$$
$$+ i \left(\frac{4\pi e^2}{q^2}\right) \sum_{m,n} \int_0^t d\tau \{\Delta(\boldsymbol{l}, \boldsymbol{n}|0^{+}, i\tau) - \Delta(\boldsymbol{l}, \boldsymbol{n}|0^{-}, i\tau)\}$$

$$\times\{\mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^+, i(t-\tau))-\mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^-, i(t-\tau))\}.$$

This equation is substituted in the numerator of the expression (7·1) for σ, where the following calculation is performed:

$$\sum_{k,l}\int_0^\infty dt\, e^{-i\omega t}\{\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^+, it)-\mathrm{F}(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}$$

$$=\sum_{k,l}\int_0^\infty dt\, e^{-i\omega t}\{\Delta(\boldsymbol{l}, \boldsymbol{k}|0^+, it)-\Delta(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}$$

$$+i\left(\frac{4\pi e^2}{q^2}\right)\sum_{k,l,m,n}\int_0^\infty dt\int_0^t d\tau\{\Delta(\boldsymbol{l}, \boldsymbol{n}|0^+, i\tau)-\Delta(\boldsymbol{l}, \boldsymbol{n}|0^-, i\tau)\}e^{-i\omega\tau}$$

$$\times\{\mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^+, i(t-\tau))-\mathrm{F}(\boldsymbol{m}, \boldsymbol{k}|0^-, i(t-\tau))\}e^{-i\omega(t-\tau)}$$

$$=\left[\sum_{k,l}\int_0^\infty d\tau\, e^{-i\omega\tau}\{\Delta(\boldsymbol{l}, \boldsymbol{k}|0^+, i\tau)-\Delta(\boldsymbol{l}, \boldsymbol{k}|0^-, i\tau)\}\right]$$

$$\times\left[1+i\left(\frac{4\pi e^2}{q^2}\right)\sum_{m,n}\int_0^\infty dt\, e^{-i\omega t}\{\mathrm{F}(\boldsymbol{m}, \boldsymbol{n}|0^+, it)-\mathrm{F}(\boldsymbol{m}, \boldsymbol{n}|0^-, it)\}\right].$$

Therefore, we get

$$\sigma(\boldsymbol{q}, \omega)=\lim_{\delta\to+0}\left(\frac{e^2\omega}{q^2}\right)\sum_{k,l}\int_0^\infty dt\, e^{-i\omega t-\delta t}\{\Delta(\boldsymbol{l}, \boldsymbol{k}|0^+, it)-\Delta(\boldsymbol{l}, \boldsymbol{k}|0^-, it)\}. \quad (7\cdot3)$$

This may be written symbolically as

$$\sigma(\boldsymbol{q}, \omega)=\lim_{\delta\to+0}\frac{\omega}{q^2}\int_0^\infty dt\, e^{-i\omega t-\delta t}\langle[\mathfrak{N}(-\boldsymbol{q}), \mathfrak{N}(\boldsymbol{q}, t)]\rangle_{proper}.$$

§ 8. Discussions

The conventional calculation by means of Kubo's formalism, in which **D** is replaced by **E** and the Coulomb interaction is neglected, corresponds to the lowest order term of Eq. (7·3) with respect to the Coulomb interaction, if the chemical potential μ is replaced by μ^0, the chemical potential of the unperturbed system. This term includes all possible diagrams whose structure is shown in Fig. 14. Therefore, it includes all the ring diagrams in Gell-Mann and Brueckner's sense[5] and, moreover, constitutes the best approximation as for the effective field which polarizes the electrons giving a net electric current.

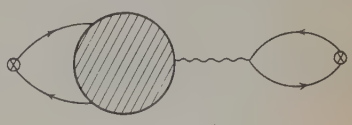


Fig. 14.

The correction terms to the simple conventional calculation, other than the correction to the chemical potential μ^0, must be afforded by proper diagrams

corresponding to the following processes. i) The polarized electron and the hole, whose current is measured as a component of the net current, collide with each other or with the electrons constituting the medium. ii) One or more phonons produced by the electric polarization due to the electric field take part in polarizing the "auf" electron whose current is observed.

The introduction of the unknown chemical potential would be a calculational disadvantage of our proper diagram expansion. It will be removed[6] in our future paper.

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Note added in proof : A diagram expansion given recently by Konstantinov and Perel (reference 2)) seems to be essentially equivalent to that given in § 4 of the present work. The diagram representation presented here is more convenient, because it is more closely related to the Feynman-Goldstone's one.

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Dispersion Relations and High Energy Limits in Quantum Field Theory. III

——*Necessary Condition for Existence of Solutions of Dispersion Relations*——

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It is shown that pion-nucleon forward scattering amplitude must have zeros on the complex energy plane. This is derived by imposing the necessary condition for the existence of solutions on the dispersion relation for pion-nucleon forward scattering. By extending this result to the case of pion propagator, discussions against perturbation theory are made. Finally by applying this necessary condition to the dispersion relation given by Goldberger, it is concluded that either the pion-nucleon total cross sections increase at very high energies or Goldberger's dispersion relation has no solution.

§ 1. Introduction

Since the appearance of the paper by Castillejo, Dalitz and Dyson,[1] it has been discussed by many authors[2] if Low's equation or dispersion relation has a unique solution. Some authors insisted that the solution of dispersion relation can be determined uniquely by requiring the possibility of power series expansion of the amplitude or propagator with respect to coupling constants. This requirement is equivalent to the exclusion of zeros from the amplitude or propagator in the complex energy planes.

In § 2 of this paper, we shall show that the pion-nucleon forward scattering amplitude must have zeros on the complex energy plane. It should be stressed that this result is independent of the assumption on the high energy limit of the amplitude. It is a mere consequence of a necessary condition for the existence of the solution of the dispersion relation for pion-nucleon forward scattering. Such a condition has been derived for pion propagator by Lehmann, Symanzik and Zimmermann[3] and a famous theorem on the damping of vertex function at high energy limit has been obtained, but problems relating to the existence of zeros remain unsolved. The reason why we can draw the answer to the problems of the existence of zeros only for the forward scattering amplitude but not for the propagator is that we have some amount of knowledge established experimentally for pion-nucleon scattering, for example, the S-wave scattering lengths, total cross sections, whereas we have not any experimental information on the propagator.

After proving the existence of zeros for pion-nucleon forward scattering amplitude, we will apply this result to two problems:

The first problem is related to the existence of zeros of propagator. Adopting an interpretation that the existence of zeros is caused by the possible existence of unstable particles capable to decay through the strong interactions, we can draw the conclusion that the propagator also has zeros. On the other hand, the perturbation theory predicts the non-existence of zeros for propagator in every order of expansion.[4] We thus find a contradiction. From this we are forced to conclude that either it is incorrect to interpret that the existence of zeros is equivalent to that of unstable particles or the perturbation theory does not reflect the correct property of the field theory. Though not yet proved, it seems to be probable that the latter case holds. These points shall be discussed in detail in § 3.

The second problem in applying the results obtained in § 2 is to study the high energy behaviours of pion-nucleon total cross sections, basing upon the dispersion relation given by Goldberger.[5] From the present high-energy experimental results[6] it now seems to be broadly accepted a conjecture that the total cross sections approach a constant value, about 30 mb, at high-energy limit. The comparisons of dispersion relations with experiments have been made with assumption of such behaviours.[7] We will attack this problem using available experimental data and it will be shown that the assumption $\sigma(\omega)\simeq 30$ mb for $\omega\gtrsim 10$ Bev, $\sigma(\omega)$ being the pion-nucleon total cross section in the laboratory system, is not consistent with the dispersion relation given by Goldberger. Therefore, we will take one or the other of the two conclusions: (1) The total cross sections must increase with energy or become some hundred times larger than of 30 mb for $\omega\gtrsim 10$ Bev, and (2) Goldberger's dispersion relation has no solution. These problems will be discussed in § 4.

§ 2. Necessary condition for the existence of solutions of dispersion relation for pion-nucleon forward scattering

As shown in the previous paper,[8] the isotopic-spin independent part of pion-nucleon forward scattering amplitude $T^{(1)}(\omega)$ (ω, the pion energy in the laboratory system) cannot increase like ω or more rapidly as $|\omega|\to\infty$ if $T^{(1)}(\omega)$ has no zero in the complex plane of ω. Furthermore, it has also been shown there that $T^{(1)}(\omega)$ cannot decrease as $|\omega|\to\infty$. Therefore, under the above assumption, it is necessary and sufficient to perform one subtraction in the dispersion relation for $T^{(1)}(\omega)$, that is, the following dispersion relation holds,

$$T^{(1)}(\omega)=T^{(1)}(\mu)+\frac{2(\omega^2-\mu^2)}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T^{(1)}(\omega')\omega'\,d\omega'}{(\omega'^2-\mu^2)(\omega'^2-(\omega+i\varepsilon)^2)}+\frac{f^2}{M}\frac{\omega^2-\mu^2}{\omega^2-(\mu^2/2M)^2}. \tag{1}$$

M and μ are the mass of nucleon and pion, respectively, f the renormalized coupling constant. Hereafter throughout this paper, we shall drop the superscript in $T^{(1)}(\omega)$ for convenience.

Since $T(\omega)$ cannot damp as $|\omega|\to\infty$, the following integral certainly exists,

$$-4i\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega)\,(x+z_0)\,\omega d\omega}{|T(\omega)|^2(\omega^2-z_0^2)(\omega^2-x^2)}=\int_C\frac{dz}{T(z)(z-z_0)(z-x)}, \tag{2}$$

where z_0 and x are respectively some real and complex parameters, and the contour of integral in the right-hand side of Eq. (2) is shown in Fig. 1.

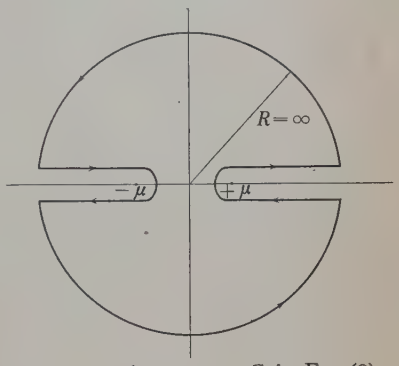


Fig. 1. The contour C in Eq. (2).

We have assumed that $T(z)$ has no zero in the complex z-plane, so the integral in the right-hand side of Eq. (2) becomes

$$2\pi i\frac{1}{(x-z_0)}\left\{\frac{1}{T(x)}-\frac{1}{T(z_0)}\right\}.$$

Then Eq. (2) can be written as follows:

$$\frac{1}{T(x)}=\frac{1}{T(z_0)}-\frac{2}{\pi}(x^2-z_0^2)\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega)}{|T(\omega)|^2}\frac{\omega d\omega}{(\omega^2-z_0^2)(\omega^2-x^2)}.$$

Choosing $z_0=\mu^2/2M$, writing $x=\omega+i\varepsilon$ and taking the real parts of the both sides in this equation, we have

$$\frac{\mathrm{Re}\,T(\omega)}{|T(\omega)|^2}=\frac{1}{T(\mu^2/2M)}$$
$$-\frac{2}{\pi}(\omega^2-(\mu^2/2M)^2)\,\mathrm{P}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')}{|T(\omega')|^2}\frac{\omega'd\omega'}{(\omega'^2-(\mu^2/2M)^2)(\omega'^2-\omega^2)}. \tag{3}$$

Making ω tend to infinity in Eq. (3), and taking account of $T(\mu^2/2M)=\infty$ ($T(\omega)$ has two poles at $\omega=\pm\mu^2/2M$), we obtain

$$\mathrm{Re}\,T(\infty)=|T(\infty)|^2\cdot\frac{2}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')}{|T(\omega')|^2}\frac{\omega'd\omega'}{(\omega'^2-(\mu^2/2M)^2)}. \tag{4}$$

The right-hand side of Eq. (4) is obviously positively definite, hence we have

$$\mathrm{Re}\,T(\infty)\geq 0. \tag{5}$$

On the other hand, taking the real parts of the both sides of Eq. (1) and making ω tend to infinity, we obtain

$$\mathrm{Re}\,T(\infty)=T(\mu)-\frac{2}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')\omega'd\omega'}{(\omega'^2-\mu^2)}+\frac{f^2}{M}. \tag{6}$$

From Eqs. (5) and (6), we obtain a necessary condition for the existence of $T(\omega)$:

$$T(\mu)-\frac{2}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')\omega'\,d\omega'}{(\omega'^2-\mu^2)}+\frac{f^2}{M}\geq 0. \tag{7}$$

This corresponds to the following equation which was obtained by Lehmann et al.[3] as a necessary condition for the existence of the pion propagator $\Delta_F'(k)$ (assuming that $\Delta_F'(k)$ has no zero) :

$$2-\int_{0}^{\infty}\frac{\mathrm{Im}\,\Delta_F'(t)}{(t+1)^2|\Delta_F'(t)|^2}\,dt\geq 0. \tag{8}$$

Lehmann et al. concluded from Eq. (8) that the vertex function should damp at high energy limit.

There is an important difference in the point that we can check Eq. (7) using experimental data, but not Eq. (8). The left-hand side of Eq. (7) was already studied in II using available experimental data and it was proved that

$$T(\mu)-\frac{2}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')\omega'\,d\omega'}{(\omega'^2-\mu^2)}+\frac{f^2}{M}<0, \tag{9}$$

that is, the necessary condition Eq. (7) cannot be satisfied.

Therefore, we can conclude that $T(\omega)$ must have zeros in the complex ω-plane.

It is worthy to note that in the Lee model in which ghost state exists the V-particle propagator and the N-θ scattering amplitude have no zero.

§ 3. Existence of zero and perturbation theory

In this section, we will compare the result obtained in § 2 with that of the analysis by perturbation theory.

It is difficult to investigate the presence of zeros in $T(\omega)$ in every order of perturbation theory, but it can be easily carried through for $\Delta_F'(k)$.

The pion propagator $\Delta_F'(k)$ is given as follows :

$$\Delta_F'(k)=1/(k^2+\mu^2-\Sigma^*(k)). \tag{10}$$

The spectral representation for $\Sigma^*(k)$, the proper self-energy part, was proved in every order of perturbation theory by Nakanishi,[4]

$$\Sigma^*(k)=\int_{(3\mu)^2}^{\infty}\frac{\rho^*(u)}{k^2+u-i\varepsilon}\,du, \tag{11}$$

where, apart from the unessential numerical factors, $\rho^*(u)$ has the following form :

$$\rho^*(u)=\int \frac{f(D)}{U^2(x)}\delta^{(\beta-1)}(\eta(x)u-M(x))dx. \qquad (12)$$

As to the meaning of the notations in Eq. (12) the reader should refer to reference 4). The singularities of the integrand of Eq. (12) come from $U(x)=0$ and $\eta(x)u-M(x)=0$. But the former are the ultra-violet divergencies and should be eliminated by the renormalization techniques. The latter cause the imaginary part of $\Sigma^*(u)$. Hence, from the inspection of the structure of Eqs. (11) and (12) we can see that $\Sigma^*(k)$ has no singularity on the complex k^2-plane except the cut from $(3\mu)^2$ to $+\infty$ on the real, positive axis. It should be noted that this result does not depend on the choice of Hamiltonian; Eqs. (11) and (12) have been proved on the basis of the general structures of Feynman diagrams, but not on the detailed structures of interactions.

Therefore we can conclude that $\Delta_F'(k)$ has no zero on the complex k^2-plane in the perturbation theory because zeros of $\Delta_F'(k)$ come from poles of $\Sigma^*(k)$.

Now, in general the dispersion relations for the quantities $M(\omega)$ have the following structures:

$$M(\omega)=B(\omega)+\frac{1}{\pi}\int_{\mu_0}^{\infty}\frac{\mathrm{Im}\,M(\omega')d\omega'}{(\omega-\omega'-i\varepsilon)}+F(\omega). \qquad (13)$$

Here, $B(\omega)$ represents the terms coming from bound states contributions, $F(\omega)$ the arbitrary polynomial of ω required to ensure the convergence of dispersion integral, and μ_0 is the threshold energy.

It was shown by Castillejo et al.[1] and by many other authors that there can be found infinitely many solutions in Eq. (13) provided that $M(\omega)$ has zeros. These solutions correspond to the variety of Hamiltonians. This can be understood physically if we introduce unstable particles capable of decaying through the strong interactions. If we should change the Hamiltonian so as to include the unstable particles, the form of Eq. (13) would remain invariant, because both $B(\omega)$ and $F(\omega)$ have no relation with the unstable particles. Therefore we can take an interpretation that the existence of zeros in $M(\omega)$ is caused by the existence of unstable particles. Such an interpretation holds for the Dyson model,[2] but naturally there may be some other interpretations.

Adopting the above interpretation and combining the result obtained in § 2, we can obtain the conclusion that $\Delta_F'(k)$ must have zeros, too. This is because the proved existence of zeros in $T(\omega)$ is caused by the existence of unstable particles, then the latter cause zeros in $\Delta_F'(k)$.

The above conclusion that $\Delta_F'(k)$ should have zeros directly contradicts the result of perturbation theory which predicts $\Delta_F'(k)$ has no zero. If the above-mentioned interpretation on the existence of zeros were accepted, it could be said that the perturbation theory cannot reflect the correct property of the exact theory.

§ 4. High energy behaviours of pion-nucleon total cross sections

As the second example of applying the result obtained in § 2, we will study the high energy behaviours of pion-nucleon total cross sections on the basis of the dispersion relation given by Goldberger.[5] This relation can be proved strictly under the additional assumption that $T(\omega)$ cannot increase like ω^2 or more strongly as $|\omega|\to\infty$. This assumption is now widely accepted because the experimental results in the Bev region seem to show that the total cross sections take the constant values, about 30 mb. Various comparisons of dispersion relation with experimental results have all been made basing on this assumption.

In this section, we will study the consistency between Goldberger's dispersion relation and the above-mentioned assumption using available experimental data.

As proved in § 2, $T(z)$ must have zeros. Now we assume the zeros are all lying on the real axis. Then Eq. (4) should be replaced as follows*:

$$\mathrm{Re}\,T(\infty)=|T(\infty)|^2\left[\frac{2}{\pi}\int_{\mu}^{\infty}\frac{\mathrm{Im}\,T(\omega')}{|T(\omega')|^2}\,\frac{\omega'\,d\omega'}{(\omega'^2-(\mu^2/2M)^2)}+\sum_i\frac{c_i}{(t_i-\mu^2/2M)}\right]. \tag{14}$$

$$c_i=\frac{1}{t_i}\left[\frac{4}{\pi}\int_{\mu}^{\infty}\frac{\omega'\,d\omega'\,\mathrm{Im}\,T(\omega')}{(\omega'^2-t_i^2)^2}+\frac{2f^2}{M}\,\frac{(\mu^2-(\mu^2/2M)^2)}{(t_i^2-(\mu^2/2M)^2)^2}\right]^{-1}\equiv d_i(t_i)/t_i. \tag{15}$$

Here, t_i are zeros of $T(z)$ and $d_i(t_i)=d(-t_i)$, $d_i(t_i)>0$.

Since t_i are real, $T(t_i)=0$ requires that

$$T(-t_i)=0$$

as obvious by Eq. (1).

Therefore zeros of $T(z)$ always appear as pairs, that is, $t_i=\pm t_i^0$, $T(t_i^0)=T(-t_i^0)=0$. Thus, the last term in brackets of the right-hand side of Eq. (14) becomes as follows:

$$\begin{aligned}\sum_i\frac{c_i}{(t_i-\mu^2/2M)}&=\sum_i d_i(t_i^0)\left\{\frac{1}{t_i^0(t_i^0-\mu^2/2M)}+\frac{1}{t_i^0(t_i^0+\mu^2/2M)}\right\}\\&=2\sum_i\frac{d_i(t_i^0)}{(t_i^{02}-(\mu^2/2M)^2)}.\end{aligned} \tag{16}$$

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* In deriving Eqs. (14) and (15), we have taken into account the fact that $T(z)$ has zero points of 1st order. This is because $T(\omega)$ can also be regarded as a function of complex ω^2, and then Eq. (1) has essentially the same mathematical structure as pion propagator of which the mathematical properties have been investigated in detail by Ford (K. W. Ford, Phys. Rev. **105** (1957), 320). He has shown that there can exist zero points of only 1st order except the point $\omega=0$ where $T(\omega)$ may have zero point of 2nd order. However, a residue calculation of integral (2) in this case gives the same result for Eq. (16) as we would calculate all zeros as 1st order.

Obviously, this term is positive if all $t_i^{02} > (\mu^2/2M)^2$.

On the other hand, Eqs. (6) and (9) require that

$$\mathrm{Re}\, T(\infty) < 0.$$

Therefore in Eq. (14) the negative sign must come from the term $\sum_i c_i/(t_i - \mu^2/2M)$, hence we know that there must be t_i which satisfy $t_i^2 < (\mu^2/2M)^2$. Namely, there are two zeros ($t_i \neq 0$) or one ($t_i = 0$) between the interval $(-\mu^2/2M, +\mu^2/2M)$ because $T(\omega)$ is monotonically decreasing for $0 > \omega > -\mu^2/2M$ and increasing for $+\mu^2/2M > \omega > 0$. From this we must conclude

$$T(0) \leq 0. \tag{17}$$

Using Eq. (1) the following inequality must hold:

$$T(\mu) - \frac{2\mu^2}{\pi} \int_\mu^\infty \frac{\mathrm{Im}\, T(\omega')\, d\omega'}{(\omega'^2 - \mu^2)\omega'} + \frac{f^2}{M}\left(\frac{2M}{\mu}\right)^2 \leq 0. \tag{18}$$

It is this Eq. (18) of which we will check its validity using experimental results.

For $T(\mu)$, in II we have obtained the value -0.03×10^{-13} cm.

For the integral $\int_\mu^\infty \mathrm{Im}\, T(\omega')\, d\omega'/(\omega'^2 - \mu^2)\omega' = 1/8\pi \int_0^\infty (\sigma_+(\omega') + \sigma_-(\omega'))\, dk'/\omega'^2$, this can be estimated assuming $\sigma_+(\omega) \simeq \sigma_-(\omega) \simeq 30$ mb for $k \gtrsim 10$ Bev. This gives about -0.34×10^{-13} cm for the second term in the left-hand side of Eq. (18).

For $(f^2/M) \cdot (2M/\mu)^2$, we have $+3.03 \times 10^{-13}$ cm.

Therefore, the left-hand side of Eq. (18) becomes

$$(-0.03 - 0.34 + 3.03) \times 10^{-13}\ \text{cm}$$

$$= +2.66 \times 10^{-13}\ \text{cm}.$$

This shows that Eq. (18) cannot be satisified. This result does not change even though we take σ some hundred times larger than 30 mb for $\omega \gtrsim 10$ Bev.

From this result we can draw the following conclusion:

(1) The pion-nucleon total cross sections increase with energy or take larger values by some hundred times than 30 mb for $\omega \gtrsim 10$ Bev. or

(2) Goldberger's dispersion relation breaks down. This is further classified into two possible cases:

(2a) The assumed high-energy behaviour of $T(\omega)$ is incorrect, in other words, $T(\omega)$ can increase like ω^2 or more strongly as $|\omega| \to \infty$, so we must construct the twice or more times subtracted dispersion relation.

(2b) The dispersion relation itself breaks down.

Thus, if we assume the consistency of the current field theory, the case (1) or (2a) can be realized.

Note added in proof : In § 4 in the text, we have assumed that the zeros of $T(\omega)$ are all lying on the real axis of complex ω-plane. We cannot, however, exclude the possibility of the existence of zeros on the imaginary axis. As is seen from the inspection of Eq. (1), there may be two and only two, zeros on the imaginary axis. If there are zeros on the imaginary axis, Eq. (17) will be replaced by $T(0)>0$, and we cannot draw the conclusion such as shown in the text. In practice, in this case the assumption $\sigma(\omega)\simeq 30$mb for $\omega\gtrsim 10$ Bev does not encounter with any contradiction. But the following argument which is due to Symanzik (K. Symanzik, Nuovo Cimento **5** (1957), 659) favours the negative conclusion on the existence of zeros on the imaginary axis. Symanzik has derived the following expression of $T(\omega)$ for the very large value of $|\omega|$:

$$T(\omega)=\omega^2\int_{-\infty}^{+\infty} dS(t)/(t^2-\omega^2), \qquad dS(t)\geq 0. \tag{N·1}$$

From this equation we can obtain the Goldberger relation if we subtract $T(\mu)$ from the both sides. Though the representation given above has been derived for a very large value of $|\omega|$, it will hold for any value because the Goldberger relation which can be obtained from Eq. (N·1) are known to hold for any ω. But $T(\omega)$, given in Eq. (N·1), does have zeros only on the real axis and this presents a favourable argument for the assumption made in the text.

If we accept the conclusion drawn in § 4 of the text, we can estimate how much influence is induced in the low-energy regions from the required very large value of the cross sections, about 24000 mb for $\omega\gtrsim 10$ Bev. A simple calculation shows that such large cross sections must increase the values of $\mathrm{Re}T_+(\omega)$ ——the amplitude for π^+-p forward scattering——for 100 Mev $\lesssim\omega\lesssim$300 Mev by the amount about ten times. This completely breaks the present good agreement of theory and experiment at low energies for π^+-p elastic scattering. Therefore, we can derive a further conclusion that the Goldberger dispersion relation does not agree with the present experimental data.

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On the Structure of Generalized Fokker-Planck Equation of a High Temperature Plasma

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The structures of the generalized Fokker-Planck equation derived in the previous publication (quoted as I) are examined in detail by considering the motion of electron beam in a high temperature electron plasma. The dynamical shielding factor provides the basis of a unified theory of the plasma oscillations and the static shielding of the particles interaction in the plasma. The friction and diffusion coefficient are calculated by taking into consideration the non-local space-time correlation effect. It is shown that the non-local correlation effect substantially modifies the contributions of the plasmon emission process. Rough estimation of the friction coefficient shows that the non-local correlation effect may increase the amount of the frictional drag at higher velocity. This effect may be essential to resolve the Langmuir paradox.

§ 1. Introduction

It is well known that the infinite force range of Coulomb interaction prevents direct application of the Boltzmann kinetic equation to a high temperature plasma. Assuming that the momentum transfer is very small in a collision process due to the long-range force, Landau[1] has derived a two-body collison term from the Boltzmann collision term. On the other hand, emphasizing effects of successive small angle scatterings due to the Coulomb interaction, use of the Fokker-Planck equation has also been proposed in some literature.[2] Recently, Enoch[3] has given a proof of equivalence of both methods, provided the two-body collision has predominant effects. In the plasma, however, the many-particle correlation is essential to give rise to the modification of the interaction between particles in the plasma.

The dynamical effects of the Coulomb interaction in the plasma have been investigated by Bohm and Pines.[4] They have separated out the plasma oscillations as collective normal modes of the system. The separation of plasma oscillations provides a natural cutoff at the limit of large distances. Following the Bohm-Pines approach to the plasma, Klimontovich[5] has formulated a kinetic theory of the composite system of plasma oscillators and incident beam. In the linear approximation he has been able to calculate the friction and diffusion

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coefficients of the Fokker-Planck equation of incident beam. These coefficients calculated by Klimontovich are due to the excitation of plasma oscillations, which accounts for the interaction effects between the incident particle and the plasma particles at large distances.

Recently, Temko[6] and Tchen[7] have examined the many-particle correlation effects in the plasma. Although they have shown that three-particles correlation has the effect to screen off contributions of the large distant part of Coulomb interaction, their formulations of the problem are not complete in that they have excluded occurrence of the plasma oscillations. Following the same approach as that used by Tchen, Rostoker and Rosenbluth[8] have developed a consistent description of motion of the test particles in the plasma, which takes account of both the excitation of plasma oscillations and the close distant two-body collisions.

Now, in the previous paper,[9] hereafter quoted as I, one of us (Y.H.I.) has developed a general theory of the many-particle correlation effects in the high temperature plasma, and has shown that the correlation effects manifest themselves in two places, one of which is the correction term in the self-consistent field and the other is the collision term. In I, we have evaluated contributions of the correlation effects to the plasma oscillations. Here we will examine the second aspect of the correlation effects by investigating the structure of generalized Fokker-Planck equation derived in I. Let us consider the motion of electron beam in the high temperature electron plasma. Applying the results obtained in I, we will write down the kinetic equation of the electron beam in the electron plasma in § 2. After analyzing the effects of the dynamical shielding factor in § 3, we will calculate the friction and diffusion coefficients in § 4 and § 5. In § 6, briefly analyzing the non-local space-time correlation effect, a possible effect will be proposed to resolve the Langmuir paradox. The last section is for summarizing the results obtained in the present paper.

§ 2. The generalized Fokker-Planck equation for the electron beam

When there occurs small amount of deviation of the electron distribution from a Maxwell distribution in the high temperature electron plasma, the space-time variation of the electron distribution can be described by the kinetic equation, Eq. (71) of I. Here, let us consider the case in which an electron beam falls into the target electron plasma. We will assume that (i) the particle density of beam is so low that the excess charge effects can be ignored and (ii) the state of target electron plasma is specified by its equilibrium parameters and the reaction of the disturbance induced in the target electron plasma to the incident beam can be ignored as in the case of discussing the Brownian motion. Then, refering to the derivation of Eq. (71) of I, it is straightforward to write down the kinetic equation of the electron beam as follows,

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right\}f(\boldsymbol{x},\boldsymbol{v};t)=\frac{1}{(2\pi)^4}\int d\omega\int d\boldsymbol{k}\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

$$\times\sum_{\alpha}\frac{1}{m}\frac{\partial}{\partial v_\alpha}\left\{A^\alpha(\boldsymbol{k},\boldsymbol{v};\omega)f(\boldsymbol{k},\boldsymbol{v};\omega)+\sum_{\beta}\frac{1}{m}B^{\alpha\beta}(\boldsymbol{k},\boldsymbol{v};\omega)\frac{\partial}{\partial v_\beta}f(\boldsymbol{k},\boldsymbol{v};\omega)\right\}\qquad(1)$$

with the abbreviation of

$$A^\alpha(\boldsymbol{k},\boldsymbol{v};\omega)=-\frac{m^{-1}}{8\pi^3}\int d\boldsymbol{l}\frac{\lambda^2V(l)^2}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}l^\alpha\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{v}'}f_0(\boldsymbol{v}')d\boldsymbol{v}',\qquad(2)$$

$$B^{\alpha\beta}(\boldsymbol{k},\boldsymbol{v};\omega)=\frac{1}{8\pi^3}\int d\boldsymbol{l}\{1+n\chi(l)\}\frac{\lambda^2V(l)^2}{\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})}l^\alpha l^\beta$$

$$\times\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')f_0(\boldsymbol{v}')d\boldsymbol{v}',\qquad(3)$$

where α, β stand for the suffices of x, y and z. For the electron plasma, we have $\lambda V(l)=4\pi e^2/l^2$. The Fourier coefficient of static binary correlation function, $\chi(k)$, is determined to be equal to $-n^{-1}k_d^2/(k^2+k_d^2)$ for the homogeneous electron plasma. Here, n is the number density of electrons and k_d is Debye's wave number defined by $k_d^2=4\pi e^2n/(\kappa T)$, where T is the temperature of electron plasma.

Now, the factor $\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})$ defined by

$$\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})=1-i\frac{\lambda V(l)}{m}\int\delta_+(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}-\boldsymbol{l}\cdot\boldsymbol{v}')\boldsymbol{l}\cdot\frac{\partial}{\partial\boldsymbol{v}'}f_0(\boldsymbol{v}')d\boldsymbol{v}'\qquad(4)$$

represents the dynamical shielding factor which modifies the bare interaction $\lambda V(l)$ between the incident electron of the velocity $\boldsymbol{v}$ and the plasma electrons to the effective interaction $\lambda V(l)/\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})$. The vector $\boldsymbol{l}$ is the wave number exchanged between the incident electron and the plasma electrons. The $\boldsymbol{k}$ and ω represent the wave number and frequency of the spectrum representation of the distribution function of the beam. The $\boldsymbol{k}$-ω dependence of the shielding factor $\Delta(\boldsymbol{k}-\boldsymbol{l},\boldsymbol{l};\omega;\boldsymbol{v})$ as well as that of the friction and diffusion coefficient, $A^\alpha(\boldsymbol{k},\boldsymbol{v};\omega)$, $B^{\alpha\beta}(\boldsymbol{k},\boldsymbol{v};\omega)$, represents non-local space-time correlation effects in the plasma.

It should be noticed that the absence of internal field term in the kinetic equation of the beam, Eq. (1), is due to the neglect of the reaction of the disturbance induced in the plasma. In order to investigate the non-local space-time correlation effects thoroughly, it is necessary to take the effects of induced disturbance of the plasma into account in the kinetic equation of the incident beam. It is expected to be true, however, that the non-local space-time correlation effects can be ignored in a region of close distance, because in the short-range region the motion of incident electrons will be determined by the local fluctuation of the plasma at the every moment. Therefore, the non-local

space-time correlation effects as well as the reactive effects of induced disturbance of the plasma upon the incident electrons are very essential to investigate the collective behaviour such as the bunching of beam and or the anomalously rapid energy exchange process taking place in the plasma.

§ 3. The dynamical shielding factor

For the analysis of the effects of dynamical shielding, let us calculate explicitly the function $\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})$. In I, we have shown that the function becomes

$$\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}) = 1 + \frac{4\pi e^2}{m} l^{-2} \cdot \mathrm{P} \int \frac{1}{\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v} - \boldsymbol{l}\cdot\boldsymbol{v}'} \boldsymbol{l} \cdot \frac{\partial}{\partial \boldsymbol{v}'} f_0(\boldsymbol{v}')\, d\boldsymbol{v}' + i\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}) \tag{5}$$

with the abbreviation of

$$\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}) = (\pi/2)^{1/2} [\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}/\omega_p] (k_d/l)^3 \times \exp[-(m/2\kappa T)\{\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}/l\}^2]. \tag{6}$$

In order to calculate the second term of Eq. (5), we will consider the following two cases, separately.

Case (I): $(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2 \gg (\kappa T/m) l^2$

Then, we approximately calculate the second term of Eq. (5) as follows,

$$\mathrm{P} \int \frac{1}{\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v} - \boldsymbol{l}\cdot\boldsymbol{v}'} \boldsymbol{l} \cdot \frac{\partial}{\partial \boldsymbol{v}'} f_0(\boldsymbol{v}')\, d\boldsymbol{v}' = -n \cdot \frac{l^2}{(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2} \left\{ 1 + 3\frac{\kappa T}{m} \cdot \frac{l^2}{(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2} + \cdots \right\}.$$

Therefore, under the condition (I), we have

$$\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}) = 1 - \frac{\omega_p^2}{(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2} \left\{ 1 + 3\frac{\kappa T}{m} \cdot \frac{l^2}{(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2} + \cdots \right\} + i\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}). \tag{7}$$

Case (II): $(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2 \ll (\kappa T/m) l^2$

In the same way, we obtain

$$\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}) = 1 + \frac{k_d^2}{l^2} \left\{ 1 + O\left(\frac{(\omega - (\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2}{(\kappa T/m) l^2} \right) + \cdots \right\} + i\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}). \tag{8}$$

The above results show that the effective interaction in the plasma is quite different for the two situations. For case (II), the Fourier coefficient of the effective interaction can be approximated by

$$\frac{\lambda V(l)}{\Delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})} \approx \frac{4\pi e^2}{l^2+k_d^2}, \tag{9}$$

which shows that the factor Δ screens off the long-range tail of the Coulomb potential and gives rise to the static Debye potential $e^2\exp(-k_d r)\cdot r^{-1}$.

On the contrary, for case (I), the factor Δ does not weaken the contributions of long-range tail of the Coulomb potential, but rather enhances the effects of its long-range tail. The enhancement of the interaction is described by the condition that

$$\mathrm{Re}\,\Delta=0. \tag{10}$$

For a fixed value of $|\boldsymbol{l}|$, this condition determines the angles between $\boldsymbol{v}$ and $\boldsymbol{l}$ along which the interaction effects reach the large distances. The amount of the enhancement is determined by the imaginary part of Δ. The small correction due to the thermal motion of plasma electrons being neglected, the condition (10) determines the angles to be

$$\cos\theta_1=\frac{|\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}|}{lv} \tag{11}$$

and

$$\cos\theta_2=-\frac{|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|}{lv}. \tag{12}$$

Since the requirement of $|\cos\theta|\leq 1$ must be fulfilled, we get the possible lower limit of l as follows, for the branch of angles determined by Eq. (11)

$$l\geq|\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}|/v \tag{13}$$

and for the angles of Eq. (12)

$$l\geq|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|/v. \tag{14}$$

The upper limit of l is determined to be k_d from the condition $(\kappa T/m)l^2<(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2=\omega_p^2$, which is obtained by substituting the condition (10) into the condition (I). The enhancement of interaction effects along the specified angles suggests that such a phenomenon is due to emission of the plasma waves, or the plasmons. We will confirm this interpretation in the following sections by calculating the friction and diffusion coefficient of the kinetic equation of incident beam.

Finally, let us remark that Tchen has obtained a static shielding factor (Eq. (23c) of reference 7)) which corresponds to our result for case (II). He has used the shielding factor throughout in his calculation for the both cases of $v^2\gg v_T^2$ and $v^2\ll v_T^2$, where $v_T^2=\kappa T/m$. Apparently, he did not realize that the static theory of Debye shielding in a plasma is not applicable to the case of $v^2\gg v_T^2$. On the other hand, in the present section, we have obtained the condition (II) under which the ordinary Debye's static shielding potential can

be used to describe the interaction between the electrons in the plasma. Although the condition obtained in the present section appears to be complicated, if we consider the particular case of $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$, the condition (II) is reduced to $v^2\ll v_T^2$. Thus, we can conclude that only if the velocity of the electron is slower than the thermal velocity of plasma electrons the long-range part of Coulomb interaction can be described by Debye's shielded potential, and that when velocity of the electron exceeds the thermal velocity of plasma electrons the long-range part of the Coulomb interaction is modified by such a process that accompanies the emission of plasmons. Of course, the short-range part of the Coulomb interaction is not affected by the screening effects for the both cases of $v^2\gg v_T^2$ and $v^2\ll v_T^2$. In the following sections, we will calculate the friction and diffusion coefficient by using the above described properties of the dynamical shielding factors.

§ 4. The frictional coefficient

Now, let us proceed to calculate the friction coefficient $A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega)$. From Eqs. (2) and (4), the frictional coefficient becomes as follows,

$$A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega)=\frac{e^2}{2\pi^3}\int d\boldsymbol{l}\frac{l^\alpha}{l^2}\cdot\frac{\delta}{|\Delta|^2}+i\frac{e^2}{2\pi^2}\int d\boldsymbol{l}\frac{l^\alpha}{l^2}\frac{\mathrm{Re}\,\Delta}{|\Delta|^2}. \tag{15}$$

The first term of Eq. (15) is the proper friction coefficient which is responsible to the irreversible process occurring in the plasma. The second term, on the contrary, can be identified as an effective field acting on the beam. Here, we will first examine the proper friction coefficient and come back to the second term in the last part of this section.

We assume that the beam falls along the z-axis, then, as it is clear from the expression of A^α, there remains only z-component of the friction coefficient $A^z(\boldsymbol{k}, \boldsymbol{v}; \omega)$. According to the discussions of the preceding section, we have to divide the region of $\boldsymbol{l}$ into two domains corresponding to the cases (I) and (II). The real part of A^z can be expressed as

$$\mathrm{Re}\,A^z(\boldsymbol{k}, \boldsymbol{v}; \omega)=A_I^z(\boldsymbol{k}, \boldsymbol{v}; \omega)+A_{II}^z(\boldsymbol{k}, \boldsymbol{v}; \omega), \tag{16}$$

$$A_I^z(\boldsymbol{k}, \boldsymbol{v}; \omega)=\frac{\omega_p^2}{8\pi^3}\frac{m}{n}\int d\boldsymbol{l}\frac{l^z}{l^2}\cdot\frac{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})}{\{1-\omega_p^2/(\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})\}^2}, \tag{17}$$

$$A_{II}^z(\boldsymbol{k}, \boldsymbol{v}; \omega)=\frac{\omega_p^2}{8\pi^3}\frac{m}{n}\int d\boldsymbol{l}\frac{l^z}{l^2}\frac{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})}{\{1+(k_d/l)^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})\}^2} \tag{18}$$

with the abbreviation given by Eq. (6). It is rather cumbersome to specify the domains of integration of A_I^z and A_{II}^z. Let us consider the two cases (i) $v\gg v_T$ and (ii) $v\ll v_T$, and specify the domains of variables l and $\cos\theta=(\boldsymbol{l}\cdot\boldsymbol{v})/\boldsymbol{lv}$ for the conditions (I) and (II).

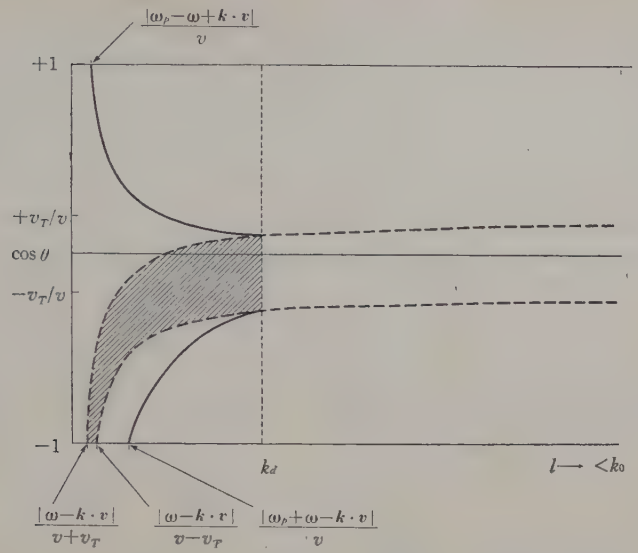


Fig. 1. Domains of l and $\cos\theta$ in which the conditions (I) and (II) are valid. (An example for the case (i) $v>v_T$.)

Firstly, when $v\gg v_T$, see Fig. 1. The broken lines cut out the domain of the condition (II) from the θ-l plane. The inside of the broken lines is the domain of integration of $A_{II}{}^{\alpha}$. According to the discussion of the preceding section, the integrand of $A_I{}^{\varepsilon}$ has a resonant character around the regions of variables determined by the condition

$$1-\frac{\omega_p{}^2}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\cos\theta)^2}=0. \tag{19}$$

This condition determines the polar angles along which the resonance occurs for each values of l. These angles are given by Eqs. (11) and (12). The resonance corresponds to the emission of plasmons of wave number l into the direction of the specified angles. The lower limits of the wave numbers of plasmons are determined by Eqs. (13) and (14), corresponding to the angles determined by Eqs. (11) and (12). The upper limit of the wave number of plasmons is k_d. In Fig. 1, the full lines represent the lines along which the resonance condition is fulfilled. At the resonance, the imaginary part of shielding factor is reduced to

$$\delta=(\pi/2)^{1/2}(k_d/l)^3\exp[-1/2(k_d/l)^2] \tag{20}$$

which represents broadening width of the resonance due to the finite lifetime of plasmons described by the Landau damping factor

$$\gamma_L=(\pi/8)^{1/2}(k_d/l)^3\exp[-1/2(k_d/l)^2]\cdot\omega_p. \tag{21}$$

One can see from Fig. 1 if $|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|>vk_d$ only the upper branch of resonances does occur. For the region of $l>k_d$, it is not essential to distinguish between the two types of the shielding factor, because at the short distance the shielding effects due to other particles are not effective.

Secondly, when $v\ll v_T$, see Fig. 2. The broken line divides the θ-l plane into the domains of the cases (I) and (II). It is one of the most interesting results of our investigation that Fig. 2 shows that even if $v<v_T$ there may occur the resonance emission of plasmons provided that $vk_d>|\omega_p-(\omega-\boldsymbol{k}\cdot\boldsymbol{v})|$.

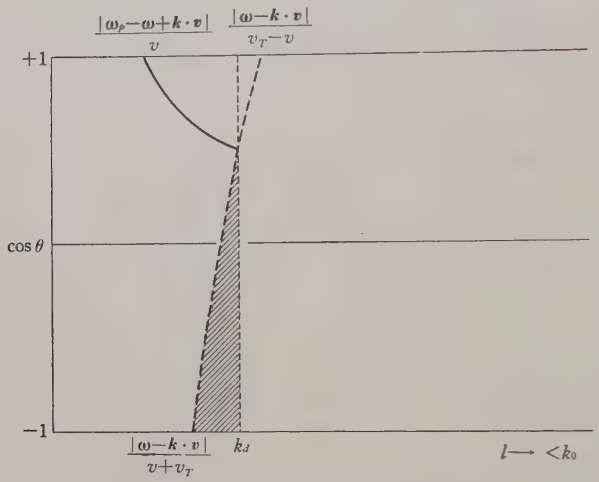


Fig. 2. Domains of l and θ in which the conditions (I) and (II) are valid. (An example for the case (ii) $v<v_T$.)

Figs. 1 and 2 have been written for the case of $\omega-\boldsymbol{k}\cdot\boldsymbol{v}>0$. It is straightforward to draw figures for the opposite case of $\omega-\boldsymbol{k}\cdot\boldsymbol{v}<0$. For this case, there occurs only the lower branch of resonance, provided that $|\omega_p-(\omega-\boldsymbol{k}\cdot\boldsymbol{v})|>k_dv$ for $v\gg v_T$ and $|\omega_p+(\omega-\boldsymbol{k}\cdot\boldsymbol{v})|\ll k_dv$ for $v\ll v_T$.

Let us calculate A_I^z defined by Eq. (17). A_I^z can be divided into two parts corresponding to such values of l that $l<k_d$ and $l>k_d$. The main contributions to the long-range part of $A_I^z(l<k_d)$ come from the regions of the resonance emission of plasmons. Hence it can be calculated as follows,

$$A_{pl}^z(\boldsymbol{k},\ \boldsymbol{v}\,;\,\omega)=\frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})/v}^{k_d}dl\cdot l\int_{v/v_T}^{1}d\xi\frac{\delta\cdot\xi}{\{1-\omega_p^2/(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi)^2\}^2+\delta^2}$$

$$+\frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})/v}^{k_d}dl\cdot l\int_{-1}^{-v/v_T}d\xi\frac{\delta\cdot\xi}{\{1-\omega_p^2/(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi)^2\}^2+\delta^2}\qquad(22)$$

where $\xi=\cos\theta$. Putting $\xi=\cos\theta_{1,2}+x$, this can be evaluated as follows, ($\xi_{1,2}=\cos\theta_{1,2}$)

$$\text{Eq. (22)} \doteqdot \frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})/v}^{k_d} dl\cdot l\int_{-\infty}^{+\infty}\frac{\delta(\xi_1)\cdot\xi_1}{(4l^2v^2/\omega_p^2)x^2+\delta(\xi_1)^2}\,dx$$

$$+\frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})/v}^{k_d} dl\cdot l\int_{-\infty}^{+\infty}\frac{\delta(\xi_2)\cdot\xi_2}{(4l^2v^2/\omega_p^2)x^2+\delta(\xi_2)^2}\,dx.$$

Accordingly, we get

$$A_{pl}{}^z(\boldsymbol{k},\boldsymbol{v};\omega)=\frac{e^2\omega_p}{2v^2}\left[(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})\log\frac{k_dv}{|\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}|}\right.$$

$$\left.+(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})\log\frac{k_dv}{|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|}\right]. \tag{23}$$

It is quite instructive to notice that if we take $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ the expression (23) reduces to the long-range part of the friction coefficient derived by Bohm and Pines and also by Klimontovich,

$$A_{pl}{}^z(v)=\frac{e^2\omega_p^2}{v^2}\log\left(\frac{v}{v_T}\right). \tag{24}$$

When $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ and $v_T>v$, the friction coefficient due to the emission of plasmons A_{pl}^z vanishes. Although we have calculated contributions of both resonance branches, in some case one of them does not contribute to the coefficient. Table I shows every case to be specified.

Table I. Contributions of resonance emission of plasmons to $A_{pl}{}^z$, $B_{pl}{}^{\parallel}$ and $B_{pl}{}^{\perp}$.

Cases		Conditions	Contributing terms
$v>v_T$	$\omega-\boldsymbol{k}\cdot\boldsymbol{v}>0$	in general	both terms
		if $k_d<\lvert\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}\rvert/v$	1st term
	$\omega-\boldsymbol{k}\cdot\boldsymbol{v}<0$	in general	both terms
		if $k_d<\lvert\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}\rvert/v$	2nd term
$v<v_T$	$\omega-\boldsymbol{k}\cdot\boldsymbol{v}>0$	in general	no
		if $k_d>\lvert\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}\rvert/v$	1st term
	$\omega=\boldsymbol{k}\cdot\boldsymbol{v}$		no
	$\omega-\boldsymbol{k}\cdot\boldsymbol{v}<0$	in general	no
		if $k_d>\lvert\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}\rvert/v$	2nd term

Next, let us consider the second part of $A_l{}^z(l>k_d)$. This corresponds to the contributions of the short-range region, where the shielding effects are irrelevant. Therefore, we may put $|\Delta|^2=1$ in the denominator of Eq. (17). Thus, we have

$$A^z_{I,short}(\boldsymbol{k}, \boldsymbol{v}; \omega) = \frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{k_d}^{k_0} dl \cdot l\left\{\int_{v_T/v}^{1} d\xi + \int_{-1}^{-v_T/v} d\xi\right\}\xi\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}). \quad (25)$$

Before going into the calculation of this term, let us consider the contribution of A_{II}^z given by Eq. (18). This term can also be divided into two parts of $l>k_d$ and $l<k_d$. To be consistent with the above evaluation of $A^z_{I,short}$, we take also $|\Delta|^2=1$ in the denominator of the short-range part of A^z_{II}. Then, we get

$$A^z_{II,short}(\boldsymbol{k}, \boldsymbol{v}; \omega) = \frac{\omega_p^2}{4\pi^2}\frac{m}{n}\int_{k_d}^{k_0} dl \cdot l\int_{-v_T/v}^{v_T/v} d\xi\, \xi\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v}). \quad (26)$$

Combining Eqs. (25) and (26), we obtain the following expression as a short-range part of $A^z(\boldsymbol{k}, \boldsymbol{v}; \omega)$,

$$A^z{}_{short}(\boldsymbol{k}, \boldsymbol{v}; \omega) = (\pi/2)^{1/2}\frac{\omega_p}{4\pi^2}\cdot\frac{m}{n}\cdot k_d^3\int_{k_d}^{k_0} dl\, l^{-2}$$
$$\times\int_{-1}^{+1} d\xi\, \xi(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi)\exp\left[-\frac{m}{2\kappa T}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi/l)^2\right]. \quad (27)$$

Now, the ω-$\boldsymbol{k}$ dependence of A^z_{short} can be ignored, because it represents the non-local space-time correlation effects which is not essential in the short-range region. Omitting the ω-$\boldsymbol{k}$ dependence of Eq. (27), we get

$$A^z{}_{short}(\boldsymbol{v}) = -(1/2)\frac{e^2\omega_p^2}{v_T^2}\frac{d}{d\beta}\{\beta^{-1}\operatorname{erf}\beta\}\cdot\log\left(\frac{k_0}{k_d}\right), \quad (28)$$

with the abbreviation of $\beta^2=v^2/(2v_T^2)$, and k_0 has been determined to be the Landau cutoff $\kappa T/e^2$ in the general theory developed by I. For collisions at closer distances than the Landau cutoff distance $e^2/\kappa T$, it is necessary to take account of effects of the secondary convection terms (the second term of the right-hand side of Eq. (22) and the last term of the right-hand side of Eq. (35) in I). It has been shown in I and also by Tchen, however, that the effects will be negligible for the high temperature plasma at low density. Therefore, the Landau cutoff has to be used as the unique cutoff at the close distance for the high temperature plasma. The result obtained as Eq. (28) is nothing but the expression obtained by Tchen (Eq. (36a) of reference 7)) and also by Rostoker and Rosenbluth (Eq. (33) of reference 8)). Although Rostoker and Rosenbluth have been confronted by the arbitrariness of the cutoff at the close distance, the above discussion together with Tchen's analysis has shown the uniqueness of the Landau cutoff.

Finally, there remains the long-range part of A^z_{II}, which is given by

$$A^z_{II,long}(\boldsymbol{k}, \boldsymbol{v}; \omega) = (\pi/2)^{1/2}\frac{\omega_p}{4\pi^2}\cdot\frac{m}{n}k_d^3\int_{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})/v_T}^{k_d} dl\frac{l^2}{(l^2+k_d^2)^2}$$
$$\times\int d\xi\, \xi(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi)\exp\left[-\frac{m}{2\kappa T}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+lv\xi/l)^2\right], \quad (29)$$

where the domain of integration is the hatched part in Figs. 1 and 2, for example. Eq. (29) is reduced from Eq. (18) by omitting $\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})$ in the denominator. Let us estimate the contribution of $A^z_{II,long}$ by considering the special case of $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$. Eq. (29) takes the following form,

$$A^z_{II\ long}(\boldsymbol{v}) = (\pi/2)^{1/2}\frac{\omega_p}{4\pi}\frac{m}{n}k_d{}^3 v\int_0^{k_d} dl \frac{l^3}{(l^2+k_d{}^2)^2} \times\int_{-1}^{+1} d\xi\, \xi^2 \exp\left[-\frac{m}{2\kappa T}v^2\,\xi^2\right]$$
$$= -\frac{1}{2}\frac{e^2\omega_p{}^2}{v_T{}^2}\frac{d}{d\beta}(\beta^{-1}\,\mathrm{erf}\,\beta)\cdot\frac{1}{2}(\log 2-1/2). \tag{30}$$

This is negligible in comparison with the contribution of the long-range region due to the resonance emission of plasmons, Eq. (24), for the case of $v\rangle\rangle v_T$ and it is also negligible in comparison with the contribution of the short-range region obtained in the above, Eq. (28). We may expect that this term can be disregarded even in the general case of $\omega\neq\boldsymbol{k}\cdot\boldsymbol{v}$.

At last, we have to examine the contribution of the imaginary part of $A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega)$. First of all, if we ignore the non-local space-time correlation effects, the imaginary part of $A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega)$ vanishes identically. Consequently, there remains the contribution of the long-range region to the imaginary part of $A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega)$. It is clear, however, that the regions of the resonance emission of plasmons do not contribute to this term, because of the condition $\mathrm{Re}\,\Delta=0$ in the resonance regions. Thus, there remains only the contribution from the long-range region of the domain (II). Therefore, we have

$$\mathrm{Im}\,A^z(\boldsymbol{k}, \boldsymbol{v}; \omega) = \frac{e^2}{\pi}\int dl \frac{l^3}{l^2+k_d{}^2}\int d\xi\,\xi, \tag{31}$$

where $\xi=\cos\theta=(\boldsymbol{l}\cdot\boldsymbol{v})/lv$. This does not vanish in general, because the domain of integration of ξ is asymmetric when the non-local space-time correlation effects are taken into account. (See Fig. 1, for example. The hatched region is the domain of integration of Eq. (31).) Although it appears to be a quite interesting problem to examine the effects of the imaginary part of $A^z(\boldsymbol{k}, \boldsymbol{v}; \omega)$, we will not discuss the problem in any details.

§ 5. The diffusion coefficients in the velocity space

The diffusion coefficients can be calculated by the same method as that used in the calculation of the friction coefficient. The real part of Eq. (3) is expressed as

$$\mathrm{Re}\ B^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) = B_I{}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) + B_{II}{}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) \tag{32}$$

$$B_I{}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) = \frac{\omega_p{}^3}{8\pi^3}\cdot\frac{m^2}{n}\int d\boldsymbol{l}\frac{l^\alpha l^\beta}{l^4}\cdot\frac{\omega_p}{\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}}\cdot\left(\frac{l}{k_d}\right)^2$$

$$\times \frac{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}{\{1-(\omega_p/\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})\}^2}, \tag{33}$$

$$B_{II}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}\,;\,\omega)=\frac{\omega_p^3}{8\pi^3}\cdot\frac{m^2}{n}\int d\boldsymbol{l}\frac{l^\alpha l^\beta}{l^4}\frac{\omega_p}{\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}}\left(\frac{l}{k_d}\right)^2$$

$$\times \frac{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})}{\{1+(k_d/l)^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}\,;\,\omega\,;\,\boldsymbol{v})\}^2}. \tag{34}$$

As a result of the axial symmetry of the present problem, there remain only the diagonal components, $B^{xx}=B^{yy}$ and B^{zz}. The $B_I^{\alpha\alpha}$ refers to the contribution of the domain (I) and the $B_{II}^{\alpha\alpha}$ to the contribution of the domain (II).

The long-range part of $B_I^{\alpha\alpha}$ can be calculated in the same way as the calculation of A_{pl}^z. The result is as follows,

$$B_{pl}^{zz}(\boldsymbol{k}, \boldsymbol{v}\,;\,\omega)=\frac{e^2}{2v^3}\kappa T\left\{(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})^2\log\frac{k_d v}{|\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}|}\right.$$

$$\left.+(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2\log\frac{k_d v}{|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|}\right\}, \tag{35}$$

$$B_{pl}^{xx}(\boldsymbol{k}, \boldsymbol{v}\,,\,\omega)=B_{pl}^{yy}(\boldsymbol{k}, \boldsymbol{v}\,;\,\omega)$$

$$=\frac{e^2}{4v^3}\kappa T\left[(k_d v)^2-(1/2)\left\{(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})^2+(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2\right\}\right.$$

$$\left.-\left\{(\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v})^2\log\frac{k_d v}{|\omega_p-\omega+\boldsymbol{k}\cdot\boldsymbol{v}|}+(\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v})^2\log\frac{k_d v}{|\omega_p+\omega-\boldsymbol{k}\cdot\boldsymbol{v}|}\right\}\right]. \tag{36}$$

The criterions of the resonance contribution are listed in Table I. Here, " the first term " and " the second term " refer to each of the two terms in curly brackets of Eqs. (35) and (36). If we take $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$, Eqs. (35) and (36) reduce to the diffusion coefficients derived by Klimontovich (Eq. (24) of reference 5)).

Next, let us consider the short-range part of $B_I^{\alpha\alpha}$ and $B_{II}^{\alpha\alpha}$, in which we disregard the shielding factors in the denominator and the non-local space-time correlation effects. Then, we get the following expressions,

$$B_{short}^{zz}(v)=\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_p^2}{4\pi^2}\frac{m^2}{n}k_d\int_{k_d}^{k_0}dl\frac{1}{l}\int_{-1}^{+1}d\xi\,\xi^2\exp\left[-\frac{m}{2\kappa T}v^2\xi^2\right]$$

$$=-(1/2)\frac{m}{v}e^2\omega_p^2\frac{d}{d\beta}(\beta^{-1}\mathrm{erf}\,\beta)\cdot\log\left(\frac{k_0}{k_d}\right), \tag{37}$$

$$B_{short}^{xx}(v)=B_{short}^{yy}(v)$$

$$=\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_p^3}{8\pi^2}\frac{m^2}{n}k_d\int_{k_d}^{k_0}dl\frac{1}{l}\int_{-1}^{+1}d\xi(1-\xi^2)\exp\left[-\frac{m}{2\kappa T}v^2\xi^2\right]$$

$$=(1/2)\frac{m}{v}e^2\omega_p^2\left\{\mathrm{erf}\,\beta+\frac{1}{2}\frac{d}{d\beta}(\beta^{-1}\,\mathrm{erf}\,\beta)\right\}\log\left(\frac{k_0}{k_d}\right). \tag{38}$$

These results confirm the expressions derived by Tchen. The long-range part of $B_{II}^{\alpha\beta}$ can be neglected in comparison with other terms presented by Eqs. (35)$\sim$(38).

Finally, let us briefly discuss the contribution of imaginary part of $B^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega)$. We obtain the following expressions,

$$\mathrm{Im}\ B^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) = \mathrm{Im}\ B_I^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) + \mathrm{Im}\ B_{II}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) \tag{39}$$

$$\begin{aligned}\mathrm{Im}\ B_I^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) = &-\frac{\omega_p^3}{8\pi^3}\frac{m^2}{n}\int d\boldsymbol{l}\frac{1}{l^2+k_d^2}\frac{l^\alpha l^\beta}{l^4}\frac{\omega_p}{\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}}\left(\frac{l}{k_d}\right)^2 \\ &\times\frac{\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})\}^2}{\{1-(\omega_p/\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})\}^2} \\ &+\frac{\omega_p^3}{8\pi^3}\cdot\frac{m^2}{n}\int d\boldsymbol{l}\frac{1}{l^2+k_d^2}\cdot\frac{l^\alpha l^\beta}{l^4}\cdot\frac{\omega_p}{\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}} \\ &\times\frac{1-(\omega_p/\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2}{\{1-(\omega_p/\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v})^2\}^2+\{\delta(\boldsymbol{k}-\boldsymbol{l}, \boldsymbol{l}; \omega; \boldsymbol{v})\}^2},\end{aligned} \tag{40}$$

$$\mathrm{Im}\ B_{II}^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega) = -\frac{\omega_p^3}{8\pi^3}\frac{m^2}{n}\int d\boldsymbol{l}\frac{l^\alpha l^\beta}{l^2(l^2+k_d^2)^2}\left(\frac{\omega}{lv_T}\right)\left(\frac{\omega-(\boldsymbol{k}-\boldsymbol{l})\cdot\boldsymbol{v}}{lv_T}\right). \tag{41}$$

Here, $\mathrm{Im}\ B_I$ and $\mathrm{Im}\ B_{II}$ represent the contributions of the domains (I) and (II), respectively. Furthermore, if the non-local space-time correlation effects are absent Eqs. (40) and (41) vanish identically, so the integrations over l in Eqs. (40) and (41) covers only the long-range region $l<k_d$. It is very interesting to observe that the first term of $\mathrm{Im}\ B_I^{\alpha\beta}$ represents non-vanishing contribution of the resonance emission of plasmons. In calculating $\mathrm{Im}\ B_I^{\alpha\beta}$, we can disregard the second term completely. According to the general pictures of the high temperature plasma, the interaction effects at the long-range region are properly described by the plasmons. Therefore, we may regard the first term of $\mathrm{Im}\ B_I^{\alpha\beta}$ as the essential contribution to the imaginary part of $B^{\alpha\beta}$. Then, comparing $\mathrm{Im}\ B_I^{\alpha\beta}$ with $B_I^{\alpha\beta}$ given by Eq. (33), we can conclude that the imaginary part of $B^{\alpha\beta}$ are of negligible order of magnitude in comparison with the real part of $B^{\alpha\beta}$.

§ 6. Effects of the non-local space-time correlation

In the foregoing sections, we have explicitly calculated the friction and diffusion coefficient. It has been shown that the contributions of the resonance emission of plasmons to these coefficients are affected considerably by the non-local space-time correlation effects. In the present section, we will briefly examine the non-local effect and also the relation between the ordinary theory and the present theory.

Let us consider the problem of solving the kinetic equation of the incident beam, Eq. (1). If we regard the right-hand side as a perturbational term, we

may expand the distribution function of the incident beam into

$$f(\boldsymbol{x}, \boldsymbol{v}; t) = f^{(0)}(\boldsymbol{x}, \boldsymbol{v}; t) + f^{(1)}(\boldsymbol{x}, \boldsymbol{v}; t). \tag{42}$$

Corresponding to the expansion of Eq. (42), Eq. (1) can be expanded into the following set of equations,

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right\} f^{(0)}(\boldsymbol{x}, \boldsymbol{v}; t) = 0, \tag{43}$$

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right\} f^{(1)}(\boldsymbol{x}, \boldsymbol{v}; t) = \frac{1}{(2\pi)^4}\int d\omega \int d\boldsymbol{k}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

$$\times \sum_{\alpha}\frac{1}{m}\frac{\partial}{\partial v_\alpha}\left\{A^\alpha(\boldsymbol{k}, \boldsymbol{v}; \omega) f^{(0)}(\boldsymbol{k}, \boldsymbol{v}; \omega) + \sum_{\beta}\frac{1}{m} B^{\alpha\beta}(\boldsymbol{k}, \boldsymbol{v}; \omega)\frac{\partial}{\partial v_\beta} f^{(0)}(\boldsymbol{k}, \boldsymbol{v}; \omega)\right\}. \tag{44}$$

Eq. (43) describes free propagation of the incident beam. We get a particular solution of the type

$$f^{(0)}(\boldsymbol{x}, \boldsymbol{v}; t) = \frac{1}{(2\pi)^4}\int d\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} f^{(0)}(\boldsymbol{k}) \int d\omega\, e^{-i\omega t}\delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}). \tag{45}$$

This solution shows that the beam moving with constant velocity $\boldsymbol{v}$ has the frequency spectrum of $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$. This relation has been used in the preceding sections to show that our results of the friction and diffusion coefficient are reduced to the coefficients obtained by Klimontovich in the special case. Substituting Eq. (45) into Eq. (44), we obtain

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right\} f^{(1)}(\boldsymbol{x}, \boldsymbol{v}; t) = \sum_{\alpha}\frac{1}{m}\frac{\partial}{\partial v_\alpha}\left\{A^\alpha(\boldsymbol{v}) f^{(0)}(\boldsymbol{x} - \boldsymbol{v}t)\right.$$

$$\left. + \sum_{\beta}\frac{1}{m} B^{\alpha\beta}(\boldsymbol{v})\frac{\partial}{\partial v_\beta} f^{(0)}(\boldsymbol{x} - \boldsymbol{v}t)\right\}, \tag{46}$$

where $A^\alpha(\boldsymbol{v})$ and $B^{\alpha\beta}(\boldsymbol{v})$ are reduced from the results of § 4 and § 5 by taking as $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$. For the convenience, let us write them down for the case in which the beam falls along the z-axis with the higher velocity than the thermal velocity v_T. Eqs. (23) and (28) give, for $v \gg v_T$,

$$A^z(v) = A^z_{plasmon}(v) + A^z_{collision}(v), \tag{47}$$

with

$$A^z_{plasmon}(v) = \frac{e^2\omega_p^2}{v^2}\log\left(\frac{v}{v_T}\right), \tag{48}$$

$$A^z_{collision}(v) = \frac{e^2\omega_p^2}{v^2}\log\left(\frac{k_0}{k_d}\right). \tag{49}$$

Eqs. (35)$\sim$(38) give, for $v \gg v_T$,

$$B^{xx}(v)=B^{yy}(v)=B^{\perp}_{plasmon}(v)+B^{\perp}_{collision}(v) \tag{50}$$

with

$$B^{\perp}_{plasmon}(v)=\frac{e^2\omega_p{}^2}{4v^3}\kappa T\left[\left(\frac{v}{v_T}\right)^2-2\log\left(\frac{v}{v_T}\right)\right], \tag{51}$$

$$B^{\perp}_{collision}(v)=\frac{e^2\omega_p{}^2}{2v^3}\kappa T\left[\left(\frac{v}{v_T}\right)^2-\log\left(\frac{k_0}{k_d}\right)\right], \tag{52}$$

and

$$B^{zz}(v)=B^{\parallel}_{plasmon}(v)+B^{\parallel}_{collision}(v) \tag{53}$$

with

$$B^{\parallel}_{plasmon}(v)=\frac{e^2\omega_p{}^2}{v^3}\kappa T\log\left(\frac{v}{v_T}\right)=m\frac{v_T{}^2}{v}\cdot A^{\parallel}_{plasmon}(v), \tag{54}$$

$$B^{\parallel}_{collision}(v)=\frac{e^2\omega_p{}^2}{v^3}\kappa T\log\left(\frac{k_0}{k_d}\right)=m\frac{v_T{}^2}{v}A^{\parallel}_{collision}(v). \tag{55}$$

As it has been mentioned in the preceding section, these coefficients agree with those obtained by Klimontovich and by Rostoker and Rosenbluth.

In contrast with Eq. (1), Eq. (46) shows that the changes of state of the beam are described by the local interaction effect. Since the perturbational expansion may be applicable only for a limited space-time interval, it is necessary to avoid the use of the perturbational expansion so that we can take into account the non-local effect in discussing overall behaviour of the incident beam.

Furthermore, there are other effects which have to be taken into account to discuss the non-local effect. When the beam falls into a quiecent plasma, the beam will be slowed down and diffused by the plasmon emission process and the collision process, which induce disturbance in the plasma. Reaction effects of the disturbance to the incident beam are described by the internal field term and the counterparts of the *A*-*B* terms of the kinetic equation (the last term of the right-hand side of Eq. (71) in I). The internal field term causes modulation of the velocity of the beam. Therefore, under the action of the internal field term, the frequency spectrum of the incident beam changes from the free propagation spectrum of $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ to the modulated frequency spectrum of $\omega\fallingdotseq\boldsymbol{k}\cdot\boldsymbol{v}$. This change of the frequency spectrum is nothing but the result of the non-local space-time correlation effect. Such internal velocity modulation effect is expected to be essential to produce bunching of the incident beam and to be a cause of the rapid exchange of energy between the incident beam and the target plasma. In fact, Eq. (23) tells us that if the velocity modulation is such that satisfies

$$\omega-\boldsymbol{k}\cdot\boldsymbol{v}=\omega_p-\frac{1}{\varepsilon}k_dv \qquad \varepsilon:\ \text{constant of natural logarithms} \tag{56}$$

$A^{z}{}_{pl}(\boldsymbol{k}, \boldsymbol{v}; \omega)$ takes the maximum value of

$$A^{z}_{plasmon}(\boldsymbol{k}, \boldsymbol{v}; \omega) = \frac{1}{2\varepsilon} \frac{e^2\omega_p{}^2}{v^2} \cdot \left(\frac{v}{v_T}\right). \tag{57}$$

When $v \gg v_T$, this is very large in comparison with the collisional friction coefficient. Therefore, the very short mean free path[10] observed may be explained by the mechanism of the velocity modulation, which is due to the internal field fluctuation of the high temperature plasma. The condition expressed by Eq. (56) is quite likely to be realized in the high temperature plasma, because the internal field fluctuates with frequency ω_p. Although the above discussion is rather qualitative, the effect appears to be essential to resolve the Langmuir paradox.

§ 7. Concluding discussions

In the present paper, the following results are obtained:

1) The effective interaction in the high temperature plasma has been analyzed in detail by investigating the dynamical shielding factor. It has been shown that the static theory of Debye shielding can be applied under the restricted condition. For a particle moving with constant velocity v, the condition is reduced to the simple form that $v^2 \ll v_T{}^2$. When $v^2 \gg v_T{}^2$, the interaction effects are not screened off but get enhancement along the specified angles. This enhancement effect is due to the emission of plasmons. Thus, our theory provides a unified view point of the two characteristic aspects of the high temperature plasma, one of which is the static shielding phenomena discussed by Debye and the other is the dynamic plasma oscillations discussed by Bohm and many others.

2) The present investigation has shown that the imaginary part of the dynamical shielding factor plays an essential role in determining the dissipation coefficients. In the long-range region, which involves such wave numbers that $0 < k < k_d$, the imaginary part is related with the Landau damping factor of the plasma oscillations. In the short-range region, which involves such wave numbers that $k_d < k < k_0$, the imaginary part determines the amount of the dissipation coefficients due to the binary collision with impact parameters between $k_d{}^{-1}$ and $k_0{}^{-1}$. In order to discuss the contribution of the extreme short-range region, which may cover the region of impact parameters between $k_0{}^{-1}$ and the nearest approach distance $\{2e^2/mv^2\}^{1/2}$, it is necessary to take account of the effects of the secondary convection terms which have been neglected in the general theory of I. It should be remembered that the Landau cutoff has been introduced to assure the omission of the secondary convection terms. Tchen has discussed the effects of the secondary convection terms and has shown the effects may be negligible for the low density plasma.

3) It is shown that the non-local space-time correlation effects are very

important in the region of the plasmon emission. In § 6, a possible resolution of the Langmuir paradox has been suggested on the basis of the non-local correlation effects due to the internal field fluctuation. In order to give a complete resolution of the Langmuir paradox, however, it is necessary to develop a consistent theory which takes account of the reaction of disturbance induced in the plasma.

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Interactions in Nucleon Core

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It is pointed out that K-N interaction rather than π-N interaction plays the important role in the phenomena within nucleon core. The processes of pion-nucleon scattering through s-wave pion-pion interaction are classified into two parts, that is, the processes in nucleon core and those in the neighborhood of pion cloud. The contributions from the former to pion-nucleon scattering are estimated and are compared with those from the processes through the mechanism via $KK\pi\pi$-interaction which may be regarded as one of the phenomena in nucleon core. Moreover, some discussion about the anomalous magnetic moments of nucleons is made.

§ 1. Introduction

So far as the phenomena in the neighborhood of pion cloud or K meson cloud are concerned, the nucleon core may be treated as if it were a point. However, such a treatment will no longer be valid in describing the phenomena within the range of $1/M$, because the effects due to the spread of nucleon core will turn out to be so remarkable that they cannot be neglected. In order to examine the interactions in nucleon core we have previously studied both the problems of s-wave pion-nucleon scattering[1] and those of s-wave K meson-nucleon scattering,[2] and tried to express the effects due to the spread of nucleon core in terms of the change of the value of coupling constants. The main results obtained previously are as follows.

(i) In the case of pion-nucleon scattering, the value of coupling constant averaged over the spread of nucleon core should be reduced in appearance to $(\mu/2M)g=f$ in spite of the fact that its value in the neighborhood of pion cloud is g, where μ and M are the masses of pion and nucleon respectively, and g is the renormalized (unrationalized) pseudoscalar coupling constant. On the basis of the above result an attempt to eliminate the divergence included in the dispersion relation was made, and the experimental results for s-wave phase shifts were satisfactorily explained.

(ii) In the case of K-N scattering the situation is more complicated because in this process the nucleon core undergoes some remarkable change in the course of collision in the meaning of perturbation. If the masses of Λ- and Σ-particles (M_Λ and M_Σ) were equal to M, the effective coupling constants in nucleon core should be reduced in appearance to $(m_K/2M)g_\Lambda=f_\Lambda$ and

$(m_K/2M)g_\Sigma=f_\Sigma$* as in the case of pion-nucleon interaction, where g_Λ and g_Σ are the renormalized (unrationalized) pseudoscalar coupling constants for $KN\Lambda$ and $KN\Sigma$ interactions, and m_K is the mass of K meson. In the practical case, however, the above conclusions are obliged to suffer a remarkable change owing to the relation $(M_\Lambda^2-M^2-m_K^2)>0$ and $(M_\Sigma^2-M^2-m_K^2)<0$, that is, the values of coupling constants averaged over the spread of nucleon core should be reduced in appearance to

$$f_\Lambda^2 \rightsquigarrow -f_\Lambda^2 \frac{(M_\Lambda^2-M^2-m_K^2)}{m_K^2-\{(M_\Lambda^2-M^2-m_K^2)/2M\}^2}=-f_\Lambda^2\bar{\zeta}_\Lambda^2, \tag{1}$$

$$f_\Sigma^2 \rightsquigarrow -f_\Sigma^2 \frac{(M_\Sigma^2-M^2-m_K^2)}{m_K^2-\{(M_\Sigma^2-M^2-m_K^2)/2M\}^2}=-f_\Sigma^2\bar{\zeta}_\Sigma^2, \tag{1}'$$

in spite of the fact that their values in the neighborhood of K meson cloud are g_Λ and g_Σ respectively.** (Of course Eqs. (1) and (1)′ are reduced to $f_\Lambda^2 \rightsquigarrow f_\Lambda^2$ and $f_\Sigma^2 \rightsquigarrow f_\Sigma^2$ when $M_\Lambda=M_\Sigma=M$.)***

Such results may give us an important clue to the true theory in nucleon core. If the expression based on the current field theory is employed, the above results may be understood in terms of the form factor for interaction in nucleon core. In treating the problem in which the divergence difficulty is inherent, it is necessary to know the shape of the form factor in nucleon core in order to get rid of the difficulty.**** But, when we try to perform the calculation for the process in which the effects of virtual pions or K mesons can be estimated without any divergence difficulty, we have only to replace g^2 and g_Λ^2(or g_Σ^2) by f^2 and $-f_\Lambda^2\bar{\zeta}_\Lambda^2$(or $-f_\Sigma^2\bar{\zeta}_\Sigma^2$) respectively in the treatment of the phenomena in nucleon core. Restricting our study within such problems we should like to examine how the prediction based on the current field theory should be modified in pion and K meson phenomena when the interactions in nucleon core have the character mentioned above. Since the effect due to the spread of nucleon core is expressed in terms of the change of coupling constant, our discussions will be developed by concentrating our attention to the problems of coupling constants. In § 2 we try to compare the properties of pion-nucleon interaction with those of K meson-nucleon interaction. And it is pointed out that K-N interaction rather than π-N interaction may play the important role in the phenomena within nucleon core. In § 3 the processes of pion-nucleon scattering

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* In the practical case of $M_\Lambda \neq M$ and $M_\Sigma \neq M$, $f_\Lambda=m_K g_\Lambda/(M_\Lambda+M)$ and $f_\Sigma=m_K g_\Sigma/(M_\Sigma+M)$.

** On the basis of these results we suggested previously[2] that the introduction of indefinite metric may be necessary in some sense to describe the phenomena in nucleon core correctly.

*** An attempt to obtain the forward scattering amplitude for K-N scattering was made on the basis of the above results for the property of K-N interaction in nucleon core. And it was shown that the results are not inconsistent with the experimental ones.

**** As the second step of our approach to the true theory in nucleon core, the problems about the form factor will be studied in some other place.

which take place through the mechanism via π-π interaction or $KK\pi\pi$-interaction are discussed by taking into account the effects due to the apparent reduction of coupling constants in nucleon core. In § 4 the anomalous magnetic moments of nucleons are discussed as one of the problems for interactions in nucleon core.

§ 2. On interactions in nucleon core

As was emphasized previously[1),2)] the determination of coupling constant in the current field theory must be performed on the basis of the phenomena in the neighborhood of pion cloud or K meson cloud. The remarkable result $f^2=0.08$ (in other words $g^2=14.5$)[3)] was derived from the study for p-wave pion-nucleon interaction at low energy. Also the similar result[4)] was obtained by applying the dispersion relation to the forward scattering amplitude for π-p scattering $D^{(2)}(\mu)=1/2[D_-(\mu)-D_+(\mu)]$ in which the effects due to the internal structure of nucleon will be almost cancelled out as was stressed in our previous paper.[1)] With regard to g_Λ^2 and g_Σ^2 we have not yet reliable values. We adopt tentatively the following values which have been obtained by Igi[5)] from studies about K-N scattering,

$$g_\Lambda^2 \cong (2.2\sim 2.6),\quad g_\Sigma^2 \cong (4.4\sim 4.8). \tag{2}$$

These may be regarded as the values of coupling constants in the neighborhood of K meson cloud. Then their values in nucleon core can be estimated by making use of the relations (1) and (1)′, that is,

$$-f_\Lambda^2\hat{\xi}_\Lambda^2=-\left(\frac{m_K}{M_\Lambda+M}\right)^2 g_\Lambda^2\hat{\xi}_\Lambda^2\cong-(0.063\sim 0.074), \tag{3}$$

$$-f_\Sigma^2\hat{\xi}_\Sigma^2=-\left(\frac{m_K}{M_\Sigma+M}\right)^2 g_\Sigma^2\hat{\xi}_\Sigma^2\cong-(0.31\sim 0.34). \tag{3}'$$

Since the pion-nucleon interaction in nucleon core may be described in terms of the coupling constant $f^2(=0.08)$, we may conclude as follows. In nucleon core K meson-nucleon interaction may be stronger than pion-nucleon interaction, although the latter is of course much stronger than the former in the neighborhood of pion cloud or K meson cloud. In the phenomena within nucleon core, for example, in the collision between proton and antiproton cores, $K\bar{K}$-pairs may be produced with comparatively large probability in nucleon core. But, as was mentioned in the previous paper,[6)] the $K\bar{K}$-pairs may not necessarily be observed directly owing to some kind of strong $KK\pi\pi$-interaction.

As is seen from Eqs. (1) and (1)′ or Eqs. (3) and (3)′, the values both of g_Λ^2 and of g_Σ^2 in nucleon core are reduced in appearance to negative ones. But this does not give rise to any difficulty in calculation for the process in

which the odd number of $KN\Lambda$ or $KN\Sigma$ interactions are contained. As an example let us take the process $\pi+N\longrightarrow K+\Sigma$ into consideration. For simplicity we make an argument about the result based on perturbation theory. When the nucleon core is treated as if it were a point, we denote the matrix element for this process through interactions within the range $1/M$ as $g_\Sigma g A(\omega)$, where ω is pion energy. If the effects due to the spread of nucleon core are taken into account, its expression will be modified as $g_\Sigma g B(\omega)$. Then our conclusion means that there may be the following relation between $A(\omega)$ and $B(\omega)$.

$$B(\omega)=\pm i(f_\Sigma\xi_\Sigma/g_\Sigma)(f/g)A(\omega). \tag{4}$$

Of course both $A(\omega)$ and $B(\omega)$ are generally complex quantities and it can be easily seen that there is no possibility of negative cross section.

The so-called cutoff theory with cutoff energy $\omega_{max}=M$ may be regarded as the theory in which the coupling constant in nucleon core is assumed to be equal to zero. As will be shown later, the contributions from pion-nucleon interactions in nucleon core are much smaller than those in the neighborhood of pion cloud, and the cutoff theory may be regarded as one of the approximate theories in pion physics. On the other hand the cutoff theory in K meson physics may not be so useful because K-N interaction in nucleon core plays the important role.

§ 3. Contributions from π-π interaction and $KK\pi\pi$-interaction to pion-nucleon scattering

Many approaches to pion-pion interaction[7] have been made from various points of view. We also studied previously[8] the effects of this interaction on pion-nucleon scattering at low energy adopting the following type Hamiltonian,

$$H_{\pi\pi}=4\pi\lambda(\phi_\alpha\phi_\alpha)^2. \tag{5}*$$

Now we should like to re-examine this problem as an approach to π-N interaction in nucleon core.

First of all we make an argument about the result obtained by a straightforward calculation based on the current field theory. The forward scattering amplitude at zero energy through the process via this pion-pion interaction is expressed as follows:

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* It must be noted that $4\pi\lambda$ in Eq. (5) corresponds to $\lambda/16$ in Eq. (11) in our previous paper.[8] Unfortunately there is a misprint in reference 8), that is, f in Eq. (11) must be replaced by $f/\sqrt{2}$ because f refers to the coupling constant between charged pion and nucleon. (If the f is interpreted as the coupling constant between neutral pion and nucleon, all the values not only in Table I but also in Table II mentioned in reference 8) should be multiplied by factor 4.) And note that the values in Table I are independent of λ and that the footnote "this value of λ may suffer some change" was added since the value of λ was regarded as an unsolved one.

To our regret, S. Ishida was too hasty to catch the meaning mentioned in our paper[8] (cf. Prog. Theor. Phys. **24** (1960), 1262).

$$D_{\pm}(\mu)_{\pi\pi}=\frac{16}{M+\mu}\left(1+\frac{1}{4}\right)g^2\lambda\left[\frac{C}{2\pi}\right], \tag{6}$$

$$C=\int_0^1 dx\int_0^\infty dt\frac{3t^2M^2x(1-x)}{[t^2+M^2x^2+\mu^2(1-x)]^{5/2}}=\int_0^1 dx[F(x,t=\infty)-F(x,t=0)]$$
$$\cong 1.193, \tag{7}$$

$$F(x,t)=\int dt\frac{3t^2M^2x(1-x)}{[t^2+M^2x^2+\mu^2(1-x)]^{5/2}}=\frac{M^2x(1-x)}{M^2x^2+\mu^2(1-x)}\cdot\frac{t^3}{[t^2+M^2x^2+\mu^2(1-x)]^{3/2}}$$
$$+(\text{constant}), \tag{8}$$

where the suffix $(\pi\pi)$ in $D_{\pm}(\mu)$ stands for the contribution from the process via the π-π interaction to the scattering amplitude and t represents the magnitude of virtual pion momentum. Although it seems that the result of $\lambda\cong 1$ has been obtained from the analysis[9] (based on perturbation calculation) of the experimental data for π-p collision at 1.4 Bev, this value cannot necessarily be regarded as an established one because of the following reasons.

(i) In such an energy region it may be possible to expect some resonance phenomena in pion-pion system. If this is the case, probably we cannot explain the resonance phenomena by means of perturbation approach. Nevertheless, if the value of λ is chosen so that the perturbation result may reproduce the experimental result for cross section of π-2π process, its value will probably be overestimated because the small magnitude of matrix element in perturbation calculation must be compensated by the largeness of coupling constant.

(ii) There may be some possibility of the existence of strong p-wave pion-pion interaction in this process in Bev region. Then, to determine the coupling constant λ so that the experimental result for cross section may be explained in terms of s-wave pion-pion interaction will probably be nonsense.

(iii) When the cross section for π^+-π^+ (or π^--π^-) scattering is estimated by adopting the $4\pi\lambda(\phi_\alpha\phi_\alpha)^2$-type interaction and by making use of the large value of $\lambda\cong 1$, the magnitude of $\sigma(\pi^+$-$\pi^+)$ (or $\sigma(\pi^-$-$\pi^-)$) at very low energy turns out to be so large that we may not regard it as a reasonable one.

At the present stage it will be natural for us to regard the value of λ as an unsolved one. Since $D_{\pm}(\mu)_{\pi\pi}$ depends not only on the magnitude of λ but also on the sign of λ, we may not now draw any definite conclusion with respect to the effects of the π-π interaction. On the other hand it may also be difficult to deny the following possibility. If the value of λ is of the order of 0.3,*

$$D^{(1)}(\mu)_{\pi\pi}=\frac{1}{2}[D_-(\mu)+D_+(\mu)]\cong\frac{g^2\lambda}{M+\mu}\left[\frac{11.93}{\pi}\right]\cong g^2/M\cong -D^{(1)}(\mu)_p. \tag{9}$$**

* Miyazawa and Kawarabayashi[10] have tried to explain the experimental results for s-wave phase shifts by adopting the coupling constant $|\lambda|=0.3$.

** $D^{(1)}(\mu)_p$ represents the value of $D^{(1)}(\mu)=1/2[D_-(\mu)+D_+(\mu)]$ which was estimated by the usual perturbation theory.

Then it may be said that s-wave phase shifts for pion-nucleon scattering may be explained by means of perturbation theory so long as the effects of pion-pion interaction are taken into account. But it may be difficult to understand this special situation without raising any question.

Now let us take into account the effects due to the internal structure of nucleon core. For this purpose it is necessary to divide the $D_{\pm}(\mu)_{\pi\pi}$ in Eq. (6) into two parts, that is, the contributions from interactions in the neighborhood of pion cloud and those from interactions in nucleon core. We divide the integral on t mentioned in Eq. (7) $\int_0^{\infty}$into $\int_0^K$ and $\int_K^{\infty}$, where K corresponds to the minimum value of virtual pion momentum in nucleon core. With regard to pion-nucleon interaction we have only to replace the coupling constant g by f in the description of the phenomena in nucleon core. Moreover the coupling constant λ also should be reduced in nucleon core to some small $\lambda'(|\lambda'|<|\lambda|)$ though we have not yet known the relation between λ' and λ. Therefore $D_{\pm}(\mu)_{\pi\pi}$ in Eq. (6) should be modified as follows:

$$D'_{\pm}(\mu)_{\pi\pi}=\frac{16}{M+\mu}\left(1+\frac{1}{4}\right)\left[\frac{C_1'+C_2'}{2\pi}\right], \tag{10}$$

$$C_1'=g^2\lambda\int_0^1 dx[F(x,\ t=K)-F(x,\ t=0)], \tag{11}$$

$$C_2'=f^2\lambda'\int_0^1 dx[F(x,\ t=\infty)-F(x,\ t=K)], \tag{12}$$

where C_1' and C_2' represent the contributions from interactions in pion cloud and those from interactions in nucleon core respectively to the scattering amplitude.

In the estimation of K, the following two cases are taken into consideration.

(i) We assume $K=M/2$ because of the interaction due to exchange of two pions whose momenta are equal to each other in our case of pion-nucleon scattering at $\omega=\mu$.

(ii) We assume $K=M$ corresponding to the cutoff theory with cutoff momentum M.

When some kind of particle interacts with a nucleon through exchange of six pions or more at the same time, as $M\cong 6\mu$ this process may be regarded as one of the phenomena in nucleon core independently of the magnitude of virtual pion momenta, that is, even in the case in which each virtual pion has zero momentum. On the basis of this consideration we may think it natural to adopt the first standpoint. Then we obtain the following result:

$$C_1'=g^2\lambda[0.731], \tag{13}$$

$$C_2'=f^2\lambda'[0.462]. \tag{14}$$

Since the magnitude of C_2' is negligibly small compared with that of C_1', we may conclude as follows. If the internal structure of nucleon core is taken into account, the contribution from the process through pion-pion interaction to pion-nucleon scattering amplitude is reduced to

$$D_{\pm}'(\mu)_{\pi\pi} \simeq 0.6\ D_{\pm}(\mu)_{\pi\pi}. \tag{15}$$

When we assume $K=M$,

$$C_1'=g^2\lambda[1.002], \tag{13}'$$

$$C_2'=f^2\lambda'[0.191]. \tag{14}'$$

In either case $C_1' \gg C_2'$ because pion-nucleon interaction turns out to be very weak in nucleon core. Thus the cutoff theory may be regarded as one of the approximate theories in pion physics.

Recently we have examined within the framework of current field theory the contributions from $KK\pi\pi$-interaction to the forward scattering amplitude of s-wave pion-nucleon scattering adopting the interaction Hamiltonian $H_1=4\pi\lambda_1 \cdot K_i^* K_i \pi_j \pi_j$ and have obtained the following results,[6]

$$D_{\pm}(\mu)_{K\pi}=\frac{1}{M+\mu}\lambda_1\left[g_{\Lambda}^2\frac{A_{\Lambda}}{2\pi}+g_{\Sigma}^2\frac{3A_{\Sigma}}{2\pi}\right], \tag{16}*$$

$$A_{\Lambda}\simeq 0.529, \qquad A_{\Sigma}\simeq 0.559. \tag{17}$$

As was mentioned in the previous paper,[6] the process of pion-nucleon scattering through the mechanism via $KK\pi\pi$-interaction may be regarded as one of the phenomena in nucleon core. Therefore g_{Λ}^2 and g_{Σ}^2 in Eq. (16) should be replaced respectively by $-f_{\Lambda}^2\xi_{\Lambda}^2$ and $-f_{\Sigma}^2\xi_{\Sigma}^2$ mentioned in Eqs. (1) and (1)'. The coupling constant λ_1** also should be reduced to some small $\lambda_1'(|\lambda_1'|<|\lambda_1|)$ though we have not yet known the relation between λ_1 and λ_1'. Therefore $D_{\pm}(\mu)_{K\pi}$ in Eq. (16) should be modified as follows.

$$D_{\pm}'(\mu)_{K\pi}=\frac{-1}{M+\mu}\lambda_1'\left[f_{\Lambda}^2\xi_{\Lambda}^2\frac{A_{\Lambda}}{2\pi}+f_{\Sigma}^2\xi_{\Sigma}^2\frac{3A_{\Sigma}}{2\pi}\right]. \tag{18}$$

Although it will be difficult to derive some definite conclusion because of the lack of knowledge about the values both of λ' and of λ_1', let us now try to compare tentatively the magnitude of this $D'_{\pm}(\mu)_{K\pi}$ with that of $D'_{\pm}(\mu)_{\pi\pi}$ due to C_2' in Eq. (14), under the crude assumption of $|\lambda_1'|\simeq|\lambda'|$. Making use of the results mentioned in Eqs. (3), (3)' and (17),

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* Note that g_{Λ}^2, g_{Σ}^2 and λ_1 in this paper correspond respectively to $g_{\Lambda}^2/4\pi$, $g_{\Sigma}^2/4\pi$ and $\lambda_1/4\pi$ in the previous paper.[6]

** Recently Igi[5] has studied K-N scattering by taking $KK\pi\pi$-interactions into consideration and estimated the value of λ_1 by means of cutoff theory with cutoff momentum of pion $k_{max}\simeq 4\mu$ or $k_{max}\simeq 6\mu$. His value $\lambda_1\simeq -0.5$ may correspond to the coupling constant in the neighborhood of pion cloud.

$$D_{\pm}'(\mu)_{K\pi}\cong\frac{-1}{M+\mu}\frac{\lambda_1'}{2\pi}[0.56\sim0.62]. \qquad (18)'$$

Comparing this with the forward scattering amplitude $D_{\pm}''(\mu)_{\pi\pi}$ due to the π-π interactions in nucleon core (cf. Eqs. (10) and (14))

$$D_{\pm}''(\mu)_{\pi\pi}=\frac{16}{M+\mu}\left(1+\frac{1}{4}\right)\left(\frac{C_2'}{2\pi}\right)\cong\frac{1}{M+\mu}\frac{\lambda'}{2\pi}\,[0.74], \qquad (19)^*$$

we may say as follows. So far as the phenomena in nucleon core are concerned, the contribution from $KK\pi\pi$-interaction to pion-nucleon scattering amplitude at low energy is comparable to that from pion-pion interaction.

§ 4. On anomalous magnetic moments of nucleons

In § 3 we have examined the processes in which the effects of virtual pions or K mesons can be estimated without any divergence difficulty. It may be said that the same kind of processes as the above ones are treated in the perturbation calculation for the anomalous magnetic moments of nucleons although some renormalization procedure is necessary to get rid of the divergence difficulty irrelevant to the magnetic moments.

As is well known the expressions for the anomalous magnetic moments of nucleons are given by the following forms :[11]

$$\left.\begin{aligned}\Delta\mu_p&=(g^2/\pi)[-(J_N/2)+J_M]=-(R_N/2)+R_M,\\ \Delta\mu_N&=(g^2/\pi)[-J_N-J_M]=-R_N-R_M,\end{aligned}\right\} \qquad (20)^{**}$$

where R_N and R_M stand for the contributions from nucleon current and meson current, respectively, to the anomalous magnetic moments of nucleons.

First of all we pay attention to the pion-nucleon interaction. In order to take into account the effects due to the spread of nucleon core, it is necessary to divide the expressions both of R_N and of R_M into two parts,

$$\left.\begin{aligned}R_N&=R_{N,1}+R_{N,2},\\ R_M&=R_{M,1}+R_{M,2},\end{aligned}\right\} \qquad (21)$$

where the suffices 1 and 2 in R_N and R_M indicate the contributions from interactions in the outer region of nucleon core and those from interactions in the inner region of nucleon core respectively. When we denote the magnitude of virtual pion momentum and its minimum value in nucleon core as t and K respectively, the integral on t from 0 to K and the integral on t from K to ∞ in the expression for R_N (or R_M) correspond respectively to the expression

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* When we assume $K=M$, $D_{\pm}''(\mu)_{\pi\pi}\cong\frac{1}{M+\mu}\frac{\lambda'}{2\pi}\,[0.31]$.

** Note that g is the unrationalized coupling constant.

for $R_{N,1}$(or $R_{M,1}$) and that for $R_{N,2}$(or $R_{M,2}$) apart from the trivial factors. As was mentioned above, the coupling constant g should be replaced by f in the description of the phenomena in nucleon core. From this it follows that

$$\left.\begin{aligned} R_{N,1} &\gg R_{N,2}, \\ R_{M,1} &\gg R_{M,2}. \end{aligned}\right\} \tag{22}$$

Thus the values of R_N and R_M are nearly equal to those of $R_{N,1}$ and $R_{M,1}$ respectively which are predicted by the cutoff theory with cutoff momentum K.*

Next let us discuss the electromagnetic interaction. In our case, not only the process in which photon interacts with the virtual pion whose momentum has a value larger than K but also the process in which photon interacts with the nucleon core during the emission of virtual pions may be regarded as one of the phenomena in nucleon core. With respect to this electromagnetic interaction also we may expect a certain reduction of the apparent value of coupling constant in nucleon core. Then not only the values of $R_{N,2}$ and $R_{M,2}$ but also the value of $R_{N,1}$ turns out to be very small. Thus it may be expected that the contributions from meson current to magnetic moment overwhelm those from nucleon current. This is of course a remarkable tendency to explain the experimental results for the anomalous magnetic moments of nucleons.

Although the so-called gauge invariance may not necessarily hold in the inner region of nucleon core, it may be necessary to satisfy apparently the condition of gauge invariance in describing the property of interaction averaged over the spread of nucleon core. Our treatment does not contradict with this theoretical requirement, because in the gauge-invariant expressions obtained beforehand we take into account the apparent reduction of coupling constant in nucleon core.**

In the Appendix we shall discuss the effects of strange particles to the anomalous magnetic moments of nucleons.

Appendix

As was given by Iso and Iwao,[13] the effects of strange particles to the anomalous magnetic moments of nucleons can be expressed by the following forms :

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* When we assume $K=M$, the values of anomalous magnetic moments of nucleons agree with those predicted by Goto.[12]

** Although it may be supposed that the apparent reduction of coupling constant makes it difficult to carry out the so-called renormalization procedure, we should like to emphasize that in the true theory there is no need to perform any renormalization procedure. But we are now obliged to adopt such a phenomenological treatment until the true theory in nucleon core is developed.

$$\Delta\mu_p' = (g_\Sigma^2/\pi)[-J_\Sigma + (J_K/2)] + (g_A^2/\pi)(J_K'/2), \qquad (A\cdot1)^*$$

$$\Delta\mu_N' = (g_\Sigma^2/\pi)[J_\Sigma + J_K], \qquad (A\cdot2)^*$$

where the suffices Σ and K indicate the contributions from hyperon current and those from K meson current respectively.

First, let us pay attention to the $KN\Lambda$ and $KN\Sigma$-interactions. We denote the minimum value of virtual K meson momentum in nucleon core as K and assume $K=\sqrt{3}\,M/2$ on the basis of the relation** $\sqrt{K^2+m_k^2} \cong M$. Then the contributions from the interactions in the outer region of nucleon core are estimated and the results are shown in Table A. It is needless to say that

Table A.

$KN\Sigma$- or $KN\Lambda$-interaction	K	J_Σ	J_K	J_K'
pseudoscalar type	∞	0.313	0.180	0.182
	$\sqrt{3}\,M/2$	0.088	0.070	0.077
scalar type	∞	−0.530	−0.494	−0.537
	$\sqrt{3}\,M/2$	−0.168	−0.213	−0.250

$K=\infty$ corresponds to the case in which the current field theory is applied. Comparing the values of J_Σ, J_K and J_K' in the case $K=\infty$*** with those in the case $K=\sqrt{3}\,M/2$, we may say that the virtual K mesons with high momenta play the more important role in J_Σ than in J_K (or J_K').

Second, the apparent reduction of coupling constant for the electromagnetic interactions in nucleon core is taken into consideration along the same line with our study in § 4. Then it may be expected that the contributions from hyperon current to magnetic moments of nucleons turn out to be very small.

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* g_Σ and g_A mean the unrationalized coupling constants for $KN\Sigma$- and $KN\Lambda$-interactions respectively and the sign of J_Σ is opposite to that in reference 13).

** With regard to this assumption, there will be enough ground for controversy.

*** Of course these values must agree with those given by Iso and Iwao[13] except for the sign of J_Σ.

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Dispersion Relations for Virtual Pion-Nucleon Scattering*

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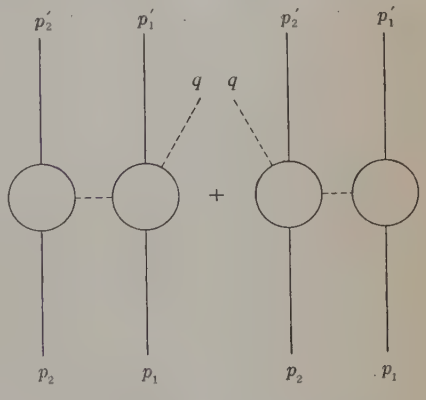
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In connection with a proposed calculation of single pion production in nucleon-nucleon collisions, the process of pion-nucleon scattering in which the incoming pion is virtual is investigated by means of the dispersion theoretic method developed by Chew, Goldberger, Low and Nambu. The static limit is studied in detail and the Chew-Low amplitude for virtual scattering is shown to result for small virtual pion masses. More generally, it is proposed that the Blankenbecler-Gartenhaus solution, which is recorded, may provide a useful approach in applications in which large virtual pion masses play an essential role.

§ 1. Introduction

In this paper, we shall study the process of pion-nucleon scattering, in which one of the pions is taken as virtual, by means of one-dimensional dispersion relations in the total energy variable. Such an investigation is clearly of interest for application to selected inelastic pion-nucleon phenomena such as single pion-production in either pion-nucleon or nucleon-nucleon collisions. Detailed application of the results of this paper to the latter process will be described in a subsequent paper by the authors. The approach used in this paper is, of course, very similar to that applied by several authors[1] to the electro-pion production process.

In these problems, the invariant amplitudes for which dispersion relations are written are functions of three parameters, since the variable mass of the virtual particle occurs in addition to the usual invariant measures of energy and momentum transfer. We shall proceed without questioning the validity of our relations as a function of the additional variable.



—(The exchange diagrams $p_1 \rightleftarrows p_2$)

Fig. 1. Diagrams for single-pion production in nucleon-nucleon collisions.

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Since the precise amplitude to be studied has been chosen with a definite application in mind, namely, single pion-production in nucleon-nucleon collisions, let us consider briefly the manner in which it enters into a simple proposed description of that process. We consider the matrix element corresponding to Fig. 1 as the most important contribution (peripheral interaction model). The matrix element in question can be written as follows:

$$\langle(-)p_2'\,p_1'\,q_\beta|\,p_1\,p_2(+)\rangle \sim (2\pi)^4\delta^4(p_1'+p_2'+q-p_1-p_2)\left(\frac{M^4}{2q_0\,p_{10}'\,p_{20}'\,p_{10}\,p_{20}}\right)^{1/2} T_{(1)}, \tag{1}$$

$$\begin{aligned} T_{(1)} = \left(\frac{2q_0\,p_{10}'\,p_{20}'\,p_{10}\,p_{20}}{M^4}\right)^{1/2} & [\langle p_2'|j_\alpha(0)|p_2\rangle\,(\Delta_F[(p_2'-p_2)^2])^2 \\ & \times \Delta_F'^{-1}[(p_2'-p_2)^2]\langle(-)\,p_1'\,q_\beta|j_\alpha(0)|p_1\rangle + (p_1'|j_\alpha(0)|p_1\rangle \\ & \times (\Delta_F[(p_1'-p_1)^2])^2\,\Delta_F'^{-1}[(p_1'-p_1)^2]\langle(-)\,p_2'\,q_\beta|j_\alpha(0)|p_2\rangle - \text{exchange terms}], \end{aligned}$$

where the p_i, p_j' and q are the 4-momenta of the initial nucleons, of the final nucleons, and of the pion, respectively. $|n(\pm)\rangle$ is the outgoing (incoming) state with character n. α and β denote the isospin indices. $j_\alpha(0)$ is defined by the equation of the pion field $\phi_\alpha(x)$, i.e.

$$(\Box-\mu^2)\phi_\alpha(x)=j_\alpha(x). \tag{2}$$

In Eq. (1) $\langle(-)p_1'q_\beta|j_\alpha(0)|p_1\rangle$ is the matrix element for the virtual-pion nucleon scattering (or we may describe this as pion-production in nucleon scattering by an external pion field). This will be studied in § 1 and § 2 by the method due to Chew, Goldberger, Low and Nambu,[2] together with the theorem found by F.N.W. In § 4 we shall also record the Blankenbecler-Gartenhaus[3] solution for this process in connection with the single pion-production by nucleon-nucleon collision.

The main result of this work is that for virtual pion masses small compared with M, the nucleon mass, the results of the Chew-Low theory for off the energy shell p-wave scattering are reproduced. It is suggested that when this parameter becomes large, as it does in the applications contemplated, the Blankenbecler-Gartenhaus solution may represent an improvement over both the Chew-Low theory as well as over another fashionable approximation which ignores completely off the energy shell effects.

§ 2. Dispersion relation and partial wave decompositions

We shall study the following relativistically invariant amplitudes,

$$H_{\beta\alpha}=\langle(-)p_2q_\beta|j_\alpha(0)|p_1\rangle\left(\frac{2q_0\,p_{20}\,p_{10}}{M^2}\right)^{1/2}(K(l^2))^{-1}, \tag{3}$$

where

$$t=p_2+q-p_1,$$

and $K(t^2)$ is the function introduced by Federbush, Goldberger and Treiman[4] which is related to the vertex function as follows:

$$K(t^2)=\Delta_F^{-1}(t^2)\Delta_F'(t^2)\Gamma_5(t^2). \tag{4}$$

Following F.N.W., we define the three invariant parameters:

$$\nu=-\frac{1}{2M}(p_1+p_2)\cdot q,\quad \nu_B=\frac{q\cdot t}{2M},\quad \text{and } t^2. \tag{5}$$

Then Eq. (3) has a form similar to the case of real pion-nucleon scattering:

$$H_{\beta\alpha}=\bar{u}(p_2)[-A(\nu,\nu_B,t^2)+i\gamma\cdot qB(\nu,\nu_B,t^2)]u(p_1), \tag{6}$$

where A and B are functions of ν, ν_B and t^2 and are operators in isospin space. The separation of isospin operators is done as usual by defining

$$\begin{aligned} A&=\delta_{\beta\alpha}A^{(+)}+\frac{1}{2}[\tau_\beta,\tau_\alpha]A^{(-)},\\ B&=\delta_{\beta\alpha}B^{(+)}+\frac{1}{2}[\tau_\beta,\tau_\alpha]B^{(-)}. \end{aligned} \tag{7}$$

The dispersion relations for $A^{(\pm)}$ and $B^{(\pm)}$ have the form:

$$\begin{aligned} A^{(\pm)}(\nu,\nu_B,t^2)&=\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\,\mathrm{Im}\,A^{(\pm)}(\nu',\nu_B,t^2)\left[\frac{1}{\nu'-\nu-i\epsilon}\pm\frac{1}{\nu'+\nu}\right],\\ B^{(\pm)}(\nu,\nu_B,t^2)&=\frac{g^2}{2M}\left[-\frac{1}{\nu-\nu_B}\mp\frac{1}{\nu+\nu_B}\right]\\ &\quad+\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\,\mathrm{Im}\,B^{(\pm)}(\nu',\nu_B,t^2)\left[\frac{1}{\nu'-\nu-i\epsilon}\mp\frac{1}{\nu'+\nu}\right],\\ \nu_0&=1+\frac{1}{2M}+\nu_B, \end{aligned} \tag{8}$$

where g is the usual rationalized renormalized pseudo-scalar coupling constant.

The above relations are meaningful only for suitably restrictive assumptions concerning the high energy behavior[5] of $\mathrm{Im}\,A^{(\pm)}$ and $\mathrm{Im}\,B^{(\pm)}$, but we shall suppose the above relation to be right. Our treatment will effectively exclude the π-π interaction from direct consideration. In principle, however, we could extend our considerations to include the effect, for example, following the method of Cini and Fubini.[6]

In terms of the total energy W in the C.M.S. of the final state, we have

$$A^{(\pm)}(W, \nu_B, t^2) = \frac{1}{\pi} \int_{(1+M)^2}^{\infty} dW'^2 \, \mathrm{Im} A^{(\pm)}(W', \nu_B, t^2)$$
$$\times \left[\frac{1}{W'^2 - W^2 - i\epsilon} \pm \frac{1}{W'^2 + W^2 - 2M^2 + 4M\nu_B} \right], \tag{9}$$
$$B^{(\pm)}(W, \nu_B, t^2) = g^2 \left[- \frac{1}{W^2 - M^2} \mp \frac{1}{W^2 - M^2 + 4M\nu_B} \right]$$
$$+ \frac{1}{\pi} \int_{(1+M)^2}^{\infty} dW'^2 \, \mathrm{Im} B^{(\pm)}(W', \nu_B, t^2) \left[\frac{1}{W'^2 - W^2 - i\epsilon} \mp \frac{1}{W'^2 + W^2 - 2M^2 + 4M\nu_B} \right].$$

Next we formally define the virtual scattering and partial wave amplitudes as follows:

$$f_{\beta\alpha}(W, x, t^2) = -\frac{M}{4\pi W} H_{\beta\alpha},$$
$$= -\frac{M}{4\pi W} \bar{u}(p_2)[-A(W, x, t^2) + i\gamma \cdot qB(W, x, t^2)]u(p_1),$$
$$f_{\beta\alpha}(W, x, t^2) = \sum_{l,T} (2l+1) P_T [h^T_{l+} P^{(+)}_l(x) + h^T_{l-} P^{(-)}_l(x)], \tag{10}$$
$$x = \cos\theta = \frac{2M\nu_B + \omega t_0}{qt},$$

where ω, q, t_0 and t are the energy and the momentum of the final pion and of the virtual pion in the C.M.S. respectively. $P^{(\pm)}_l(x)$ is an eigenfunction of the total angular momentum $J = l \pm 1/2$, and P_T is the projection operator for isospin T. $h^T_{l\pm}$ denotes an amplitude with the total angular momentum $J = l \pm 1/2$ and the total isospin T. $h^T_{l\pm}$ is a function of t^2 and W. From Eq. (10), we have

$$f = f_1 + \boldsymbol{\sigma} \cdot \boldsymbol{q} \boldsymbol{\sigma} \cdot \boldsymbol{t} f_2,$$
$$f_1 = \sum_{T,l=1}^{\infty} P_T h^T_{l+} P'_{l+1}(x) - \sum_{T,l=2}^{\infty} P_T h^T_{l-} P'_{l-1}(x), \tag{11}$$
$$f_2 = \sum_{T,l=1}^{\infty} P_T (h^T_{l-} - h^T_{l+}) P'_l(x).$$

From Eqs. (10) and (11), we now find

$$f_1 = \frac{[(E_2+M)(E_1+M)]^{1/2}}{2W} \frac{1}{4\pi} [A + (W-M)B],$$
$$f_2 = \frac{[(E_2-M)(E_1-M)]^{1/2}}{2W} \frac{1}{4\pi} [-A + (W+M)B], \tag{12}$$

where E_i is the energy of the nucleon i in the C.M.S. and is connected with W and t^2 as follows:

$$E_2=\frac{W^2+M^2-1}{2W},\quad E_1=\frac{W^2+M^2+t^2}{2W}. \tag{13}$$

Finally we find the following relations by using Eqs. (11) and (12),

$$\begin{aligned}
A^{(\pm)}&=4\pi\left[\frac{W+M}{\sqrt{(E_2+M)(E_1+M)}}f_1^{(\pm)}-\frac{W-M}{\sqrt{(E_2-M)(E_1-M)}}f_2^{(\pm)}\right],\\
B^{(\pm)}&=4\pi\left[\frac{1}{\sqrt{(E_2+M)(E_1+M)}}f_1^{(\pm)}+\frac{1}{\sqrt{(E_2-M)(E_1-M)}}f_2^{(\pm)}\right],\\
f_1^{(\pm)}&=\sum_{l=0}^{\infty}h_{l+}^{(\pm)}P'_{l+1}(x)-\sum_{l=2}^{\infty}h_{l-}^{(\pm)}P'_{l-1}(x),\\
f_2^{(\pm)}&=\sum_{l=1}^{\infty}(h_{l-}^{(\pm)}-h_{l+}^{(\pm)})P_l'(x),
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
h_{l\pm}^{(+)}&=\frac{1}{3}[h_{l\pm}^{(1/2)}+2h_{l\pm}^{(3/2)}],\\
h_{l\pm}^{(-)}&=\frac{1}{3}[h_{l\pm}^{(1/2)}-h_{l\pm}^{(3/2)}].
\end{aligned} \tag{15}$$

To obtain the approximate value of the imaginary part of $h_{l\pm}^T$, we use the one-pion approximation. (This approximation is, of course, valid for energies below the pion production threshold.) Then, according to the theorem due to F.N.W., we have the relations

$$\mathrm{Im}\,h_{l\pm}^T=qf_{l\pm}^{T*}h_{l\pm}^T, \tag{16}$$

where

$$f_{l\pm}^T=\frac{\sin\delta_{l\pm}^T}{q}e^{i\delta_{l\pm}^T}, \tag{17}$$

and $\delta_{l\pm}^T$ is the pion-nucleon scattering phase shift with the total angular momentum $J=l\pm 1/2$, and isospin T. This relation will be exploited in the next section.

§ 3. Integral equation for $h_{l\pm}^T$

In this section we shall obtain integral equations for the amplitudes $h_{l\pm}^T$ in the low energy region. C.G.L.N. have shown one way to obtain such integral equations for $h_{l\pm}^T$ from the dispersion relations. Blankenbecler and Gartenhaus have recently suggested another more powerful procedure. We shall first discuss our problem by the method of C.G.L.N. The assumptions we make are:

i) One-pion approximation.
ii) $t^2<M$.
iii) An expansion in power of M^{-1} is permissible.
iv) Only s and p waves are included.

Let us now introduce the parameter z for convenience.

$$2M\nu_B=z, \quad qt-\omega t_0=z_0. \tag{18}$$

Then we have

$$\cos\theta=\frac{z+\omega t_0}{qt}.$$

Thus when z is equal to z_0, $\cos\theta=1$, namely, the scattering amplitude $f(z_0)$ corresponds to the forward case. Now we may write Eq. (14) as follows:

$$f_1^{(\pm)}(z)=h_S{}^{(\pm)}-h_{D(3/2)}^{(\pm)}+3\left(\frac{z+\omega q_0}{qt}\right)h_{P(3/2)}^{(\pm)}+\frac{1}{2}\left[15\left(\frac{z+\omega t_0}{qt}\right)^2-3\right]h_{D(5/2)}^{(\pm)}+\cdots, \tag{19a}$$

$$f_2^{(\pm)}(z)=h_{P(1/2)}^{(\pm)}-h_{P(3/2)}^{(\pm)}+3\left(\frac{z+\omega t_0}{qt}\right)(h_{D(3/2)}^{(\pm)}-h_{D(5/2)}^{(\pm)})+\cdots. \tag{19b}$$

Differentiating Eq. (19a) with respect to z and setting $z=z_0$, we have

$$\begin{aligned} f_1^{(\pm)\prime}(z_0)&=\frac{3}{qt}h_{P(3/2)}^{(\pm)}+\frac{15}{qt}h_{D(5/2)}^{(\pm)}+\cdots, \\ f_1^{(\pm)\prime\prime}(z_0)&=\frac{15}{q^2t^2}h_{D(5/2)}^{(\pm)}+\cdots. \end{aligned} \tag{20}$$

Thus from the above equations, we find

$$\begin{aligned} h_S{}^{(\pm)}&=f_1^{(\pm)}(z_0)-qtf_1^{(\pm)\prime}(z_0)+D\text{ waves}+\cdots, \\ \frac{3}{qt}h_{P(3/2)}^{(\pm)}&=f_1^{(\pm)\prime}(z_0)-qtf_1^{(\pm)\prime\prime}(z_0)+\cdots. \end{aligned} \tag{21}$$

By similar manipulations for the amplitude f_2, we find

$$h_{P(1/2)}^{(\pm)}-h_{P(3/2)}^{(\pm)}=f_2^{(\pm)}(z_0)-qtf_2^{(\pm)\prime}(z_0)+\cdots, \tag{22}$$

etc.

Our next task is to connect these equations with the dispersion relations, Eq. (9). Before going on to this work, we consider the contributions to the dispersion integral in Eq. (9). Because of the relation Eq. (16) in the one-pion approximation, we may infer that the (3/2, 3/2) amplitude is quite large compared with the other amplitudes. Therefore we can make the following approximations for $\mathrm{Im}\,A^{(\pm)}$ and $\mathrm{Im}\,B^{(\pm)}$ in the dispersion integral by considering Eq. (14), i.e.

$$\begin{aligned} \mathrm{Im}\,A^{(\pm)}(W', z, t^2)\cong 4\pi\Bigg[&\frac{W'+M}{\sqrt{(E_2'+M)(E_1'+M)}}\frac{3(z+\omega' t_0')}{q't'}\,\mathrm{Im}\,h_{P(3/2)}^{(\pm)} \\ &-\frac{W'-M}{\sqrt{(E_2'-M)(E_1'-M)}}\,\mathrm{Im}(h_{P(1/2)}^{(\pm)}-h_{P(3/2)}^{(\pm)})\Bigg], \end{aligned} \tag{23}$$

$$\mathrm{Im}\,B^{(\pm)}(W',z,t^2)\cong 4\pi\Big[\frac{1}{\sqrt{(E_2'+M)(E_1'+M)}}\frac{3(z+\omega' t_0')}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}$$

$$+\frac{1}{\sqrt{(E_2'-M)(E_1'-M)}}\,\mathrm{Im}\,(h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)})\Big].$$

Then considering Eq. (9), (21), (22) and (23), we reach the desired integral equations. As an example, we have for $h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)}$,

$$\mathrm{Re}(h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)})=\frac{f^2}{4\pi}\frac{\sqrt{(E_2-M)(E_1-M)}}{2W}\Big[-\frac{1}{W-M}\mp\frac{W+M}{W^2-M^2+2z_0}$$

$$\mp\frac{2qt(W+M)}{(W^2-M^2+2z_0)^2}\Big]-\frac{\sqrt{(E_2-M)(E_1-M)}}{2W}\frac{P}{\pi}\int_{(1+M)^2}^{\infty}dW'^2$$

$$\times\Big[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+2z_0}\pm\frac{2qt}{(W'^2+W^2-2M^2+2z_0)^2}\Big]$$

$$\times\Big[\frac{W'+M}{\sqrt{(E_2'+M)(E_1'+M)}}\frac{3(z_0+\omega' t_0')}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}-\frac{W'-M}{\sqrt{(E_2'-M)(E_1'-M)}}$$

$$\times\mathrm{Im}(h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)})\Big]+\frac{\sqrt{(E_2-M)(E_1-M)}}{2W}\frac{P}{\pi}\int_{(1+M)^2}^{\infty}dW'^2$$

$$\times\Big[\frac{1}{W'^2-W^2}\mp\frac{1}{W'^2+W^2-2M^2+2z_0}\mp\frac{2qt}{(W'^2+W^2-2M^2+2z_0)^2}\Big]$$

$$\times\Big[\frac{W+M}{\sqrt{(E_2'+M)(E_1'+M)}}\frac{3(z_0+\omega' t_0')}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}+\frac{W+M}{\sqrt{(E_2'-M)(E_1'-M)}}$$

$$\times\mathrm{Im}(h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)})\Big]+\frac{qt\sqrt{(E_2-M)(E_1-M)}}{2W}\frac{P}{\pi}\int_{(1+M)^2}^{\infty}dW'^2$$

$$\times\Big[\frac{1}{W'^2-W^2}\pm\frac{1}{W'^2+W^2-2M^2+2z_0}\Big]\frac{W'+M}{\sqrt{(E_2'+M)(E_1'+M)}}\frac{3}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}$$

$$-\frac{qt\sqrt{(E_2-M)(E_1-M)}}{2W}\frac{P}{\pi}\int_{(1+M)^2}^{\infty}dW'^2\Big[\frac{1}{W'^2-W^2}\mp\frac{1}{W'^2+W^2-2M^2+2z_0}\Big]$$

$$\times\frac{W+M}{\sqrt{(E_2'+M)(E_1+M)}}\frac{3}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}. \qquad (24)$$

The other amplitudes also have a structure quite similar to that of the above equation. These expressions in all are too complicated to analyze as they stand. We therefore study the static approximation to see whether or not any differences exist between this case and the real scattering one.

Noting the relations

$$W \cong M+\omega,$$
$$W' \cong M+\omega', \tag{25}$$
$$t_0=\omega-\frac{1+t^2}{2W}\approx\omega,$$

and neglecting terms of order M^{-2} in Eq. (24), we find results quite similar to those of C.G.L.N. For example, we have

$$\mathrm{Re}\left(\frac{h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)}}{qt}\right)=-\frac{1}{4\pi}\left(\frac{g}{2M}\right)^2\left[\left(1-\frac{\omega}{2M}\right)\pm\left(1+\frac{\omega}{2M}\right)\right]$$
$$\times\left(1-\frac{\omega}{2M}\right)+\frac{P}{\pi}\int_1^\infty d\omega'\left[\frac{1}{\omega'-\omega}\mp\frac{1}{\omega'+\omega}+\frac{1}{M}\right]$$
$$\times\frac{1}{q't'}\,\mathrm{Im}\,(h^{(\pm)}_{P(1/2)}-h^{(\pm)}_{P(3/2)}), \tag{26}$$

and for $h^{(\pm)}_{P(3/2)}$,

$$\mathrm{Re}\left(\frac{3}{qt}h^{(\pm)}_{P(3/2)}\right)=\mp\frac{1}{4\pi}\left(\frac{g}{2M}\right)^2\frac{2}{\omega}$$
$$+\frac{P}{\pi}\int_1^\infty d\omega'\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}+\frac{3}{M}\right]\frac{1}{q't'}\,\mathrm{Im}\,h^{(\pm)}_{P(3/2)}. \tag{27}$$

We do not record the corresponding equations for the s-wave, since we know that s-wave certainly cannot be represented accurately by this method.

The solution of these equations is

$$h^{(\pm)}_{P(3/2,1/2)}=\frac{t}{q}f^{(\pm)}_{P(3/2,1/2)}, \tag{28}$$

where $f^{(\pm)}_{P(3/2,1/2)}$ is the solution given by the corresponding equation for the real pion-nucleon scattering. To the extent that these coincide with the static Chew-Low equations[7] we have achieved a partial justification for the use of the latter in off-the-energy-shell applications. This will be justified for cases for which t^2 remains effectively small compared to M. Where this is not so, a more accurate approach which takes account of nucleon recoil must be used. Such a method, that of Blankenbecler and Gartenhaus, is considered in the next section.

§ 4. On the Blankenbecler-Gartenhaus solution

According to this method,[3] we have

$$A^{(\pm)}(\xi,\nu_B,t^2)=\frac{1}{\pi}e^{i\delta_{33}(\xi)}\int_{1+(1/2M)}^{\infty}d\xi' e^{\rho(\xi)-\rho(\xi')}A_{33}^{(\pm)}(\xi',\nu_B,t^2)$$
$$\times\sin\delta_{33}(\xi')\left[\frac{1}{\xi'-\xi-i\epsilon}\pm\frac{1}{\xi'+\xi+2\nu_B}\right],$$
$$B^{(\pm)}(\xi,\nu_B,t^2)=\frac{g^2}{2M}\left(-\frac{1}{\xi}\mp\frac{1}{\xi+2\nu_B}\right)+\frac{1}{\pi}e^{i\delta_{33}(\xi)}$$
$$\times\int_{1+(1/2M)}^{\infty}d\xi' e^{\rho(\xi)-\rho(\xi')}B_{33}^{(\pm)}(\xi',\nu_B,t^2)\sin\delta_{33}(\xi')$$
$$\times\left[\frac{1}{\xi'-\xi-i\epsilon}\mp\frac{1}{\xi'+\xi+2\nu_B}\right], \tag{29}$$

where

$$\rho(\xi)=\frac{P}{\pi}\int_{1+(1/2M)}^{\infty}d\xi'\,\delta_{33}(\xi')\left[\frac{1}{\xi'-\xi}+\frac{1}{\xi'+\xi+2\nu_B}\right],$$
$$A_{33}^{(\pm)}(\xi,\nu_B,t^2)=-\begin{pmatrix}2/3\\-1/3\end{pmatrix}\frac{g^2}{2W}\left[\frac{W+M}{\sqrt{(E_2+M)(E_1+M)}}3\left(\frac{2M\nu_B+\omega t_0}{qt}\right)\right.$$
$$\left.+\frac{W-M}{\sqrt{(E_2-M)(E_1-M)}}\right]\left[-\frac{W-M}{\sqrt{(E_2-M)(E_1-M)}}Q_1(\zeta)\right.$$
$$\left.+\frac{W+M}{\sqrt{(E_2+M)(E_1+M)}}Q_2(\zeta)\right],$$
$$B^{(\pm)}(\xi,\nu_B,t^2)=-\begin{pmatrix}2/3\\-1/3\end{pmatrix}\frac{g^2}{2W}\left[\frac{1}{\sqrt{(E_2+M)(E_1+M)}}3\left(\frac{2M\nu_B+\omega t_0}{qt}\right)\right.$$
$$\left.-\frac{1}{\sqrt{(E_2-M)(E_1-M)}}\right]\left[-\frac{W-M}{\sqrt{(E_2-M)(E_1-M)}}Q_1(\zeta)\right.$$
$$\left.+\frac{W+M}{\sqrt{(E_2+M)(E_1+M)}}Q_2(\zeta)\right],$$

and

$$\xi=\nu-\nu_B,\quad \zeta=\frac{W^2-M^2-2\omega t_0}{2qt},$$
$$W=2M\xi+M^2,$$
$$E_1=\frac{W^2+M^2+t^2}{2W},\quad E_2=\frac{W^2+M^2+t^2}{2W},$$

and $Q_i(\zeta)$ denotes the Legendre function of the second kind. These solutions are obtained on the assumption that $A^{(\pm)}$ and $B^{(\pm)}$ have, in good approximation, the phase $\delta_{33}(\xi)$, after we subtract the small phase shifts. This assump-

tion may be correct for small values of t^2. But even for large t^2, the above solutions will give us some clue as to how to treat the processes including large momentum transfers. The interesting feature of these solutions is that we may have some hope that $h_{l\pm}^{T}(W, t^2)$ defined in the preceding section shows a certain damping effect for large t^2, because Eq. (30) includes t^2 in the denominator which cannot be canceled by the spinor normalization factor (see Eq. (12)). The detailed evaluation of these expressions will be left for a future publication.

In this connection, we may note that current treatment of single pion-production in nucleon-nucleon collisions utilizes the following assumptions (see also § 1).

i) The one-pion exchange mechanism between nucleons is mainly responsible for the process.

ii) Virtual-pion nucleon scattering is accurately represented by the energy shell approximation.

According to this method, it has been found that the calculated cross section[8] (we are mainly interested in $p+p\rightarrow p+n+\pi^+$) is always larger than the observed one in the high energy region (beyond 1.4 Bev). But this general picture of the production mechanism is expected to be increasingly good for the high energy region, because (i) the interaction time is shorter, (ii) the final state interaction between nucleons becomes less important, (iii) the multi-pion exchange process likely contributes to multiple production processes. Therefore a major reason[9] for the excessively large theoretical cross section might be the inadequacy of the energy shell approximation. Thus it may be of interest to work out the virtual-pion nucleon scattering for the large t^2 case using the method of this section.

§ 5. Summary

In the preceding sections we first studied the virtual-pion scattering by the method of C.G.L.N. With the restriction $t^2 < M$, the integral equations for the partial wave amplitudes have almost the same form as in C.G.L.N. in the one-pion and static approximation and have as solutions the Chew-Low form of the virtual amplitudes. These results seem hardly surprising. The reasons are firstly because our dispersion relations are quite similar to those of the real scattering case and secondly because the restriction on t^2, namely, $t^2 < M$ does not permit any qualitative differences for the dependence on the energy and momentum of the corresponding particles between our case and that of C.G.L.N.

These statements can be easily understood by observing the relationship between the energies of the initial and the final particles, i.e.

$$E_1 = E_2 + \frac{1+t^2}{2W},$$

$$t_0=\omega-\frac{1+t^2}{2W}, \quad |\vec{t}|\cong\sqrt{|\vec{q}|^2+t^2}.$$

Thus the appreciable effects of the virtual nature of the external source occur when $t_0\lesssim 0$, or $t^2\gtrsim$ a few M.

For the large t^2 case, so far no promising theory has been proposed. As a possible approach we suggest the Blankenbecler-Gartenhaus solution, which is recorded here, but not studied in detail.

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The Σ-Λ Relative Parity and the $\bar{K}$-N Reaction. II

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According to Dalitz and Tuan's phenomenological analysis of K^--p reactions the recently discovered π-Λ resonance may be interpreted as caused by the existence of a quasi-bound state of $\bar{K}$-N which decays into π-Λ. Granting this interpretation and considering dispersion relations together with experimental information, it is conjectured that the K-Λ (K-Σ) parity relative to N is odd (even) and hence the Σ-Λ relative parity is odd. Arguments for our conjecture are given concerning branching ratios of the decay of the quasi-bound state and also of the hyperon production in the K^--p absorption near the threshold.

§ 1. Introduction

Recently a sharp resonance in the π-Λ system with a Q-value of about 130 Mev was found by analysing the process $K^-+p \to \Lambda+\pi^++\pi^-$.[1] To our knowledge the parity and the angular momentum of the resonant system have not been determined as yet. This may be the $P_{3/2}$ resonance which has been anticipated on the analogy of the 3-3 resonance in the π-N scattering. However, another interesting possibility has been suggested by Dalitz and Tuan through a phenomenological analysis of the low-energy K^--p reactions.[2] They obtained four sets of scattering amplitudes, $(a\pm)$ and $(b\pm)$, which are consistent with the present data on the K^--p reactions. In the case of $(a-)$ solution there exists a quasi-bound state of $I=1$ which decays into π-Y, Y being Λ or Σ. This may be identified with the observed resonance. Assuming the situation implied by DT's $(a-)$ solution to be correct, we wish to gain further insight into the mechanism of the strong interactions.

In § 2, coordinating the dispersion theoretical analyses with experimental information from our viewpoint, we shall be led to the conjecture that $P_{K\Lambda}$, the K-Λ parity relative to N, is odd and $P_{K\Sigma}$ is even and hence $P_{\Sigma\Lambda}$ is odd.* Arguments for this combination of parities will be given in the subsequent sections.

It will be inferred in § 3 that the quasi-bound state decays much more frequently into π-Λ than into π-Σ. Experimental data seem to indicate such a tendency. § 4 will be concerned with the branching ratio of $K^-+p \to \pi+Y$ at

* The same conjecture was once made by Barshay[3] on the basis of perturbational estimates including certain higher order effects. His argument is insufficient in that relativistic effects such as those due to baryon pairs are disregarded, yet it contains significant inferences especially as to the role of a P-wave in the K-N scattering.

low energies. The infrequency of the Λ-production and the large phase difference between $I=0$ and $I=1$ states of the π-Σ system is favorably interpreted in terms of final interactions. In § 5 the main points of urgent experimental interest will be summarized and related problems will be discussed.

§ 2. Conjecture

DT reexamined the analyses by Matthews and Salam[4] and by Goebel,[5] using their four sets of solutions. Assuming $P_{\Sigma\Lambda}$ to be even, they showed (though not conclusively) that the $(a-)$ solution* is compatible only with even $P_{K\Lambda}$ and even $P_{K\Sigma}$. On the other hand there is an evidence concerning the K^--absorption in He^4 which strongly favors odd $P_{K\Lambda}$.[6] Accepting this evidence, we may say that the $(a-)$ solution and eventually DT's interpretation of the π-Λ resonance will not hold if $P_{\Sigma\Lambda}$ is even. This interpretation will be restored by dropping the assumption of even $P_{\Sigma\Lambda}$.

In the case of odd $P_{\Sigma\Lambda}$, S-wave $\bar{K}$-N absorption will produce one of π-Λ and π-Σ system in an S-state and the other in a P-state. Then the values of parameters will be somewhat modified from those given in DT because of the asymmetry between the π-Λ and π-Σ channels as to the energy dependence. It is our feeling, however, that the qualitative features of DT's solutions still hold in the case of odd $P_{\Sigma\Lambda}$ and the interpretation of the π-Λ resonance based on the $(a-)$ type solution is possible.** In dispersion relations for the $K^{\pm}$-p scatterings the bound state contributions of Λ and Σ appear with different signs, if $P_{\Sigma\Lambda}$ is odd. The sign of the bound state contribution as a whole need not be definite. It may therefore be expected that dispersion relations for the $K^{\pm}$-p scatterings are compatible with odd $P_{K\Lambda}$ and even $P_{K\Sigma}$. We are now free to conjecture that $P_{K\Lambda}$ is odd and $P_{K\Sigma}$ is even.

As is well known, in dispersion relations for charge exchange scatterings the bound state contributions of Λ and Σ appear with signs in a combination different from that for the $K^{\pm}$-p scatterings. Combination of these two types of dispersion relations enables us to determine $P_{K\Lambda}$ and $P_{K\Sigma}$, respectively. Such an attempt was made by Sugano and Komatsuzawa[7] and is now being reexamined from our viewpoint. Though a definite conclusion can hardly be reached because of yet poor experimental information and of the ambiguity concerning the unphysical region, a preliminary analysis seems to favor our conjecture.

Using some dispersion relations, Galzenati and Vitale[8] and Amati[9] inferred that $P_{K\Lambda}$ and $P_{K\Sigma}$ are both likely to be odd. Their analyses are interesting in that no assumption is needed on the dispersive part of the elastic K^--p scattering amplitude. The result, however, seems to be somewhat dependent on the behavior of the K^+-p cross section at low energies.[10] Igi's dispersion rela-

* In the following we shall consider only the $(a-)$ solution unless otherwise mentioned.

** However, see § 5 where a remark is given on DT's analysis.

tion also suffers a similar ambiguity.[11] This point deserves more attention, especially of experimentalists.

§ 3. Decay of quasi-bound state

According to our picture the quasi-bound state, which will be denoted by Y^*, has isospin 1, spin 1/2 and odd parity relative to Λ. Y^* decays into π-Λ of $S_{1/2}$ or into π-Σ of $P_{1/2}$, provided that the parity is conserved. The Q-value is about 130 Mev for π-Λ and 60 Mev for π-Σ. As is well known in the theory of nuclear reactions,[12] the decay probability of a compound system or the partial width Γ is written as

$$\Gamma=(2kRv_l)\cdot(\text{reduced width}), \tag{3·1}$$

where k is the momentum of the emerging pion, R is the force range and v_l is the penetrability of the centrifugal barrier which is given by

$$v_0=1 \quad \text{and} \quad v_1=(kR)^2/(1+(kR)^2). \tag{3·2}$$

The factor $(2kRv_l)$ in Eq. (3·1) depends on the conditions outside the compound system while the reduced width incorporates all the properties of the interior. The Σ-Λ branching ratio of the decay of Y^* will be determined by the former. For the π-Σ system, $k=0.89\mu$ and for the π-Λ system, $k=1.48\mu$.* Tentatively assuming $R\approx 1/2\mu$, we have $v_1\approx 0.17$ and the branching ratio $(\Sigma/\Lambda)_{I=1}\approx 10\%$. If $P_{\Sigma\Lambda}$ is even the π-Σ system is emitted also in an $S_{1/2}$ state. Then the branching ratio is $(\Sigma/\Lambda)_{I=1}\approx 60\%$. For example, we have

$$\frac{Y^*\to\Sigma^0+\pi^\pm}{Y^*\to\Lambda+\pi^\pm}\approx\begin{cases}5\% \\ 30\%\end{cases}\quad\text{if } P_{\Sigma\Lambda}\text{ is }\begin{cases}\text{odd}\\ \text{even.}\end{cases} \tag{3·3}$$

Here the factor 1/2 which comes from the isospin of the π-Σ system has been taken into account.

According to Alston et al.[1] the distribution of the observed events were as follows.

		No. of events
$K^-+p\to$	$\Sigma^0 \quad +\pi^++\pi^-$	27
	$\Lambda \quad +\pi^++\pi^-$	49
	$(\Sigma^0 \text{ or } \Lambda)+\pi^++\pi^-$	92

It was inferred that the overwhelming majority of the events in the third group were also reactions of the second type. They treated the 141 events of the second and the third groups as examples of the reaction of the second type but estimated that 10 to 15% were actually Σ^0 events. Because a

* $k=(\omega^2-\mu^2)^{1/2}$, $\omega=(W^2-M^2+\mu^2)/2W$. μ, M and W are respectively the masses of π, Y and Y^*. The unit $c=\hbar=1$ is used.

nonresonant background is present, the substantial (Σ/Λ) ratio of the decay of Y^* will be much smaller than that literally given by the above distribution of events. Therefore the case of odd $P_{\Sigma\Lambda}$ seems to be favorable. If a very precise measurement is carried out, it will turn out that the Σ events are mostly due to nonresonant processes.

§ 4. Final interaction in $\bar{K}+N\to\pi+Y$

According to Alvarez et al.[13] the branching ratio of $K^-+p\to\pi+Y$ at rest is $\Sigma^-:\Sigma^+:\Sigma^0:\Lambda\approx 45:21:27:7$. The Λ-production is remarkably infrequent at rest, but it becomes more frequent as the energy is raised. It was inferred from the above branching ratio that the π-Σ scattering phase shift difference is $\delta_{I=0}-\delta_{I=1}\approx\pm 62^\circ$ at the energy ($\sim$100 Mev) corresponding to the K^--p threshold.[13,14]

Previously we investigated the π-Y scattering in the case of odd $P_{\Sigma\Lambda}$, in terms of the conventional Yukawa type interaction.[15] In such an approach the π-Y scattering is roughly described as the repetition of a Born term. The sign of the Born term indicates that the π-Λ force in $S_{1/2}$ is repulsive, the π-Σ force in $P_{1/2}$ is repulsive for $I=1$ and attractive for $I=0$. As was shown in I, the $P_{1/2}$, $I=0$ π-Σ scattering is strongly enhanced by higher order effect, and even a resonance analogous to the 3-3 resonance may occur. On the contrary, the $P_{1/2}$, $I=1$ π-Σ scattering is suppressed. These features are all favorable to the experimental data. Namely, the π-Λ production will be suppressed by the repulsive force in $S_{1/2}$. Its energy dependence may be due to the admixture of a P-wave.[3] The π-Σ phase shift $\delta_{I=0}$ will be positive and very large while $\delta_{I=1}$ will be negative.

Effect of the existence of Y^*, which was not considered in I, should be superposed upon the above final interaction. The former will certainly be predominant at low energies around the level of Y^*, but will be relatively unimportant in the physical energy region of $K^-+p\to\pi+Y$. Since the level of Y^* is very sharp, Y^* may be approximately regarded as an "elementary" particle. The effect will be estimated by means of such a diagram as shown in Fig. 1(a). To the diagrams (a) and (b) of Fig. 1, the spin and the isospin factors are common and consequently the sign of the matrix elements is also common for energies above the level of Y^*. The inclusion of Y^* does not alter but rather reinforce the conclusion of I.

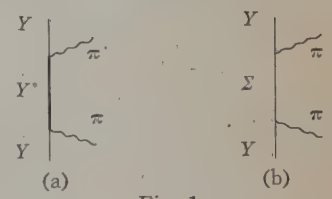


Fig. 1.

§ 5. Discussion

Needless to say, the experimental determination of the orbital as well as total angular momentum of the relevant π-Y system is vital. In view of the

argument made in § 3 special attention should be given to the Σ-Λ branching ratio of the decay of Y^*.

As is well known, DT's (+) and (−) type solutions are discriminated by the Coulomb-nuclear interference in the elastic K^--p scattering. It is constructive for the (+) type and destructive for the (−) type. Alvarez et al.[13] reported that it was observed as constructive. If this should be unambiguously established, the (−) type solution and eventually our conjecture would be eliminated. We feel, however, that accumulation of further experimental verification is needed.

In the dispersion theoretical analysis, as was stressed in § 2, the energy dependence of the K^+-p cross section at very low energies is important. Experiments on the charge exchange scatterings and also on the K^--n scattering should be urged. Information on the latter is especially desired since in its dispersion relations the bound state contribution comes only from Σ.

It may be added that, as was shown in I and the preceding section, P-wave resonances are likely to occur in the π-Σ scatterings with $(I, J)=(0, 1/2)$, $(1, 3/2)$ and $(2, 3/2)$, provided that the $\pi\Sigma\Sigma$ interaction is sufficiently strong. The first one, $(0, 1/2)$, is especially interesting since it is expected only for odd $P_{\Sigma\Lambda}$. The other two will not serve as a test of $P_{\Sigma\Lambda}$ because they are expected in both cases of even and odd $P_{\Sigma\Lambda}$. In DT's analysis and consequently in making our conjecture, however, it has been implicitly granted that there is no resonance other than the π-Λ one in the low-energy $\bar{K}$-N reactions. Strictly speaking, if the resonance, $(0, 1/2)$, and/or any other anomaly are found in the low-energy $\bar{K}$-N reactions, DT's analysis ought to be revised. Then the effective range approach may not be justified. In this sense our argument may turn out to be not selfcontained.

Now let us speculate on the binding mechanism of Y^*. In the Born approximation of the Yukawa interaction theory the $\bar{K}$-N force is repulsive both for scalar and pseudoscalar KYN couplings. The virtual pion effect may act as an attractive force,[16] but presumably it will not be the main agency, as the width of the level of Y^* is very small. Conversely speaking, Y^* will not be mainly responsible for the large cross section for the inelastic process $K^-+p\rightarrow \pi+Y$.[17] It is quite conceivable that Y^* would still exist if the π-Y channel were switched off. Some π-K interaction may induce a strong attractive force for the $\bar{K}$-N system but it will then be a delicate problem whether the K-N force is reproduced as repulsive in conformity with the experimental evidence. This possibility should be subject to further investigation. However, the origin of Y^* may be beyond the conventional theory. At the present stage, it may be appropriate to regard Y^* as "elementary" and put in the same category with other elementary particles.

Incidentally, we wish to remark on the Λ-N force. The long-range part of the Λ-N force will be given by the two pion exchange potential which can be

calculated by using the π-Λ scattering amplitude.[18] In the calculations of the Λ-N force so far made the π-Λ scattering amplitude has been treated by the Born approximation but it is now clear that such an approximation is wrong and the effect of Y^* must be taken into account. A rough estimation will be given by including the graph in Fig. 1(a).*

If the π-Λ scattering occurs only in $S_{1/2}$, the two pion exchange Λ-N force will be spin-independent in the adiabatic approximation. On the other hand, phenomenological analyses of hypernuclei show that the Λ-N force has considerable spin-dependence.[19] This is, however, not directly confronted with our conjecture because it may well be that sufficiently strong spin-dependence comes out from the short-range part of the potential. It should be remembered here that the binding of the deuteron has not yet been explained in terms of the meson theoretical potential.

Finally, we should like to note another interpretation of the π-Λ resonance. Suppose that the Σ-hyperon is not "elementary" but a bound state of π-Λ. Then the π-Λ scattering phase shift starts from 180° at the zero energy and drops down to zero, passing through 90°, as the energy increases. A similar possibility was once noted by Gell-Mann[20] as to the π-N scattering in the $P_{1/2}$, $I=1/2$ state. As was discussed by Ida[21] a resonance of this type has characters different from the ordinary resonance where the phase shift goes up through 90°. For the width of the former a lower bound exists which comes from the requirement of causality. Therefore it is questionable whether such a sharp resonance as observed can be interpreted in this way. This possibility will be discussed in a seperate article.

In conclusion the author wishes to express his sincere thanks to Professor H. Yukawa, Dr. M. Ida, Dr. J. Iwadare and the members of the Yukawa Laboratory for enlightening discussions.

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Note added in proof: Recently Nambu and Sakurai (Phys. Rev. Letters **6** (1961), 337) summarized the recent experimental developments which are indicative of *odd* $P_{\Sigma\Lambda}$. They inferred that, among other evidences, the nonexistence of the P-wave resonances with $(I, J) = (1, 3/2)$ and $(2, 3/2)$ might be taken as an argument against even $P_{\Sigma\Lambda}$. As was noted in §§ 4 and 5 of the present paper, however, these resonances are likely to occur in both cases of even and odd $P_{\Sigma\Lambda}$ if the $\pi\Sigma\Sigma$ interaction is sufficiently strong. The nonexistence of these resonances, if confirmed, may imply that $P_{\Sigma\Lambda}$ is odd and that the $\pi\Sigma\Sigma$ interaction is fairly weak.

As for the role of the final π-Σ interaction in the reaction $K^- + p \to \pi + \Sigma$, see also Capps' paper (Phys. Rev. Letters **6** (1961), 375).

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Note on the Pairing Correlation in Nuclear Matter

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A difficulty in literally applying the BCS-Bogoliubov formalism to the nuclear system with singular interaction is pointed out. A prescription to overcome this difficulty is proposed. According to this method, it is possible to consistently join the BCS-Bogoliubov formalism with the "reaction matrix" formalism in a unified manner. The results obtained verify the validity of the basic conjecture underlying Cooper-Mills-Sessler's procedure.

§ 1. Introduction

After the appearance of the new theory of superconductivity, many attempts[1] have been made to extend the Bardeen-Cooper-Schrieffer theory[2] or the method of Bogoliubov[3] to the nuclear system.

In the case of practical application, they have been made by introducing the appropriate "effective interaction" phenomenologically.[4] However, the literal application of the BCS-Bogoliubov formalism to the actual nuclear system encounters a difficulty as will be discussed in § 2, because of the presence of a "hard core" in the nucleon-nucleon interaction. It has not yet been attempted to relate the "effective interaction" with the nucleon-nucleon interaction without ambiguities.

On the other hand, Brueckner[5] has shown by using the "reaction matrix" formalism that the strong two-body scattering correlation due to the singular nature of the nuclear interaction is mainly responsible for the saturation phenomena. Although the concept of "reaction matrix" is very useful in treating the singular interaction, this theory is constructed on the assumption that the ground state of nuclear matter is not drastically different from the Fermi-sphere.

It is the main purpose of this paper to join the BCS-Bogoliubov formalism with the "reaction matrix" formalism consistently and to treat the pairing

correlation and the scattering correlation in a unified manner. Our basic idea is as follows: i) We use the Bogoliubov transformation in order to treat the pairing correlation and to introduce the possibility of the modification of the Fermi distribution, ii) the reaction matrix formalism is extended to such a system in order to take the scattering correlation properly into account and iii) with the assistance of the "generalized reaction matrix" thus obtained, the coefficients of the Bogoliubov transformation and the energy gap are given. Thus we obtain a set of equations to be solved self-consistently. To formulate the above idea, the method of eigenmodes developed by Sawada[6] is used.

§ 2. Approximate eigenmodes

The Hamiltonian of the Fermion system here considered is

$$H=\sum_{p}\epsilon_{p}c_{p}^{+}c_{p}+\sum_{pqrs}c_{p}^{+}c_{q}^{+}\langle pq|v|rs\rangle c_{s}c_{r}, \tag{2·1}$$

where $\epsilon_{p}=(p^{2}/2m)-\lambda$, λ being the chemical potential, and c_{p} and c_{p}^{+} are the annihilation and creation operators of a fermion with momentum p and satisfy the usual commutation relation of fermions.* $\langle pq|v|rs\rangle$ is the antisymmetrized matrix element of the two-body interaction:

$$\langle pq|v|rs\rangle=\frac{1}{8}(\langle pq|V|rs\rangle-\langle pq|V|sr\rangle-\langle qp|V|rs\rangle+\langle qp|V|sr\rangle), \tag{2·2}$$

where

$$\langle pq|V|rs\rangle=\int\psi_{p}^{*}(x)\psi_{q}^{*}(x')V(x,x')\psi_{r}(x)\psi_{s}(x')dxdx',$$

$V(x, x')$ being the two-body interaction and $\psi_{p}(x)$ being a plane wave with momentum p in a large box. V contains a hard sphere since we want to treat the nuclear system.

In order to treat the pairing correlation and to take into account the possibility to deform the Fermi sea, we introduce a new Fermi operator,

$$\alpha_{p}=u_{p}c_{p}-v_{p}c_{-p}^{+}. \tag{2·3}$$

We expect that this operator becomes the approximate one-particle (quasi-particle) eigenmode. u_{p} and v_{p} are real numbers which satisfy the condition

$$\left.\begin{aligned}&u_{p}^{2}+v_{p}^{2}=1,\\&u_{p}=u_{-p},\quad v_{p}=-v_{-p}.\end{aligned}\right\} \tag{2·4}$$

From (2·1) and (2·3) we get an equation of the following type,

$$[\alpha_{p}^{+},H]=-\omega_{p}\alpha_{p}^{+}-\tilde{\omega}_{p}\alpha_{-p}+Y_{p}. \tag{2·5}$$

* For the sake of simlicity we shall neglect the spin- and τ-spin dependence in the following arguments.

If α_p^+ was an exact eigenmode, $\tilde{\omega}_p$ and Y_p should vanish. Bogoliubov's method is as follows: i) The "interaction" Y_p is expressed by the normal product with respect to α, ii) u_p and v_p are determined so that $\tilde{\omega}_p$ vanishes and iii) it is assumed that the ground state is given approximately by the vacuum $|\rangle$ with respect to α. Then the ground state energy $\langle|H|\rangle$ and the excitation energy ω_p are obtained.

How well the vacuum $|\rangle$ reproduces the ground state which is an eigenstate of the Hamiltonian depends on how well α_p reproduces an exact eigenmode. Therefore, whether α_p is a "good" approximate eigenmode or a "bad" one is essentially important for Bogoliubov's formalism.

The literal application of the Bogoliubov formalism to nuclear matter gives the "interaction" Y_p which includes the original nuclear interaction V with a hard core. In this case Y_p is singular and cannot be neglected, so that α_p^+ is a "bad" approximate eigenmode. This is the essential difficulty pointed out in § 1. It must be emphasized that the "interaction" Y_p should be replaced by a less singular one in order that α_p becomes a "good" approximate eigenmode.

To overcome the above-mentioned difficulty and to treat the singular interaction appropriately, we must take into account the scattering correlation which is very big because of the singular nature of the interaction. For this purpose it is convenient to set up the approximate two-particle scattering eigenmode in the following way:

$$X_n^+ = \sum_{pq} \Psi^n(p, q) : c_p^+ c_q^+ :, \quad \Psi^n(p, q) = -\Psi^n(q, p). \qquad (2\cdot6)^*$$

where the symbol : : means the normal product with respect to the operator α. $\Psi^n(p, q)$ should be determined so that X_n^+ becomes a "good" approximate eigenmode satisfying the relation

$$[X_n^+, H] = -W^n X_n^+ + Z_n, \qquad (2\cdot7)$$

where "interaction" Z_n contains only the small interaction which is less singular than the original V.

§ 3. Generalized Bethe-Goldstone equation

In this section we examine the approximate two-particle scattering eigenmode in the deformed Fermi sea and construct the "generalized reaction matrix". With the help of the "generalized reaction matrix" thus obtained, the coefficients of the Bogoliubov transformation will be determined.

Using (2·1) and (2·6), we have

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* In general, the definition $X_n^+ = \sum_{pq} \Psi^n(p, q) c_p^+ c_q^+$ might be chosen as the two-particle scattering mode. It should be noted, however, that Eq. (3·4) holds. Therefore, we choose the definition (2·6).

$$
\begin{aligned}
W^n X_n^+ + [X_n^+, H] = & - W^n \sum_p \Psi^n(p, -p) u_p v_p \\
& + \sum_{pq} (: c_p^+ c_q^+ : + v_p u_q \delta_{-p,q}) [\{W^n - (E_p + E_q)\} \Psi^n(p, q) \\
& \qquad - \sum_{rs} \langle pq|v|rs\rangle \theta(r, s) \Psi^n(r, s)] \\
& + \sum_{pq} (: c_p^+ c_q^+ : + v_p u_q \delta_{-p,q}) \{(E_p - \epsilon_p) + (E_q - \epsilon_q)\} \Psi^n(p, q) \\
& + 4 \sum_{pq} \Psi^n(p, q) \sum_{p'q'r} (: c_{p'}^+ c_{q'}^+ : + v_{p'} u_{q'} \delta_{-p',q'}) \langle p'q'|v|pr\rangle : c_q^+ c_r :
\end{aligned}
\tag{3·1}
$$

where

$$
\theta(r, s) \equiv u_r^2 + u_s^2 - v_r^2 - v_s^2. \tag{3·2}
$$

E_p is the renormalized one-particle energy with momentum p and is determined later self-consistently (cf. Eq. (3·17)).

Putting the first and the second term respectively in the right-hand side of Eq. (3·1) equal to zero, we obtain the following generalized Bethe-Goldstone equation:

$$
\{W^n - (E_p + E_q)\} \Psi^n(p, q) = \sum_{rs} \langle pq|v|rs\rangle \theta(r, s) \Psi^n(r, s), \tag{3·3}
$$

and

$$
W^n \sum_p \Psi^n(p, -p) u_p v_p = 0. \tag{3·4}
$$

We can prove in § 4 that Eq. (3·4) is valid.* In the case of sharp Fermi distribution, Eq. (3·3) is reduced to that obtained by Sawada (Eq. (12) in ref. 6)), which consists of the Bethe-Goldstone equation[7] including the hole motion[8] and an equation for the "attached field". From Eqs. (3·3) and (3·4) we obtain

$$
\sum_p \{2 u_p v_p E_p - (u_p^2 - v_p^2) \Delta_p\} \Psi^n(p, -p) = 0, \tag{3·5}
$$

where

$$
\Delta_p = -2 \sum_q u_q v_q \langle q - q|v|p - p\rangle. \tag{3·6}
$$

The operator $v\theta$ is a self-adjoint operator with respect to the definition of inner product:

$$
(\Psi^n \circ \Psi^m) = \sum_{pq} \Psi^{n*}(p, q) \theta(p, q) \Psi^m(p, q). \tag{3·7}
$$

When a pair (p_0, q_0) satisfies the equation $\theta(p_0, q_0) = 0$,** the component $\Psi^n(p_0, q_0)$

---

* Using Eq. (4·5) we immediately get Eq. (3·5). Thus we can prove, by the use of Eqs. (3·3), (3·5) and (3·6), that Eq. (3·4) holds.

We are very much thankful to Prof. K. Sawada for pointing out to us the importance of this equation.

** Hereafter the momentum pair with suffix 0, (p_0, q_0), means the one satisfying $\theta(p_0, q_0) = 0$.

does not contribute to the norm of Ψ^n. If we construct the state Ψ^{n_0} of which (p_0, q_0)-component is the only non-vanishing one, i.e.

$$\Psi^{n_0}(p, q)=c(p_0, q_0)\,(\delta_{pp_0}\delta_{qq_0}-\delta_{pq_0}\delta_{qp_0}), \tag{3·8}$$

this Ψ^{n_0} satisfies Eq. (3·3) with the eigenvalue

$$W^{n_0}=E_{p_0}+E_{q_0}, \tag{3·9}$$

and represents the free two-particle motion. Thus this solution should not be interpreted as the scattering eigenmode in its proper sense.

The orthonormality relation is obtained from Eq. (3·3):

$$(\Psi^n\circ\Psi^m)=\epsilon_n\delta_{nm} \qquad \epsilon_n=\begin{cases} 0, & \text{for } n=n_0 \\ \pm 1 & \text{for } n\neq n_0. \end{cases} \tag{3·10}$$

Then we can define, as usual, the creation and destruction operator, A_n^+ and B_n, of two-particle scattering eigenmode:

$$X_n^+=\begin{cases} A_n^+ & \text{for } \epsilon_n=+1 \\ B_n & \text{for } \epsilon_n=-1, \end{cases} \tag{3·11}$$

because of the relation

$$\langle|[X_n, X_m^+]|\rangle=\epsilon_n\delta_{nm} \qquad \text{for } \alpha_p|\rangle=0. \tag{3·12}$$

The (p_0, q_0)-component of non-trivial solution $\Psi^n(n\neq n_0)$ is expressed in terms of the other components from (3·3):

$$\Psi^n(p_0, q_0)=\frac{1}{W^n-(E_{p_0}+E_{q_0})}\sum_{pq}\langle p_0q_0|v|pq\rangle\theta(p, q)\Psi^n(p, q). \tag{3·13}$$

In the case of sharp Fermi distribution, this becomes the "attached field" in Sawada's formalism.[6]

Assuming the completeness relation:

$$\sum_n\Psi^n(p, q)\,\epsilon_n\Psi^{n*}(p', q')\,\theta(p', q')=\frac{1}{2}(\delta_{pp'}\delta_{qq'}-\delta_{pq'}\delta_{qp'}) \qquad \text{for } \theta(p, q)\neq 0, \tag{3·14}$$

we can find the expansion of: $c_p^+c_q^+$: (for $\theta(p, q)\neq 0$) into eigenmodes X_n^+:

$$:c_p^+c_q^+:=\sum_n\Psi^{n*}(p, q)\,\theta(p, q)\,(A_n^+-B_n)$$
$$-\sum_{p_0q_0}:c_{p_0}^+c_{q_0}^+:\sum_n\Psi^{n*}(p, q)\,\theta(p, q)\,\epsilon_n\Psi^n(p_0, q_0). \tag{3·15}$$

Now we examine the operator Z_n in Eq. (2·7), the expression of which is given by the right-hand side of Eq. (3·1). Using (3·3), (3·4) and (3·15), we find after some calculations

$$Z_n=4\sum_{pq}\Psi^n(p,q)\Big[\sum_{\epsilon_n>0}\sum_r R^n_{,pr}A_n^+ : c_q^+ c_r : -\sum_{\epsilon_n<0}\sum_r R^n_{,pr} : c_q^+ c_r : B_n$$
$$+\sum_{p_0q_0}\sum_r \widetilde{R}^{p_0q_0}_{,pr} : c_{p_0}^+ c_{q_0}^+ c_q^+ c_r : \Big]-2\sum_{pq}\Psi^n(p,q)\Delta_p : c_q^+ c_{-p} :$$
$$+8\sum_{pq}\sum_{p_0q_0}\Psi^n(p,-q_0)\widetilde{R}^{p_0q_0}_{,pr}u_{q_0}v_{q_0} : c_{p_0}^+ c_q : , \tag{3·16}$$

with the equation for E_p:

$$E_p=\epsilon_p-4\sum_{\epsilon_n<0}\sum_q R^n_{,pq}\Psi^n(p,q)+4\sum_q \widetilde{R}^{pq}_{,pq}v_q^2, \tag{3·17}$$

where the following generalized reaction matrices R and $\widetilde{R}$ are used:

$$R^n_{,pq}=\sum_{p'q'}\Psi^{n*}(p',q')\theta(p',q')\langle p'q'|v|pq\rangle=(R^n_{pq,})^*, \tag{3·18}$$

$$\widetilde{R}^{pq}_{,rs}=\langle pq|v|rs\rangle-\sum_n R^n_{,rs}\epsilon_n\Psi^n(p,q)$$
$$=\begin{cases}0 & \text{for } \theta(p,q)\neq 0\\ \langle pq|v|rs\rangle-\sum_n \dfrac{R^n_{pq,}\epsilon_n R^n_{,rs}}{W^n-(E_p+E_q)} & \text{for } \theta(p,q)=0\end{cases}$$
$$=(\widetilde{R}^{pq}_{rs,})^*. \tag{3·19}$$

In the case of sharp Fermi distribution, Eq. (3·17) corresponds to Eq. (20′) in Sawada's paper[6] which is analogous to that obtained by Brueckner and Gammel.[5]

Eq. (3·16) shows that Z_n is expressed in terms of the reaction matrix and Δ which are less singular than the original interaction.

§ 4. Approximate one-particle (quasi-particle) eigenmodes and the pairing correlation

We now come to the stage to examine the approximate one-particle (quasi-particle) eigenmode.

A calculation similar to that in the preceding section leads to

$$[\alpha_p^+, H]=-\omega_p\alpha_p^+-\tilde{\omega}_p\alpha_{-p}$$
$$-2u_p\Big[\sum_{\epsilon_n>0}\sum_q R^n_{,pq}A_n^+ c_q-\sum_{\epsilon_n<0}\sum_q R^n_{,pq}c_q B_n+\sum_{p_0q_0}\sum_r \widetilde{R}^{p_0q_0}_{,pr} : c_{p_0}^+ c_{q_0}^+ c_r :\Big]$$
$$-2v_p\Big[\sum_{\epsilon_n>0}\sum_q R^n_{-pq,}c_q^+ A_n-\sum_{\epsilon_n<0}\sum_q R^n_{-pq,}B_n^+ c_q^+ +\sum_{p_0q_0}\sum_r \widetilde{R}^{p_0q_0}_{-pr,} : c_r^+ c_{q_0}c_{p_0} :\Big], \tag{4·1}$$

where

$$\omega_p\equiv(u_p^2-v_p^2)E_p+2u_pv_p\Delta_p, \tag{4·2}$$

$$\tilde{\omega}_p\equiv 2u_pv_pE_p-(u_p^2-v_p^2)\Delta_p, \tag{4·3}$$

$$\Delta_p=-2\sum_q u_qv_q\langle q-q|v|p-p\rangle. \tag{4·4}$$

Eq. (4·1) shows that the "interaction" Y_p is expressed only by the "reaction matrix" which is less singular than the original nuclear interaction. Thus, the difficulty discussed in § 2 has been overcome, and α_p becomes a "good" approximate eigenmode under the condition

$$\tilde{\omega}_p \equiv 2u_p v_p E_p - (u_p^2 - v_p^2)\Delta_p = 0, \tag{4·5}*$$

which is equivalent to Eq. (3·5).

Using Eqs. (4·5) and (2·4), we obtain

$$\begin{aligned} &2u_p v_p = \frac{\Delta_p}{\sqrt{E_p^2+\Delta_p^2}}, \quad u_p^2 - v_p^2 = \frac{E_p}{\sqrt{E_p^2+\Delta_p^2}}, \\ &u_p^2 = \frac{1}{2}\left\{1+\frac{E_p}{\sqrt{E_p^2+\Delta_p^2}}\right\}, \quad v_p^2 = \frac{1}{2}\left\{1-\frac{E_p}{\sqrt{E_p^2+\Delta_p^2}}\right\}, \\ &\omega_p = \sqrt{E_p^2+\Delta_p^2}, \end{aligned} \tag{4·6}$$

which are analogous to those given by the BCS-Bogoliubov formalism.

§ 5. Summary

According to our basic idea mentioned in § 1, we have obtained a set of equations, (3·3), (3·17) and (4·6) to be solved self-consistently. This results verifies the validity of the basic conjecture underlying Cooper-Mills-Sessler's procedure.[1] Thus it has become possible to join the BCS-Bogoliubov formalism with the "reaction matrix" formalism consistently and to treat the pairing correlation and the scattering correlation in a unified manner.

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* Putting

$$\theta(p,q)\Psi_B^0(p,q) \propto v_p u_q \delta_{-pq}, \tag{A}$$

we obtain from Eq. (4·5)

$$-(E_p+E_q)\Psi_B^0(p,q) = \sum \langle pq|v|rs\rangle \theta(r,s)\Psi_B^0(r,s). \tag{B}$$

Comparing the above equation with Eq. (3·3), we see that $\Psi_B^0(p,q)$ is a solution of the generalized Bethe-Goldstone equation (3·3) with $W^n=0$. This means that states Ψ^n, in general, not only represent the scattering states but also include the bound state (Ψ_B^0) with zero energy. Moreover, by the use of Eq. (A), Eq. (3·4) can be written as

$$W^n \sum_{pq} \Psi^n(p,q)\,\theta(p,q)\,\Psi_B^0(p,q) = 0$$

which shows the orthogonality condition of Ψ_B^0 with Ψ^n.

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Letters to the Editor

The opinions expressed in these columns do not necessarily reflect those of the Board of Editors. Communications should be submitted in duplicate and should be held to within 100 lines (pica type) on standard size letter paper (approx. 21×30 cm.), so that each letter may be arranged into two pages when printed. Do not forget to count in enough space for formulas, figures or tables.

On the Green Function of the Spin System. I

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March 13, 1961

The two-time Green function method was applied to the study of ferromagnets by Bogolyubov and Tjablikov,[1] who derived an equation which determines the magnetization in the entire region of temperature. We used the same technique to calculate the spatial correlation of spins in a ferromagnet in the lowest order approximation. Suppose the direction of the spontaneous magnetization to be in the positive z-axis. The derivation of the correlation function of the x-y components of spins is straightforward and we simply quote the results. Denoting the Fourier transforms of the correlations by $\phi^{xx}(q)$, etc. we have

$$\phi^{xx}(q)=\phi^{yy}(q)=\frac{\sigma}{2}\frac{e^{2\beta\sigma J(0,q)}+1}{e^{2\beta\sigma J(0,q)}-1}, T<T_c \quad (1)$$

$$=\frac{k_BT}{2J(0,q)}, T\approx T_c \quad (2)$$

$$=\frac{k_BT}{(\gamma^2/\chi)+2J(0,q)}, T>T_c, \quad (3)$$

where $J(0,q)=J(0)-J(q)$, $J(q)$ being the Fourier transform of the exchange energy, and $\sigma=\langle s_m^z\rangle$. γ is the gyromagnetic ratio and χ the susceptibility. This agrees with the results of van Hove[2] and de Gennes.[3]

In order to calculate the correlation of fluctuations of the z-components of spins, we apply a weak magnetic field H in the z-direction to a particular spin, say, the one situated at the origin. Then, the thermal average σ_g of s_g^z situated at the lattice point g is given by

$$\sigma_g=\sigma+\gamma H\Phi_g$$

where $\Phi_g\equiv\frac{1}{\beta}$

$$\times\frac{\mathrm{Tr}\int_0^\beta dx e^{-\beta H_0}(s_g^z-\sigma)e^{-xH_0}(s_0^z-\sigma)e^{xH_0}}{\mathrm{Tr}\, e^{-\beta H_0}}, \quad (4)$$

H_0 being the system Hamiltonian. Φ_g is not quite the same as the correlation function, but the difference is not important near and above the Curie point.[3] Thus the problem is reduced to the calculation of σ_g, which can be done by employing Bogolyubov and Tjablikov's technique. The Green function $G_{mg}(t)=\langle s_m^+(t), s_g^-\rangle$ being introduced, the Fourier transform of this with respect to time, $G_{mg}(E)$, satisfies the following equation in the first approximation,

$$EG_{mg}=\frac{\sigma_m}{\pi}\delta_{mg}+[\gamma H\delta_{m0}+2\sum_f J_{mf}\sigma_f]G_{mg}$$
$$-2\lambda\sigma_m\sum_f J_{mf}G_{fg} \qquad (5)$$

where λ is a parameter. For $\lambda=1$, we have the full Hamiltonian, but for $\lambda=0$, our system reduces to the Ising ferromagnet, in which case, Bogolyubov and Tjablikov's first approximation is identical with the Weiss approximation. Here we introduce the matrix notation,

$$(\boldsymbol{G})_{mg}=G_{mg},\ (\boldsymbol{\sigma})_{mg}=\delta_{mg}\sigma_m,\ (\boldsymbol{J})_{mg}=J_{mg},$$
$$(\boldsymbol{e})_{mg}=e_m\delta_{mg},\ e_m\equiv\gamma H\delta_{m0}+2\sum_f J_{mf}\sigma_f. \qquad (6)$$

For spin 1/2 case, Eq. (5) can be solved to yield, with the application of the spectral theorem and the identity, $\langle s_g^- s_g^+\rangle=1/2-\sigma_g$, the following equation which determines σ_g.

$$\frac{1}{2}-\sigma_g=2\sigma_g\left[\frac{1}{e^{\beta[e-2\lambda\sigma\cdot J]}-1}\right]_{gg} \qquad (7)$$

i) $T>T_c$, $H\to 0$

Eq. (7) reduces to the following,

$$\frac{1}{2}=2k_BT\frac{\sigma_g}{e_g}\left[\frac{1}{1-2\lambda\dfrac{\boldsymbol{\sigma}}{\boldsymbol{e}}\cdot\boldsymbol{J}}\right]_{gg}. \qquad (8)$$

If one notices from this equation that the value of σ_g/e_g is the same when a weak uniform field is applied to the system, we obtain an expression for $\Phi(\boldsymbol{q})$, the Fourier transform of Φ_g, which is identical with Eq. (3), as one would expect. For $\lambda=0$ and near the Curie point, $\chi=\gamma^2/4k_B(T-T_c)$.

ii) $T<T_c$, $H\to 0$

Substituting Eq. (4) in Eq. (7) and extracting the terms linear in H, we obtain the following expression for $\Phi(\boldsymbol{q})$ after some calculation,

$$\Phi(\boldsymbol{q})=\frac{2k_BT\dfrac{\sigma^2}{N}\sum_{\boldsymbol{k}}F_\lambda(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{q})}{\dfrac{1}{2}-\dfrac{4\sigma^2}{N}\sum_{\boldsymbol{k}}F_\lambda(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{q})J_\lambda(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})} \qquad (9)$$

where

$$F_\lambda(\boldsymbol{k},\boldsymbol{k}')\equiv\frac{1}{2\lambda\sigma J(\boldsymbol{k},\boldsymbol{k}')}\left[\frac{1}{e^{2\beta\sigma J_\lambda(0,k)}-1}-\frac{1}{e^{2\beta\sigma J_\lambda(0,k')}-1}\right], \qquad (10)$$
$$J_\lambda(\boldsymbol{k},\boldsymbol{k}')\equiv J(\boldsymbol{k})-\lambda J(\boldsymbol{k}').$$

For $\lambda=0$, and near the Curie point, $\Phi(\boldsymbol{q})$ becomes similar to Eq. (3) except that χ is replaced by $\gamma^2/8k_B(T_c-T)$. This agrees with the results already known.[3),4)] However, $\Phi(\boldsymbol{q})$ shows a quite different behavior for $\lambda=1$. For small q, the preliminary examination gives

$$\Phi(\boldsymbol{q})=\frac{1}{q+\delta}\frac{k_BT}{2J_0a^2q}, \qquad (11)$$

where δ is a positive constant vanishing at the Curie point, and we put $J(0,\boldsymbol{q})\approx J_0a^2q^2$. This peculiar behavior is associated with the divergence of the parallel susceptibility in this approximation below the Curie point, which is given by

$$\chi=\gamma^2\frac{16J_0a^2\sigma^2}{k_BT\delta}\frac{1}{N}\sum_{\boldsymbol{k}}F_1(\boldsymbol{k},\boldsymbol{k}). \qquad (12)$$

However, this point needs further investigation. We would like to express our deep appreciation of many helpful discussions with Prof. T. Matsubara on this subject.

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On the Green Function of the Spin System. II

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The Green function method of Bogolyubov and Tjablikov quoted in the foregoing article has been successful only for the case with the spin 1/2. We have extended their method to the cases with an arbitrary magnitude of spin. For simplicity, we describe here a case with spin 1 only.

First, note that the following identities for the spin operators at every lattice site are satisfied:

$$(s_m^z-1)s_m^z(s_m^z+1)=0 \text{ or } s_m^{z3}=s_m^z, \quad (1)$$

$$s_m^- s_m^+=2-s_m^z-s_m^{z2}. \quad (2)$$

From these we can deduce the following:

$$s_m^{-2}s_m^{+2}=2(s_m^{z2}-s_m^z). \quad (3)$$

For the Hamiltonian

$$H=-\gamma H\sum_m s_m^z-\sum_{mf}J_{mf}\Big[\frac{1}{2}s_m^+ s_f^- +\frac{1}{2}s_m^- s_f^+ +s_m^z s_f^z\Big], \quad (4)$$

the equation of motion is

$$i(d/dt)s_m^+=\gamma Hs_m^+-2\sum_f J_{mf}[s_m^z s_f^+ -s_f^z s_m^+]. \quad (5)$$

Introducing the two kinds of Green's functions

$$G_{mg}(t-t')=\langle\langle s_m^+(t),\ s_g^-(t')\rangle\rangle, \quad (6)$$

$$G_{mg,h}(t-t')=\langle\langle s_m^+(t)s_g^+(t),\ s_h^{-2}(t')\rangle\rangle, \quad (7)$$

we have the following equations for them,

$$i(d/dt)G_{mg}=2\sigma\delta_{mg}\delta(t-t')+\gamma HG_{mg} -2\sum_f J_{mf}[\langle\langle s_m^z s_f^+,\ s_g^-\rangle\rangle-\langle\langle s_f^z s_m^+,\ s_g^-\rangle\rangle], \quad (8)$$

$$i(d/dt)G_{mgh}=\langle[s_m^+ s_g^+,\ s_h^{-2}]\rangle\delta(t-t') +2\gamma HG_{mgh}-2\sum_f J_{mf}[\langle\langle s_m^z s_f^+ s_g,\ s_h^{-2}\rangle\rangle -\langle\langle s_f^z s_m^+ s_g^+,\ s_h^{-2}\rangle\rangle]-2\sum_f J_{fg}[\langle\langle s_m^+ s_g^z s_f^+,\ s_h^{-2}\rangle\rangle -\langle\langle s_m^+ s_f^z s_g^+,\ s_h^{-2}\rangle\rangle]. \quad (9)$$

The Green function appearing on the right-hand side of these equations can be approximated as follows:

$$\langle\langle s_m^z s_f^+,\ s_g^-\rangle\rangle\simeq\sigma G_{fg}, \quad (10)$$

$$\langle\langle s_m^z s_f^+ s_g^+,\ s_h^{-2}\rangle\rangle\simeq\sigma G_{fg,h}. \quad (11)$$

Thus we obtain two sets of equations for G_{mg} and $G_{mg,h}$.

These equations are solved by introducing the Fourier transforms of the Green functions,

$$G_{mg}(t)=(1/N)\sum_k\int_{-\infty}^{\infty}dEG_k(E)e^{-iEt}e^{i(m-g,k)}, \quad (12)$$

$$G_{mg,h}(t)=(1/N^2)\sum_{k,q}^{\infty}\int_{-\infty}^{\infty}dEG_{kq}(E)e^{-iEt} \times e^{i(m-h,k)+i(g-h,q)}. \quad (13)$$

Thus we obtain:

$$G_k(E)=(\sigma/\pi)/[E-\gamma H-2\sigma J(0,\boldsymbol{k})], \tag{14}$$

$$G_{kq}(E)=(\varphi_{kq}/\pi)/[E-2\gamma H -2\sigma J(0,\boldsymbol{k})-2\sigma J(0,\boldsymbol{q})], \tag{15}$$

where

$$\langle[s_m^+ s_g^+,\ s_h^{-2}]\rangle=(1/N^2)\sum_{k,q}\varphi_{kq} \times e^{i(m-h,k)+i(g-h,q)}. \tag{16}$$

Now, the spectral theorem is applied to G_h and G_{kq} to yield the following relations:

$$\langle s_m^- s_m^+\rangle=\frac{2\sigma}{N}\sum_k\frac{1}{e^{\beta[\gamma H+2\sigma J(0,k)]}-1}, \tag{17}$$

$$\langle s_m^{-2} s_m^{+2}\rangle=\frac{1}{N^2}\sum_{kq}\frac{\varphi_{kq}}{e^{\beta[\gamma H+2\sigma J(0,k)+2\sigma J(0,q)]}-1}. \tag{18}$$

These can be combined with Eqs. (2) and (3) to eliminate $\langle s_m^{z2}\rangle$. Thus we obtain

$$\frac{1-\sigma}{\sigma}=\frac{1}{N}\sum_k\frac{1}{e^{\beta[\gamma H+2\sigma J(0,k)]}-1} + \frac{1}{N^2}\sum_{k,q}\frac{\varphi_{k,q}/4\sigma}{e^{\beta[2\gamma H+2\sigma J(0,k)+2\sigma J(0,q)]}-1}. \tag{19}$$

On the other hand, the φ_{kq} defined by Eq. (16) satisfies

$$\varphi_{kq}=4\sigma+4\sum_{m\neq g}\langle s_m^+ s_g^- s_g^\circ\rangle[e^{-i(m-g,k)}+e^{-i(m-g,q)}]. \tag{20}$$

With the same approximation with Eq. (10), we have

$$\langle s_m^+ s_g^- s_g^\circ\rangle\simeq\sigma\langle s_m^+ s_g^-\rangle=\sigma\langle s_g^- s_m^+\rangle \quad (m\neq g). \tag{21}$$

For $\langle s_g^- s_m^+\rangle$ there is a relation similar to Eq. (17) deduced by applying the spectral theorem to G_k, which is

$$\langle s_g^- s_m^+\rangle=\frac{2\sigma}{N}\sum_k\frac{1}{e^{\beta[\gamma H+2\sigma J(0,k)]}-1}e^{i(m-g,k)}. \tag{22}$$

Thus, combining Eqs. (19), (20), (21) and (22), we obtain the following equation which determines σ at any temperature.

$$\frac{1-\sigma}{\sigma}=\frac{1}{N}\sum_k\frac{1}{e^{\beta[\gamma H+2\sigma J(0,k)]}-1} +\left[1-\frac{4\sigma}{N}\sum_p\frac{1}{e^{\beta[\gamma H+2\sigma J(0,p)}-1^]}\right] \times\frac{1}{N^2}\sum_{kq}\frac{1}{e^{\beta[2\gamma H+2\sigma J(0,k)+2\sigma J(0,q)]}-1} +\frac{4\sigma}{N^2}\sum_{kq}\frac{1}{e^{\beta[\gamma H+2\sigma J(0,k)]}-1} \times\frac{1}{e^{\beta[2\gamma H+2\sigma J(0,k)+2\sigma J(0,q)]}-1} \tag{23}$$

For sufficiently low temperatures, $\langle s_m^{-2} s_m^{+2}\rangle\cong 0$, since this is proportional to the probability that the spin on the site m lies in the opposite direction from most of other spins. Thus the right-hand side of Eq. (23) drops out except the first term. This means that the spin wave picture holds at low temperatures. Near the Curie point, the spontaneous magnetization can be shown to be proportional to $(T_c-T)^{1/2}$. Therefore, Eq. (23) can be looked upon as giving an interpolation between high and low temperatures as it was in the spin 1/2 case. In particular, if we replace $J(0,\boldsymbol{k})$ by $J(0)$ (which is equivalent to choosing $\lambda=0$ in our previous article), Eq. (23) reduces to the ordinary Weiss approximation.

The authors wish to express their

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Remarks on the Mass Formula in the Sakata Model

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In 1956, Sakata[1] proposed the model in which all particles belonging to baryon and meson families are interpreted as the entities constructed from proton p, neutron n and Λ-particle Λ and their antiparticles. He compared the role of Λ-particle in his model with that of neutron's in the early stage of theory of nuclear structure and suggested a way based on the correspondence to unclear theory.

In the early stage of the theory of nuclear structure, Heisenberg[2] and Weizsäcker[3] investigated the regularities of masses of nucleus and found the mass formula and analysed the stability of *all possible nucleus* based on the mass formula. Correspondingly, the author[4] investigated the regularities of masses of baryons and mesons and found the mass formula and analysed the stability of *the baryons and mesons which were experimentally undiscovered but derived from the Sakata model.* The conclusions from this formula are the following:[4]

Table. Sakata's Model.

Particle	Composition (lowest configuration)	Mass of ground state (with unit of $\alpha^{-1}m_e \fallingdotseq 137\, m_e$)
N	(N)	$13\frac{1}{2}$
Λ	(Λ)	16
π	$(N\bar{N})$	2
K	$(N\bar{\Lambda})$	7
Σ	$(N\Lambda\bar{N})$	17
Ξ	$(\Lambda\Lambda\bar{N})$	$18\frac{1}{2}\sim 19$
$F(0, \frac{3}{2})$	$(NN\bar{N})$	$17\frac{1}{2}$
$F(0\ \frac{1}{2})$ (angular momentum $\neq \frac{1}{2}^+$)	$\alpha(NN\bar{N})+(1-\alpha^2)^{1/2}(N\Lambda\bar{\Lambda})$	
$B(0, 0)$	$\alpha'(N\bar{N})+(1-\alpha'^2)^{1/2}(\Lambda\bar{\Lambda})$	
$F(+1, 1 \text{ or } 0)$	$(NN\bar{\Lambda})$	?
undiscovered $B(-2, 1 \text{ or } 0)$	$(\Lambda\Lambda\bar{N}\bar{N})$	
$F(-3, 1 \text{ or } 0)$	$(\Lambda\Lambda\Lambda\bar{N}N)$	

* $F(b, a)$, $B(b, a)$ are the baryon and meson having $I=a$ and $S(\text{strangeness})=b$ respectively.

** α, α' are unknown except $0\leq\alpha^2, \alpha'^2\leq 1$.

*** $F(0, \frac{3}{2})$ is the well-known $(\frac{3}{2}, \frac{3}{2})$ resonance state at 190 Mev. laboratory pion kinetic energy in π–N scattering. We take the total energy in the center of mass system at resonance as the mass of $F(0, \frac{3}{2})$.

1) Unobserved particles derived from the Sakata model are almost unstable against the strong interaction.

2) Some "new" stable particles against the strong interaction can be predicted.

3) Some lower levels in unstable particles exist and appear as resonance state in reaction processes.

The detailed analyses[5] were given in connection with full symmetry theory[6] among p, n and Λ.

In this note, we would like to make some further remarks on the mass formula.

i) In the mass formula proposed there,[4] namely,

$$\begin{aligned} M=&m_N(n_N+n_{\bar{N}})+m_\Lambda(n_\Lambda+n_{\bar{\Lambda}}) \\ &-V_{N\bar{N}}n_{N\bar{N}}-V_{NN}(n_{NN}+n_{\bar{N}\bar{N}}) \\ &-V_{N\bar{\Lambda}}(n_{N\bar{\Lambda}}+n_{\Lambda\bar{N}})-V_{N\Lambda}(n_{N\Lambda}+n_{\bar{N}\bar{\Lambda}}) \\ &-V_{\Lambda\bar{\Lambda}}n_{\Lambda\bar{\Lambda}}-V_{\Lambda\Lambda}(n_{\Lambda\Lambda}+n_{\bar{\Lambda}\bar{\Lambda}}), \end{aligned} \tag{1}$$

where $n_N, \cdots n_\Lambda$, $n_{N\bar{N}}, \cdots$, $n_{\bar{\Lambda}\bar{\Lambda}}$ are the number of N, $\cdots$, Λ, $N\bar{N}$ pair, $\cdots$, $\bar{\Lambda}\bar{\Lambda}$ pair in the composite particle (lowest configuration; see Table), we assumed,

$$\begin{aligned} &V_{N\bar{N}}=-V_{NN}, \\ &N_{N\bar{\Lambda}}=-V_{N\Lambda}, \\ &V_{N\bar{\Lambda}}=V_{N\bar{N}}-\Delta V \\ &\qquad \text{with } \Delta V/V_{N\bar{N}}\simeq 0.10, \\ &V_{\Lambda\Lambda}=V_{N\Lambda}-\Delta V' \\ &\qquad \text{with } \Delta V'/V_{N\Lambda}\simeq 0.14, \end{aligned} \tag{2}$$

and

$$V_{N\bar{N}}=2m_N-m_\pi(\simeq 2m_N).$$

However, releasing the above restrictions, we could determine all the coefficients to fit with the observed mass values, and found that the newly determined coefficients have rather remarkable fits with the previous ones in very small errors. It shows that the assumed regularites may have some significient physical background.

ii) It may also be suggestive that the mass formula can be expressed in the following from,

$$\begin{aligned} M\approx& m(n_B-n_{\bar{B}})^2+\Delta m\{2(n_\Lambda+n_{\bar{\Lambda}}) \\ &-(n_B-n_{\bar{B}})(n_\Lambda-n_{\bar{\Lambda}})\} \\ &+\text{small term}, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &n_B=n_N+n_\Lambda, \\ &m\simeq m_N, \\ &\Delta m\simeq m_\Lambda-m_N. \end{aligned}$$

The main contribution of the mass formula comes from the first term, which depends only on the baryon number and is independent of the lowest configuration, while the second term brings the nucleon Λ-particle difference and asymmetry for the sign of strangeness quantum number with fixed $n_B-n_{\bar{B}}$. *A noticeable point is that the main part of the mass formula is due to the baryonic charge and involves a constant* ($m\simeq m_N$) *only.** This suggests that *the masses of the composite particles* (*for instance, the vanishing pion mass*) *are due to the same origin of the mass of fundamental baryon*, which must be considered in future and seems to be

* In the case of nucleus, the main part of the mass defect shows the saturation of the nuclear force which is the fundamental problem in the theory of nuclear structure.

connected with the axial vector current conservation in the weak interaction as pointed out by Nambu.[7]

The mass formula has many defects, especially in the prediction of the intrinsic augular momentum of the particle. The situation is similar to the case of the mass formula for nucleus. In order to correct this defect, we must introduce some effect besides the Pauli principle into the formula, on an analogy of the introduction of pairing energy, spin dependence and spin-orbit coupling in nuclear theory. The work along this line will be reported in a later issue of this journal.

The auther is greatly indebted to Prof. Y. Katayama for his kind interest and valuable discussions. He also expresses his gratitude to Prof. S. Sakata for his guidance and valuable suggestions.

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Expansion of the Linear Boltzmann Operator

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1. *The "CD expansion"*. Suppose the distribution function $P(y, t)$ of the thermal fluctuations of a variable y to satisfy the partial differential equation

$$\partial P/\partial t = B' P, \qquad (1)$$

where B' is a linear operator (linearized Boltzmann or other) on the y variable. We take B' to be given in the form of the Kramers expansion

$$B' = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial y}\right)^n \alpha_n(y), \qquad (2)$$

which may usually be derived from integral-operator or other forms. It is known that the successive terms of this expansion cannot readily be interpreted as corresponding to successive kinetic-theoretical corrections; for example, the Fokker-Planck equation cannot be obtained by simply taking the lowest term or terms of (2) intact, as they stand. The purpose of this paper is to present a rearrangement of this expansion for which it *is* possible to claim such an interpretation. Some discussion, and an application to the calculation of noise, spectra will

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also be given.

Let us assume the equilibrium distribution to have the Maxwell-Boltzmann form $(2\pi)^{-1/2}\exp(-y^2/2)$, and for convenience transform according to the equations:

$$B=e^{+y^2/4}B'e^{-y^2/4},\ \varphi=e^{+y^2/4}P, \quad (3)$$

by which Eq. (1) is transformed into

$$\partial\varphi/\partial t=B\varphi. \quad (4)$$

Then the proposed expansion—we shall call it the "*CD* expansion"—is

$$B=\sum_{m=1}^{\infty}\sum_{l=1}^{m}A_{lm}C^{m-l+1}D^l, \quad (5)$$

where C and D are the creation and destruction operators on Hermite functions of y, and the A_{lm} are constant coefficients:

$$A_{lm}=\sum_{k=l}^{m}\frac{1}{(m-k+1)!}\binom{k}{l}\alpha_{m-k+l,k}, \quad (6)$$

which can be shown to follow from the form (2).

2. *Use of the CD expansion in obtaining approximations to B.* In the usual derivation of the Fokker-Planck equation from the Kramers expansion, using the random walk model, one takes for α_1 and α_2 their limiting values as $y\to 0$, these being proportional to y and a constant, respectively. All other α_k are put equal to zero. But the random walk model furnishes no guide for improving this approximation. The *CD* expansion will yield such a guide. One can see this by the following argument.

Further terms will become important in addition to the Fokker-Planck terms as the distribution P deviates increasingly from its equilibrium form. Thus one needs an appropriate measure for this deviation. The following measure for this purpose has been proposed by the author:[1]

$$\chi_0^2=(2\pi)^{1/2}(\varphi-(2\pi)^{-1/2}e^{-y^2/4}, \varphi-(2\pi)^{-1/2}e^{-y^2/4}). \quad (7)$$

The expansion (5) gives an expansion of the rate of decrease of χ_0^2, as follows:

$$\frac{d\chi_0^2}{dt}=2\sqrt{2\pi}\sum_{m=1}^{\infty}\left[\sum_{r,s=0}^{\infty}a_r(b_m)_{rs}a_s\right], \quad (8)$$

where

$$b_m=\sum_{l=1}^{m}A_{lm}C^{m-l+1}D^l, \quad (9)$$

which will be recognized as the summand of the sum over m in Eq. (5); the a_n are the expansion coefficients of φ in Hermite functions; and $(b_m)_{rs}$ is the matrix element of b_m between the r th and s th normalized Hermite functions.

Eq. (8) makes possible a clear distinguishing of order of magnitude in the trend to equilibrium, since the physical parameters on which B depends affect the convergence of (8) entirely through the constants A_{lm}, the matrix elements of C and D being dimensionless. A rapid decrease in the successive terms of the m-summation of Eq. (8) is to be expected under conditions which are commonly satisfied in physical kinetic processes.

In particular, the expansion of B becomes especially simple when the A_{lm} are homogeneous expressions of m th order (or of order m plus a constant) in a small physical parameter characteristic of the system under consideration. In such a case

the terms in the sum over m in the B expansion, and in the related deviation from equilibrium (8), are just the successive terms of a power series in terms of this parameter. Two systems are in fact now known which have this property: (1) The Rayleigh model,[1] where the A_{lm} are proportional to $[\mu(1+\mu)]^{m/2}$, where μ is the (presumably very small) ratio of the mass of a molecule of the medium to that of the colloidal particle whose velocity distribution is being studied. (2) The diode (nonlinear ohmic resistance plus capacitance) of Alkemade:[2] Here van Kampen[3] has found that the A_{lm} are proportional to $\varepsilon^{(m+1)/2}$ where $\varepsilon=e^2/kTC$, e being the electronic charge and C the capacitance of the diode.

The fact that in both of these systems, so different physically, the CD expansion takes the form of a power series, encourages the speculation that this expansion has fundamental validity as a power-series expansion of the linear Boltzmann operator. Further study, toward a more general understanding, is obviously indicated.

It is evident that the CD expansion does give the Fokker-Planck operator in lowest order. Taking only the leading term in Eq. (5), we have

$$B \cong A_{11}CD. \qquad (10)$$

This is precisely the Fokker-Planck operator with linear friction and constant force spectrum, as can be inverting the transformation (3).

3. *Corrections to the spectrum due to perturbation of the Fokker-Planck operator.* The operater $A_{11}CD$ gives, of course, the well-known $1/(A_{11}^2+\omega^2)$ spectrum for the random variable $y(t)$. Using the perturbation method of Davies,[4] the first order correction to this spectrum is

$$4(2\pi)^{-1/2}\mathrm{Re}\frac{(h_1, B_1h_1)}{(-A_{11}+i\omega)^2}, \qquad (11)$$

where B_1 is the perturbation $B_1=B-A_{11}CD$. But the matrix element in (11) vanishes identically for all the higher terms of the CD expansion. We therefore find in general that corrections to the Fokker-Planck spectrum are of second order in the perturbation.

A more complete account of this work will be published in the Proceedings of the 2nd International Symposium of Rarefied Gas Dynamics, to appear in 1961.

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Vibrations of Spherical Nuclei

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Several authors[1] applied the Sawada method[2] in describing the collective vibrational states in spherical nuclei. Under the random phase approximation, it has been shown that the

energy of the collective eigenmode is obtained by solving a very simple dispersion relation. In all of these works, however, the dispersion relation is derived by treating nuclei to consist only of one kind of nucleons. Needless to say, actual nuclei consist of protons and neutrons, and, except for singly closed-shell nuclei[3] it is expected that the polarization of protons induces that of neutrons and vise versa. Therefore it will be quite natural to assume that there exists a long-range correlation between protons, and neutrons, of essentially the same type as is assumed among protons and among neutrons. Then the dispersion relation would have to be modified accordingly and such a modified dispersion relation is given in the present note.

The extended Q-Q interaction is defined as

$$H_{Q-Q}=-\sum_{i,j}(V_0+V_1(\tau_i\cdot\tau_j))r_i^2 r_j^2 \times\sum_{\mu}(-)^{\mu}Y_{2\mu}(i)\,Y_{2-\mu}(j). \quad (1)$$

Then the relevant dispersion relation is given as

$$1-G_0(X_p+X_n)+(G_0^2-G_1^2)X_pX_n=0, \quad (2)$$

where

$$G_0=(V_0+V_1) \text{ and } G_1=(V_0-V_1), \quad (3)$$

while X_p is given by

$$X_p=\sum_{(a,b)}2E_{ab}g_{ab}^2/(E_{ab}^2-\omega^2). \quad (4)$$

$E_{ab}=E_a+E_b$, where E_a is the energy of a quasi-proton[3),4)] specified by a set a of quantum numbers. By using the energy gap Δ and the Fermi energy λ, E_a is written as $E_a=\sqrt{(\varepsilon_a-\lambda)^2+\Delta^2}$, ε_a being the energy of a free particle in the state a. E_b is written similarly and so E_{ab} is the energy of a quasi-proton pair.

In (4) g_{ab} is the double-bar part of the matrix element of the operator r^2Y_2 between the single-proton states a and b, and is explicitly given by

$$g_{ab}=\langle a\|r^2\,Y_2\|b\rangle(u_av_b+v_au_b) \times\{5(1+\delta_{ab})\}^{-1/2}, \quad (5)$$

where u_a, v_a, etc., describe the fractional occupation of the relevant single proton state.[3] X_n of (2) is given in a similar way as X_p is given by (4).

It is interesting to consider in (2) the case with $G_1=0$. As is seen from the definitions (3) and (1), it is easy to see that this corresponds to the case where there is no interaction between proton pairs and neutron pairs. (Because of the V_1 term in (1) there still remains interaction between protons and neutrons, but it is easy to see that such terms have non-vanishing contributions only in odd-odd nuclei.) If $G_1=0$ in (2), the latter is reduced to

$$(1-G_0X_p)(1-G_0X_n)=0, \quad (6)$$

which means that either $(1-G_0X_p)=0$ or $(1-G_0X_n)=0$. Clearly, these are the dispersion relations obtained previously for one kind of nucleons, and thus (6) means that there exist two independent collective states corresponding respectively to the excitation of proton pairs and neutron pairs. If G_1 is increased in (2), which means that a stronger proton-neutron correlation is considered, it is easy to see

that the smallest eigenvalue of this equation gets smaller than any of the solutions of (6). This means that the above two independent collective states begin to be coupled and thus one of the eigenvalues is suppressed (while the other is pushed up and if G_1 is sufficiently large, it will merge into the continuum of the quasi-pair states). Thus it is seen that the new dispersion relation (2) is of a quite reasonable form.

The eigenfunction of the collective state is given by

$$|\Psi_{coll}\rangle=\frac{1}{N}\Big[C_p\sum_{(a,b)}\Big(\frac{g_{ab}}{E_{ab}-\omega}A_{ab}^{+} - \frac{g_{ab}}{E_{ab}+\omega}A_{ab}\Big)+C_n\sum_{(\alpha,\beta)}\Big(\frac{g_{\alpha\beta}}{E_{\alpha\beta}-\omega}A_{\alpha\beta}^{+} - \frac{g_{\alpha\beta}}{E_{\alpha\beta}+\omega}A_{\alpha\beta}\Big)\Big]|\Psi_0\rangle, \quad (7)$$

where the quantities with Roman suffices refer to protons and the Greek ones to neutrons. A_{ab}^{+} is the creation operator of a quasi-proton pair with angular momentum of two units, while A_{ab} is the corresponding annihilation operator. $C_p=\sqrt{G_1X_n(1-G_0X_n)}$ and $C_n=\sqrt{G_1X_p(1-G_0X_p)}$ are constants and using them and $X_p'=dX_p/d\omega$ and $C_n'=dX_n/d\omega$, the normalization constant N is given as $N=\sqrt{C_p^2X_p'+C_n^2X_n'}$. $|\Psi_0\rangle$ is the normalized wave function of the true ground state.

Finally, the reduced matrix element of the $E2$ transition is given, by defining the effective charges[3] e_p and e_n for protons and neutrons, as

$$\langle\Psi_0\|\mathfrak{m}(E2)\|\Psi_{coll}\rangle = 1/N\cdot(e_pC_pX_p+e_nC_nX_n). \quad (8)$$

A more detailed account of this note will be published elsewhere together with the numerical solutions of (2).

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Evolution of a Star with Intermediate Mass after Hydrogen Burning

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The evolution of a star of $4M_\odot$, after the formation of a dehydrogenized core and a hydrogen burning shell, is calculated up to the stage when helium has almost burnt out in its central region. The mass $4M_\odot$ is chosen to get a full interpretation of the whole H-R diagram together with the results which have been obtained for a massive star of $15.6M_\odot$[1),2)] and a less massive star of $1.2M$.[3),4)]

The original chemical composition of the star is taken to be

$$X_e=0.61,\ Y_e=0.37,\ Z_e=0.02 \text{ and } X_{\text{CNO}}=0.008, \quad (1)$$

where X, Y and Z denote concentrations by weight of hydrogen, helium and metal respectively. According to the results by Henyey et al. on the previous phase of hydrogen burning,[5] the distribution of the hydrogen content after the formation of a helium core is approximately expressed as

$$X(q)=3.81(q-0.07), \quad (0.23\geq q\geq 0.07), \quad (2)$$

where q denotes the mass fraction $M(r)/M$. There will be some un-

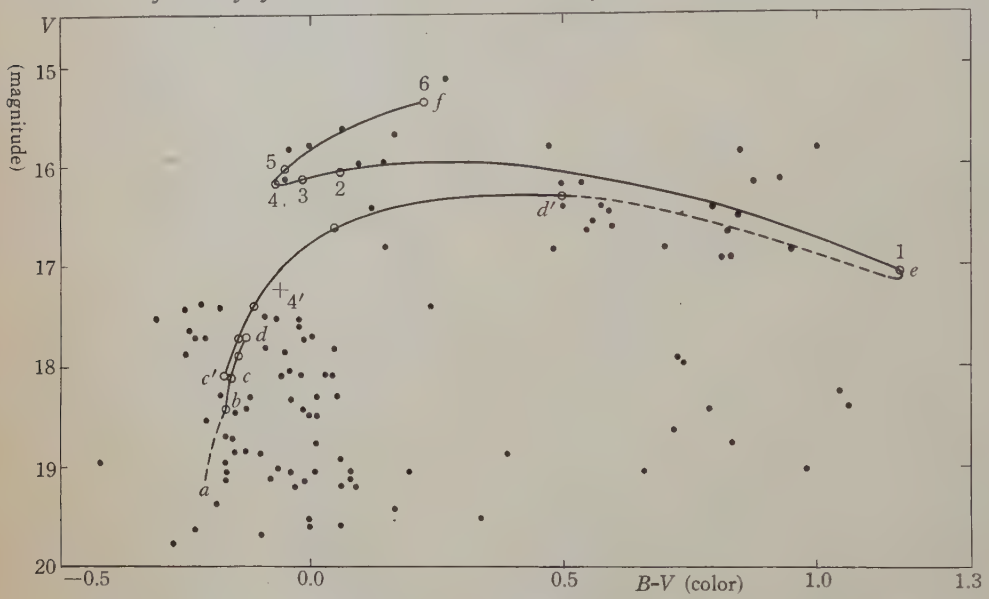


Fig. 1. Evolutionary track in the H-R diagram. Solid lines with open circles and broken lines show the calculated and estimated results, respectively. Dots represent stars in NGC 458 observed by Arp.

Table. I. Structures in the phase of helium burning.

Stage	1	2	3	4	5	6	4′
Y_c	1.0	0.7	0.5	0.3	0.1	≈0.0	0.3
q_1	0.133	0.192	0.216	0.232	0.244	0.250	0.232
X_{1e}	0.24	0.47	0.56	X_e	X_e	X_e	X_e
$\log L/L_\odot$	2.98	3.30	3.40	3.46	3.50	3.52	3.08
$\log R/R_\odot$	1.83	1.22	1.12	1.07	1.18	1.48	0.87
$\log T_e$	3.59	3.98	4.05	4.09	4.08	3.90	4.10
$\log T_c$	8.10	8.13	8.16	8.20	8.25	8.36	8.20
$\log \rho_c$	4.15	3.91	3.84	3.83	3.89	4.20	3.82
$\log q_d$*	−1.40	−1.26	−1.22	−1.20	−1.19	−1.18	−1.19
$L_{3\alpha}/L$	0.048	0.085	0.11	0.13	0.16	0.16	0.35

* q_d is the mass fraction of the convective core.

certainty in the above form of $X(q)$, especially in the mass fraction of the helium core, but the results are not sensitive to it except for the very early phase in which the isothermal core grows. Calculations of an evolutional sequence of stellar structures, which start with the above $X(q)$, are made using the Kramers' opacity and that by electron scattering along the general line of our previous works.

In an early phase the helium core grows and contracts, but the release of the gravitational energy is so small that the core remains nearly isothermal until the hydrogen burning shell moves up to $q_1=0.120$. The difference between the central temperature T_c and that of the hydrogen burning shell T_1 is found to be 10 percent at most. The time scale of this phase is 2.1×10^6 years, during which the star stays near the main sequence, as shown by b to c in Fig. 1.

After q_1 exceeds 0.120, there exists no solution with an isothermal core. The gravitational contraction of the core proceeds rapidly and T_c/T_1 increases until helium begins to burn in the central region. This phase is calculated in two cases, $q_1=0.120$ and $q_1=0.132$, under the assumption that q_1, the mass fraction of the core, is constant and that the release of the gravitational energy is uniform throughout the core. It is found that the envelope expands rapidly, and the star moves to the red giant region in a time as short as 1.1×10^6 years, as shown in Fig. 1 by c to d for $q_1=0.120$ and by c' to d' for $q_1=0.132$. The actual evolutional path lies between these two curves. In this phase the degree of electron degeneracy at the center increases slowly, but its effect is not important since ψ_c reaches only to $+0.6$.

In the helium burning phase, the core expands and the envelope shrinks slowly as Y_c, the helium content in the central region, decreases due to the conversion of helium into carbon, oxygen and neon. As Y_c becomes less than about 0.3, the envelope begins to expand again. The numerical results for the successive six stages are given in Table I and the track is shown by e to f in Fig. 1. The time scale of this phase is 2.1×10^7 years and the star spends half of this time as A-type giant. To know the dependence of the results on the original chemical composition, a calculation is made for the same values of q_1 and Y_c as the stage 4 but for a different composition, $X_e=0.90$, $Y_e=0.10$, $Z_e=0.003$ and $X_{CNO}=0.001$, which may correspond to a population II star. The result, which is given in the last column, 4′, of Table I and shown by a cross in Fig. 1, shows that the effective temperature of the turning point of the evolutional track is not much sensitive to the composition if both the helium and metal contents are reduced.

The ratio of the abundances by weight of the final products C^{12}, O^{16} and Ne^{20} after helium has been exhausted is found to be 0.32 : 0.09 : 0.59 if the reaction rate of $O^{16}(\alpha,\gamma)Ne^{20}$ given by Hayakawa et al.[6] is used

and to be 0.23 : 0.72 : 0.04 if a 100-th value for this rate is assumed to allow for its uncertainty.

The calculated evolutional path is compared in Fig. 1 with the H–R diagram of NGC 458, the blue globular cluster in the Small Magellanic Cloud, observed by Arp.[7] The broken line a to b in Fig. 1 shows the path in the phase of hydrogen burning with a lifetime 5.8×10^7 years, which is estimated by using the homologous transformation from the results by Hoyle[8] for a star of $3.89M_\odot$. A separate group of A-type giants in NGC 458, which lies about one magnitude above the main sequence, can be explained as the stars in the phase of helium burning, since there is a close agreement between the ratio of the number of these stars to that of main sequence stars, 9 : 52, and the ratio of the calculated lifetimes, 12×10^6 years : 58×10^6 years. On the other hand, the yellow giants in NGC 458 will be interpreted as in the later phases since the calculated lifetime in this region is too short to account for the number of these stars. The narrowness of the Hertzsprung gap in NGC 458 will be explained as due to its lower metal content, which will increase the effective temperature of stars with surface convection zones, as seen in the difference between M3 and M67.

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Note on the Integral Representation of Absorptive Part of Vertex Function

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In the previous paper,[1] on the basis of the Lorentz invariance, local commutativity and mass spectral conditions, it was proved that the absorptive part $A(z_1, z_2, \sigma^2)$ of the vertex function $F(z_1, z_2, z_3)$ has the integral representation in the form

$$A(z_1, z_2, \sigma^2) = \int d^3m\varphi(\sigma, m_1, m_2, m_3) \times A^p(z_1, z_2, \sigma^2; m_1, m_2, m_3) \qquad (1)$$

where A^p is that of the lowest order perturbation theory and m_i is the mass of the virtual particle.

In § 2 of the previous paper,[1] it was conjectured that Jost's example[2] has not the integral representation (1). Furthermore, in B) of the notes added in proof, it was stated that this conjecture was correct hence Jost's example contradicts the local commutativity. However, these are incorrect. The analyticity of $F(z_1, z_2, z_3)$ dis-

cussed by Källén and Wightman[3] is sufficient to assure that the absorptive part of $F(z_1, z_2, z_3)$ has the integral representation (1).

From the discussion in § 2 of the previous paper, it is supposed that the weight function φ for Jost's example is singular as a function of σ. In order to see such a situation more clearly, it is worthwhile to note that the absorptive part of

$$f_0(z, r)=[\log(r-z_1)+\log(r-z_2)+\log(r-z_3)-\log(-z_1)-\log(-z_2)-\log(-z_3)][r^2-r(z_1+z_2+z_3)+z_1z_2+z_2z_3+z_3z_1]^{-1} \quad (2)$$

is given by

$$\frac{\pi\theta(r-\sigma^2)\theta(\sigma^2)}{r^2-r(\sigma^2+z_1+z_2)+(z_1+z_2)\sigma^2+z_1z_2}=\pi\int dm_3^2\,\delta'(r-\sigma^2-m_3^2)\times A^p(z_1, z_2, \sigma^2; 0, 0, m_3^2). \quad (3)$$

Here $f_0(z, r)$ is the function that appeared in the integral representation of Källén and Toll,[4] and has not the simple connection with the result of the perturbation theory.

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